Transient Conduction:
The Lumped Capacitance Method

Transient Conduction

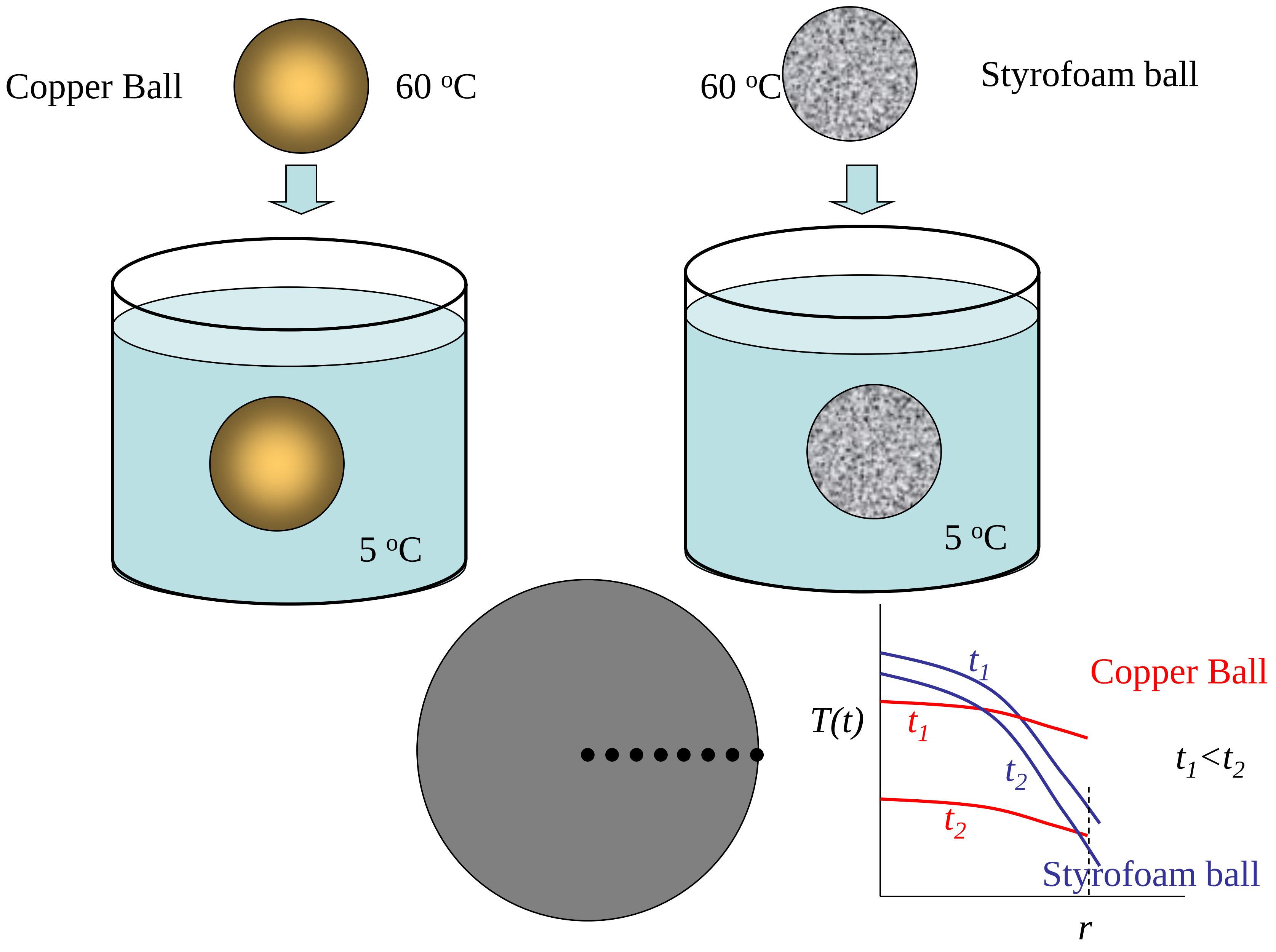
- **A heat transfer process for which the temperature varies with time, as well as location within a solid.**

- **It is initiated whenever a system experiences a change in operating conditions and proceeds until a new steady state (thermal equilibrium) is achieved.**

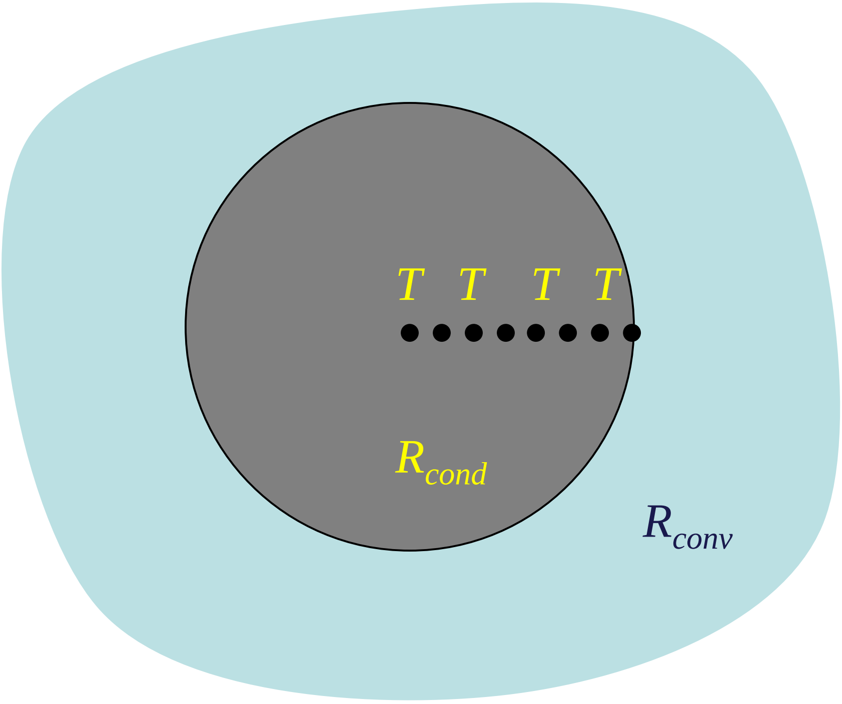
- **It can be induced by changes in:**
  - surface convection conditions (h, T_∞),
  - surface radiation conditions (h_r, T_{sur}),
  - a surface temperature or heat flux, and/or
  - internal energy generation.

- **Solution Techniques**
  - The Lumped Capacitance Method
  - Exact Solutions
  - The Finite-Difference Method

The Lumped Capacitance Method



Lumped body

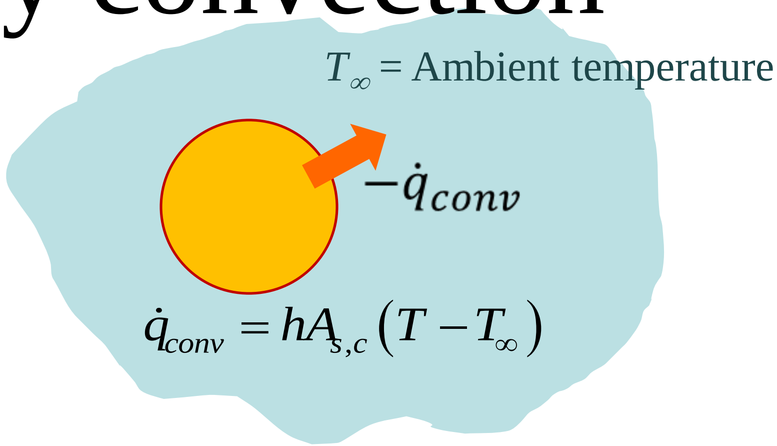


- **Temperature nearly homogeneous everywhere inside the solid**
- **Valid when internal thermal resistance is negligible compared to the external thermal resistance**
  - $\boldsymbol{R_{conv} >> R_{cond}}$

$$\frac{1}{hA} \gg \frac{d}{k_s A}$$

$$\frac{hd}{k_s} \ll 1$$

Transient cooling by convection



$$\frac{dE}{dt} = -\dot{q}_{conv}$$

First Law/
Energy equation

$$\rho \forall c \frac{dT}{dt} = -hA_{s,c}(T - T_\infty)$$

$$\frac{\rho \forall c}{hA_{s,c}} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_{o}^{t} dt$$

$\theta = (T - T_\infty)$

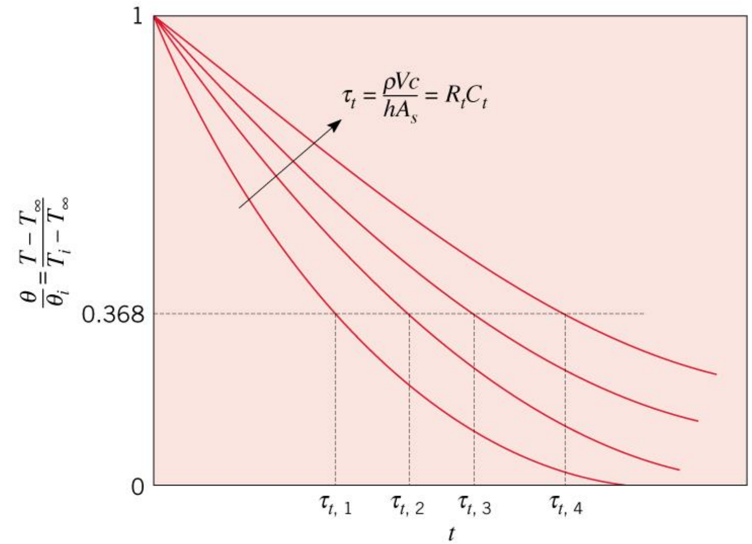
$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-\left(\frac{hA_{s,c}}{\rho \forall c}\right)t\right] = \exp\left[-\frac{t}{\tau_t}\right]$$

The thermal time constant is defined as

$$\tau_t \equiv \left(\frac{1}{hA_{s,c}}\right)(\rho \forall c)$$

Thermal Resistance, R_t

Lumped Thermal Capacitance, C_t



The change in thermal energy storage due to the transient process is

$$\Delta E_{st} \equiv -Q = -\int_{o}^{t} \dot{E}_{out} dt = -hA_{s,c} \int_{o}^{t} \theta dt = -(\rho \forall c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right]$$

Transient cooling by radiation only

Assuming radiation exchange with large surroundings,

$$\rho \forall c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma \left(T^4 - T_{sur}^4 \right)$$

$$\frac{\varepsilon A_{s,r} \sigma}{\rho \forall c} \int_o^t dt = \int_{T_i}^{T} \frac{dT}{T_{sur}^4 - T^4}$$

$$t = \frac{\rho \forall c}{4 \varepsilon A_{s,r} \sigma T_{sur}^3} \left\{ 1n \left| \frac{T_{sur} + T}{T_{sur} - T} \right| - 1n \left| \frac{T_{sur} + T_i}{T_{sur} - T_i} \right| + 2 \left[\tan^{-1} \left(\frac{T}{T_{sur}} \right) - \tan^{-1} \left(\frac{T_i}{T_{sur}} \right) \right] \right\}$$

Result necessitates implicit evaluation of *T*(*t*).

The Biot Number and Validity of The Lumped Capacitance Method

- The Biot Number: The first of many dimensionless parameters to be considered.
  - Definition:

$$Bi \equiv \frac{hL_c}{k}$$

$h \rightarrow$ convection or radiation coefficient

$k \rightarrow$ thermal conductivity of the solid

$L_c \rightarrow$ characteristic length of the solid ($\forall / A_s$ or coordinate associated with maximum spatial temperature difference)

  - Physical Interpretation:

$$Bi = \frac{L_c / kA_s}{1/\mathrm{h}A_s} \sim \frac{R_{cond}}{R_{conv}} \sim \frac{\Delta T_{solid}}{\Delta T_{solid\,/\,fluid}}$$

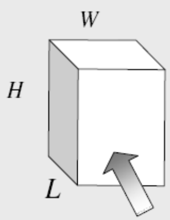
  - Criterion for Applicability of Lumped Capacitance Method:

$$Bi << 1$$

Characteristic Length

$$L_c = \frac{V}{A_s}$$

Plane Wall



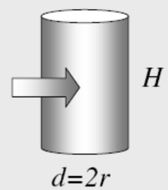
$$A_s = 2HW$$

$$V = LHW$$

$$L_c = \frac{L}{2}$$

$$Bi = \frac{h}{k}\frac{L}{2}$$

Cylinder



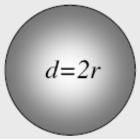
$$A_s = \pi dH = 2\pi rH$$

$$V = \frac{\pi d^2}{4}H = \pi r^2 H$$

$$L_c = \frac{d}{4} = \frac{r}{2}$$

$$Bi = \frac{h}{k}\frac{r}{2}$$

Sphere



$$A_s = \pi d^2 = 4\pi r^2$$

$$V = \frac{\pi d^3}{6} = \frac{4}{3}\pi r^3$$

$$L_c = \frac{d}{6} = \frac{r}{3}$$

$$Bi = \frac{h}{k}\frac{r}{3}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-\left(\frac{hA_{s,c}}{\rho\forall c}\right)t\right]$$

$$\tau_t \equiv \left(\frac{1}{hA_{s,c}}\right)(\rho\forall c) = \frac{\rho c}{k}\frac{k}{h}\frac{L_c}{1} = \frac{L_c^2}{\alpha}\frac{k}{hL_c} = \frac{1}{Fo}\frac{1}{Bi}t$$

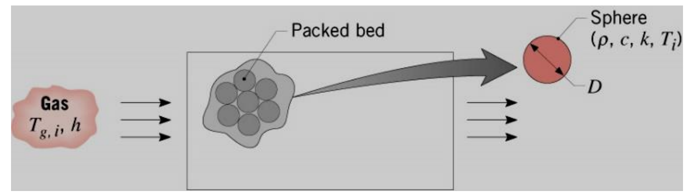
$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-Bi \cdot Fo\right]$$

$$Fo = \frac{t\alpha}{L_c^2}$$ Fourier Number, the nondimensional time

Sample Problem

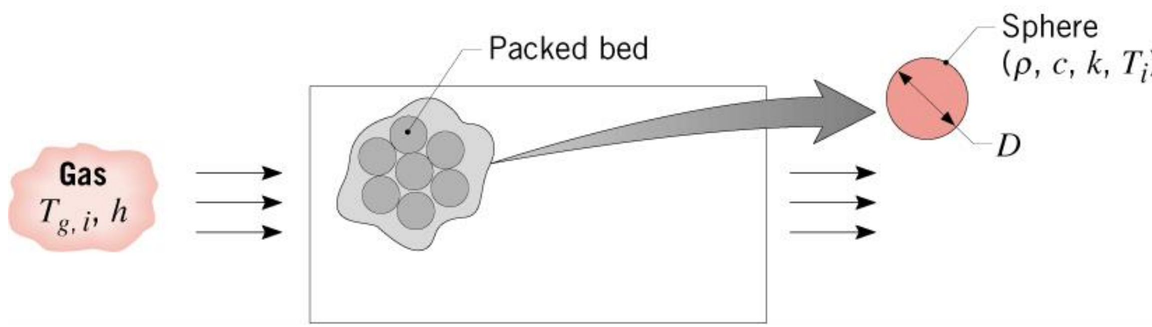
P 5.11 From Incropera & Dewitt, 5th Edition

Thermal energy storage systems commonly involve a *packed bed* of solid spheres, through which a hot gas flows if the system is being charged, or a cold gas if it is being discharged. In a charging process, heat transfer from the hot gas increases thermal energy stored within the colder spheres; during discharge, the stored energy decreases as heat is transferred from the warmer spheres to the cooler gas.



Consider a packed bed of 75-mm-diameter aluminum spheres ($\rho = 2700$ kg/m^3, $c = 950$ J/kg · K, $k = 240$ W/m · K) and a charging process for which gas enters the storage unit at a temperature of $T_{g,i} = 300°C$. If the initial temperature of the spheres is $T_i = 25°C$ and the convection coefficient is $h = 75$ W/m^2 · K, how long does it take a sphere near the inlet of the system to accumulate 90% of the maximum possible thermal energy? What is the corresponding temperature at the center of the sphere? Is there any advantage to using copper instead of aluminum?

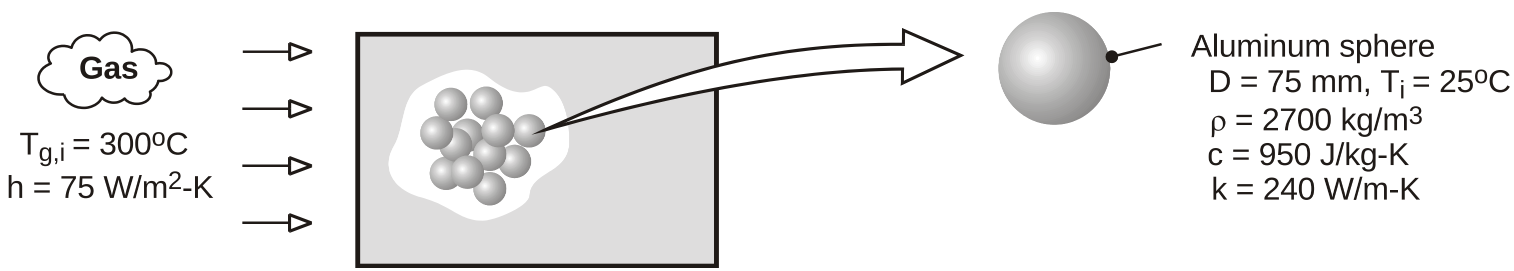
Charging a thermal energy storage system consisting of a packed bed of aluminum spheres.



KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere at inlet to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

Schematic:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute $Bi = h(r_o/3)/k = 75\ W/m^2 \cdot K\ (0.025m)/150\ W/m \cdot K = 0.013 < 0.1$.

Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time.

From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$-\frac{\Delta E_{st}}{\rho c V \theta_i} = 0.90 = 1 - \exp(-t/\tau_t) \Rightarrow \left(1 - \frac{\theta}{\theta_i}\right) = 0.9 \Rightarrow \left(\frac{\theta}{\theta_i}\right) = 0.1$$

$$\tau_t = \rho V c / hA_s = \rho D c / 6h = \frac{2700\ kg/m^3 \times 0.075m \times 950\ J/kg \cdot K}{6 \times 75\ W/m^2 \cdot K} = 427s.$$

$$t = -\tau_t \ln(0.1) = 427s \times 2.30 = 984s$$

For the corresponding metal temperature,

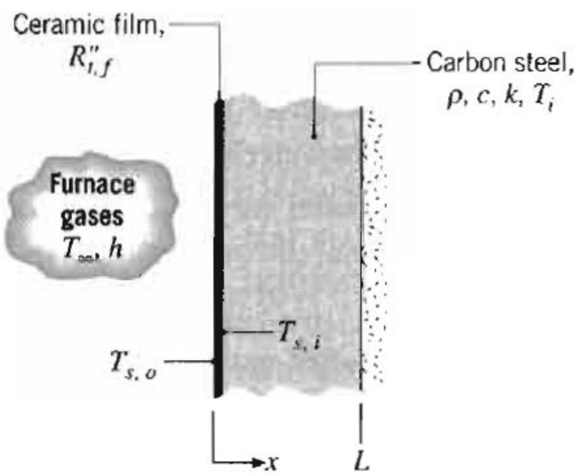
$$\left(\frac{\theta}{\theta_i}\right) = 0.1 \Rightarrow \frac{(T-300)}{(25-300)} = 0.1 \Rightarrow T = 272.5$$

If the product of the density and specific heat of copper is $(\rho c)_{Cu} \approx 8900\ kg/m^3 \times 400\ J/kg \cdot K = 3.56 \times 10^6\ J/m^3 \cdot K$, is there any advantage to using copper spheres of equivalent diameter in lieu of aluminum spheres?

Does the time required for a sphere to reach a prescribed state of thermal energy storage change with increasing distance from the bed inlet? If so, how and why?

Sample Problem

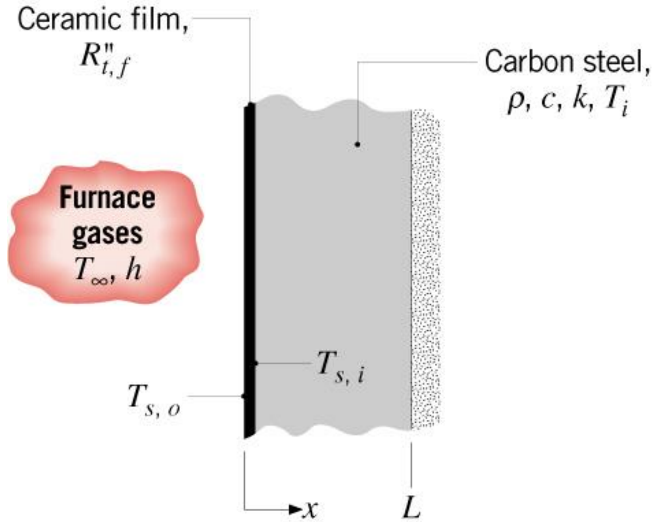
P 5.15 From Incropera & Dewitt, 5th Edition



A plane wall of a furnace is fabricated from plain carbon steel ($k = 60\ W/m \cdot K$, $\rho = 7850\ kg/m^3$, $c = 430\ J/kg \cdot K$) and is of thickness $L = 10$ mm. To protect it from the corrosive effects of the furnace combustion gases, one surface of the wall is coated with a thin ceramic film that, for a unit surface area, has a thermal resistance of $R''_{t,f} = 0.01\ m^2 \cdot K/W$. The opposite surface is well insulated from the surroundings.

At furnace start-up the wall is at an initial temperature of $T_i = 300$ K, and combustion gases at $T_\infty = 1300$ K enter the furnace, providing a convection coefficient of $h = 25\ W/m^2 \cdot K$ at the ceramic film. Assuming the film to have negligible thermal capacitance, how long will it take for the inner surface of the steel to achieve a temperature of $T_{s,i} = 1200$ K? What is the temperature $T_{s,o}$ of the exposed surface of the ceramic film at this time?

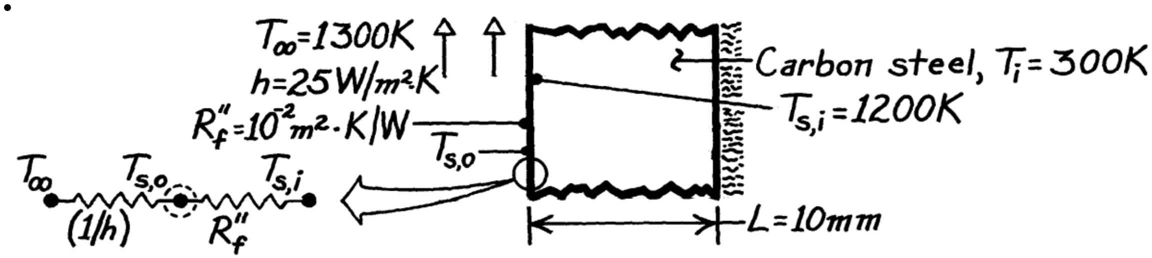
Heating of coated furnace wall during start-up.



KNOWN: Thickness and properties of furnace wall. Thermal resistance of ceramic coating on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of coating surface temperature.

Schematic:



ASSUMPTIONS: (1) Constant properties, (2) Negligible coating thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel: ρ = 7850 kg/m^3, c = 430 J/kg·K, k = 60 W/m·K.

ANALYSIS: Heat transfer to the wall is determined by the total resistance to heat transfer from the gas to the surface of the steel, and not simply by the convection resistance.

Hence, with

$$U = \left(R''_{tot}\right)^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \mathrm{W/m^2\cdot K}} + 10^{-2}\ \mathrm{m^2\cdot K/W}\right)^{-1} = 20\ \mathrm{W/m^2\cdot K}.$$

$$Bi = \frac{UL}{k} = \frac{20\ \mathrm{W/m^2\cdot K} \times 0.01\ \mathrm{m}}{60\ \mathrm{W/m\cdot K}} = 0.0033 << 1$$

and the lumped capacitance method can be used.

(a) From Eqs. (5.6) and (5.7),

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-t/\tau_t\right) = \exp\left(-t/R_t C_t\right) = \exp\left(-Ut/\rho Lc\right)$$

$$t = -\frac{\rho Lc}{U} \ln\frac{T - T_\infty}{T_i - T_\infty} = -\frac{7850\ \mathrm{kg/m^3}\left(0.01\ \mathrm{m}\right) 430\ \mathrm{J/kg\cdot K}}{20\ \mathrm{W/m^2\cdot K}} \ln\frac{1200 - 1300}{300 - 1300}$$

$$t = 3886\mathrm{s} = 1.08\mathrm{h}.$$

(b) Performing an energy balance at the outer surface (s,o),

$$h\left(T_\infty - T_{s,o}\right) = \left(T_{s,o} - T_{s,i}\right)/R''_f$$

$$T_{s,o} = \frac{hT_\infty + T_{s,i}/R''_f}{h + \left(1/R''_f\right)} = \frac{25\ \text{W/m}^2 \cdot \text{K} \times 1300\ \text{K} + 1200\ \text{K}/10^{-2}\text{m}^2 \cdot \text{K/W}}{\left(25 + 100\right)\text{W/m}^2 \cdot \text{K}}$$

$$T_{s,o} = 1220\ \text{K}.$$

How does the coating affect the thermal time constant?