Radiation Part 2: View Factor Algebra

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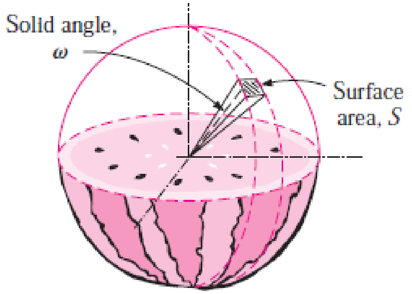
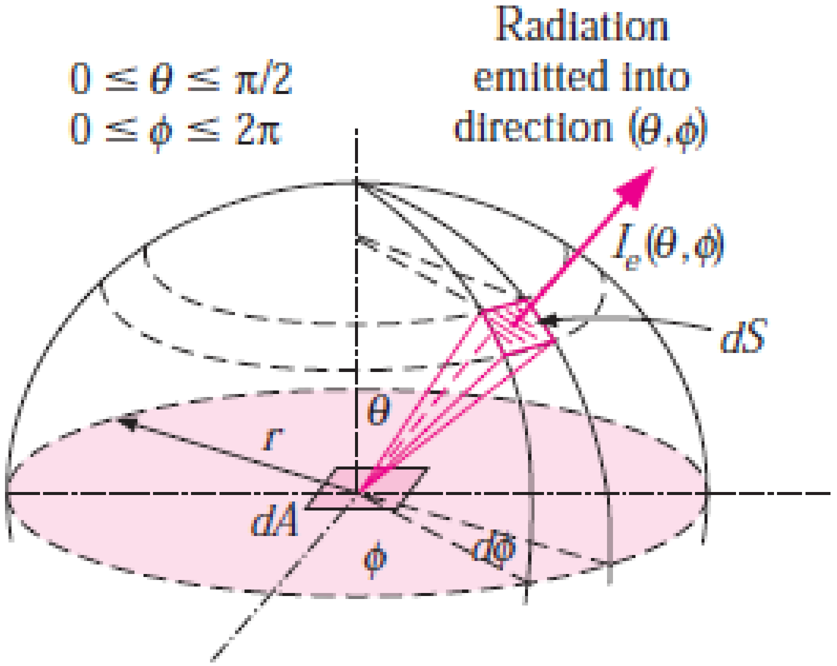
Radiation exchange between two surfaces

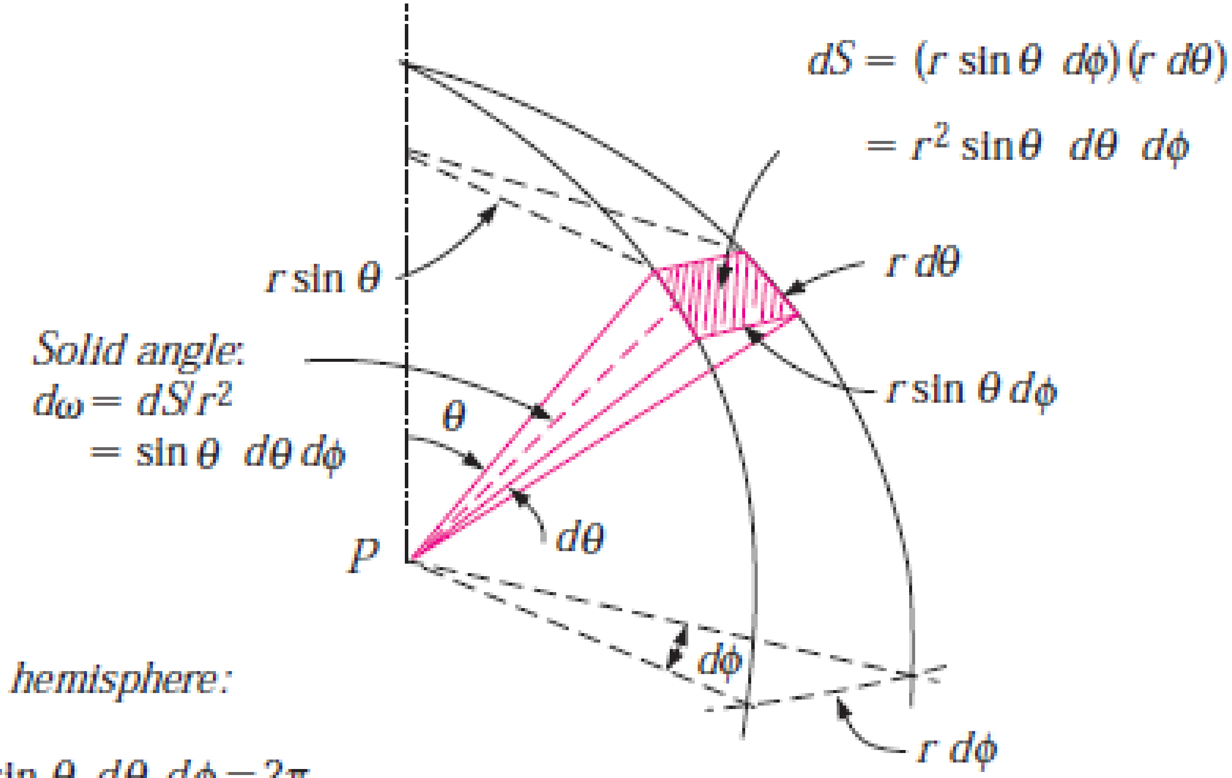
- Radiation exchange between two or more surfaces depends strongly on
  - Temperatures of the surfaces
  - their radiative properties
  - the surface geometries and orientations
- We already know about the first two factors
- How does the shape and relative orientation of the surfaces??
  - Need to introduce the concept of view factor/ shape factor/ configuration factor

The concept of solid angle



$$d\omega = \frac{dS}{r^2} = \sin\theta \, d\theta \, d\phi$$



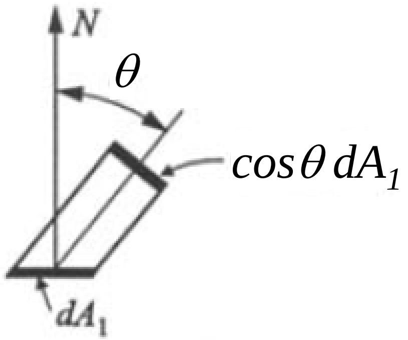


Solid angle for a hemisphere:

$$\omega = \int_{\text{Hemisphere}} d\omega = \int_{0=\theta}^{\pi/2} \int_{0=\phi}^{2\pi} \sin\theta \, d\theta \, d\phi = 2\pi$$

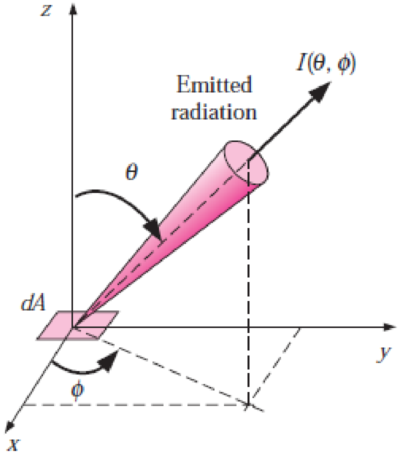
Intensity of emitted radiation

- Radiant power $d\dot{Q}_e$ emitted per unit solid angle in a direction (θ,ϕ), per unit area of the emitter projected normal to the line of view of the receiver from the radiating element



$$d\omega = \frac{dS}{r^2} = \sin\theta\, d\theta\, d\phi$$

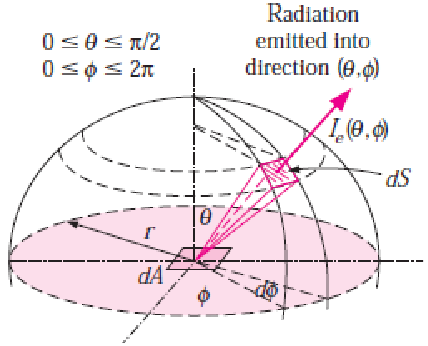
$$I_e(\theta,\phi) = \frac{d\dot{Q}_e}{dA\cos\theta\cdot d\omega} = \frac{d\dot{Q}_e}{dA\cos\theta\sin\theta\, d\theta\, d\phi} \qquad (\text{W/m}^2\cdot\text{sr})$$



Radiation flux: $dE = \frac{d\dot{Q}_e}{dA} = I_e(\theta,\phi)\cos\theta\sin\theta\, d\theta\, d\phi$

Hemispherical emission

$$E = \int_{\text{hemisphere}} dE = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_e(\theta,\phi)\cos\theta\sin\theta\, d\theta\, d\phi \qquad (\text{W/m}^2)$$

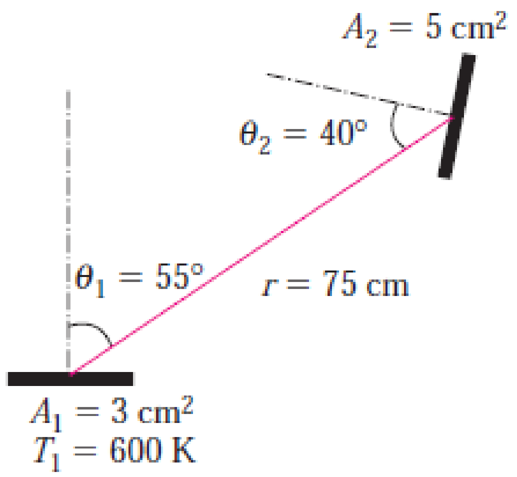


Diffusely emitting surface: $I_e = \text{constant} \Rightarrow E = \pi I_e \qquad (\text{W/m}^2)$

For blackbody surface: $I_b(T) = \frac{E_b(T)}{\pi} = \frac{\sigma T^4}{\pi} \qquad (\text{W/m}^2\cdot\text{sr})$

Example 2

A small surface of area $A_1 = 3\ \text{cm}^2$ emits radiation as a blackbody at $T_1 = 600\ \text{K}$. Part of the radiation emitted by A_1 strikes another small surface of area $A_2 = 5\ \text{cm}^2$ oriented as shown in Fig. 21–23. Determine the solid angle subtended by A_2 when viewed from A_1, and the rate at which radiation emitted by A_1 strikes A_2.

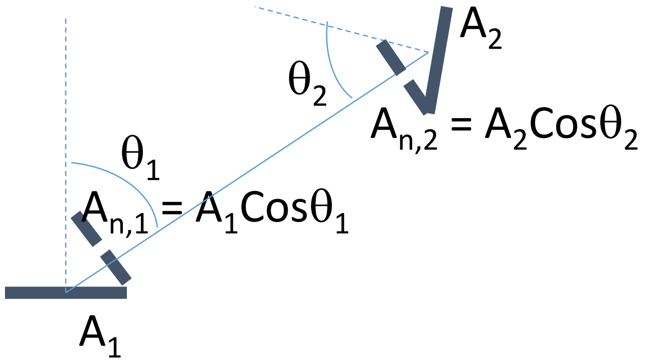


<u>Assumptions:</u>

1. A_1 emits as blackbody (diffuse)
2. Both surface dimensions << r; surfaces may be treated as differential areas

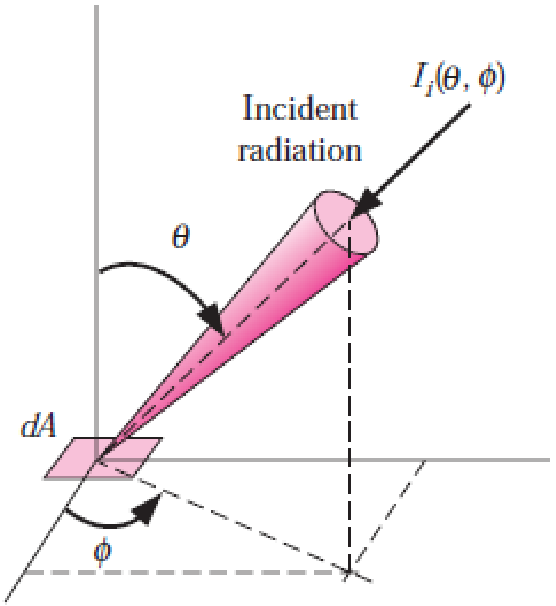
$$I_1 = \frac{E_b(T_1)}{\pi} = \frac{\sigma T_1^4}{\pi} = \frac{(5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4)(600\ \text{K})^4}{\pi} = 2339\ \text{W/m}^2 \cdot \text{sr}$$

$$\omega_{2-1} \cong \frac{A_{n,2}}{r^2} = \frac{A_2 \cos\theta_2}{r^2} = \frac{(5\ \text{cm}^2)\cos 40°}{(75\ \text{cm})^2} = \mathbf{6.81 \times 10^{-4}\ sr}$$



$$\dot{Q}_{1-2} = I_1(A_1 \cos\theta_1)\omega_{2-1}$$
$$= (2339\ \text{W/m}^2 \cdot \text{sr})(3 \times 10^{-4} \cos 55°\ \text{m}^2)(6.81 \times 10^{-4}\ \text{sr})$$
$$= \mathbf{2.74 \times 10^{-4}\ W}$$

Incident radiation and Irradiation



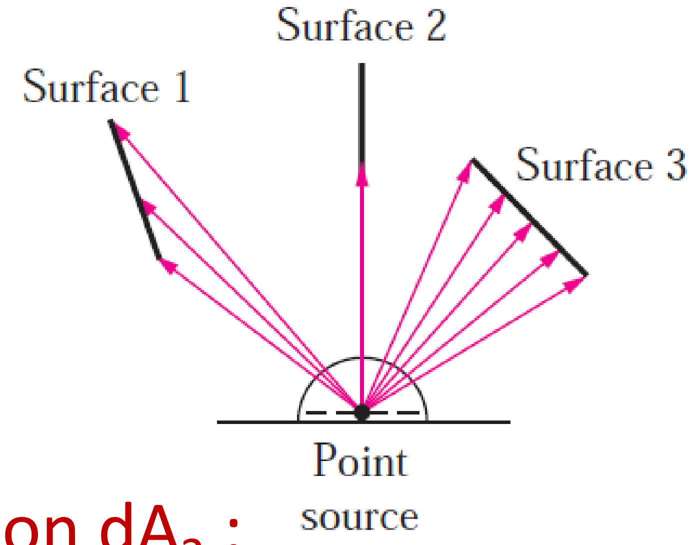
- Intensity of incident radiation (I_i) is the rate at which radiation energy dG is incident from the (θ,ϕ) direction per unit area of the receiving surface normal to this direction and per unit solid angle about this direction

Irradiation:

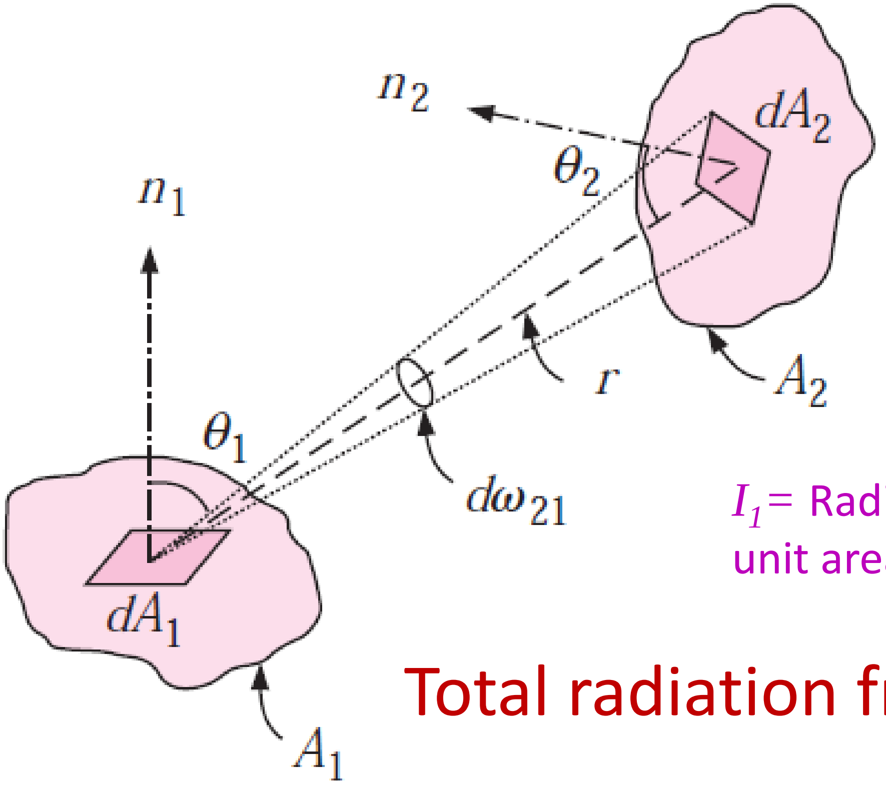
$$G = \int_{\text{hemisphere}} dG = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_i(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi \qquad (\text{W/m}^2)$$

Diffusely incident radiation: $I_i = \text{constant}$ $\qquad G = \pi I_i \qquad (\text{W/m}^2)$

View factor/ Shape factor/ Configuration factor*



☠ **Applicable for Blackbody and Diffuse Grey Surfaces only**



Radiation from dA_1 falling on dA_2 :

$$\dot{Q}_{dA_1 \to dA_2} = I_1 \cos\theta_1 \, dA_1 \, d\omega_{21} = I_1 \cos\theta_1 \, dA_1 \frac{dA_2 \cos\theta_2}{r^2}$$

I_1= Radiation Intensity (magnitude of radiation emitted in a particular direction, per unit area, from A_1

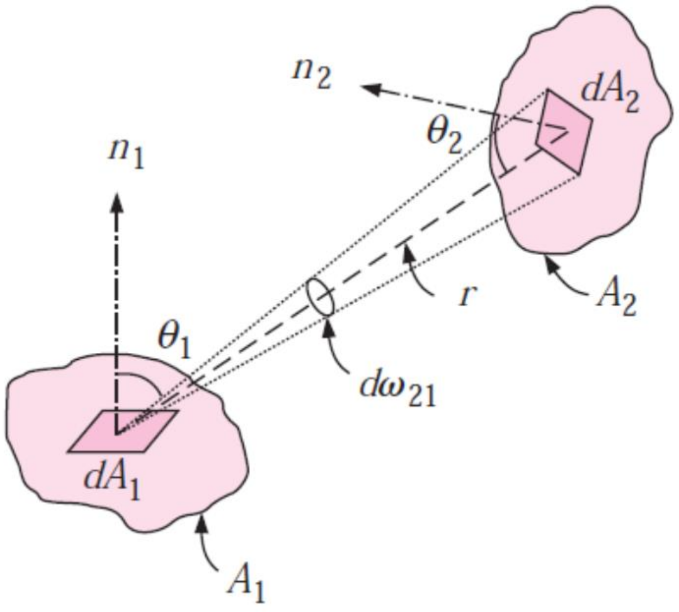
Total radiation from dA_1 : $\dot{Q}_{dA_1} = J_1 \, dA_1 = \pi I_1 \, dA_1$

Its fraction falling on dA_2 : $dF_{dA_1 \to dA_2} = \frac{\dot{Q}_{dA_1 \to dA_2}}{\dot{Q}_{dA_1}} = \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_2$

Integrating over surface A_2: $F_{dA_1 \to A_2} = \int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_2$

Fraction of radiation emitted from dA_1 that falls on A_2

$F_{ij} =$ *the fraction of the radiation leaving surface i that strikes surface j directly*



View factor (contd…)

Radiation leaving the ENTIRE A_1 :

$$\dot{Q}_{A_1} = J_1 A_1 = \pi I_1 A_1$$

Radiation falling on dA_2 :

$$\dot{Q}_{A_1 \rightarrow dA_2} = \int_{A_1} \dot{Q}_{dA_1 \rightarrow dA_2} = \int_{A_1} \frac{I_1 \cos\theta_1 \cos\theta_2 \, dA_2}{r^2} \, dA_1$$

Integrating over A_2 :

$$\dot{Q}_{A_1 \rightarrow A_2} = \int_{A_2} \dot{Q}_{A_1 \rightarrow dA_2} = \int_{A_2} \int_{A_1} \frac{I_1 \cos\theta_1 \cos\theta_2}{r^2} \, dA_1 \, dA_2$$

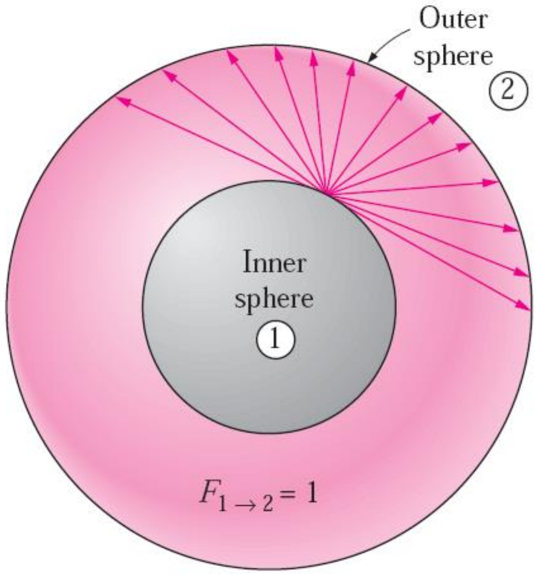
$$F_{12} = F_{A_1 \rightarrow A_2} = \frac{\dot{Q}_{A_1 \rightarrow A_2}}{\dot{Q}_{A_1}} = \frac{1}{A_1} \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} \, dA_1 \, dA_2$$

F_{ij} = the fraction of the radiation leaving surface i that strikes surface j directly

View factor (contd…)

$$F_{12} = F_{A_1 \rightarrow A_2} = \frac{\dot{Q}_{A_1 \rightarrow A_2}}{\dot{Q}_{A_1}} = \frac{1}{A_1} \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_1 \, dA_2$$

$$F_{21} = F_{A_2 \rightarrow A_1} = \frac{\dot{Q}_{A_2 \rightarrow A_1}}{\dot{Q}_{A_2}} = \frac{1}{A_2} \int_{A_2} \int_{A_1} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2} dA_1 \, dA_2$$

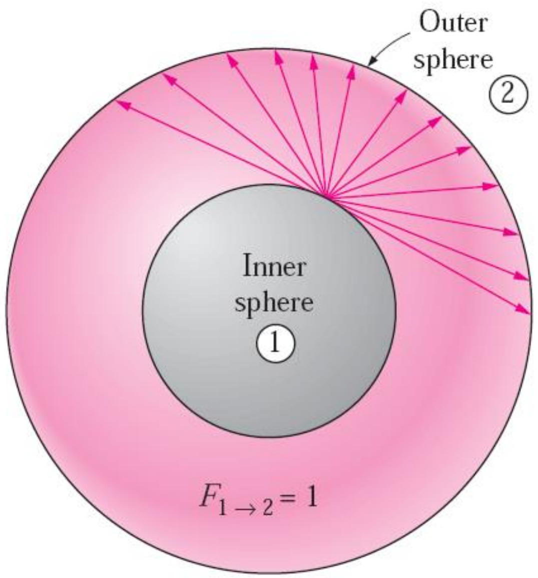


$$A_1 F_{12} = A_2 F_{21}$$ **Reciprocity relation**

What is F_{21}?

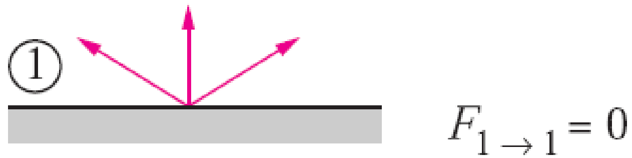
Self view factor

$F_{i\to i}$ = *the fraction of radiation leaving surface i that strikes itself directly*

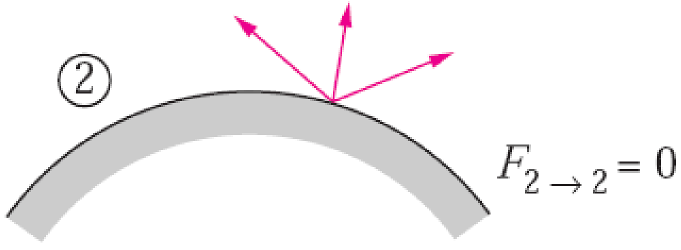


$F_{1\to 1} = 0$

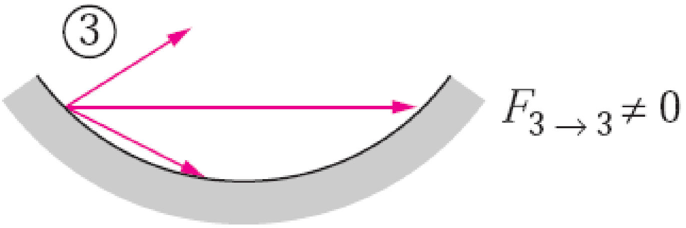
$F_{2\to 2} = ?$



(*a*) Plane surface



(*b*) Convex surface



(*c*) Concave surface

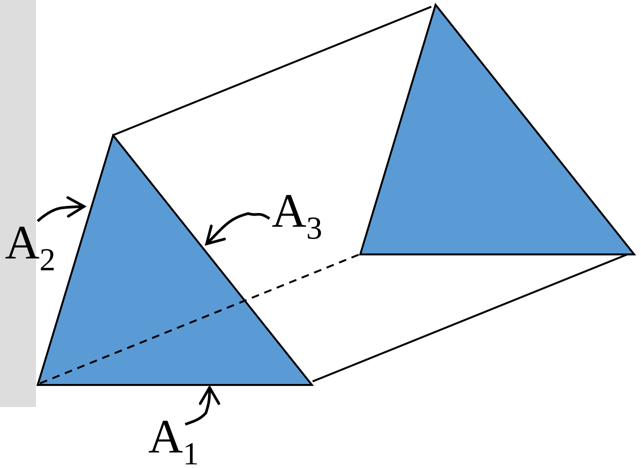
View Factor Algebra: Summation Rule

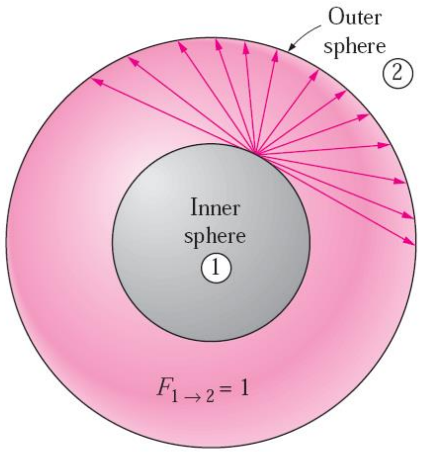
The sum of the view factors from surface i of an enclosure to all surfaces of the enclosure, including to itself, must equal unity.

Radiation leaving any surface *i* of an enclosure must be intercepted completely by the surfaces of the enclosure. Therefore, the sum of the view factors from surface *i* to each one of the surfaces of the enclosure must be unity.

$$\sum_{j=1}^{N} F_{i \to j} = 1$$

$$\sum_{j=1}^{3} F_{1 \to j} = F_{1 \to 1} + F_{1 \to 2} + F_{1 \to 3} = 1$$





Flat Surface

$$F_{11} = 0;$$

Reciprocity rule

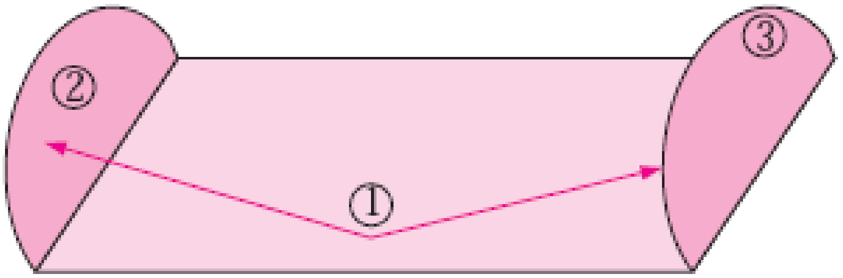
$$A_1F_{12} = A_2F_{21} \Rightarrow F_{21} = \frac{A_1}{A_2}F_{12}$$

Summation rule

$$F_{11} + F_{12} = 1 \Rightarrow F_{12} = 1 \Rightarrow F_{21} = \frac{A_1}{A_2}$$

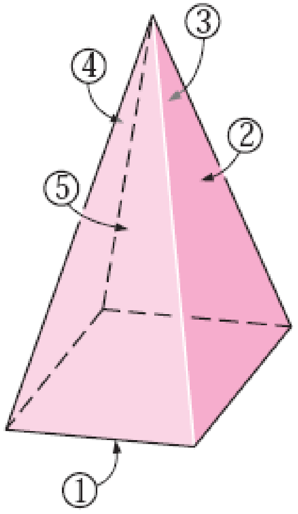
$$F_{21} + F_{22} = 1 \Rightarrow F_{22} = 1 - F_{21} = \left(1 - \frac{A_1}{A_2}\right)$$

View factor algebra: Symmetry rule



$F_{1\to 2} = F_{1\to 3}$

(Also, $F_{2\to 1} = F_{3\to 1}$)



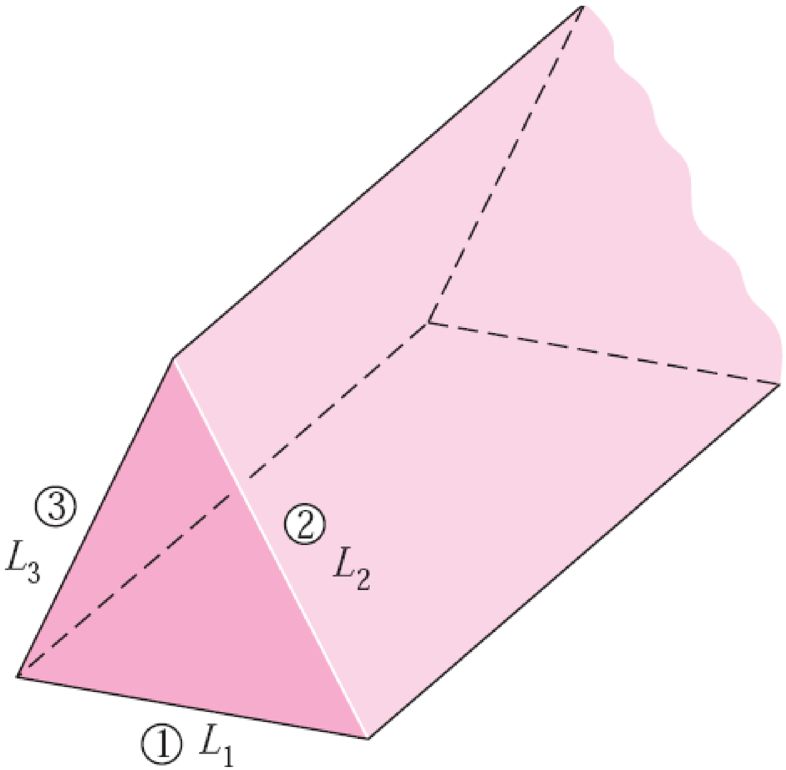
$$F_{12} = F_{13} = F_{14} = F_{15}$$

$$\sum_{j=1}^{5} F_{1j} = F_{11} + F_{12} + F_{13} + F_{14} + F_{15} = 1$$

$$F_{11} = 0$$

$$F_{12} = F_{13} = F_{14} = F_{15} = \mathbf{0.25}$$

Assignment



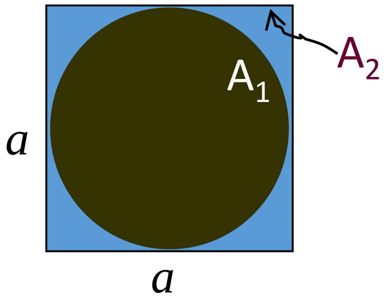
Show that:

$$F_{12} = \frac{A_1 + A_2 - A_3}{2A_1} = \frac{L_1 + L_2 - L_3}{2L_1}$$

$$F_{13} = \frac{A_1 + A_3 - A_2}{2A_1} = \frac{L_1 + L_3 - L_2}{2L_1}$$

$$F_{23} = \frac{A_2 + A_3 - A_1}{2A_2} = \frac{L_2 + L_3 - L_1}{2L_2}$$

Examples

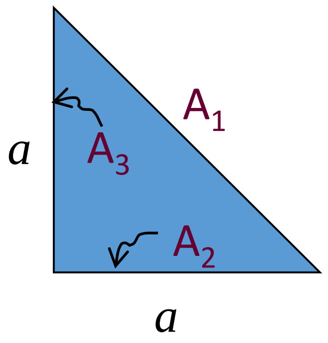


Find F_{12} and F_{21}

Sphere within a cube

$$F_{11} = 0; \quad F_{12} = 1$$

$$A_1 F_{12} = A_2 F_{21} \Rightarrow F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\pi a^2}{6a^2} = \frac{\pi}{6}$$



Find F_{12}, F_{13}, F_{21}, F_{31} F_{23} and F_{32}

Infinitely long right angle triangular prism

By observation: $F_{11} = 0;$

Summation Rule: $F_{11} + F_{12} + F_{13} = 1 \Rightarrow F_{12} + F_{13} = 1$

By observation, A_3 and A_2 are symmetrically placed $\quad F_{12} = F_{13}$

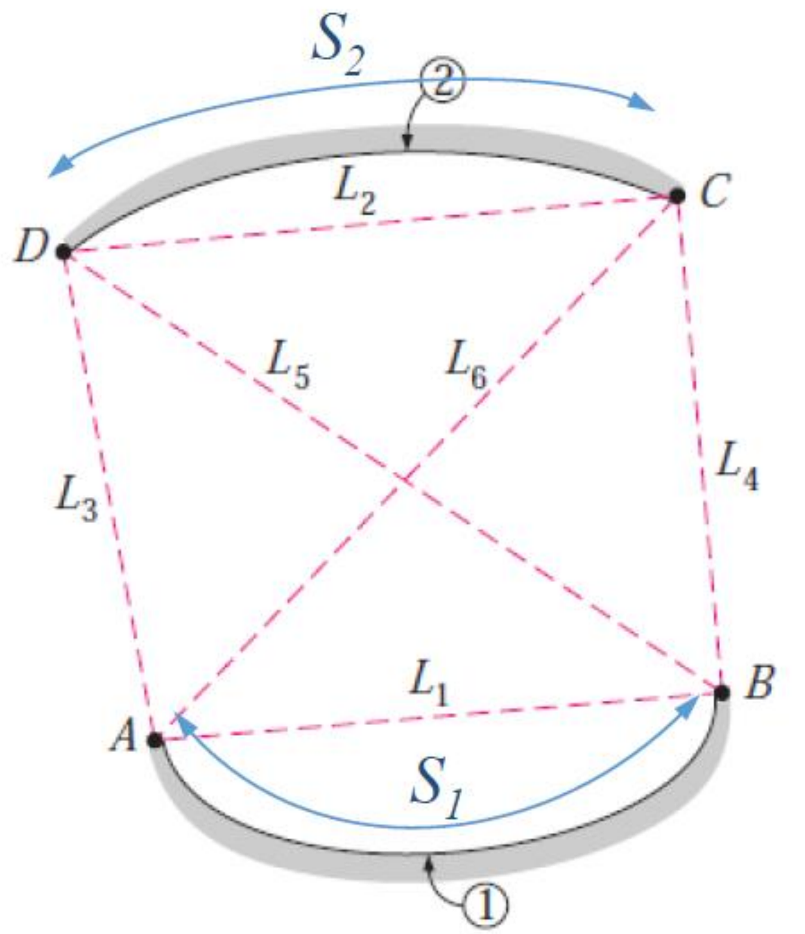
$$\therefore F_{12} = F_{13} = \frac{1}{2}$$

Reciprocity Rule:

$$A_1 F_{12} = A_2 F_{21} \Rightarrow \sqrt{2} a \frac{1}{2} = a F_{21} \Rightarrow F_{21} = \frac{1}{\sqrt{2}}$$

For Surface 2: $\quad F_{21} + F_{22} + F_{23} = 1; \; F_{22} = 0; \quad F_{23} = 1 - F_{21} = 1 - \frac{1}{\sqrt{2}}$

Hottel's Crossed-string Method



$$F_{1 \rightarrow 2} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2S_1}$$

View Factors between Infinitely Long Surfaces

developed by H. C. Hottel in the 1950s

Wrong expression in Cengel and Ozisik

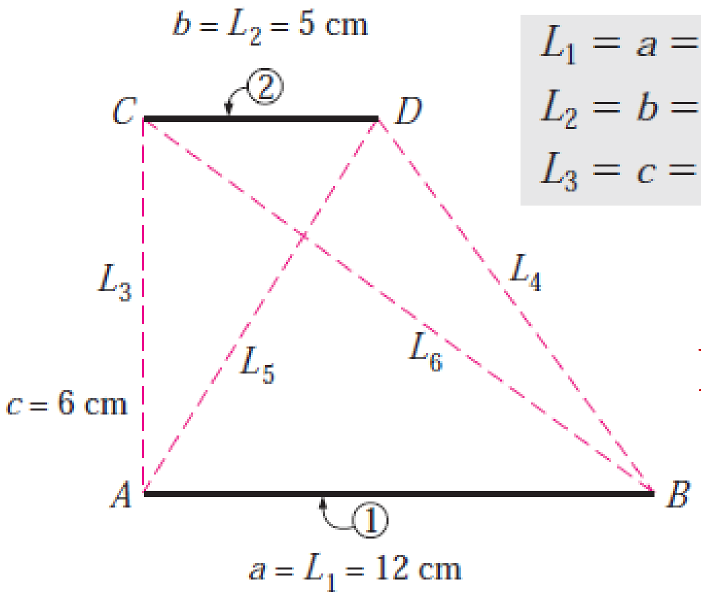
$$F_{i \rightarrow j} = \frac{\Sigma \text{ (Crossed strings)} - \Sigma \text{ (Uncrossed strings)}}{2 \times \text{(String on surface } i)}$$

Correct Expression:

$$F_{i \rightarrow j} = \frac{\Sigma \text{ (Crossed strings)} - \Sigma \text{ (Uncrossed strings)}}{2 \times \textit{Curved Length of surface 1}}$$

Hottel's Crossed-string Method

$$F_{i \to j} = \frac{\Sigma\,(\text{Crossed strings}) - \Sigma\,(\text{Uncrossed strings})}{2 \times (\text{String on surface } i)}$$



$L_1 = a = 12$ cm $\quad L_4 = \sqrt{7^2 + 6^2} = 9.22$ cm

$L_2 = b = 5$ cm $\quad L_5 = \sqrt{5^2 + 6^2} = 7.81$ cm

$L_3 = c = 6$ cm $\quad L_6 = \sqrt{12^2 + 6^2} = 13.42$ cm

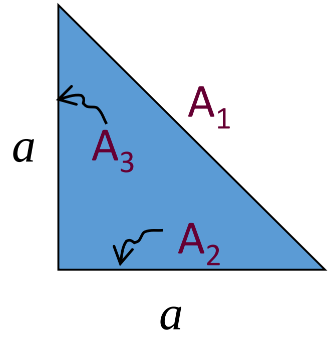
Find F_{12}, F_{21}, $F_{1\infty}$ and $F_{2\infty}$

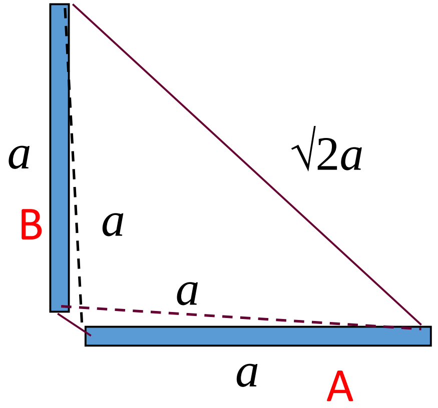
$$F_{12} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1} = 0.25$$

$$F_{21} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_2} = 0.6$$ Verify reciprocity relation

View Factor of the background with respect to the plates:

$$F_{1\infty} = 1 - F_{12} = 0.75 \qquad F_{2\infty} = 1 - F_{21} = 0.4$$

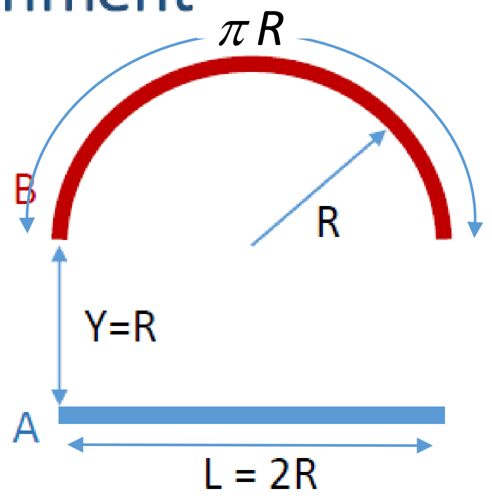




$$F_{AB} = \frac{(a + a) - (\sqrt{2}a + 0)}{2a} = 1 - \frac{1}{\sqrt{2}}$$

Attention: Use the correct form of Hottel's crossed-string method!

Assignment



$$F_{AB} = \frac{\left(\sqrt{5}R + \sqrt{5}R\right) - (R + R)}{4R} = 0.618$$

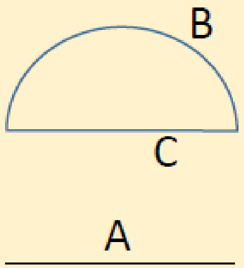
$$F_{BA} = \frac{\left(\sqrt{5}R + \sqrt{5}R\right) - (R + R)}{2\pi R} = 0.393$$

Find F_{AB}, F_{BA}, F_{AA} and F_{BB}. Also find $F_{A\infty}$ and $F_{B\infty}$

Summation
$$F_{AA} + F_{AB} + F_{A\infty} = 1$$
$$F_{BB} + F_{BA} + F_{B\infty} = 1$$

Observation $F_{AA} = 0$

Reciprocity relation between Curved surface B and the hypothetical open face (C)



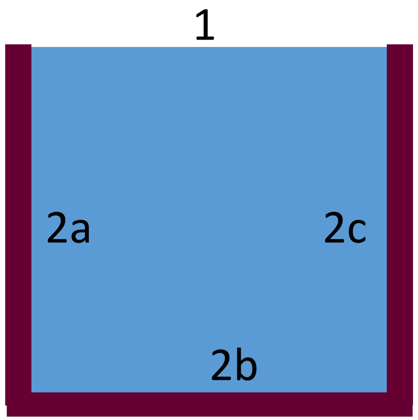
$$\pi R F_{BC} = 2R F_{CB} \Rightarrow F_{BC} = \frac{2}{\pi} F_{CB} = \frac{2}{\pi}$$

$$F_{BB} = 1 - 2/\pi = 0.363$$

$$F_{A\infty} = 1 - F_{AB} = 1 - 0.618 = 0.382$$

$$F_{B\infty} = 1 - F_{BB} - F_{BA} = 1 - 0.393 - 0.363$$
$$= 0.244$$

Assignment



What fraction of radiation leaves from the open lid of the infinitely long square cavity?

$F_{21} = ?$

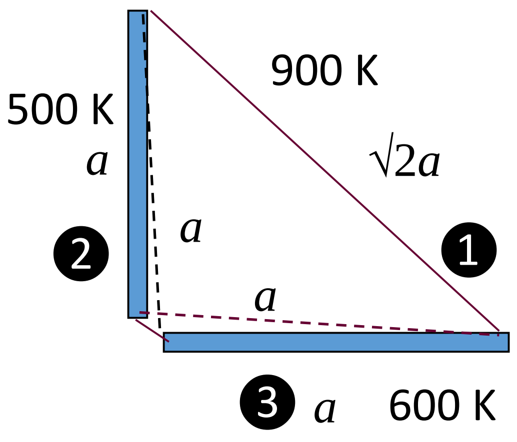
View factor algebra: how many equations do we need?

- Radiation exchange in an enclosure of N surfaces: N^2 view factors required

$$\begin{bmatrix} F_{11} & F_{12} & -- & F_{1N} \\ F_{21} & F_{22} & -- & F_{2N} \\ - & - & - & - \\ F_{N1} & F_{N2} & -- & F_{NN} \end{bmatrix}$$

- Summation rule can be applied to get N equations which gives N view factors
- Application of Reciprocity relation for N(N-1)/2 times gives N(N-1)/2 view factors
- So we need essentially N^2-N-N(N-1)/2

N=3, N^2=9
Summation Rule: 3
Reciprocity Rule = 3 (F_{12} & F_{21}, F_{23} & F_{32}, and F_{13} & F_{31})
Remaining: 3



$$F_{12} = F_{13} = \frac{1}{2}$$

$$F_{11} = F_{22} = F_{33} = 0,$$

$$F_{21} = F_{31} = \frac{1}{\sqrt{2}},$$

$$F_{23} = F_{32} = 1 - \frac{1}{\sqrt{2}}$$

View factor algebra: Additive property of view factor

Radiation falling on a composite surface

$$F_{i(j)} = \sum_{k=1}^{n} F_{ik}$$

Multiply A_i on both sides,

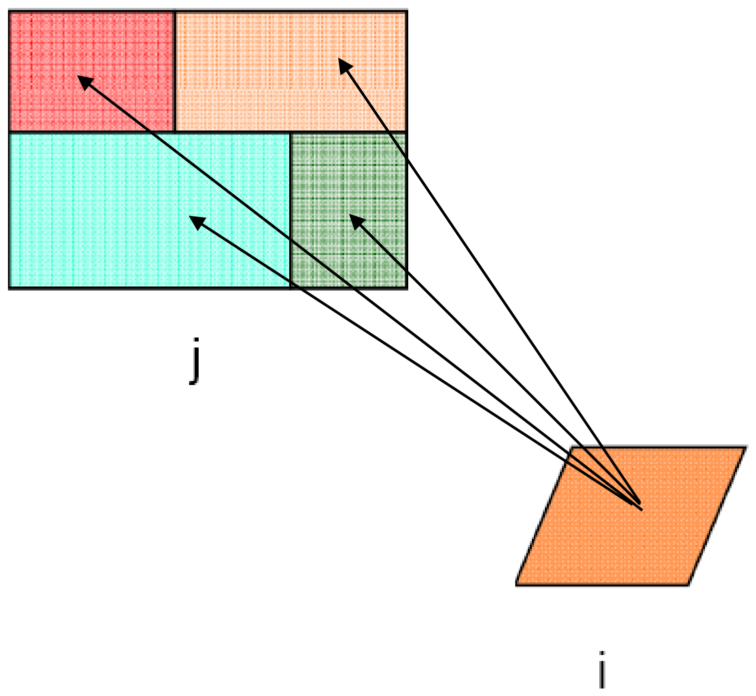
$$A_i F_{i(j)} = A_i \sum_{k=1}^{n} F_{ik} = A_i F_{i1} + A_i F_{i2} + A_i F_{i3} + - - - - - + A_i F_{in}$$

Reciprocity

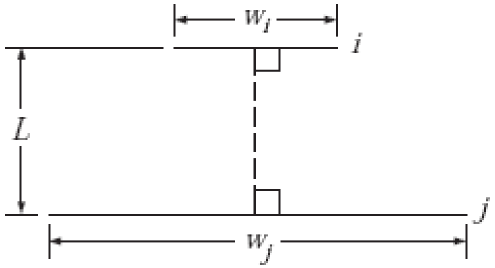
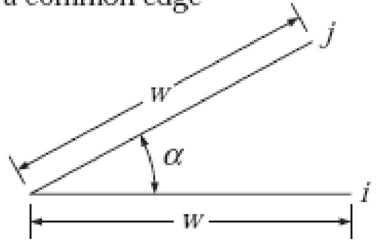
$$A_j F_{j(i)} = A_1 F_{1i} + A_2 F_{2i} + A_3 F_{3i} + - - - - - + A_n F_{ni}$$

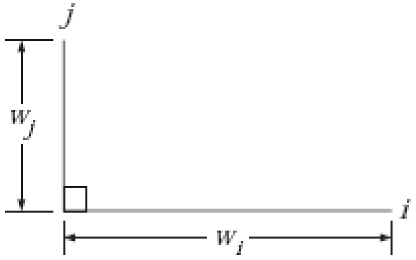
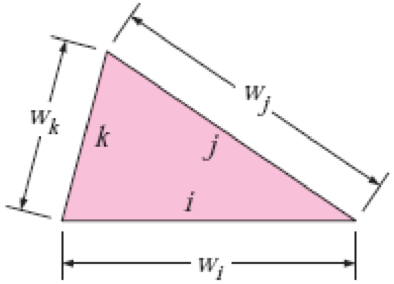
$$A_j F_{j(i)} = \sum_{k=1}^{n} A_k F_{ki}$$

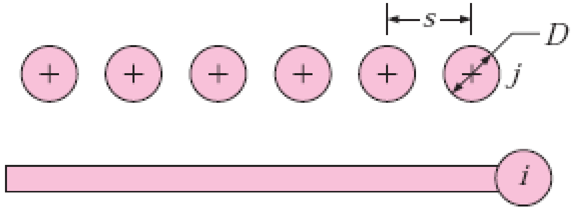
$$F_{j(i)} = \frac{\sum_{k=1}^{n} A_k F_{ki}}{\sum_{k=1}^{n} A_k}$$



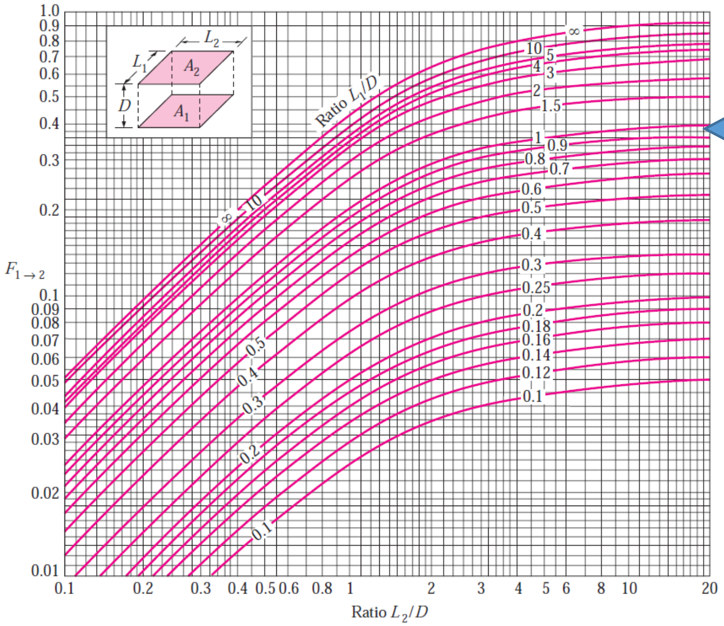
View factor of infinitely long parallel surfaces

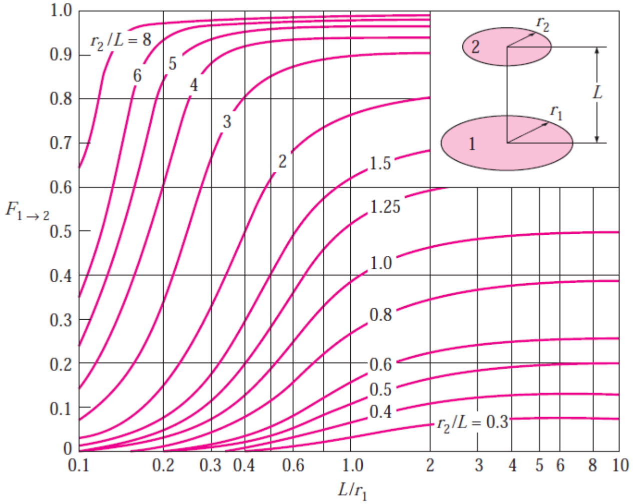
Geometry	Relation
Parallel plates with midlines connected by perpendicular line 	$W_i = w_i/L$ and $W_j = w_j/L$ $F_{i \to j} = \frac{[(W_i + W_j)^2 + 4]^{1/2} - (W_j - W_i)^2 + 4]^{1/2}}{2W_i}$
Inclined plates of equal width and with a common edge 	$F_{i \to j} = 1 - \sin \frac{1}{2}\alpha$

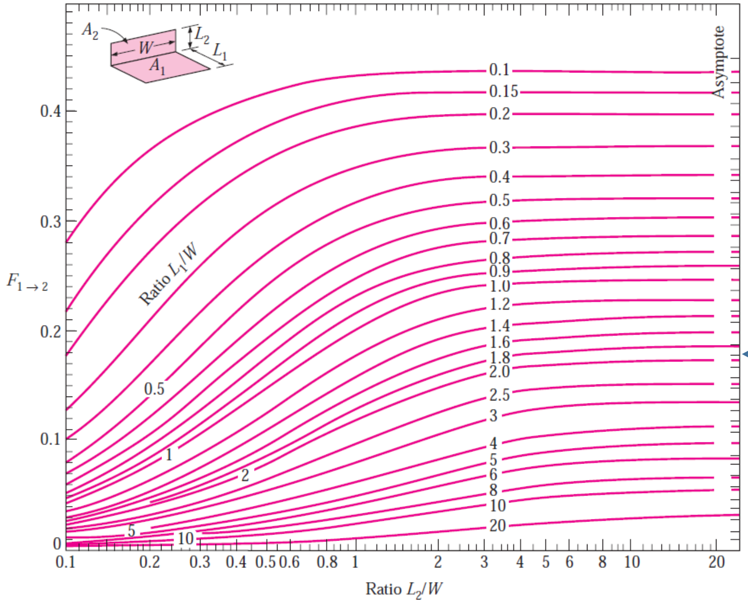
Perpendicular plates with a common edge 	$F_{i \to j} = \frac{1}{2}\left\{1 + \frac{w_j}{w_i} - \left[1 + \left(\frac{w_j}{w_i}\right)^2\right]^{1/2}\right\}$
Three-sided enclosure 	$F_{i \to j} = \frac{w_i + w_j - w_k}{2w_i}$

Infinite plane and row of cylinders 	$F_{i \to j} = 1 - \left[1 - \left(\frac{D}{s}\right)^2\right]^{1/2} + \frac{D}{s} \tan^{-1}\left(\frac{s^2 - D^2}{D^2}\right)^{1/2}$

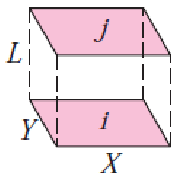
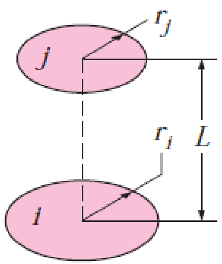
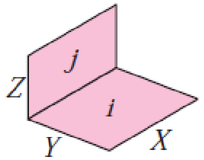
View Factors of more complicated 3-D surface (analytical and graphical methods)



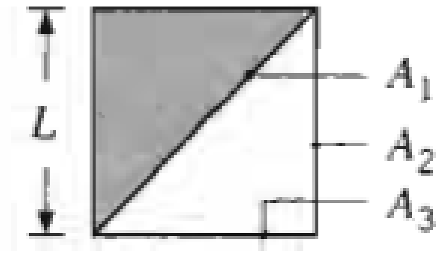




View factor expressions for some common geometries of finite size (3D)

Geometry	Relation
Aligned parallel rectangles 	$\bar{X} = X/L$ and $\bar{Y} = Y/L$ $F_{i\to j} = \frac{2}{\pi \bar{X}\bar{Y}}\Big\{\ln\Big[\frac{(1+\bar{X}^2)(1+\bar{Y}^2)}{1+\bar{X}^2+\bar{Y}^2}\Big]^{1/2} + \bar{X}(1+\bar{Y}^2)^{1/2}\tan^{-1}\frac{\bar{X}}{(1+\bar{Y}^2)^{1/2}} + \bar{Y}(1+\bar{X}^2)^{1/2}\tan^{-1}\frac{\bar{Y}}{(1+\bar{X}^2)^{1/2}} - \bar{X}\tan^{-1}\bar{X} - \bar{Y}\tan^{-1}\bar{Y}\Big\}$
Coaxial parallel disks 	$R_i = r_i/L$ and $R_j = r_j/L$ $S = 1 + \frac{1+R_j^2}{R_i^2}$ $F_{i\to j} = \frac{1}{2}\Big\{S - \Big[S^2 - 4\Big(\frac{r_j}{r_i}\Big)^2\Big]^{1/2}\Big\}$
Perpendicular rectangles with a common edge 	$H = Z/X$ and $W = Y/X$ $F_{i\to j} = \frac{1}{\pi W}\Big(W\tan^{-1}\frac{1}{W} + H\tan^{-1}\frac{1}{H} - (H^2+W^2)^{1/2}\tan^{-1}\frac{1}{(H^2+W^2)^{1/2}} + \frac{1}{4}\ln\Big\{\frac{(1+W^2)(1+H^2)}{1+W^2+H^2} \times \Big[\frac{W^2(1+W^2+H^2)}{(1+W^2)(W^2+H^2)}\Big]^{W^2} \times \Big[\frac{H^2(1+H^2+W^2)}{(1+H^2)(H^2+W^2)}\Big]^{H^2}\Big\}\Big)$

Examples: Determine the view factors F_{12} and F_{21}



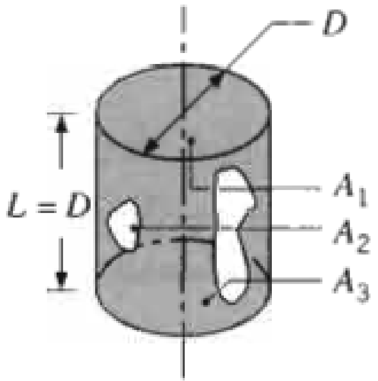
From summation rule, $F_{11} + F_{12} + F_{13} = 1$

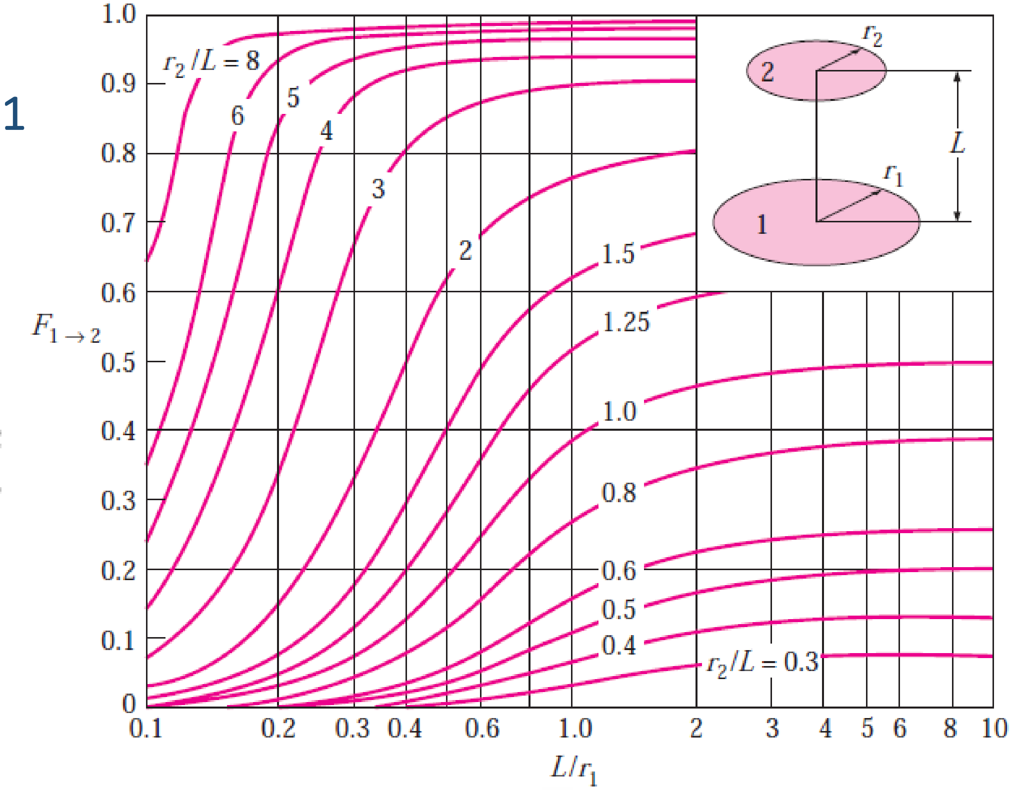
where $F_{11} = 0$

By symmetry, $F_{12} = F_{13}$

Hence $F_{12} = 0.50$

By reciprocity, $F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\sqrt{2}\,L}{L} \times 0.5 = 0.71$





with $(r_3/L) = 0.5$ and $(L/r_1) = 2$, $F_{13} = 0.172$

From summation rule, $F_{11} + F_{12} + F_{13} = 1$

or, with $F_{11} = 0$, $F_{12} = 1 - F_{13} = 0.828$

From reciprocity, $F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\pi D^2/4}{\pi D L} \times 0.828 = 0.207$