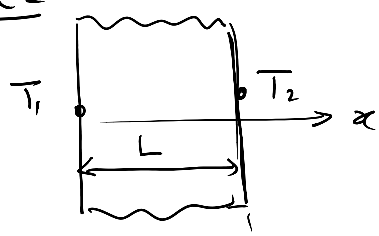
Tuesday, March 9, 2021 3:34 PM

1-D Steady State Heat Conduction (k=cont)

$$\frac{\partial}{\partial t}(\rho C T) = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial T}{\partial r}\right) + \dot{q}''$$

1-D Cartesian System with No heat Generation wt

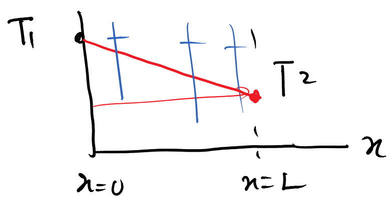
BC1



$$0 = \frac{d}{dr}\left(\frac{dT}{dr}\right) + 0$$

$$\boxed{\frac{d^2T}{dx^2} = 0}$$

$$\Rightarrow \boxed{T = C_1 x + C_2} \quad (1)$$



BC1 ⇒

$x=0 \Rightarrow T = T_1$

$x=L \Rightarrow T = T_2$

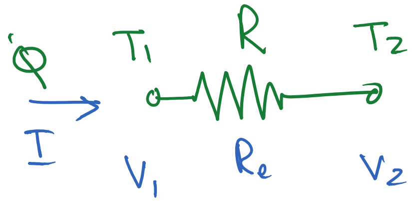
$C_2 = T_1$

$C_1 = \dfrac{T_2 - T_1}{L}$

$$T(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

$$\boxed{T(x) = T_1 - \frac{T_1 - T_2}{L}\, x} \quad (2)$$

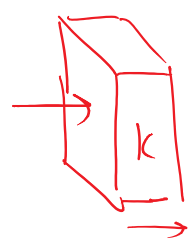
$$\dot{Q} = -kA\frac{dT}{dx}\Big|_x = -kAC_1 = kA\frac{T_1 - T_2}{L}$$



$$\dot{Q} = \frac{T_1 - T_2}{\left(\frac{L}{kA}\right)}$$ thermal resistance

$$\rightarrow I = \frac{V_1 - V_2}{R}$$ OHM's Law

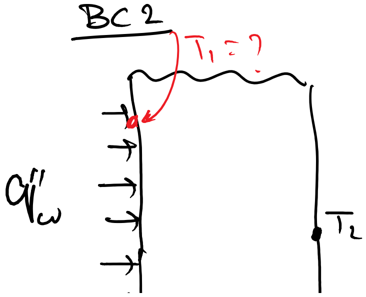
Thermal resistance $R = \dfrac{L}{kA}$



Thermal Resistance analog is valid ONLY →

1. 1-D
2. steady
3. No heat generation ($\dot{q}''=0$)
4. k = const.

BC 2

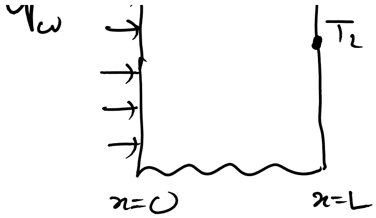


at $x=0$, $-k\dfrac{dT}{dx} = q''_w$

$x = L \qquad T = T_2$,

$T = C_1 x + C_2$

$-k\dfrac{dT}{dx} = -kC_1 = q'' \Rightarrow C_1 = -\dfrac{q''_w}{k}$



$$T = C_1 x + C_2$$

$$-k\frac{dT}{dx} = -kC_1 = q''_w \Rightarrow \check{C}_1 = -\frac{q''_w}{k}$$

$$T_2 = C_1 L + C_2 \Rightarrow \check{C}_2 = T_2 + \frac{q''_w}{k} L$$

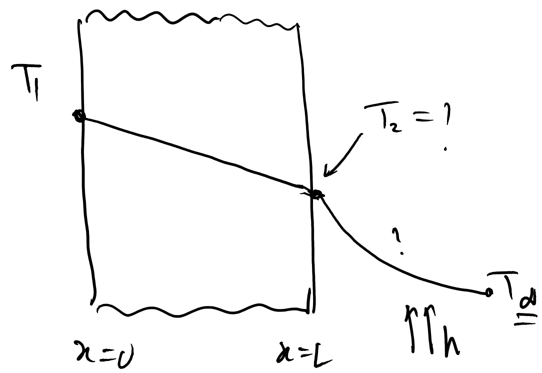
$$\boxed{T(x) = T_2 + \frac{q''_w}{k}(L - x)}$$

at $x=0$, $$T_1 = T_2 + \frac{q''_w L}{k}$$

$$A k \frac{T_1 - T_2}{L} = q''_w \times A = \dot{Q}$$

$$\dot{Q} = \frac{T_1 - T_2}{L/kA} = \frac{\Delta T}{R}$$

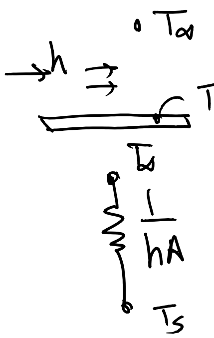
BC 3



HW

$$\left.\begin{aligned} T(x) &= T_1 - \frac{hx}{k + hL}(T_1 - T_\infty) \\ T_2 &= \frac{kT_1 + hL T_\infty}{k + hL} \\ \dot{Q} = q'' A &= \frac{T_1 - T_\infty}{\frac{1}{hA} + \frac{L}{kA}} \end{aligned}\right\}$$

Convection resistance — Conduction resistance



$$q''_w = h(T_s - T_\infty) \quad W/m^2$$

$$\dot{Q} = hA(T_s - T_\infty) \quad W$$

$$= \frac{T_s - T_\infty}{\frac{1}{hA}}$$

$$\boxed{R_{conv} = \frac{1}{hA}}$$

① We can express the conduction HT. in terms of a resistance network if it is 1-D, steady, heat transfer with $k =$ const & $\dot{q}''' = 0$

$$\left\{ R_{cond} = \frac{L}{kA} \quad \left[\frac{K}{W}\right] \right. \qquad \frac{\cancel{m}\,\cancel{m}\,K}{W\,\cancel{m^2}}$$

$$R \ldots = \frac{1}{\ldots} \quad [K/\ldots] \qquad \frac{m^2 K}{\ldots}$$

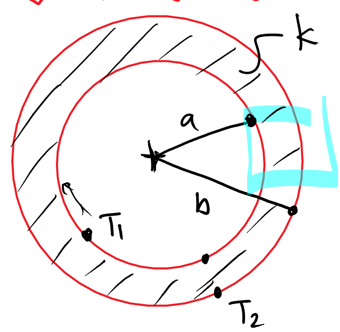
$$\left\{\begin{array}{l} R_{cond} = \dfrac{\quad}{kA} \quad \left[\dfrac{\quad}{W}\right] \\ R_{conv} = \dfrac{1}{hA} \quad [K/W] \end{array}\right.$$

W $\cancel{m^2}$

$$\frac{\cancel{m^2}K}{W\,\cancel{m^2}}$$

steady

1-D, Heat Transfer across a hollow cylinder with $k = \text{const}$, $\dot{q}''' = 0$



GDE $\quad \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = 0 \quad (1) \left[a \leq r \leq b\right]$

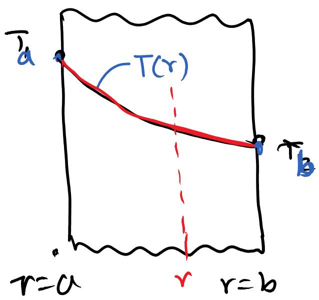
or $\quad \dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = 0$

Integrate once w.r.t. r

$$r\frac{dT}{dr} = C_1 \quad \text{——— (2)}$$

Cannot impose the B.C. at $r = 0$

Integrating once more $\quad \boxed{T = C_1 \ln r + C_2} \quad \text{——— (3)}$



BC $\quad$ at $r = a$, $\quad T = T_a$

$\qquad\quad r = b \qquad T = T_b$

Apply this in (3) $\Rightarrow$

$$\left.\begin{array}{l} T_a = C_1 \ln a + C_2 \\ T_b = C_1 \ln b + C_2 \end{array}\right\} \text{——— (4)}$$

From (4)

$$C_1 = \frac{T_a - T_b}{\ln a - \ln b} = \frac{T_b - T_a}{\ln b/a}$$

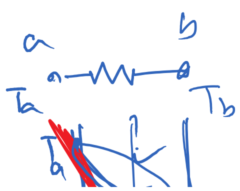
$$C_2 = \left[T_a - \frac{T_b - T_a}{\ln b/a} \ln a\right]$$

Using C_1 & C_2 in eqn (3)

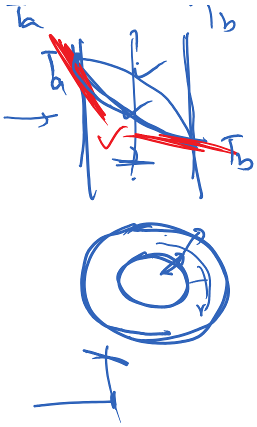
$$T(r) = C_1 \ln r + C_2$$

$$= \frac{T_b - T_a}{\ln b/a} \cdot \ln r + T_a - \frac{T_b - T_a}{\ln b/a} \ln a$$

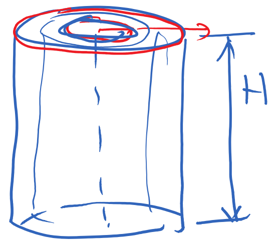
$$T(r) - T_a = \frac{T_b - T_a}{\ln b/a} \ln r/a$$



or $\quad \boxed{\dfrac{T(r) - T_a}{T_b - T_a} = \dfrac{\ln r/a}{\ln b/a}} \quad \text{——— (4)}$



$$\boxed{T_b - T_a \qquad \ln b/a}$$



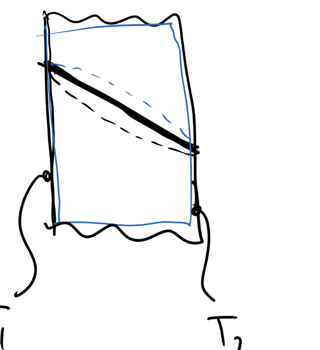
$$\dot{Q} = A \times q'' = 2\pi r H \times \left(-k\frac{dT}{dr}\Big|_r\right)$$

$$= -2\pi H k \left(r\frac{dT}{dr}\right)_r$$

$$= -2\pi H k \, \frac{T_b - T_a}{\ln b/a}$$

$$r\frac{dT}{dr} = C_1 = \frac{T_b - T_a}{\ln b/a}$$

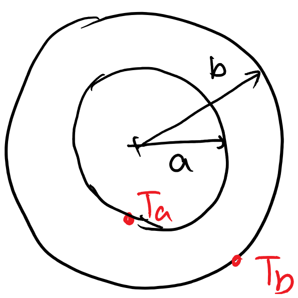
$$\boxed{\dot{Q} = \frac{T_a - T_b}{\left(\frac{\ln b/a}{2\pi k H}\right)}} = \frac{T_a - T_b}{R}$$



Thermal resistance for a cylindrical wall

$$R = \frac{\ln\frac{b}{a}}{2\pi k H} \quad \left[\frac{mK}{W\,m}\right]$$

1-D, steady heat conduction for a spherical shell without $\dot{q}'''$ & with $k = const$



<u>GDE</u> $\quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

$$\Rightarrow r^2\frac{dT}{dr} = C_1$$

$$\Rightarrow \boxed{T(r) = -\frac{C_1}{r} + C_2} \quad \text{—— (1)}$$

<u>B.C.</u> $\quad T(a) = T_a$, $\quad T(b) = T_b$

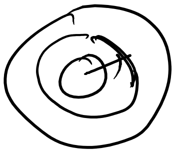
$$\left.\begin{array}{l} T_a = -\frac{C_1}{a} + C_2 \\ T_b = -\frac{C_1}{b} + C_2 \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} C_1 = -\frac{ab}{b-a}(T_a - T_b) \\ C_2 = \frac{bT_b - aT_a}{b-a} \end{array}\right\} \quad (3)$$

(*) I was missing this -ve sign

$$C_2 = \frac{b T_b - a T_a}{b-a}$$

$$\boxed{T(r) = \frac{a}{r}\left(\frac{b-r}{b-a}\right)\cdot T_a + \frac{b}{r}\cdot\left(\frac{r-a}{b-a}\right) T_b} \quad (4)$$



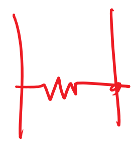
$$\dot{Q} = -kA \times \left.\frac{dT}{dr}\right|_r = -k\, \underbrace{4\pi r^2 \frac{dT}{dr}}_{C_1}\Big|_r$$

$$\dot{Q} = -k \times 4\pi\, C_1$$

$$= +4\pi k \frac{(T_a - T_b)}{\left(\frac{b-a}{ab}\right)} = \frac{T_a - T_b}{\frac{1}{4\pi k}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$\dot{Q} = \frac{T_a - T_b}{R}$$

$$R = \frac{1}{4\pi k}\left[\frac{1}{a} - \frac{1}{b}\right] \qquad \left[\frac{\cancel{m}K}{W\ \cancel{m}}\right]$$



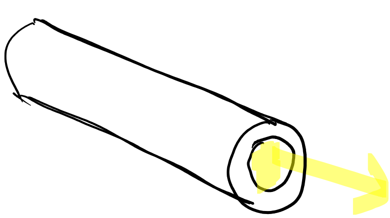
Thermal resistances for

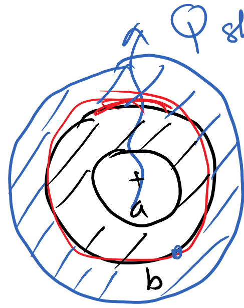
1. Plane wall of thickness L $\qquad R_{plane} = \dfrac{L}{kA}$

2. Cylindrical shell of height H and inner & outer radii of a & b $\qquad R_{cyl.} = \dfrac{\ln b/a}{2\pi k H}$

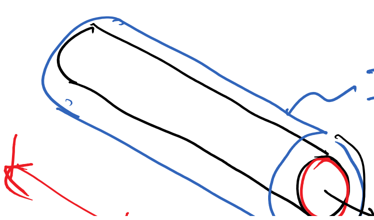
3. Spherical shell of inner & outer radii a & b $\qquad R_{sph} = \dfrac{1}{4\pi k}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

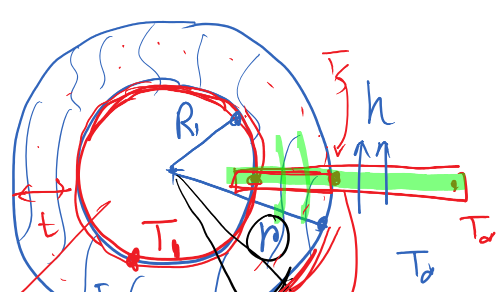
$\dot{Q}$ should be minimum

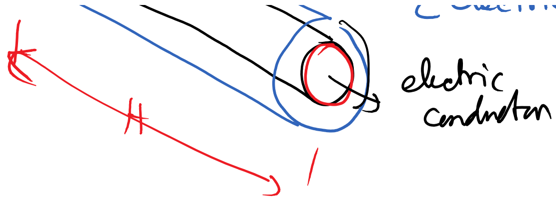


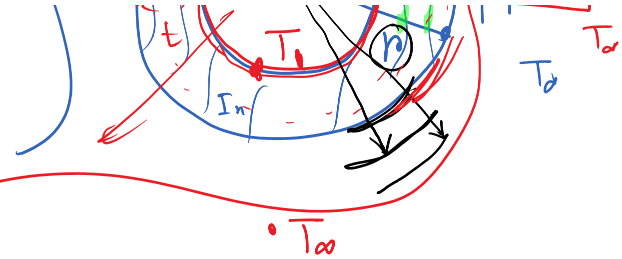


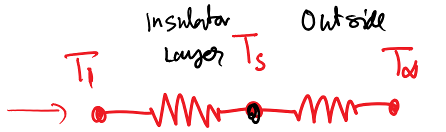
T_∞







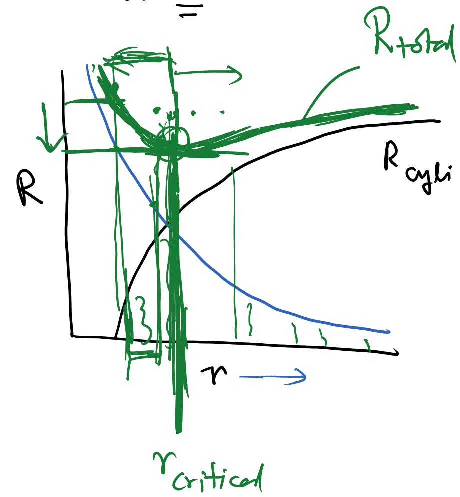




$$R_{cyl} = \frac{\ln r/R}{2\pi k H} \qquad R_{conv} = \frac{1}{hA} = \frac{1}{2\pi r H\, h}$$

$$R_{tot} = R_{cyl.} + R_{conv}$$

$$\boxed{R_{tot} = \frac{\ln r/R}{2\pi k H} + \frac{1}{2\pi H h\, r}}$$



$$\frac{dR_{tot}}{dr} = \frac{1}{r}\,\frac{1}{2\pi k H} - \frac{1}{2\pi H h}\,\frac{1}{r^2}$$

$$\frac{d^2R_{tot}}{dr^2} = -\frac{1}{r^2}\,\frac{1}{2\pi k H} + \frac{1}{\pi H h\, r^3}$$

For extreme value of R_{tot} $\quad \frac{dR_{tot}}{dr} = 0 \Rightarrow \frac{1}{r}\,\frac{1}{2\pi k \cancel{H}} = \frac{1}{2\pi \cancel{H} h}\,\frac{1}{r^2}$

$$\Rightarrow \boxed{r_{crit} = \frac{k}{h}} \qquad \left[m\,\frac{\cancel{W}}{m\cancel{K}} \cdot \frac{m^2 \cancel{K}}{\cancel{W}} \right]$$

put $r = k/H$

$$\frac{d^2R_{tot}}{dr^2} = -\frac{h^2}{k^2} \times \frac{1}{2\pi k H} + \frac{1}{\pi H \cancel{k}}\,\frac{h^{\cancel{3}\,2}}{\cdot k^3}$$

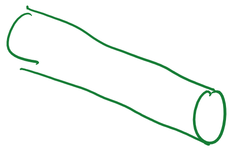
$$= +\frac{1}{2\pi}\,\frac{h^2}{k^3 H} \qquad \text{+ve}$$

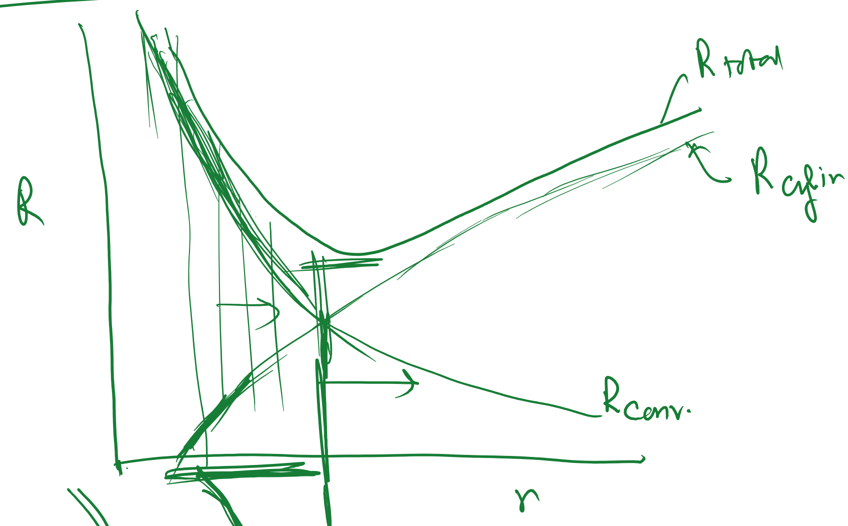
R_{tot} is minimum when $\boxed{r = \frac{k}{h}}$

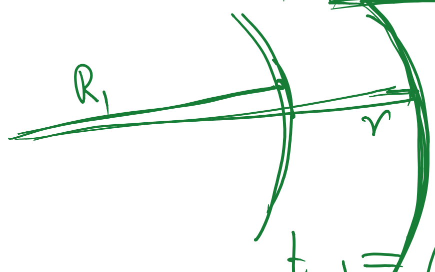
r_{crit} is the critical thickness of insulation

R_{total}

r_{crit} is the



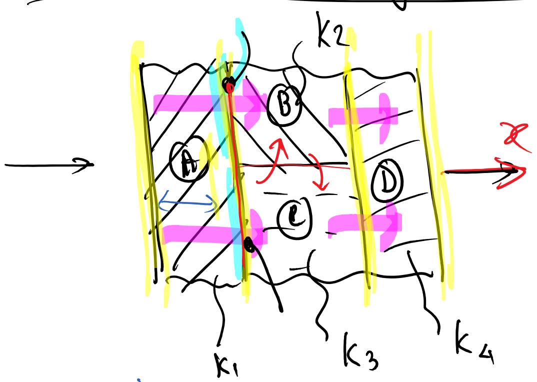




$$R = \frac{1}{hA}$$

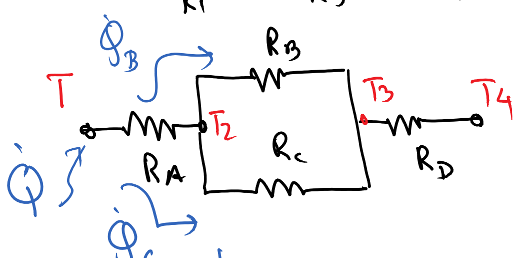
$$t_{crit} = (r_{crit} - R_1)$$

Thermal resistance by a composite wall



1. Heat Transfer is 1D, steady
2. No heat generation
3. Materials have different k but they are homogeneous

(1a) all surfaces ⊥r to the heat transfer are isothermal



$L_C = L_B$

$$R_A = \frac{L_A}{k_A A_A} \qquad R_B = \frac{L_B}{k_B A_B} \qquad R_D = \frac{L_D}{k_D A_D}$$

$$R_C = \frac{L_C}{k_C A_C}$$

$$\dot{Q}_B = \frac{T_3 - T_2}{R_B} \qquad (\dot{Q}_B + \dot{Q}_C) = \dot{Q}_A = \dot{Q}_D$$

$$\dot{Q}_c = \frac{T_3 - T_2}{R_c}$$