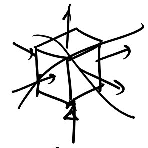
Generalized Conduction Heat Transfer Equation

$$\boxed{\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}'''}$$



$T(x,y,z,t)$

1-D Heat conduction

$$\frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \ll \frac{\partial T}{\partial x}$$

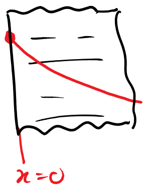
$$\boxed{\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q}'''}$$

pde
2nd order in space
1st order in time

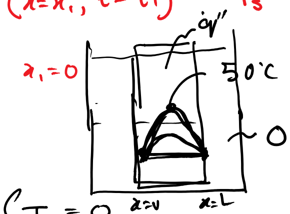
Differential eqn. $\longrightarrow$

Boundary Conditions

1) Specified temperature (Dirichlet type BC)
[BC of Type 1]



$T(x=x_1, t=t_1) = T_s$



$T_{x=0} = 0$
$T_{x=L} = 0$

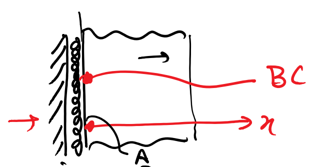
Initial cond. ~~at~~ $t=0$
$T(x,\cancel{0}) = 50$

2) Neumann condn.
[Specified heat flux BC]
B.C. of Type 2

A) Non-zero, heat flux

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=x_1} = q''_s$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q''_s$$

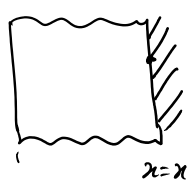


$\dot{Q} = V \cdot I$
$q''_s = (V \cdot I / A)$

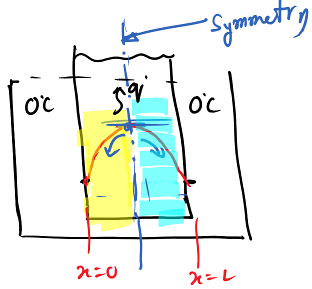
B) Zero Heat flux

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=x_2} = 0$$

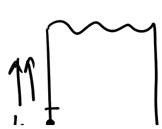
$$\boxed{\left.\frac{\partial T}{\partial x}\right|_{x=x_2} = 0}$$



$\frac{dT}{dx} = 0$ if symmetry prevails

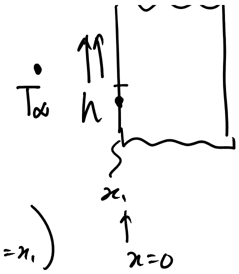


3) Convective B.C.
BC of type 3



BC of type 3
(Robin BC)

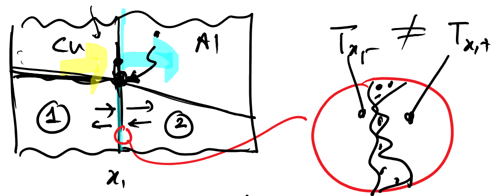
$$q''_w = h(T_\infty - T_{x=x_1})$$



$$-k \frac{\partial T}{\partial x}\Big|_{x=x_1} = h(T_\infty - T_{x=x_1})$$

$$\boxed{-k \frac{\partial T}{\partial x}\Big|_{x=0} = h(T_\infty - T_{x=0})}$$

4) Conjugat H.T. B.C.
(B.C. of Type 4)



continuity/equality of heat flux

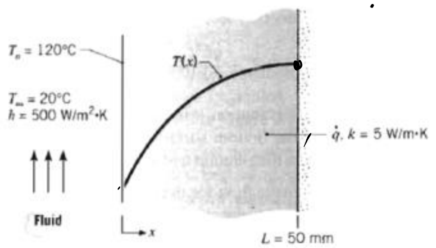
$$k_1 \frac{\partial T}{\partial x}\Big|_{x_1-} = k_2 \frac{\partial T}{\partial x}\Big|_{x_1+}$$

$T_{x_1-} = T_{x_1+}$ } if the contact is intimate

$T_{x_1-} \neq T_{x_1+}$ } if there is contact gap

4. One-dimensional, steady-state conduction with uniform internal energy generation occurs in a plane wall with a thickness of 50 mm and a constant thermal conductivity of 5 W/m · K. For these conditions, the temperature distribution has the form, $T(x) = a + bx + cx^2$. The surface at $x = 0$ has a temperature of $T(0) \equiv T_o = 120°C$ and experiences convection with a fluid for which $T_\infty = 20°C$ and $h = 500\ W/m^2 \cdot K$. The surface at $x = L$ is well insulated.

steady-state



(a) Applying an overall energy balance to the wall, calculate the internal energy generation rate, $\dot{q}$.

(b) Determine the coefficients a, b, and c by applying the boundary conditions to the prescribed temperature distribution. Use the results to calculate and plot the temperature distribution.

(c) Consider conditions for which the convection coefficient is halved, but the internal energy generation rate remains unchanged. Determine the new values of a, b, and c, and use the results to plot the temperature distribution. *Hint:* recognize that $T(0)$ is no longer 120°C.

(d) Under conditions for which the internal energy generation rate is doubled, and the convection coefficient remains unchanged ($h = 500\ W/m^2 \cdot K$), determine the new values of a, b, and c and plot the corresponding temperature distribution. Referring to the results of parts (b), (c), and (d) as Cases 1, 2, and 3, respectively, compare the temperature distributions for the three cases and discuss the effects of h and $\dot{q}$ on the distributions.

Home Work

$T(x)$

Generalized 1-d Conduction eqn.

$$\boxed{\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \dot{q}'''}$$

Steady: → $\frac{\partial}{\partial t} = 0$

$k =$ const →

$$0 = k\frac{d^2T}{dx^2} + \dot{q}'''$$

$$\frac{d^2T}{dx^2} = \left(-\frac{\dot{q}'''}{k}\right) \quad \text{— (1)}$$

$$\boxed{T = \left(-\frac{\dot{q}'''}{k} \times \frac{x^2}{2}\right) + C_1 x + C_2} \quad \text{— (2)}$$

General Sol. of (1)

$$T = a + bx + cx^2$$

$a \to\ ?$, $b \to\ ?$, $c \to \left(-\dot{q}'''/2k\right)$

BC

$x = 0$, $T_{x=0} = 120°C$

$$\boxed{a = 120°C} \quad \text{— (2)}$$

$$\boxed{x = L, \quad \frac{dT}{dx} = 0}$$

$$T = a + bx + cx^2$$

$$\boxed{b + 2cL = 0}$$

$$\dot{q}''_s = h(T_\infty - T_s)$$

$$-k\left.\frac{dT}{dx}\right|_{x=0} = -h\Delta T$$

$\boxed{\Delta T = +ve = 100°C}$

20 → 120

$$T = a + bx + cx^2$$

$$\left.\frac{dT}{dx}\right|_{x=0} = b$$

$$\cancel{-}kb = \cancel{-}h\Delta T$$

or $\boxed{b = \frac{h\Delta T}{k}}$

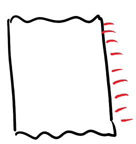
(a) = T_s
(b) = $h\Delta T/k$

(c) = $-b/2L = -\frac{h\Delta T}{2kL}$

$$c = -\frac{\dot{q}'''}{2\cancel{k}} = -\frac{h\Delta T}{2\cancel{k}L}$$

$$\Rightarrow \boxed{\dot{q}''' = \frac{h}{L}\Delta T = \frac{h(T_s - T_\infty)}{L}}$$

Eq. 1
$$\int_{x=0}^{L}\left(\frac{d^2T}{dx^2} + \frac{\dot{q}'''}{k}\right)dx = 0$$

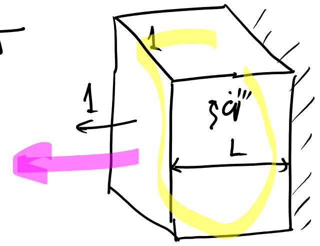


$$\Rightarrow \left[\cancel{\left.\frac{dT}{dx}\right|_{x=L}} - \left.\frac{dT}{dx}\right|_{x=0}\right] + \frac{\dot{q}'''L}{k} = 0$$

$$0 - \frac{h\Delta T}{k} + \frac{\dot{q}'''L}{k} = 0$$

$$\Rightarrow \dot{q}''' = \frac{h\Delta T}{L}$$

Alternately



Heat generated within the block

$1 \times 1 \times L \times \dot{q}'''$

Heat lost through the left face $= h \times A \times (T_s - T_\infty)$

$= h \Delta T$

Under steady state

$\dot{q}''' L = h \Delta T \Rightarrow \boxed{\dot{q}''' = \frac{h\Delta T}{L}}$

If h is halved

ΔT should be doubled

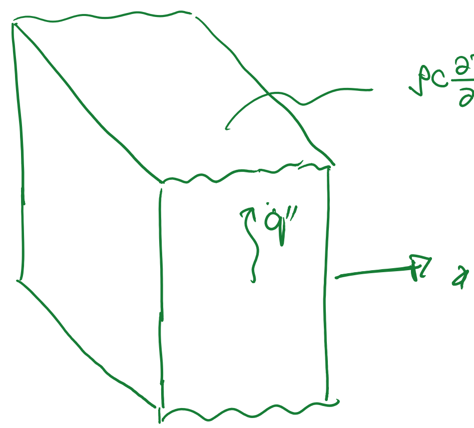
Plotting

Use standard Graph plotting tools

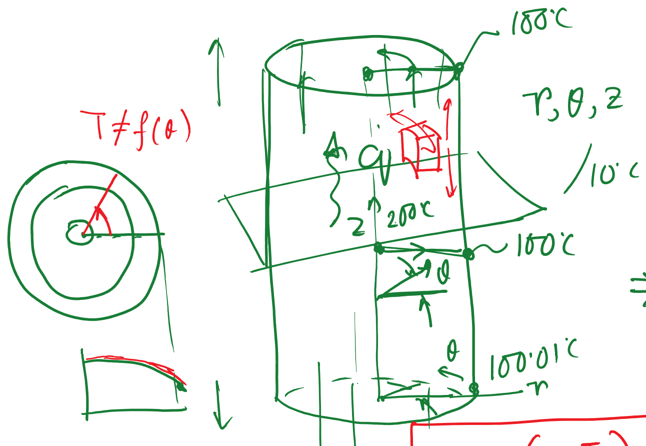
Excel

Matlab

1D Conduction Equation for cylindrical & spherical geometries



$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \dot{q}'''$$



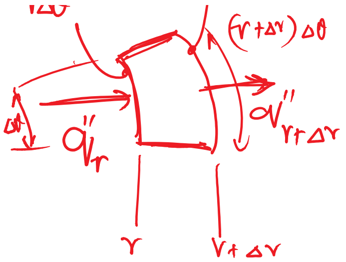
If $\frac{\partial T}{\partial z} \ll \frac{\partial T}{\partial r}$

& $\cancel{\frac{\partial T}{\partial \theta}} = 0$ (Axisymmetric)

$\Rightarrow$ 1-D Heat transfer in cylindrical coordinates system

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \dot{q}''' = \rho c \frac{\partial T}{\partial t}}$$

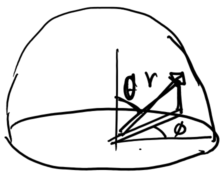
1 D Heat Conduction eqn in cylindrical coordinates



cylindrical coordinates

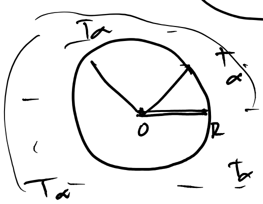
Heat Conduction Eqn. in Spherical Coordinates

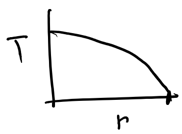
$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial \theta} = 0$$



Spherico-symmetric configuration

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(k r^2 \frac{\partial T}{\partial r}\right) + \dot{q}''' = \rho C \frac{\partial T}{\partial t}}$$





Cartesian System $\boxed{\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \dot{q}''' = \rho C \frac{\partial T}{\partial t}}$

Short hand

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left[k r^n \frac{\partial T}{\partial r}\right] + \dot{q}''' = \rho C \frac{\partial T}{\partial t}$$

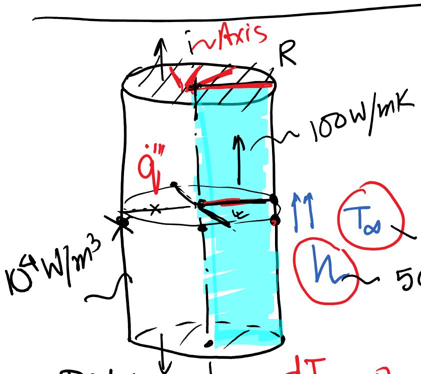
$n=0$ for Cartesian coordinates systems

$n=1$ " Cylindrical " "

$n=2$ " Spherical " "

Under steady state $\boxed{\frac{1}{r^n}\frac{d}{dr}\left[k r^n \frac{dT}{dr}\right] + \dot{q}''' = 0}$

for $k =$ Const. $\boxed{\frac{1}{r^n}\frac{d}{dr}\left(r^n \frac{dT}{dr}\right) + \frac{\dot{q}'''}{k} = 0}$



Find out the temp. distribution inside the conductor under steady state (Axi-symmetric)

Governing Eqn. $\frac{1}{r^n}\frac{d}{dr}\left(r^n \frac{dT}{dr}\right) + \frac{\dot{q}'''}{k} = 0$

$n=1$

$$\Rightarrow \boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \dot{q}'''/k = 0} \quad (1)$$

BC1: $r=0$ $\frac{dT}{dr} = 0$ —— (A)

BC2: $k\frac{dT}{...} = h(T-T_\infty)$ (B) $\frac{d}{...}\left(r\frac{dT}{...}\right) = -\frac{\dot{q}''' r}{...}$

BC1: $r = 0 \quad \frac{dT}{dr} = 0$ —— (A)

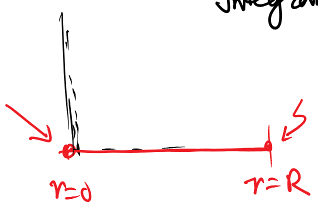
BC2: $-k\frac{dT}{dr}\Big|_{r=R} = h(T - T_\infty)$ —— (B)

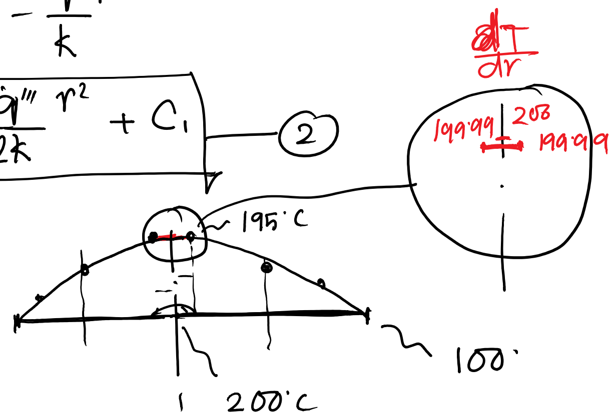
T at R

Integrate w.r.t. r

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}''' r}{k} \quad —— (1)$$

$$\boxed{r\frac{dT}{dr} = -\frac{\dot{q}'''}{2k}r^2 + C_1} \quad —— (2)$$





Apply BC – A in eqn. (2)

$$r\cancel{\frac{dT}{dr}}^{\,0} = -\frac{\dot{q}'''}{2k}\times 0^2 + C_1 \Rightarrow C_1 = 0$$

$$r\frac{dT}{dr} = -\frac{\dot{q}'''}{2k}r^2$$

or $$\boxed{\frac{dT}{dr} = -\frac{\dot{q}'''}{2k}r} \quad —— (3)$$

Integrate w.r.t. r

$$\boxed{T(r) = -\frac{\dot{q}'''}{4k}r^2 + C_2} \quad —— (4)$$

$$\boxed{-k\frac{dT}{dr}\Big|_{r=R} = h(T - T_\infty)}$$

r=R

$$\cancel{-}\cancel{k}\frac{\dot{q}''' R}{2\cancel{k}} = h(T_{r=R} - T_\infty)$$

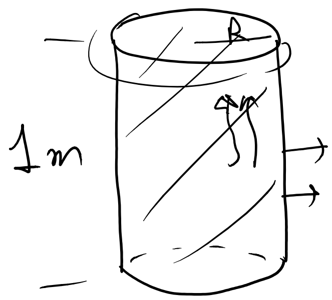
$$\Rightarrow \boxed{T_R = -\frac{\dot{q}''' R}{2h} + T_\infty} \quad —— (5)$$

$$T_R = -\frac{\dot{q}''' R^2}{4k} + C_2 = -\frac{\dot{q}''' R}{2h} + T_\infty$$

$$\Rightarrow \quad C_2 = -\dot{q}'''\left(\frac{R}{2h} - \frac{R^2}{4k}\right) + T_\infty$$

$$T(r) = -\frac{\dot{q}'''}{4k}(r^2 - R^2) + \frac{\dot{q}''' R}{2h} + T_\infty$$

$$\boxed{T(r) = \frac{\dot{q}''' R^2}{4k}\left(1 - \frac{r^2}{R^2}\right) + \frac{\dot{q}''' R}{2h} + T_\infty}$$

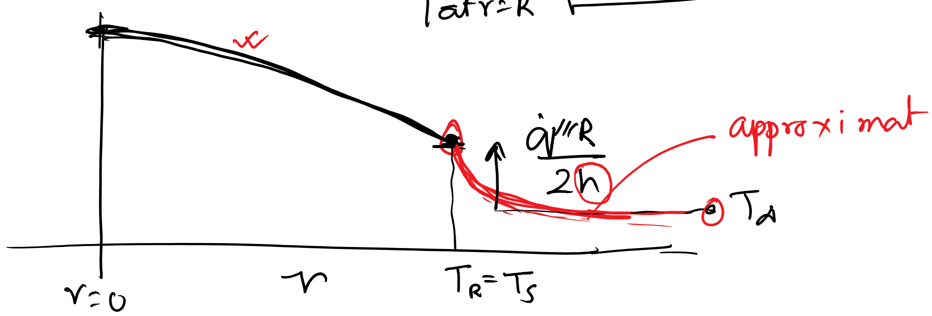


$$\dot{q}_{cyl.} = \dot{q}''' \times \cancel{\pi} R^{\cancel{2}}\, 1 = h(2\cancel{\pi}\cancel{R}\,1)(T_s - T_\infty)$$

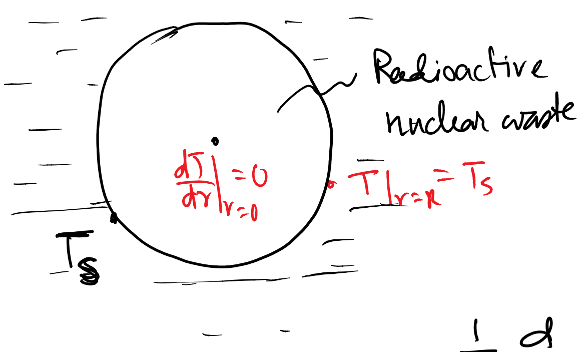
$$\Rightarrow \frac{\dot{q}''' R}{2h} = T_s - T_\infty$$

or $\boxed{T_s = \frac{\dot{q}''' R}{2h} + T_\infty}$

T at r=R



1 D heat conduction in sphere



The sphere surface temp is T_s

Find $T(r)$ and T_{max} at the Centre in terms of T_s and $\dot{q}'''$

Consider steady state

$$\frac{1}{r^n}\frac{d}{dr}\left(r^n \frac{dT}{dr}\right) + \frac{\dot{q}'''}{k} = 0$$

$n = 2$
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \dot{q}'''/k = 0$$

$$\Rightarrow \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = -\frac{\dot{q}'''}{k} r^2$$

integrating w.r.t. r
$$r^2 \frac{dT}{dr} = -\frac{\dot{q}'''}{k}\frac{r^3}{3} + C_1$$

at $r=0$ $\frac{dT}{dr} = 0 \Rightarrow$ $C_1 = 0$

$$\cancel{r^2}\frac{dT}{dr} = -\frac{\dot{q}''' r^{\cancel{3}}}{3k}$$

$$\frac{dT}{dr} = -\frac{\dot{q}''' r}{3k}$$

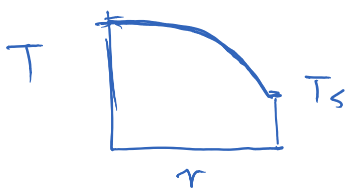
$$\frac{\quad}{dr} = \frac{\quad}{3k}$$

$$T = -\frac{\dot{q}''' r^2}{6k} + C_2$$

at $r = R$, $T = T_s \Rightarrow T_s = -\frac{\dot{q}''' R^2}{6k} + C_2$

$$\Rightarrow C_2 = T_s + \frac{\dot{q}''' R^2}{6k} \quad \text{or} \quad \left[\frac{\cancel{W}}{\cancel{m^3}} \cdot \frac{\cancel{m^2}\ \cancel{m}K}{\cancel{W}}\right]$$

$$T(r) = T_s + \frac{\dot{q}''' R^2}{6k}\left(1 - r^2/R^2\right)$$



$$\boxed{\underline{T_o} = T_s + \frac{\dot{q}''' R^2}{6k}}$$