Heat Transfer from Extended Surfaces: Fins

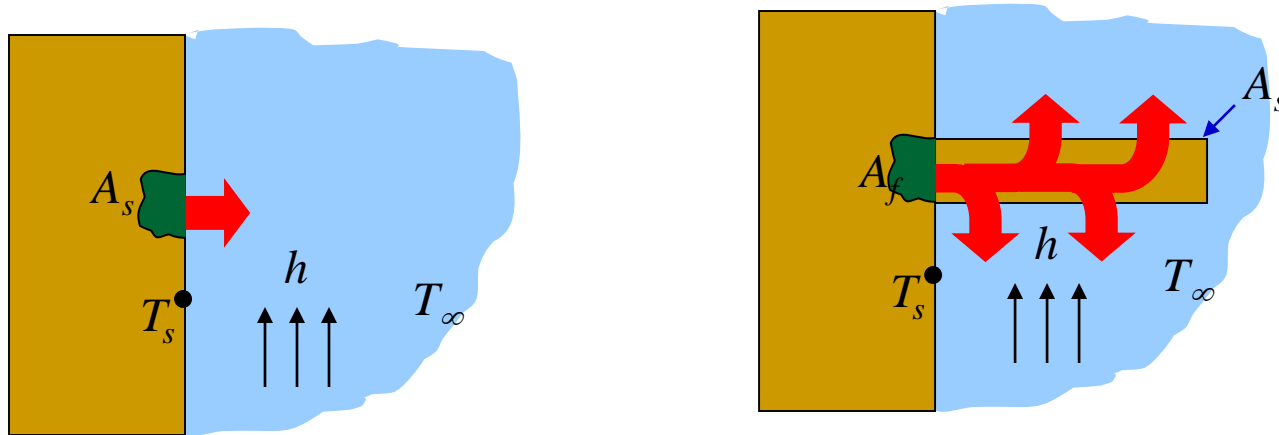Ranjan Ganguly

Strategies of increasing heat transfer from a surface

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

- Increase the temperature difference $(T_s - T_\infty)$ [may lead to surface overheating]
- Increase the heat transfer coefficient h [there is always a practical upper bound*]
- Increase area A_s
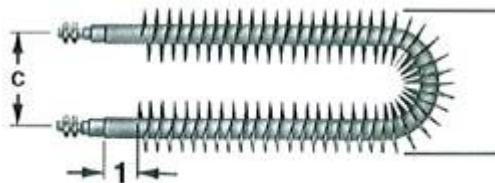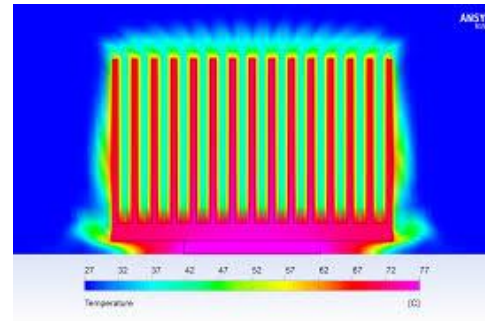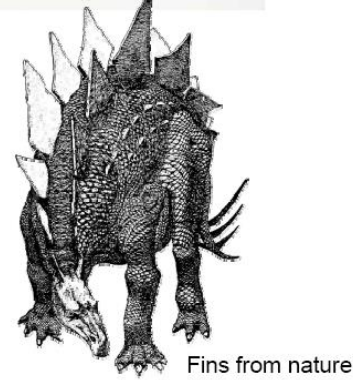

But the fin surface temperature is not T_s (because the fin material has a finite k)

[* Would require installing a larger fan/ pump, would entail larger pressure drop, may not be economically viable]

Fins used in engineering applications

- Electronics cooling
- Heat exchanger tubes
- Car radiators
- Engine cylinders
- AC…

Fins from nature

Radial fins

Longitudinal fins

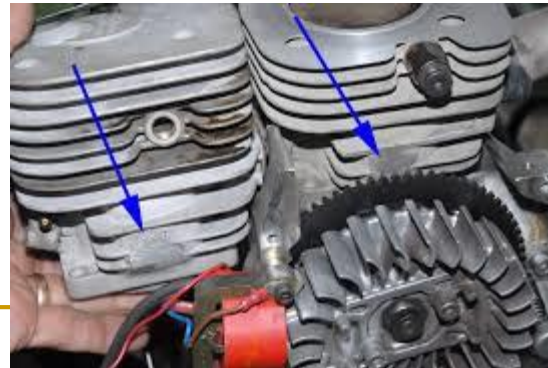
Fins in car radiators
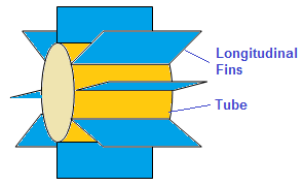
Fins on engine cylinders

Plate Fin Heat Sink

Radial Fin Heat Sink With Circular Fins
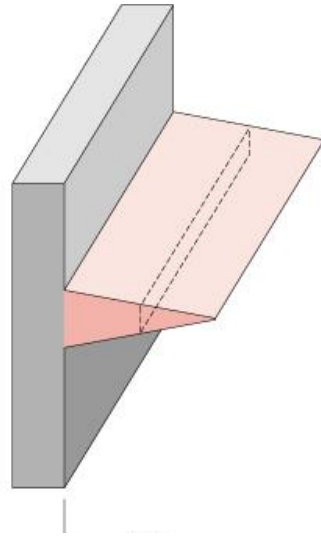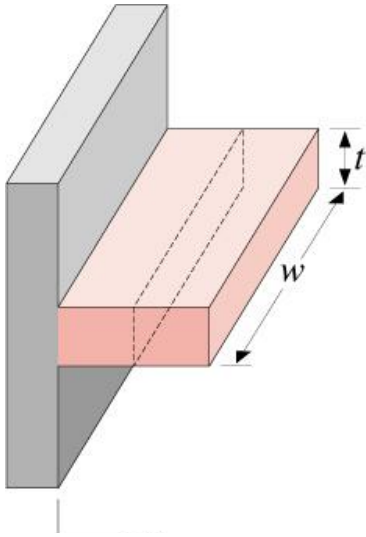
Radial Fin Heat Sink With Square Fins

Different types of straight fins

Longitudinal Fins
(a) uniform and (b) non-uniform cross sections
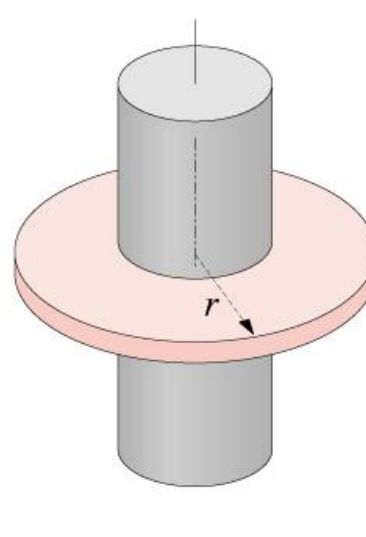
Annular fin
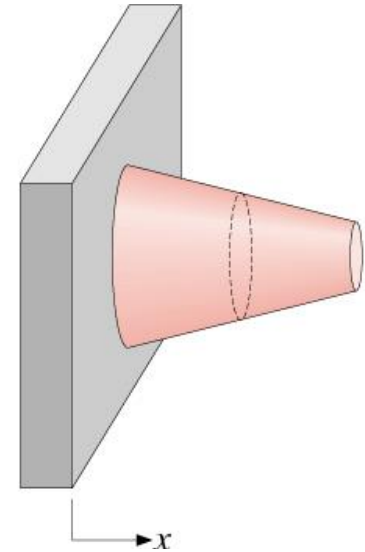
Pin fin of non-uniform cross section

Pin fin of uniform cross section

Temperature distribution on the extended surface

T_b

T_∞

x
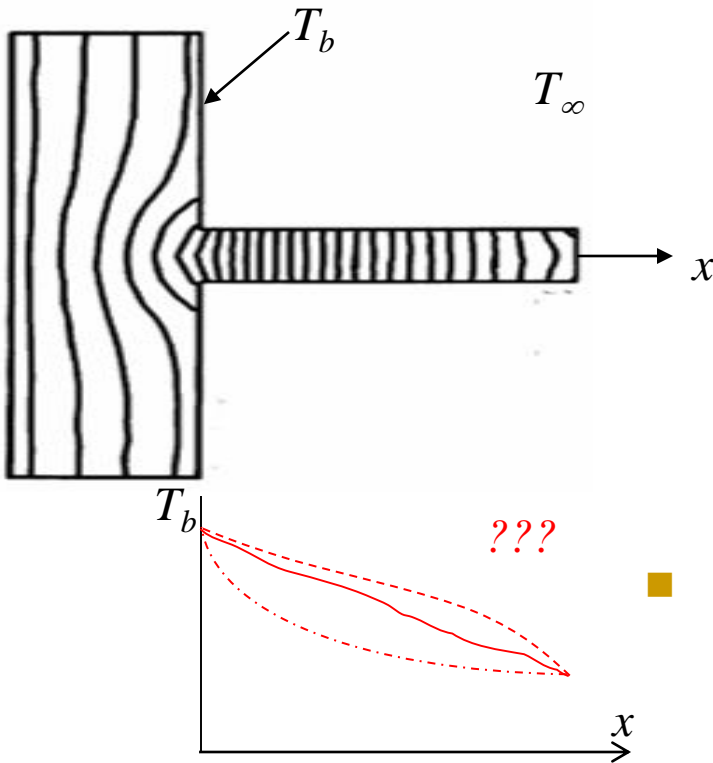
Fin with low k Fin with high k

T_b

???

x
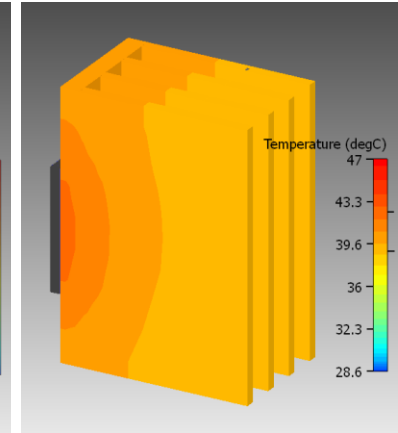
- **1-D heat transfer**
  - Temperature within the fin varies primarily in the x direction
  - Nearly homogeneous $T(x)$ within the fin at a particular cross-section

1-D analysis of pin fin

Assumptions:
- 1-D, steady state heat conduction
- Uniform k and h along the x
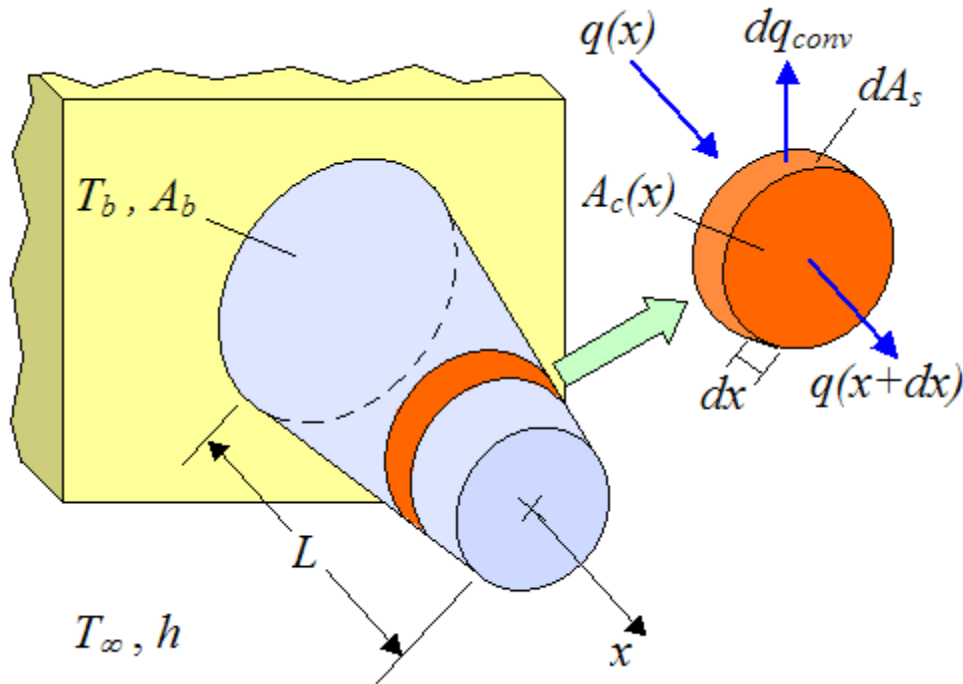
Energy balance across the slice

$$q(x) = q(x + dx) + dq_{conv} \quad \text{(i)}$$

Fourier's Law of Heat Conduction

$$q(x) = -kA_c(x)\frac{dT}{dx}$$
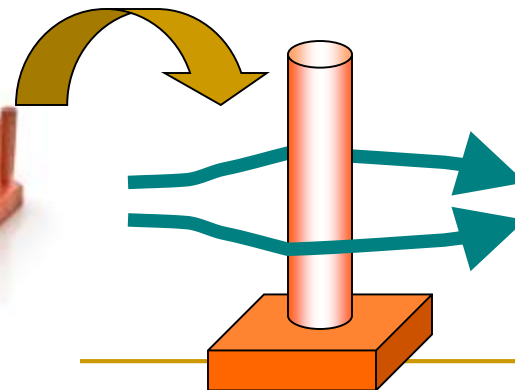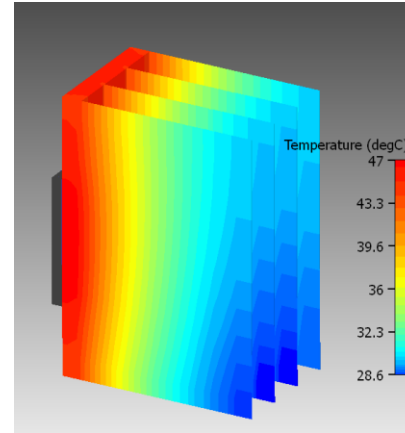
Taylor series expansion

$$q(x + dx) = q(x) + \frac{d}{dx}\{q(x)\}dx$$

$$= q(x) + \frac{d}{dx}\left\{-kA_c(x)\frac{dT}{dx}\right\}dx \quad \text{(ii)}$$

$$dq_{conv} = hdA_s(x)(T - T_\infty) \quad \text{(iii)}$$

From (i), (ii) and (iii)

$$q(x) = q(x) + \frac{d}{dx}\left\{-kA_c(x)\frac{dT}{dx}\right\}dx + hdA_s(x)(T - T_\infty)$$

1-D analysis of pin fin, contd…
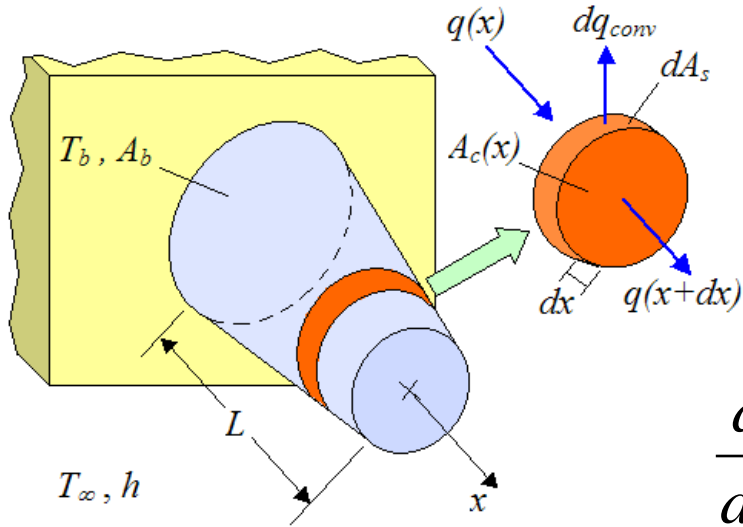
$$\frac{d}{dx}\left\{-kA_c\left(x\right)\frac{dT}{dx}\right\}+h\frac{dA_s\left(x\right)}{dx}\left(T-T_\infty\right)=0$$

$$\frac{d}{dx}\left\{A_c\left(x\right)\frac{dT}{dx}\right\}-\frac{h}{k}\frac{dA_s\left(x\right)}{dx}\left(T-T_\infty\right)=0$$

$$\left\{\frac{d^2T}{dx^2}\right\}+\frac{1}{A_c\left(x\right)}\left\{\frac{dA_c\left(x\right)}{dx}\right\}\left\{\frac{dT}{dx}\right\}-\frac{1}{A_c\left(x\right)}\frac{h}{k}\frac{dA_s\left(x\right)}{dx}\left(T-T_\infty\right)=0$$

For fins of uniform cross se $dA_s = P(x)\,dx = Pdx$
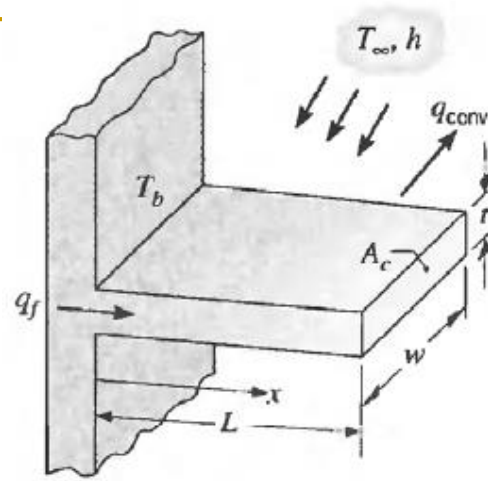
$$P = 2w + 2t$$
$$A_c = wt$$
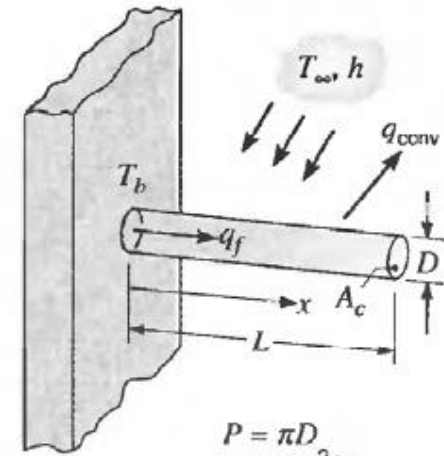
$$P = \pi D$$
$$A_c = \pi D^2/4$$

$$\left\{ \frac{d^2T}{dx^2} \right\} + \left(-\frac{1}{A_c} \cdot \frac{h}{k} \cdot P \right) (T - T_\infty) = 0$$

$$\left\{ \frac{d^2T}{dx^2} \right\} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$$\theta = (T - T_\infty); \quad m = \sqrt{\frac{hP}{kA_c}}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

θ can be looked up as the "excess temperature"; *m* as the ratio of convective to conductive HT

Fin Equation

General Equation

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Boundary Conditions

At the Fin Base

$$\theta(0) = \theta_b = T_b - T_\infty$$

At the Fin Tip???
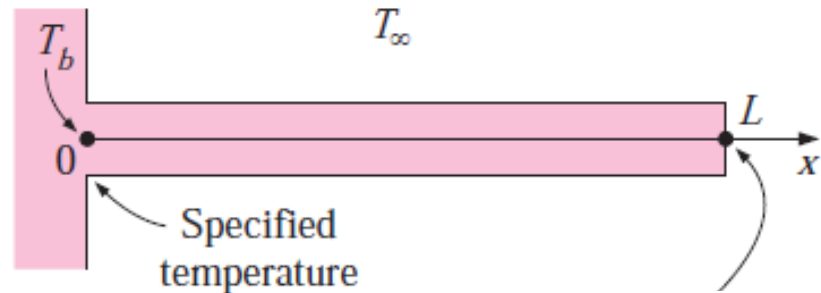
Fin Tip BC

A. *Infinitely long Fin*
B. *Prescribed fin tip temperature*
C. *Adiabatic fin tip*
D. *Convective BC at fin tip also*

Infinitely long fin

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Boundary Conditions:

$$x = 0 \Rightarrow \theta(0) = \theta_b$$

$$L \to \infty \Rightarrow \theta_L = (T_L - T_\infty) = \theta_L \to 0$$

Imposing the two BCs

$$\theta(x) = \theta_b e^{-mx}$$

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}}$$

$$\dot{Q}_{\text{long fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$

T_b h, T_∞

k D

$A_b = A_c$
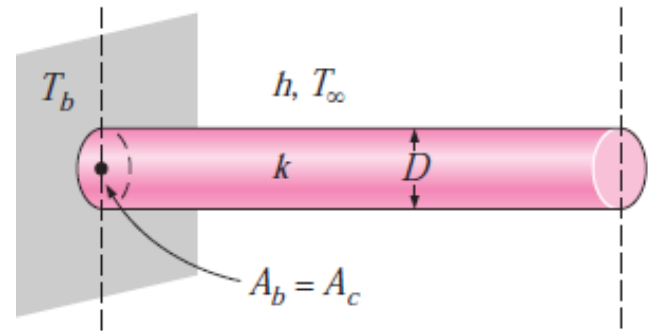
$(p = \pi D, A_c = \pi D^2/4$ for a cylindrical fin$)$

$$T(x) = T_\infty + (T_b - T_\infty)e^{-x\sqrt{\frac{hp}{kA_c}}}$$
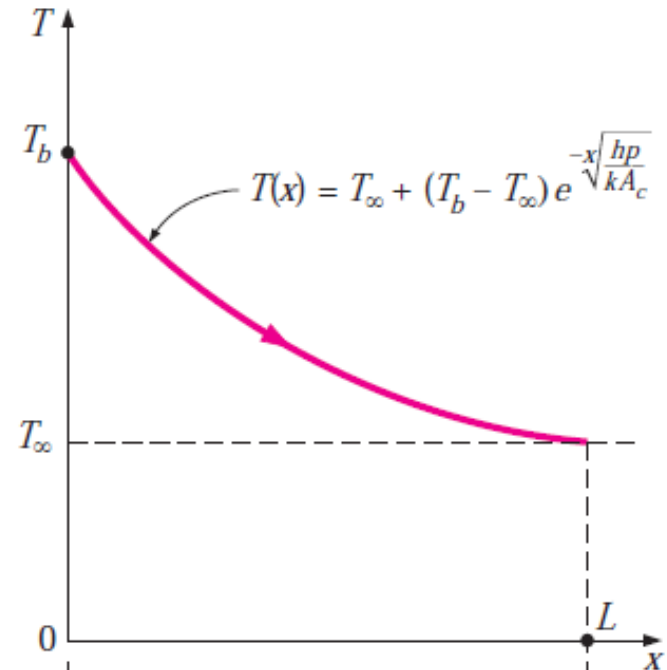
Prescribed fin tip temperature

$$\theta(0) = \theta_b \qquad \boxed{\theta_b = C_1 + C_2}$$

$$\theta(L) = \theta_L \qquad \boxed{\theta_L = C_1 e^{mL} + C_2 e^{-mL}}$$

$$\frac{\theta}{\theta_b} = \frac{(\theta_L / \theta_b) \sinh\, mx + \sinh\, m(L-x)}{\sinh\, mL}$$

$$\boxed{\dot{Q}_{f,T_L} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hPkA_c}\; \theta_b \frac{(\cosh\, mL - \theta_L / \theta_b)}{\sinh\, mL}}$$

Insulated fin tip

General Sol: $\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$

BC 1: $\theta(0) = T_b - T_\infty \equiv \theta_b$

BC 2: $\dfrac{d\theta}{dx}\Big|_{x=L} = 0$

$$\frac{\theta}{\theta_b} = \frac{\cosh\ m(L-x)}{\cosh\ mL}$$
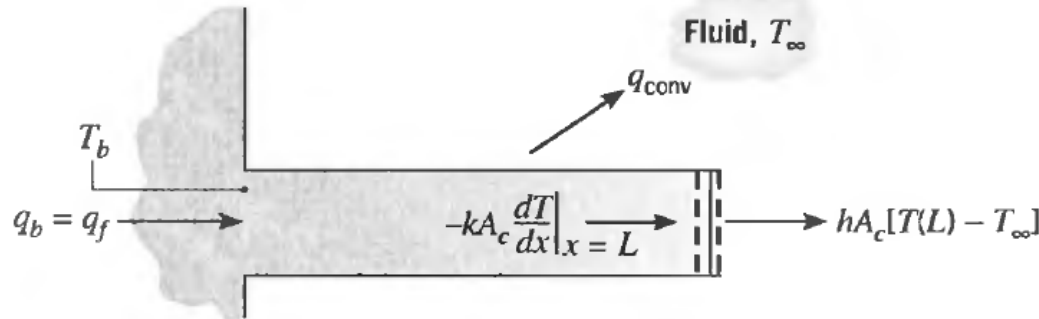
$$\dot{Q}_{f,adiabatic} = -kA_c \frac{dT}{dx}\Big|_{x=0} = \sqrt{hPkA_c}\ \theta_b\ \tanh\ mL$$

When do you have insulated fin tip?

Convective fin tip boundary condition

Fluid, T_∞

q_{conv}

q denotes $\dot{Q}$

T_b

$q_b = q_f \longrightarrow$

$-kA_c \dfrac{dT}{dx}\Big|_{x=L} \longrightarrow$

$hA_c[T(L) - T_\infty]$

BCs

$T(0) = T_\infty$

$\theta(0) = T_b - T_\infty \equiv \theta_b$

$hA_c\big[T(L) - T_\infty\big] = -kA_c\dfrac{dT}{dx}\Big|_{x=L}$

$h\theta(L) = -k\dfrac{d\theta}{dx}\Big|_{x=L}$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

θ_b

$\theta(x)$

0

0 L
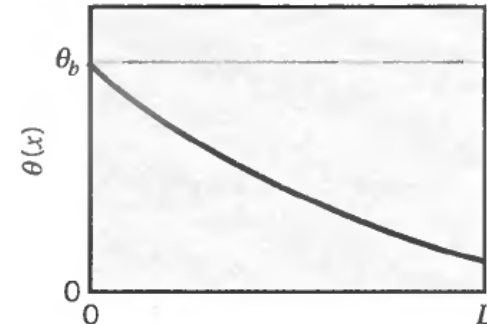
$$\dot{Q}_{f,convective} = -kA_c\frac{dT}{dx}\bigg|_{x=0} = \sqrt{hPkA_c}\ \theta_b\ \frac{\{sinh\ mL + (h/mk)\cosh\ mL\}}{\{cosh\ mL + (h/mk)\sinh\ mL\}}$$

Fin Performance

$\dot{Q}_{\text{fin}}$

$\dot{Q}_{\text{base}}$

$\dot{Q}_{\text{base}} = \dot{Q}_{\text{fin}}$
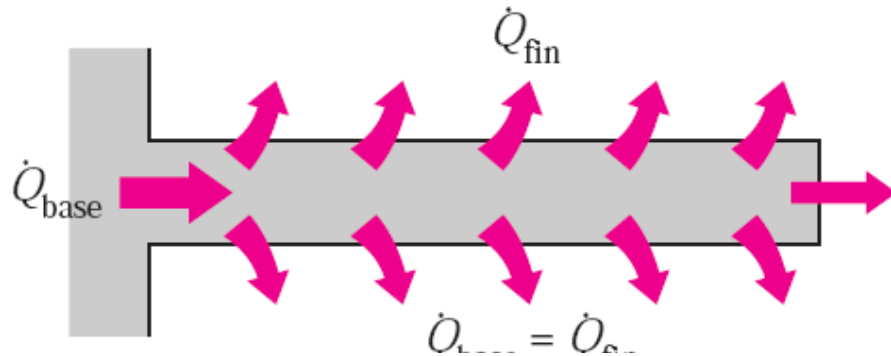
$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty] \, dA_{\text{fin}} = \int_{A_{\text{fin}}} h\theta(x) \, dA_{\text{fin}}$$

Fin Effectiveness

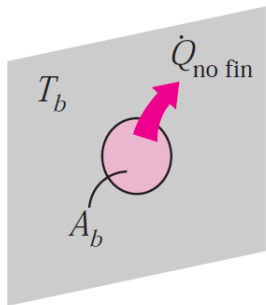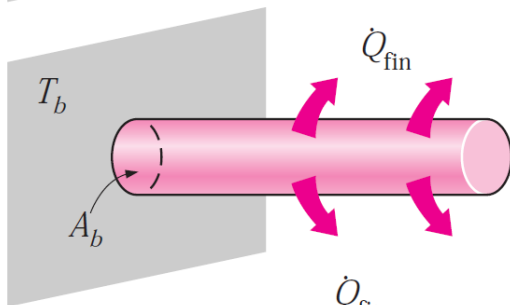$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b}$$

$$\text{longfin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)}{hA_b\,(T_b - T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}}$$

What is the right effectiveness?

$\varepsilon_{fin} = 1$ Does not affect the heat transfer at all.

$\varepsilon_{fin} < 1$ Fin act as insulation (if low k material is used)

$\varepsilon_{fin} > 1$ Enhancing heat transfer (use of fins justified if $\varepsilon_{\text{fin}} > 2$)

Factors leading to high fin effectiveness

1. k should be as high as possible, (copper, aluminum, iron).
   Aluminum is preferred: low cost and weight, resistance to corrosion.

2. p/A_c should be as high as possible. (Thin plate fins and slender pin fins)

3. Most effective in applications where h is low. (Use of fins justified if when the medium is gas and heat transfer is by natural convection).
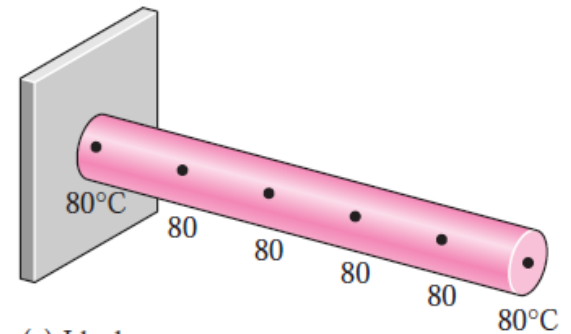
Fin Efficiency

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\,max}} == \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin}}$$
$$\text{if the entire fin were at base temperature}$$

$$\dot{Q}_{fin} = \eta_{fin}\,\dot{Q}_{fin,\,max} = \eta_{fin}\,hA_{fin}\,(T_b - T_\infty)$$

$$\eta_{long\,fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\,max}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)}{hA_{fin}\,(T_b - T_\infty)}$$

$$= \frac{1}{L}\sqrt{\frac{kA_c}{hP}} = \frac{1}{mL}$$

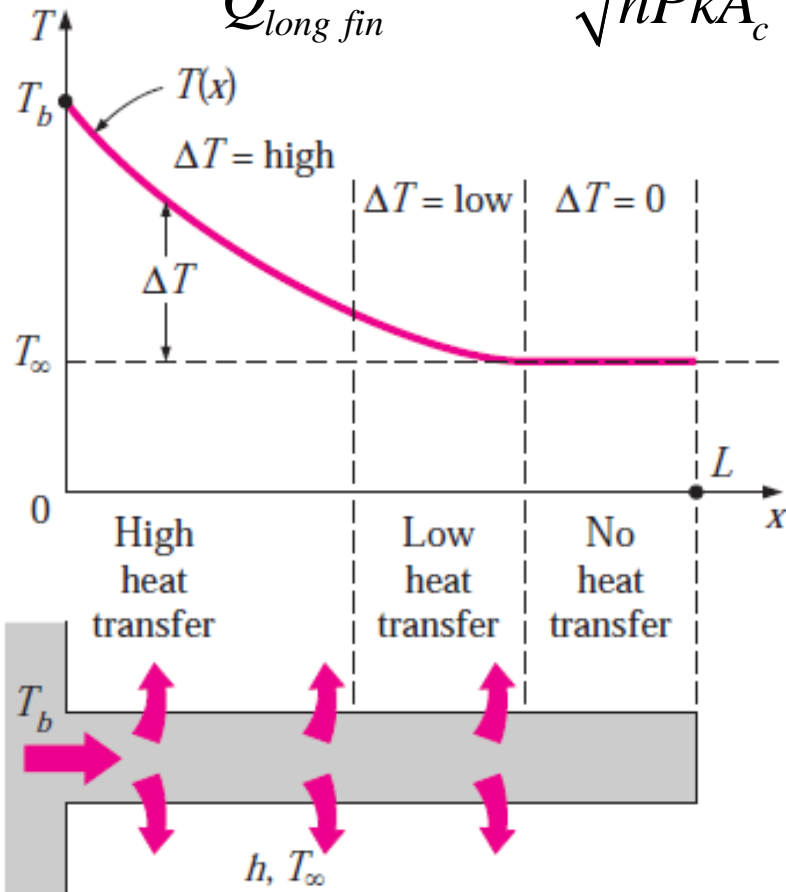80°C 80 80 80 80 80°C

(a) Ideal

80°C 70 65 61 58 56°C
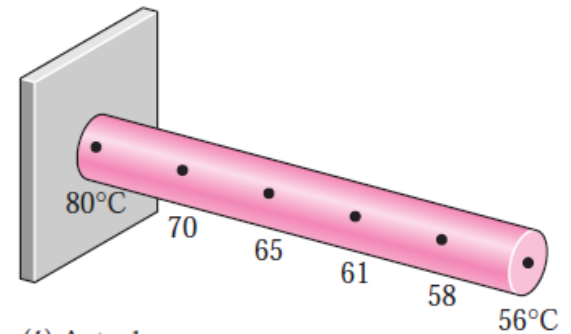
(b) Actual

Proper length of a Fin

$$\frac{Q_{fin}}{Q_{long\ fin}} = \frac{\sqrt{hPkA_c}\left(T_b - T_\infty\right)\tanh mL}{\sqrt{hPkA_c}\left(T_b - T_\infty\right)} = \tanh mL$$
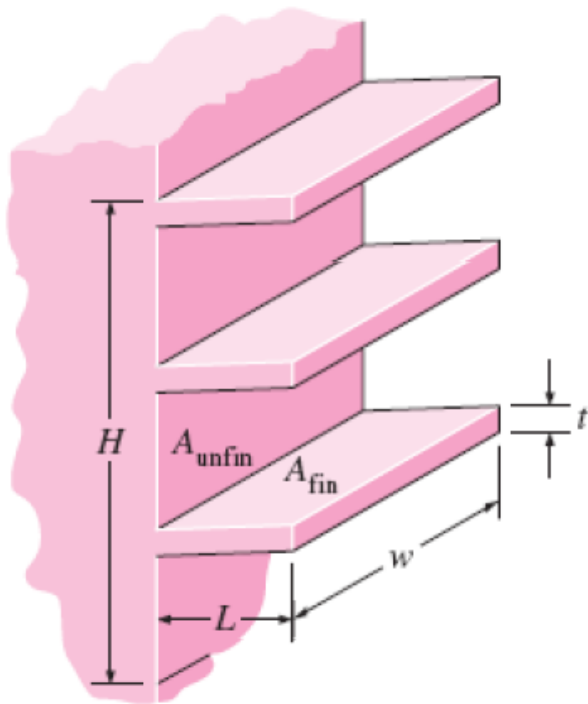

The variation of heat transfer from a fin relative to that from an infinitely long fin

mL	$\dfrac{\dot{Q}_{fin}}{\dot{Q}_{long\ fin}} = \tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

mL> 5 is not necessary

Overall fin efficiency

When determining the rate of heat transfer from a finned surface, we must consider the unfinned portion of the surface as well as the fins. Therefore, the rate of heat transfer for a surface containing n fins can be expressed as

$$\dot{Q}_{total,fin} = \dot{Q}_{unfin} + \dot{Q}_{fin}$$
$$= hA_{unfin}(T_b - T_\infty) + \eta_{fin}hA_{fin}(T_b - T_\infty)$$
$$= h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)$$

We can also define an **overall effectiveness for a finned surface as the ratio** of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins

$$A_{no\ fin} = w \times H$$
$$A_{unfin} = w \times H - 3 \times (t \times w)$$
$$A_{fin} = 2 \times L \times w + t \times w \text{ (one fin)}$$
$$\approx 2 \times L \times w$$

$$\varepsilon_{fin,overall} = \frac{\dot{Q}_{total,fin}}{\dot{Q}_{total,nofin}} = \frac{h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)}{hA_{nofin}(T_b - T_\infty)}$$

Fin Effectiveness vs Efficiency

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\left(T_b - T_\infty\right)} = \frac{\eta_{\text{fin}}\, hA_{\text{fin}}\left(T_b - T_\infty\right)}{hA_b\left(T_b - T_\infty\right)} = \frac{A_{\text{fin}}}{A_b}\, \eta_{\text{fin}}$$