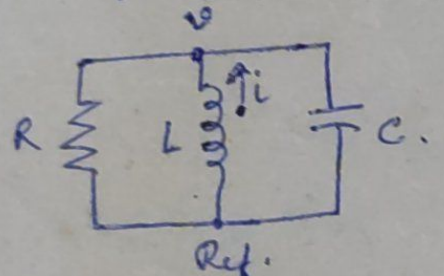
RLC circuits : Second order ~~ckt~~ system. ∴ two arbit. constants to be evaluated.

Source free parallel ckt.



— analogous to mass-spring-damper system
— vertical motion of automobile,
~~pendulum (torsional or simple)~~

— useful in communications networks
— part of electronic amplifiers in radio receivers. — frequency selectivity.
— also in multiplexing filters, harmonic suppression filters etc.

≡ physical inductor + capacitor in parallel.

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v\,dt - i(t_0) + C\frac{dv}{dt} = 0. \qquad i(0^+) = I_0 \,,\; v(0^+) = V_0.$$

Differentiating,

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0. \quad \ldots \text{ⓐ}$$

Assume $v(t) = Ae^{st}$

$$\therefore\ Ae^{st}\left(Cs^2 + \frac{1}{R}s + \frac{1}{L}\right) = 0. \Rightarrow s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0.$$

CHARACTERISTIC EQN. / AUXILIARY EQN..

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$\therefore v = v_1 + v_2 = A_1e^{s_1t} + A_2e^{s_2t}$ is the form of the natural response.

$s_{1,2}$ as well as $\frac{1}{2RC}$ and $\frac{1}{\sqrt{LC}}$ must have units s^{-1} for $s_{1,2}t$ to be dimensionless. $\xrightarrow{\therefore}$ FREQUENCIES

$$\omega_0 = \frac{1}{\sqrt{LC}} \triangleq \text{RESONANT FREQUENCY} \qquad \alpha = \frac{1}{2RC} \triangleq \text{EXPONENTIAL DAMPING COEFFICIENT or NEPER FREQUENCY}$$

s, s_1, s_2 : COMPLEX FREQUENCIES.

$$\therefore \quad v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad ; \quad \alpha = 1/2RC \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

$\zeta = \frac{\alpha}{\omega_0} \triangleq$ DAMPING RATIO. $= \frac{1}{2R}\sqrt{\frac{L}{C}}$ $\quad A_1, A_2$ obtained from i.c.s.

(zeta)

Note : $i_{L,R,C}$ are also of similar forms

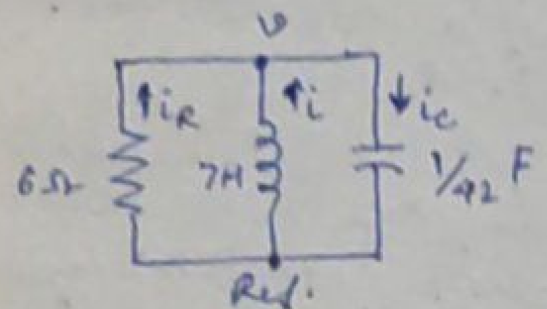
$\zeta >, =, < 1$ determines 3 kinds of responses.

Overdamped ∥ RLC ckt : $\alpha > \omega_0 \Rightarrow \frac{1}{2RC} > \frac{1}{\sqrt{LC}} \Rightarrow LC > 4R^2C^2$.

$\therefore s_{1,2}$ both real and $\sqrt{\alpha^2 - \omega_0^2} < \alpha$

$\therefore s_1, s_2 < 0$.

$\therefore v(t) \to A_1 e^{s_1 t} \to 0$ as $t \to \infty$ (s_2 being more negative, $A_2 e^{s_2 t}$ dies out faster).



$v(0) = 0$, $i(0) = 10\,A$.

$\alpha = 3.5\,s^{-1}$ $\omega_0 = \sqrt{6}\,s^{-1}$, $s_1 = -1\,s^{-1}$ $s_2 = -6\,s^{-1}$.

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ .... ① $= A_1 e^{-t} + A_2 e^{-6t}$

$v(0) = 0 = A_1 + A_2$.

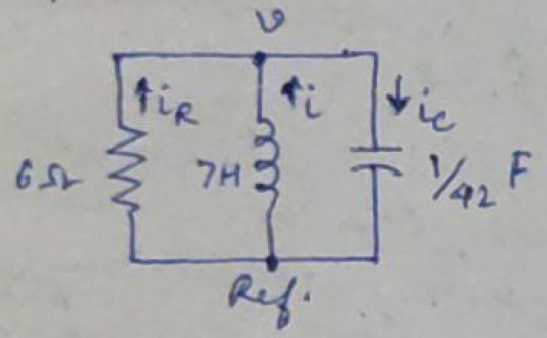
Take derivative of ①, $\therefore \frac{dv}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$

$\therefore \left.\frac{dv}{dt}\right|_{t=0} = -A_1 - 6A_2$

$i_C = C\frac{dv}{dt}$ $\therefore \frac{i_C(0)}{C} = \left.\frac{dv}{dt}\right|_0 = \frac{i(0) + i_R(0)}{C} = \frac{i(0)}{C} = 420\ V/s$

$\therefore A_1 = 84$, $A_2 = -84$.

$\therefore v(t) = 84\left(e^{-t} - e^{-6t}\right)$.



$v(0) = 0$, $i(0) = 10\,A$.

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$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ .... ① $= A_1 e^{-t} + A_2 e^{-6t}$

$v(0) = 0 = A_1 + A_2$.

Take derivative of ①, $\therefore$ $\dfrac{dv}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$ — ②

$\therefore \left.\dfrac{dv}{dt}\right|_{t=0} = -A_1 - 6A_2$

$i_C = C\dfrac{dv}{dt}$ $\therefore$ $\dfrac{i_C(0)}{C} = \left.\dfrac{dv}{dt}\right|_0 = \dfrac{i(0) + i_R(0)}{C} = \dfrac{i(0)}{C} = 420\,V/s$

$\therefore A_1 = 84$, $A_2 = -84$.

$\therefore v(t) = 84\left(e^{-t} - e^{-6t}\right)$.

Note:
① $\tau_1 = 1s$, $\tau_2 = 1/6\,s$.

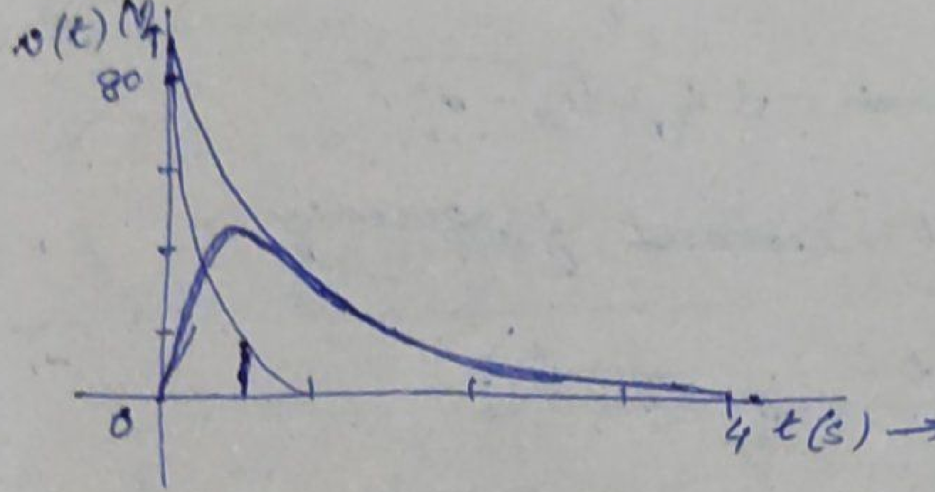
② $\Rightarrow \dfrac{v}{V_0} = e^{-t} - e^{-6t}$ $\therefore$ Both exponential terms with unity amp. but 2nd term dies out faster

② Total response never negative ③ as $t \to \infty$, both $\to 0$.

③ Eqn. ② yields maxima. ~~after differentiating~~. $0 = -e^{-t_m} + 6e^{-6t_m}$

$\ln[e^{-t_m}] = \ln[6e^{-6t_m}]$ $\therefore \dfrac{\ln 6}{5} \doteq t_m = 0.358\,s$, $v(t_m) = 48.9\,V$.

(4) Plot by taking difference of two curves.



(5) Settling time (t_s) : length of time it takes for transient part of response to die out.

→ 1% or 2% or even 5% level.

when $v \leq x\%$ of $|v_m|$.

For 1% → $\underline{t_s = 5.15 s}$

Critical damping: $\alpha = \omega_0$ Practically impossible.

$\therefore LC = 4R^2C^2$ $\therefore R = \frac{1}{2}\sqrt{\frac{L}{C}}$

Say change R to obtain critical damping: $R = 7\sqrt{6}/2\ \Omega$.

$\therefore \alpha = \omega_0 = \sqrt{6}$, $s_{1,2} = -\sqrt{6}$.

$\therefore v(t) = Ae^{-\sqrt{6}t} \longrightarrow$ but then only one i.c. would be satisfied. $\therefore v(t)$ has another form.

<u>Note</u>: For $\alpha = \omega_0$, eqn. (a) becomes.

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0.$$

has soln. of form $v(t) = e^{-\alpha t}(A_1 t + A_2)$.

$\therefore\ v_s = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t}$

$v(0) = 0 \quad \therefore \quad A_2 = 0.$

$\frac{dv}{dt} = A_1 t(-\sqrt{6}) e^{-\sqrt{6}t} + A_1 e^{-\sqrt{6}t} \quad [\because A_2 \text{ is } 0]$

$\therefore \left.\frac{dv}{dt}\right|_{t=0} = A_1 = \frac{i_c(0)}{C} = \frac{i(0)}{C} = 420$

$\therefore v(t) = 420t\, e^{-2.45t}$

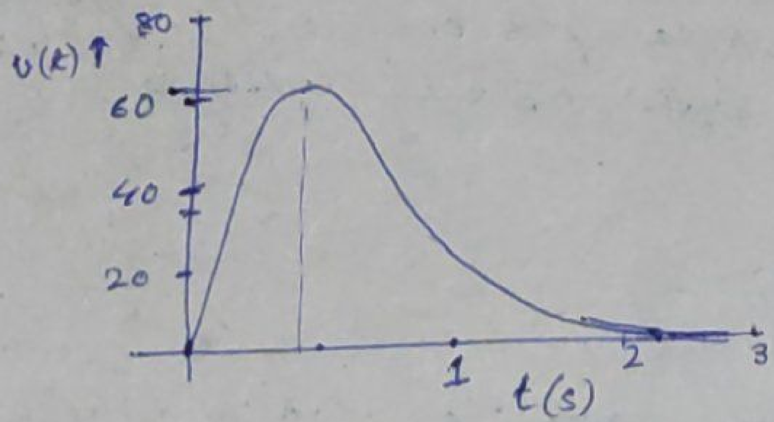
At $t=0$, $v(t)=0$ and, at $t=\infty$, $v(t) = 420 \lim_{t\to\infty} \frac{t}{e^{2.45t}} = 420 \lim_{t\to\infty} \frac{1}{2.45e^{2.45t}}$

$= 0.$

(L' Hôpital's rule).

v_m (at $t_m = 0.408s$) $= 63.1V$ $\longrightarrow$ larger than overdamped case.

$\frac{v_m}{100} = 420\, t_s\, e^{-2.45 t_s}$ yields $t_s = 3.12s$.



Underdamped ∥ RLC ckt. $\alpha < \omega_0$

$$\therefore \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \triangleq \text{natural resonant frequency.}$$

$$\therefore \quad v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(A_1 + A_2)\left(\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right) + j(A_1 - A_2)\left(\frac{e^{+j\omega_d t} - e^{-j\omega_d t}}{j2}\right)\right]$$

$$= e^{-\alpha t}\left[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right]$$

$$= e^{-\alpha t}\left[B_1 \cos\omega_d t + B_2 \sin\omega_d t\right].$$

Note : since $v(t)$ must be real $\therefore$ B_2 is real.

Let $R = 10.5\,\Omega$ $\quad\therefore\ \alpha = 2,\ \omega_0 = \sqrt{6},\ \omega_d = \sqrt{2}$ (rad/s all)

$\therefore\ v(t) = e^{-2t}\left(B_1 \cos\sqrt{2}t + B_2 \sin\sqrt{2}t\right)$

For $v(0) = 0$, $B_1 = 0$ $\quad\therefore\ v(t) = B_2 e^{-2t}\sin\sqrt{2}t$

$i(0) = 10$ $\quad\therefore\ \left.\frac{dv}{dt}\right|_{t=0} = \left.\left(\sqrt{2}B_2 e^{-2t}\cos\sqrt{2}t - 2B_2 e^{-2t}\sin\sqrt{2}t\right)\right|_{t=0}$

$$= \sqrt{2}\,B_2 = 420$$

$\therefore\ B_2 = 210\sqrt{2}$ $\quad\therefore\ v(t) = 210\sqrt{2}\,e^{-2t}\sin\sqrt{2}t.$

Note: (1) $v(0) = 0$ $\because \sin t = 0$, $v(\infty) = 0$ $[\because e^{-2t} \to 0]$

(2) t_m is before $\sin\sqrt{2}t$ attains max.

when $t = \pi/\sqrt{2}$, $\sin\sqrt{2}t = 0$ $\therefore\ v(t) = 0$.

$\pi/\sqrt{2} < t < \sqrt{2}\pi \rightarrow v(t)$ is negative

Thus $v(t)$ is an oscillatory fn. of time and crosses the real axis for all $t = n\pi/\sqrt{2}$.

More noticeable for $\alpha \ll \omega_0$.

For $\alpha = 0$, $v(t)$ is an undamped sinusoid $\therefore\ \omega_0$ = resonant frequency
oscillating with constant amplitude.

$v_{m_1} = 71.8$ V at $t_{m_1} = 0.435$ s

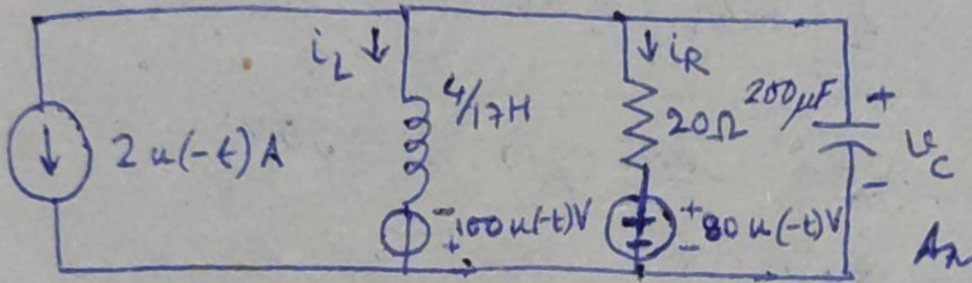
$v_{m_2} = -0.845$ V at $t_{m_2} = 2.657$ s.

$t_s = 2.92$ s $\rightarrow$ slightly lower than that for critical damping

Observations:

1. As ξ decreases, v_m ↑.
2. $\zeta < 1$ → oscillatory response.
3. $t_{s\,min.}$ for slight underdamping.

Prob. 7-5 pg 210

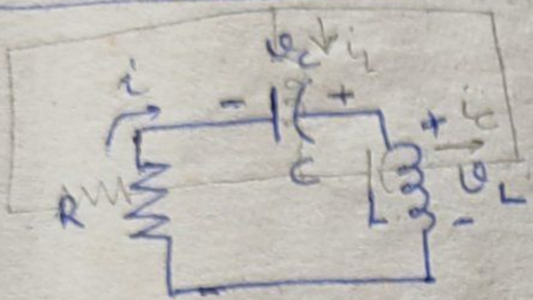


All sources turn off at $t = 0$.

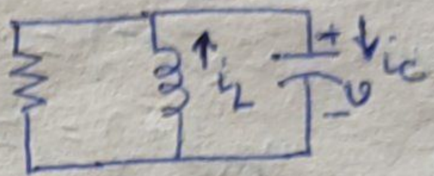
$v_C(0)$, $i_L(0)$, $i_R(0^+)$, $i_L(10ms)$

Ans: −100 V, 7A, −5A, 2.64A.

Source free series RLC ckt:



dual of



overdamped:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

critically damped case, $i(t) = e^{-\alpha t}(A_1 t + A_2)$

Underdamped: $i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

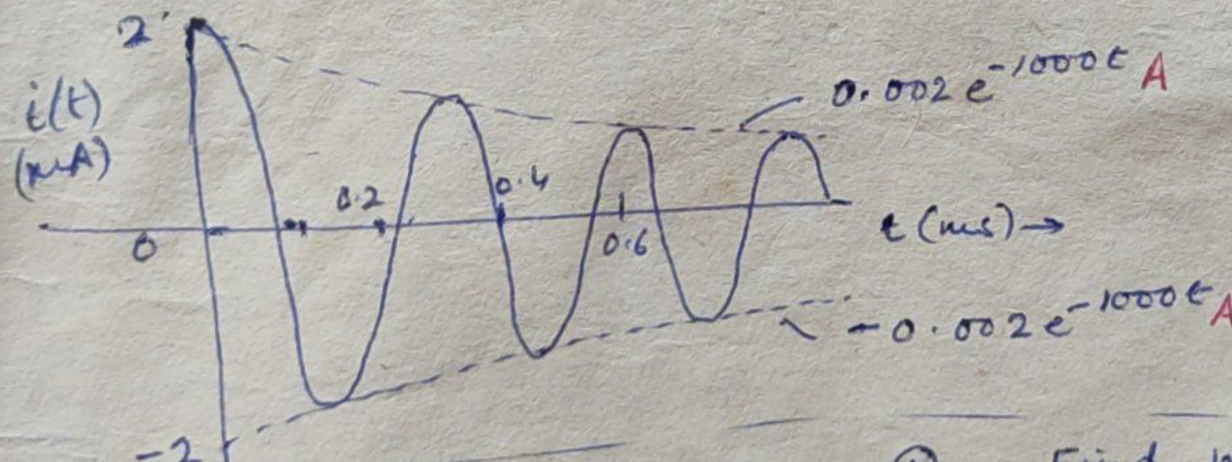
Ex: $R = 2\,k\Omega$, $C = 1/401\,\mu F$, $L = 1H$, $i(0) = 2\,mA$, $v_c(0) = 2V$

So $\alpha = 1000$ $\quad \omega_0 = 20\,025$, $\quad \omega_d = 20\,000$.

$$i(t) = e^{-1000t}\left(B_1 \cos 20000t + B_2 \sin 20000t\right).$$

$B_1 = 0.002$, $B_2 = 0$.

$\therefore\; i(t) = 2e^{-1000t} \cos 20\,000t\ mA$.

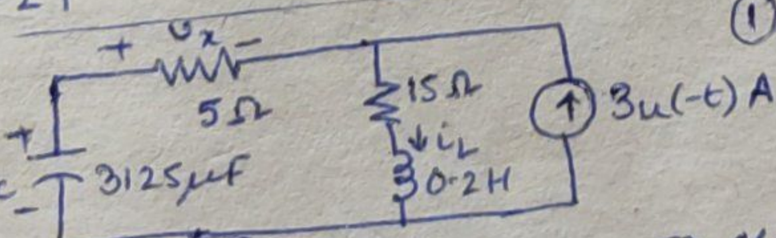


$20{,}000\,t = n\pi/2$

or $t = 0.07854\,n\ ms$

$t_s = 461\,\mu s$

Prob. 7-6, 7-7 pg 213

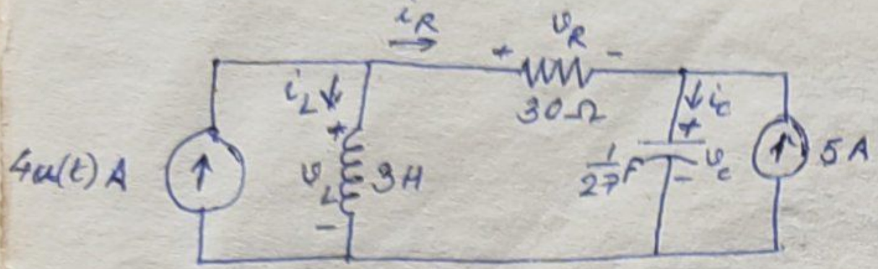


① Find $v_c(0)$, $i_L(0)$, $v_x(0^-)$, $v_x(0^+)$, $v_c(2\ldots$

Ans: 45V, 3A, 0, 15V, 29.7V.

② Change 15Ω resistor to 4.6 Ω & repeat

Ans: 13.8V, 3A, 0, 15V, −0.412V.



$v(t) = V_f + Ae^{s_1 t} + Be^{s_2 t}$

$\frac{dv}{dt} = 0 + s_1 A e^{s_1 t} + s_2 B e^{s_2 t}$

$v_L(0^-) = 0$ [$\overset{*}{\text{s.s.}}$ voltage across L has to be 0 − short to dc − else unrealistic infinite value of i_L or Illy v_c at $t=0^-$]

~ly $i_c(0^-) = 0$

KCL to right node : $i_R(0^-) = -5A$. $\Rightarrow v_R(0^-) = -150V$

KVL @ central mesh : $v_C(0^-) = 150V$

KCL at left node : $i_L(0^-) = 5A$.

At $t = 0^+$, $v_c(0^+) = v_c(0^-) = 150V$, ~ly $i_L(0^+) = 5A$.

$\therefore i_R(0^+) = -1A$, $v_R(0^+) = -30V$

$\therefore i_c(0^+) = 4A$ $v_L(0^+) = 120V$.

$v_L = L\frac{di_L}{dt}$ $\therefore \frac{di_L}{dt}\Big|_{t=0^+} = \frac{v_L(0^+)}{L} = 40$ A/s

~ly $\frac{dv_c}{dt}\Big|_{t=0^+} = \frac{i_C(0^+)}{C} = 108$ V/s

For other derivatives, KCL at left node : $4 - i_L - i_R = 0$ $t > 0$.

$\Rightarrow -\frac{di_L}{dt} - \frac{di_R}{dt} = 0 \Rightarrow \frac{di_R}{dt}\Big|_{t=0^+} = -40$ A/s.

~ly $\frac{dv_R}{dt}\Big|_{t=0^+} = -1200$ V/s, $\frac{dv_L}{dt}\Big|_{t=0^+} = -1092$ V/s, $\frac{di_c}{dt}\Big|_{t=0^+} = -40$ A/s.

$[v_R + v_C = v_L]$ $[i_R + 5 = i_c]$

For dead network, series RLC : $s_1 = -1$, $s_2 = -9$

$v_{c_f} = 150V$ (from dc equivalent of ckt.)

$v_c(t) = 150 + Ae^{-t} + Be^{-9t}$ / $\frac{dv_c}{dt} = -Ae^{-t} - 9Be^{-9t}$

$v_c(0^+) = 150 = 150 + A + B$ $\quad \frac{dv_c}{dt}\Big|_{t=0^+} = 108 = -A - 9B$

$\therefore A = 13.5 = -B$ $\quad \therefore v_c(t) = 150 + 13.5(e^{-t} - e^{-9t})$.