HURWITZ (1895)

ROUTH- STABILITY ; ROBUST CONTROL

(1892)

STABLE LINEAR SYS iff all poles of system TF have (-)ve real parts.

—STRONG condn. since roots on $j\omega$ axis disallowed

Let $F(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$ be charac. polynomial of sys.

∴ (1) if all roots have (-) real parts, then NECESSARY – all coeffs of $F(s)$ have SAME SIGN

(2) No coeff. of $F(s) = 0$. Absence of a_0 or both a_0 & a_1 etc. ⇒ one or more roots at origin.

Note : NECESSARY but NOT sufficient conditions

Hence R–H criterion

STABILITY : (i) Asymptotic : if impulse response → 0 as $t \to \infty$.

(ii) BIBO

iii) Marginal if DISTINCT roots on $j\omega$ axis (NOT MULTIPLE !!)

Note 1: certain bounded i/ps like $\sin t$ produce unbounded o/p.

$s^2+1=0$ roots : $\pm j$ For $u(t) = \sin t$, $y(t) = t \sin t$ UNBOUNDED

Note 2 : Complex conjugate poles on $j\omega$ axis : $\frac{c}{s^2+\omega_0^2} \to \left(\frac{c}{\omega_0}\right) \sin \omega_0 t$ – NO EXP. DAMPING

∴ UNSTABLE for $\sin \omega_0 t$ i/p

~ly a) simple pole at origin driven by step → ramp o/p

b) repeated poles – UNBOUNDED

SO, CAUSAL TI system with TF $G(s)$ is stable if

1. All poles of $G(s)$ are in left-half of s-plane.

2. $\Theta(D(s)) \geq \Theta(N(s))$ where $G(s) = N(s)/D(s)$ → RATIONAL T.F.

$$D(s) = a_0 \prod_{i=1}^{p} (s+\alpha_i) \prod_{l=1}^{q} (s^2 + 2\beta_l s + \beta_l^2 + \phi_l^2)$$

ROUTH ARRAY for $F(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...
s^{n-2}	c_1	c_2	c_3	c_4	...
s^{n-3}	d_1	d_2	d_3	d_4	...
⋮	⋮				
s^3	e_1	e_2	e_3		
s^2	f_1	f_2			
s^1	g_1				
s^0	h_1				

$$c_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$c_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

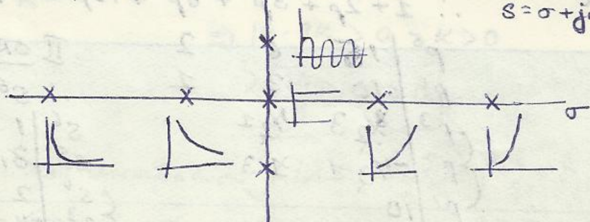
⋮

$$d_1 = \frac{c_1 a_{n-3} - c_2 a_{n-1}}{c_1}$$

$$d_2 = \frac{c_1 a_{n-5} - c_3 a_{n-1}}{c_1}$$

⋮

No. of sign changes in 1st column = no. of roots of $F(s)$ with +ve real parts.



Ex 5.1 $F(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$

Routh array:

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & \frac{a_1 a_2 - a_3 a_0}{a_2} & \\ s^0 & a_0 & \end{array}$$

$\therefore$ Condn. for all roots to have −ve real parts

$a_1 a_2 - a_3 a_0 > 0$

$a_1, a_3, a_2, a_0 > 0$

$\therefore a_1 a_2 > a_3 a_0$

$\Rightarrow a_1 > \frac{a_3 a_0}{a_2} > 0$

Ex 5.2 $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

$$\begin{array}{c|cc} s^4 & 1 & 3 & 5 \\ s^3 & \cancel{2}\,1 & \cancel{4}\,2 & \\ s^2 & \frac{3-2}{1} = 1 & 5 & \\ s^1 & 2-5 = -3 & & \\ s^0 & 5 & & \end{array}$$

2 sign changes $1 \to -3 \to 5$

$\therefore$ UNSTABLE with 2 roots in RHP.

Ex 5.3 $s^4 + (2\times10^3)s^3 + (3\times10^6)s^2 + (4\times10^9)s + (5\times10^{12}) = 0$

Let $s = 10^3 p \Rightarrow 10^{12}\left[p^4 + 2p^3 + 3p^2 + 4p + 5\right] = 0$

↳ same as in previous case

$\therefore$ UNSTABLE.

Ex 5.4 $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & \cancel{2}\,1 & \cancel{2}\,1 \\ s^1 & 0 \simeq \varepsilon & \\ s^0 & 1 & \end{array}$$

If 1st column element zero while others nonzero/no remaining term, then replace by ε and evaluate rest of array assuming $\varepsilon \to 0^+$

No poles in rt. half $\because$ no sign change but s^1 row 0 $\therefore$

$s^2 + 1 = 0 \Rightarrow s = \pm j$ are roots of the ch. eqn.

$\therefore s^3 + 2s^2 + s + 2 = (s^2+1)(s+2)$

Ex 5.5 $s^4 + s^3 + 2s^2 + 2s + 3 = 0$

$$\begin{array}{c|cc} s^4 & 1 & 2 & 3 \\ s^3 & 1 & 2 & \\ s^2 & 0 \simeq \varepsilon & 3 & \\ s^1 & \frac{2\varepsilon - 3}{\varepsilon} & & \\ s^0 & 3 & & \end{array}$$

For $\varepsilon \to 0^+$, $2\varepsilon - \frac{3}{\varepsilon}$ is −ve $\therefore$ 2 sign changes $\therefore$ 2 roots in RHP

Ex 5.6 $s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15 = 0$

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 10 \\ s^4 & 2 & 6 & 15 \\ s^3 & 0 \simeq \varepsilon & 5/2 & \\ s^2 & 6 - 5/\varepsilon & 15 & \\ s^1 & \frac{30\varepsilon - 25 - 30\varepsilon^2}{12\varepsilon - 10} & & \\ s^0 & 15 & & \end{array}$$

2 sign changes

$\therefore$ 2 roots in RHP

UNSTABLE

$$\frac{\frac{6\varepsilon-5}{\varepsilon}\cdot\frac{5}{2} - \frac{30\varepsilon^2}{2\varepsilon}}{\frac{(6\varepsilon-5)2}{2\varepsilon}}$$

I OR use $s = 1/p$

$\therefore 1 + 2p + 3p^2 + 6p^3 + 10p^4 + 15p^5 = 0$

$$\begin{array}{c|ccc} p^5 & 15 & 6 & 2 \\ p^4 & 10 & 3 & 1 \\ p^3 & \cancel{3/2}\,3 & \cancel{1/2}\,1 & \\ p^2 & \cancel{-1/3}\,-1 & \cancel{1}\,3 & \\ p^1 & 10 & & \\ p^0 & 3 & & \end{array}$$

II OR multiply by $(s+1)$ say

$\therefore s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 3s + 1$

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 8 & 1 \\ s^5 & \cancel{3}\,1 & \cancel{9}\,3 & \cancel{3}\,1 & \\ s^4 & 2 & 7 & 1 & \\ s^3 & \cancel{-1/2}\,-1 & \cancel{1/2}\,1 & & \\ s^2 & 9 & 1 & & \\ s^1 & 10/9 & & & \\ s^0 & 1 & & & \end{array}$$

Ex 5.8 $F(s) = s^6+5s^5+11s^4+25s^3+36s^2+30s+36$

$$\begin{array}{c|cccc}
s^6 & 1 & 11 & 36 & 36 \\
s^5 & \cancel{5}\,1 & \cancel{25}\,5 & \cancel{30}\,6 \\
s^4 & \cancel{6}\,1 & \cancel{30}\,5 & \cancel{36}\,6 \\
s^3 & \cancel{0}\,\cancel{4}\,2 & \cancel{0}\,\cancel{10}\,5 \\
s^2 & \cancel{5/2}\,5 & \cancel{6}\,12 \\
s^1 & 1/5 \\
s^0 & 12
\end{array}$$

$\to A(s) = s^4+5s^2+6 = (s^2+2)(s^2+3)$

$\Rightarrow s = \pm j\sqrt{2}, \pm j\sqrt{3}$

$\dfrac{dA(s)}{ds} = 4s^3+10s$

$\therefore$ 4 poles on $j\omega$ axis
MARGINALLY STABLE

Note: $F(s) = A(s)\,(s+2)(s+3)$

Ex 5.8a $F(s) = s(s^2-1)(s^2+2s+4) = s^5+2s^4+3s^3-2s^2-4s = 0$

$$\begin{array}{c|ccc}
s^5 & 1 & 3 & -4 \\
s^4 & \cancel{2}\,1 & \cancel{-2}\,-1 & 0 \\
s^3 & \cancel{4}\,1 & \cancel{-4}\,-1 \\
s^2 & \cancel{0}\,3 & \cancel{0}\,-1 \\
s^1 & -2/3 \\
s^0 & -1
\end{array}$$

$\therefore A(s) = s^3 - s = s(s^2-1) = 0 \Rightarrow s = 0, \pm 1$

$\dfrac{dA(s)}{ds} = 3s^2 - 1$

$\underline{1 \text{ poles in RHP}}$

$s^2+2s+4 = P(s) = 0$

$\therefore s = \dfrac{-2 \pm \sqrt{4-16}}{2} = -1 \pm j\sqrt{3}$

Ex 5.9 $F(s) = s^6+4s^5+12s^4+16s^3+41s^2+36s+72 = A(s)\cdot(s^2+4s+8)$

$$\begin{array}{c|cccc}
s^6 & 1 & 12 & 41 & 72 \\
s^5 & \cancel{4}\,1 & \cancel{16}\,4 & \cancel{36}\,9 \\
s^4 & \cancel{8}\,1 & \cancel{32}\,4 & \cancel{72}\,9 \\
s^3 & \cancel{0}\,\cancel{4}\,1 & \cancel{0}\,\cancel{8}\,2 \\
s^2 & 2 & 9 \\
s^1 & -5 \\
s^0 & 9
\end{array}$$

$A(s) = s^4+4s^2+9 = 0 \quad \therefore \dfrac{dA(s)}{ds} = 4s^3+8s$

$\Rightarrow s^2 = \dfrac{-4 \pm \sqrt{16-36}}{2} = -2 \pm j\sqrt{5}$

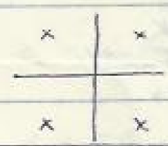
2 poles in RHP

** or $A(s) = s^4+6s^2+9-2s^2 = 0$

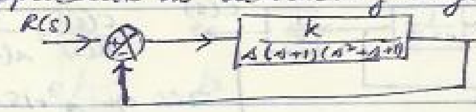
$= (s^2+3)^2 - (\sqrt{2}s)^2 = (s^2-\sqrt{2}s+3)(s^2+\sqrt{2}s+3)$

$\therefore s = \dfrac{\pm\sqrt{2} \pm \sqrt{2-12}}{2} = \pm 1/\sqrt{2} \pm j\sqrt{\dfrac{5}{2}}$

$= \pm \dfrac{1}{\sqrt{2}} \pm j\sqrt{\dfrac{5}{2}}$



Application in determining range of gain parameter K for stability.



$\therefore$ C.E. T.F. gives $F(s) = s^4+3s^3+3s^2+2s+K$

$\dfrac{C(s)}{R(s)} = \dfrac{K}{s^4+3s^3+3s^2+2s+K}$

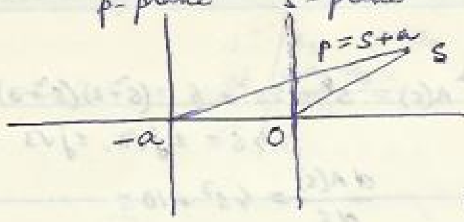
$$\begin{array}{c|cc}
s^4 & 1 & 3 \quad K \\
s^3 & 3 & 2 \\
s^2 & \cancel{7/3}\,7 & \cancel{K}\,3K \\
s^1 & \dfrac{14-9K}{7} \\
s^0 & K
\end{array}$$

$\Rightarrow 14-9K > 0,\ K > 0$

$\Rightarrow \boxed{\dfrac{14}{9} > K > 0}$

** State-space representation : $|\lambda I - A| = \Delta(s) = 0$

Relative Stability:



$F(s) \rightarrow F(p)$ where $p = s+a$

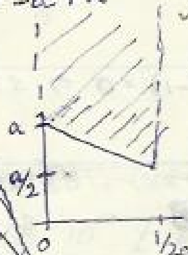
Ex 5.11 $G(s) = \dfrac{K}{s(sT+1)}$

Desired: All roots of Q sys. to left of $s=-a$ for min. damping. Determine K, T.

$F(s) = 1 + G(s)H(s) = 0 = Ts^2 + s + K$

Let $s = p - a$ $\therefore$ $T(p-a)^2 + (p-a) + K = Tp^2 + (1-2Ta)p + (a^2T - a + K) = 0$.

$$\begin{array}{c|cc} p^2 & T & a^2T-a+K \\ p^1 & 1-2aT & \\ p^0 & a^2T-a+K & \end{array}$$



$\therefore T > 0$

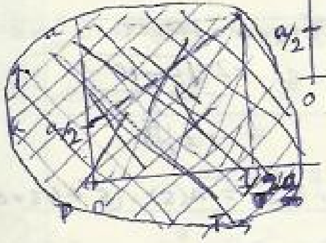
$1 - 2aT > 0 \Rightarrow \frac{1}{2a} > T > 0$

$a^2T - a + K > 0$

$\Rightarrow K > a(1-aT) > \frac{a}{2}$

$a > a(1-aT) > a\left(1 - a\cdot\frac{1}{2a}\right)$

$= \frac{a}{2}$



Ex. 5.12 For $F(s) = s^3 + 9s^2 + 26s + K$

Determine K s.t. dominant time constant ≤ 0.5. System should be slightly UD.

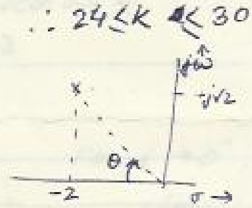
$T = 0.5 \Rightarrow s + \frac{1}{0.5} = 0 \Rightarrow s = -2$ is the relative stability line for $p = 0$

$\Rightarrow s = p - 2$.

$F(p) = (p-2)^3 + 9(p-2)^2 + 26(p-2) + K = p^3 + 3p^2 + 2p + (K-24)$

$$\begin{array}{c|cc} p^3 & 1 & 2 \\ p^2 & 3 & K-24 \\ p^1 & \frac{30-K}{3} & \\ p^0 & K-24 & \end{array}$$

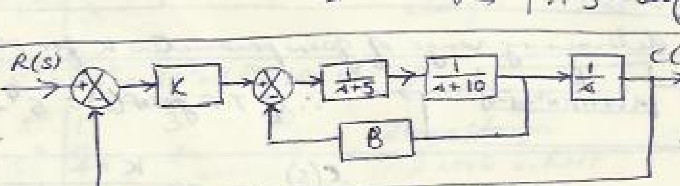
$\therefore 24 \leq K \leq 30$



At $K = 24$, single pole at origin of p-plane $\Rightarrow$ at $s = -2$.

At $K = 30$,
$3p^2 + 6 = 0 \Rightarrow p^2 + 2 = 0$
$\Rightarrow p = s + 2 = \pm j\sqrt{2}$.
$\Rightarrow s = -2 \pm j\sqrt{2}$.
$\therefore \zeta = \cos\left(\tan^{-1}\frac{\sqrt{2}}{2}\right) = 0.816$

Ex 5.13

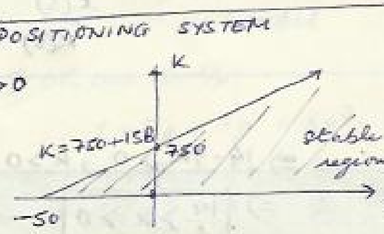


$\dfrac{C(s)}{R(s)} = \dfrac{K}{s(s+5)(s+10) + Bs + K}$

$\Rightarrow$ Ch. eqn: $s^3 + 15s^2 + (50+B)s + K = 0$.

Routh array: $750 + 15B > K > 0$

$$\begin{array}{c|cc} s^3 & 1 & 50+B \\ s^2 & 15 & K \\ s^1 & \frac{750+15B-K}{15} & \\ s^0 & K & \end{array}$$



If $B = 0$, $0 < K < 750$

But for velocity fb, $B \neq 0 \Rightarrow K$ accordingly adjustable

ROOT LOCUS TECHNIQUE — I

W.R. EVANS (1948)

— complete information of the poles from the pole-zero info w/o evaluating roots of the system when one or more parameters (eg gain K) of the TF varied.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$ for negative f/b system where

$$G(s)H(s) = K\frac{P(s)}{Q(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = KG_1(s)H_1(s)$$

$m \le n$
CAUSAL

Ch. eqn. $1 + G(s)H(s) = 0$ or $G(s)H(s) = -1$

∴ (i) Magnitude condn. $|G(s)H(s)| = 1$

$$\therefore |G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{1}{|K|} \;;\; 0 \le K < \infty.$$

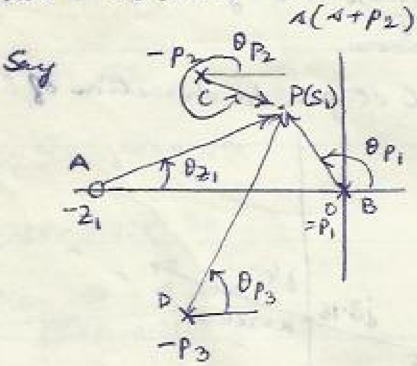
(ii) Phase condn. or $\angle$ condn. $\angle G(s)H(s) = (2k+1)\pi \;;\; k = 0,1,2\dots$

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = (2k+1)\pi \quad \because \angle K = 0$$

∴ Any pt. satisfying both (i) & (ii) is a pt. on the root loci.

Let $G(s)H(s) = \dfrac{K(s+z_1)}{s(s+p_2)(s+p_3)}$.

Say



∴ (i) Mag. condn. $|G_1(s_1)H_1(s_1)| = \dfrac{1}{|K|}$

$$= \frac{AP}{BP \cdot CP \cdot DP}$$

(ii) $\angle$ condn. : $\theta_{z_1} - (\theta_{P_1} + \theta_{P_2} + \theta_{P_3}) = (2k+1)\pi$.

Rules for constructing root loci

1. Ch. eqn. in pole-zero form s.t. K appears as $1 + KG_1(s)H_1(s) = 0$.
2. Locate the poles and zeros of $G_1(s)H_1(s) = 0$.
3. Locate segments of real axis where root loci exist.
4. Determine no. of separate root loci
5. Locate angles of asymptotes and intersection of the asymptotes
7. Determine the break-away (in) pt. on the real axis (if any). + elsewhere
6. Using Routh-Hurwitz condn., determine pt. at which locus crosses imaginary axis (if any)
8. Estimate $\angle$ of departure (arrival) of root loci from (at) complex poles (zeros).

Go to (12a).

Ex 6.1 Draw root loci for ch. eqn. of a sys. given as
$s(s+4)(s^2+4s+20)+K=0$. Find Gain margin for $K_1=26$, $K_2=2600$.

1. Rewrite $G(s)H(s) = 1 + \dfrac{K}{s(s+4)(s^2+4s+20)} = 0 \quad \therefore G_1H_1(s) = \dfrac{1}{s(s+4)(s^2+4s+20)}$

2. Poles at $s = 0, -4, -2 \pm j4$ $\quad : K=0 \quad P=4$
zeros at ∞ $\quad K=\infty \quad Z=0 \quad \therefore$ 4 asymptotes $= N = P - Z$

5. $\angle$ of asymptotes $\pm 45°, \pm 135°$, Centroid $\sigma_A = \dfrac{-4-2-2}{4} = -2$

6. Ch. eqn: $s^4 + 8s^3 + 36s^2 + 80s + K = 0$.

R-H array

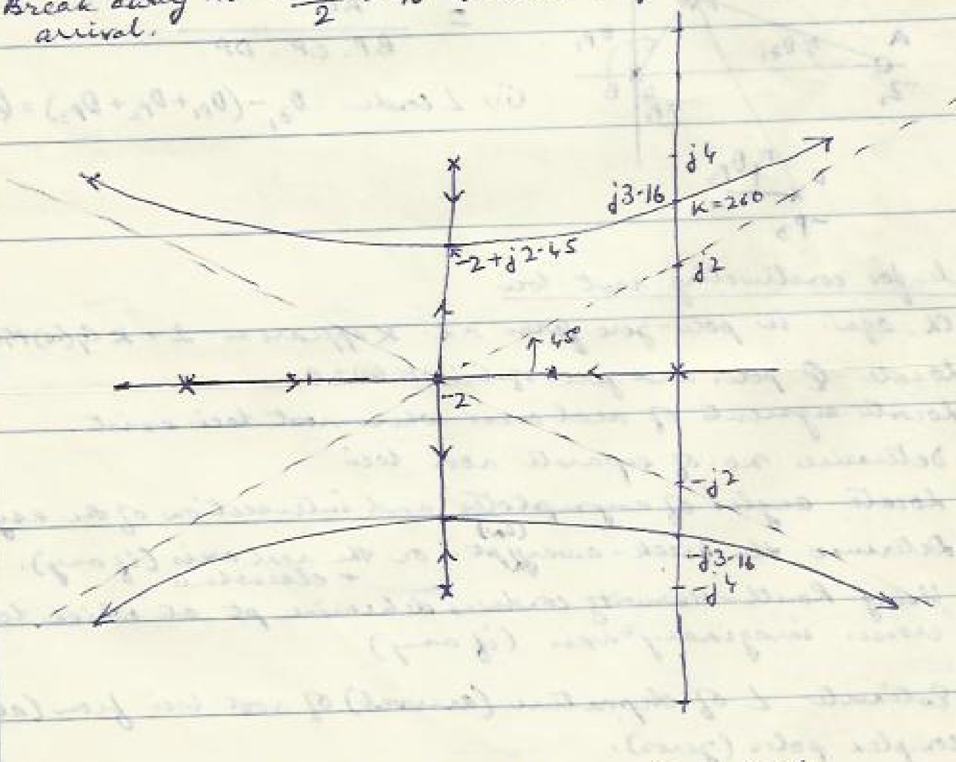
s^4	1	36	K
s^3	8 1	~~80~~ 10	
s^2	26	K	
s^1	$10 - \frac{K}{26}$		
s^0	K		

$\therefore 0 < K < 260$

Aux. eqn: $s^2 + 10 = 0$
$\Rightarrow s = \pm j3.16$

7. $K = -(s^4 + 8s^3 + 36s^2 + 80s)$
$\therefore \dfrac{dK}{ds} = 0 = 4s^3 + 24s^2 + 72s + 80$
$\Rightarrow s^3 + 6s^2 + 18s + 20 = 0 = (s+2)(s+2 \pm j2.45) = 0$.

- Break away at $\pm \dfrac{180°}{2} = 90°$ at each stage to direction of arrival.



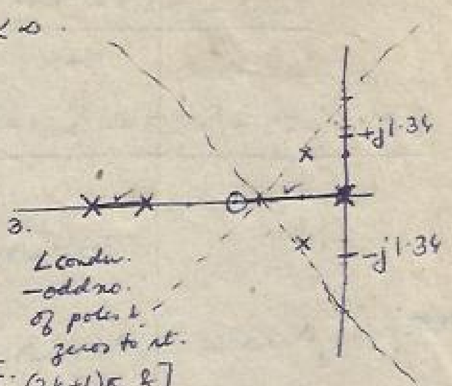
Gain margin at $K_1 = 26$ is $+20$dB : system stable
$\left(-20 \log_{10} \dfrac{K}{K_u}\right)$ $\quad K_2 = 2600 \quad GM = -20\log_{10}\dfrac{2600}{260} = -20\log_{10} 10 = -20$dB : UNSTABLE.

12a (13)

Q T.F. $G(s)H(s) = \dfrac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$ for $0<K<\infty$.

1. Now, $G_1(s)H_1(s) = \dfrac{s+3}{s(s+5)(s+6)(s^2+2s+2)} = \dfrac{1}{K}$

2. Q poles START: $s = 0, -5, -6, -1\pm j1$ $\quad P=5$
Q zeros END: $s=-3$ rest 4 zeros at infinity, $Z=1$
$= P-Z$



L condn. - odd no. of poles & zeros to rt. [$\therefore (2k+1)\pi$ & neg. f/b]

4. $\therefore$ No. of root loci $N = P = 5 \quad \because P > Z$.

5. $(P-Z)$ no. of infinite zeros $\therefore$ $(P-Z)$ asymptotes intersecting at centroid σ_A on real axis s.t.

$$\sigma_A = \frac{\sum p_j - \sum z_i}{m-n} \quad \text{with angles } \phi_k = \frac{(2k+1)\pi}{P-Z}$$

$$\therefore \sigma_A = \frac{(0-5-6-1-1)-(-3)}{4} = -2.5 \quad ; \quad \phi_k = \frac{(2k+1)\pi}{4} = \pm 45^\circ, \pm 135^\circ$$

6. $K(s+3) + s(s^2+11s+30)(s^2+2s+2) = s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$ ch. eqn

s^5	1	54	$60+K$
s^4	13	82	$3K$
s^3	47.7	$60+0.769K$	
s^2	$65.6-0.212K$	$3K$	
s^1	$\dfrac{3940-105K-0.163K^2}{65.6-0.212K}$		
s^0	$3K$		

$\therefore K \geq 0$
$65.6 > 0.212K \Rightarrow K < 309$
$3940 - 105K - 0.163K^2 > 0$
$\Rightarrow K < 35.$
$\therefore 0 < K < 35$ for stability
Aux eqn.: $58.2s^2 + 105 = 0$
$\Rightarrow s = \pm j1.34 \quad \therefore \omega = \pm 1.34$ rad/s are the intersection pts. on imaginary axis.

7. At multiple roots. If r root loci branches meet at a pt., they break away at $\pm 180^\circ/r$ w.r.t. angle at which they arrive. Usually on real axis.

I $\quad F(s) = 1 + GH(s) = 0$ has multiple roots at $\dfrac{d}{ds}F(s) = 0$.

$\therefore \dfrac{d}{ds}GH(s) = \dfrac{d}{ds}(G_1H_1)(s) = 0$ NECESSARY not sufficient
$\therefore$ all solns. not valid - need testing.

II $\quad F(s) = 1 + K\dfrac{P(s)}{Q(s)} = 0 \Rightarrow Q(s) + KP(s) = 0.$

$\therefore \dfrac{dF(s)}{ds} = Q'(s) + KP'(s) = 0 \quad \therefore K = -\dfrac{Q'(s)}{P'(s)}$ & $F(s)=0 \Rightarrow QP' - Q'P = 0.$

$\therefore K = -\dfrac{Q(s)}{P(s)}$ Note $\dfrac{dK}{ds} = -\dfrac{Q'P - QP'}{P^2} = 0$ is same as $\therefore \dfrac{dK}{ds} = 0$ get $s = s_1$

s.t. $\dfrac{d^2K}{ds^2} < 0$ for break away and $\dfrac{d^2K}{ds^2} > 0$ for break in.

and $K = \left.\dfrac{1}{G_1H_1}\right|_{s=s_1}$

$$\frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} = -1 \Rightarrow K = \frac{-(s^5+13s^4+54s^3+82s^2+60s)}{s+3}$$

$$\therefore \frac{dK}{ds} = \frac{-(s+3)(5s^4+52s^3+162s^2+164s+60) + (s^5+13s^4+54s^3+82s^2+60s)}{(s+3)^2} = 0$$

$$\Rightarrow -4s^5 - 54s^4 - 264s^3 - 568s^2 - 492s - 180 = 0$$

$$\Rightarrow s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$$

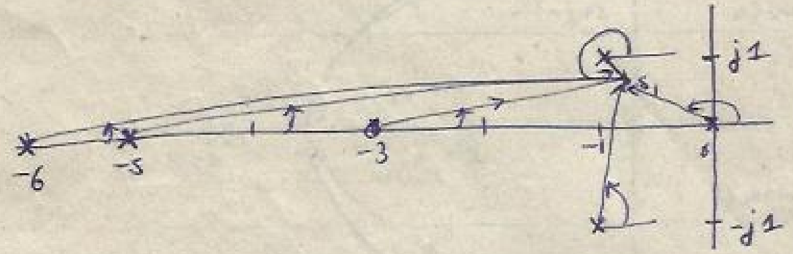
From root locus plot, expected $-6 < s_1 < -5$ $\therefore$ Bisection search gives

$$s_1 = -5.33.$$

Other roots are $s = -0.656 \pm j0.468 ; -3.33 \pm j1.204$. not SUFFICIENT $\therefore$ extra.

8. $\angle$ of departure from $-1 \pm j1$.
Say $s_1 = -1 \pm j1 + \delta$ (very close to pole) and on root locus leaving the pole.

$$\therefore \angle G_1H_1(s_1) = \angle s_1+3 - \Big[\angle s_1 + \underbrace{\angle s_1+1+j1}_{=\theta} + \angle s_1+1-j1 + \angle s_1+5 + \angle s_1+6\Big] = \pm(2k+1)\pi$$



$$= \left(\tan^{-1}\frac{1}{2}\right) - \left(135^\circ + 90^\circ + \theta + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5}\right)$$

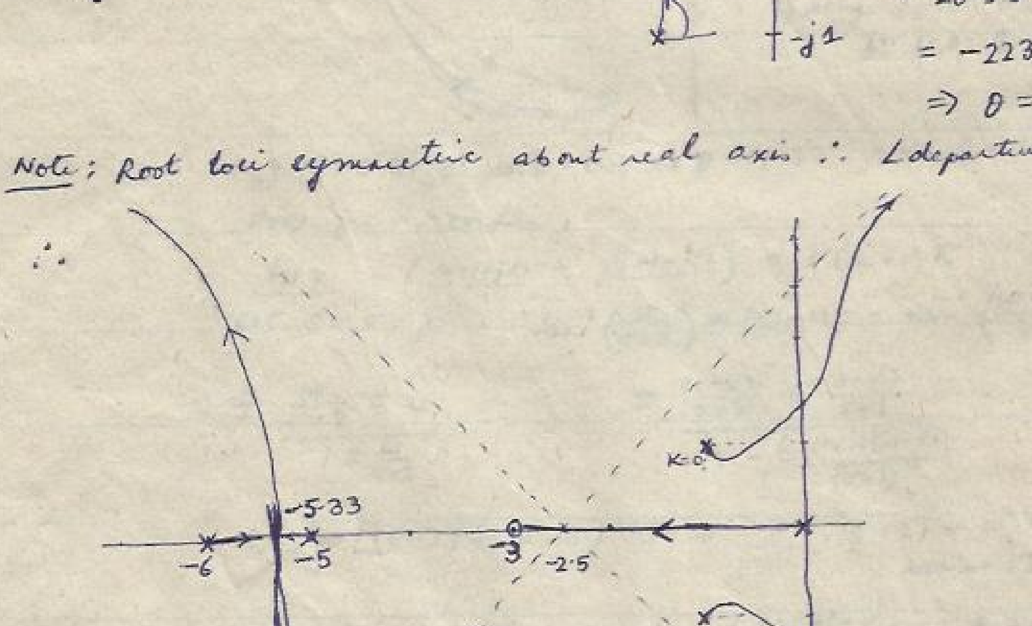
$$= 26.56^\circ - (225^\circ + 14^\circ + 11.3^\circ + \theta)$$

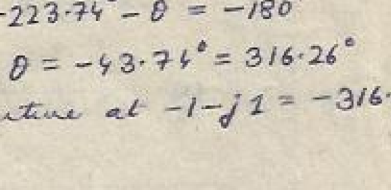
$$= -223.74^\circ - \theta = -180^\circ$$

$$\Rightarrow \theta = -43.74^\circ = 316.26^\circ$$

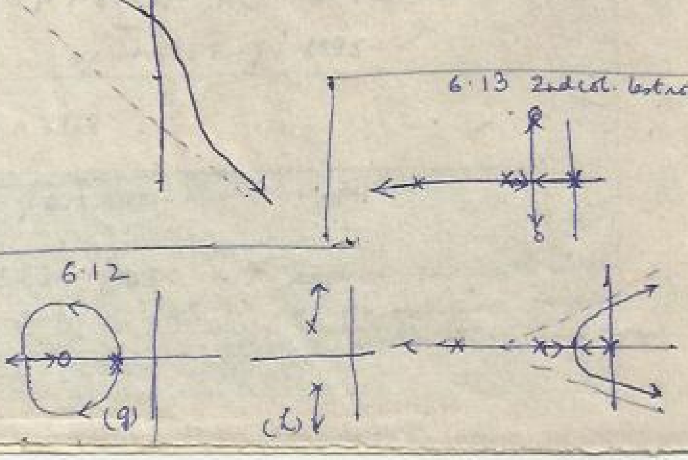
Note: Root loci symmetric about real axis $\therefore$ $\angle$departure at $-1-j1 = -316.26^\circ$

$\therefore$





6.12



To do
Fig 6.12, 6.13 (Drill prob. 6.6) : To do.

Ex 6.2. Draw root loci of unity f/b system with Q T.F.

$$GH(s) = \frac{K(s+2)}{s^2+2s+3}$$

Determine value of K for a) repetitive roots b) range of K for UD sys. c) $\zeta = 0.7$

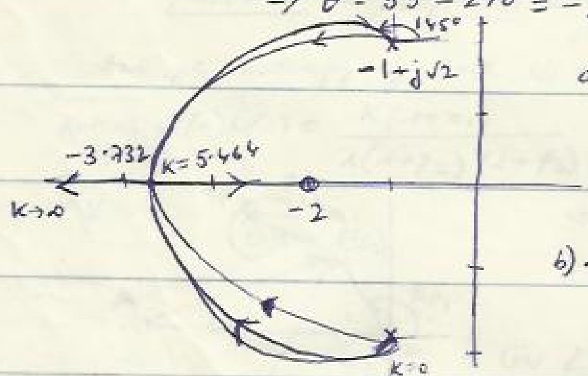
1. $GH(s) = \dfrac{K(s+2)}{(s+1 \pm j\sqrt{2})}$

2. Q poles: $-1 \pm j\sqrt{2}$ $P=2$: $K=0$
   zeros: -2 : $K=\infty$ $Z=1$ $\therefore N = P - Z = 1$ infinite zero, 1 asymptote
4.
5. $\angle$ of asymptote: $\theta = \pi$ 3. existence of root loci bet. -2 & $-\infty$
6. No root loci to rt. $\therefore$ R-H array not needed. – no intersection with $j\omega$ axis
7. Break-in pt. $K = \dfrac{-(s^2+2s+3)}{s+2}$ $\therefore \dfrac{dK}{ds} = \dfrac{-(2s+2)(s+2)+(s^2+2s+3)}{(s+2)^2} = 0$

$\Rightarrow s^2+4s+1=0 \Rightarrow s = -3.732, -0.268 \rightarrow$ NOT VALID PT. ON ROOT LOCI

8. $\angle$ of departure: $\theta_z - (\theta_p + \theta_p^*) = 180° = \tan^{-1}\frac{\sqrt{2}}{1} - (\theta + 90°) = 55° - \theta - 90°$

$\Rightarrow \theta = 55° - 270° = -215° = 145°$ (360)



a) repetitive roots at break in pt.
$\therefore s_1 = -3.732$. Note: $\zeta = 1$
$\therefore K = -\dfrac{(s^2+2s+3)}{s+2}\Big|_{s=s_1} = 5.464$

b) range of K $0 < K < 5.464$ for U.D. Q system.

c) $\zeta = 0.7 \Rightarrow \cos\theta = 0.7 \Rightarrow \theta = 45.57° \Rightarrow \tan\theta = 1.0202 = \frac{\omega}{\sigma}$ for $s = \sigma + j\omega$

Now, from $\angle$ condn.,

$\angle s+2 - (\angle s+1+j\sqrt{2} + \angle s+1-j\sqrt{2}) = \pm(2k+1)\pi$

Let $s = \sigma + j\omega$, $\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \pm (2k+1)\pi = \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) + \tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right)$

$$\Rightarrow \frac{\frac{\omega}{\sigma+2} \pm 0}{1 \mp \frac{\omega}{\sigma+2}\cdot 0} = \frac{\frac{\omega+\sqrt{2}}{\sigma+1} + \frac{\omega-\sqrt{2}}{\sigma+1}}{1 - \frac{(\omega+\sqrt{2})(\omega-\sqrt{2})}{(\sigma+1)^2}}$$

$\boxed{\tan x \pm y = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}}$

$\Rightarrow \omega[(\sigma+2)^2 + \omega^2 - 3] = 0 \Rightarrow \omega = 0, (\sigma+2)^2 + \omega^2 = (\sqrt{3})^2 \rightarrow$ Eqn. of circle radius $\sqrt{3}$, centre $(-2, 0)$

$\therefore \omega = 1.0202\sigma \Rightarrow |\sigma| = |-1.6659| = \zeta\omega_n = 0.7\omega_n \Rightarrow \omega_n = 2.38$

$\omega = \omega_n\sqrt{1-\zeta^2} = 1.6995 \therefore s_2 = -1.6659 + j1.6995$

$\therefore K = -\dfrac{(s^2+2s+3)}{s+2}\Big|_{s=s_2} = 1.3318$

Ex 6.8, 6.9, 6.10 (to correct), 6.11 / 6.12, 6.13 – Nyquist, compensator

*Note: Pole-zero cancellation DOES NOT reflect in root locus plot $\therefore$ Any varn. in pole – not accounted – large instability may occur.

Rules for root locus

1. Write characteristic eqn. in pole-zero form in terms of parameter of interest K as

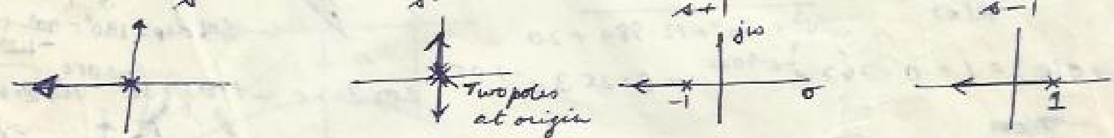
   $\therefore\ 1 + K\,G_1(s)\,H_1(s) = 0$

   $\therefore$ Magnitude condn. $|G_1(s)H_1(s)| = \frac{1}{|K|}$

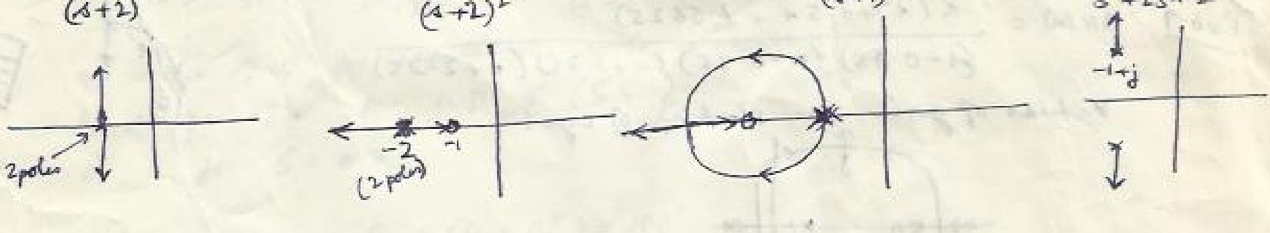
   Phase condn. $\angle G_1(s)H_1(s) = (2k+1)\pi$

2. Q poles and zeros of $G_1(s)\,H_1(s) = 0$
3. Locate segments of real axis where root loci exist.
4. Determine no. of separate root loci
5. Locate angles of asymptotes and intersection of asymptotes
6. Determine break away pt. on real axis and elsewhere.
7. Determine imaginary axis crossover pts. using R-H array.
8. Estimate angle of departure of root locus from complex poles and angle of arrival at complex zeros.

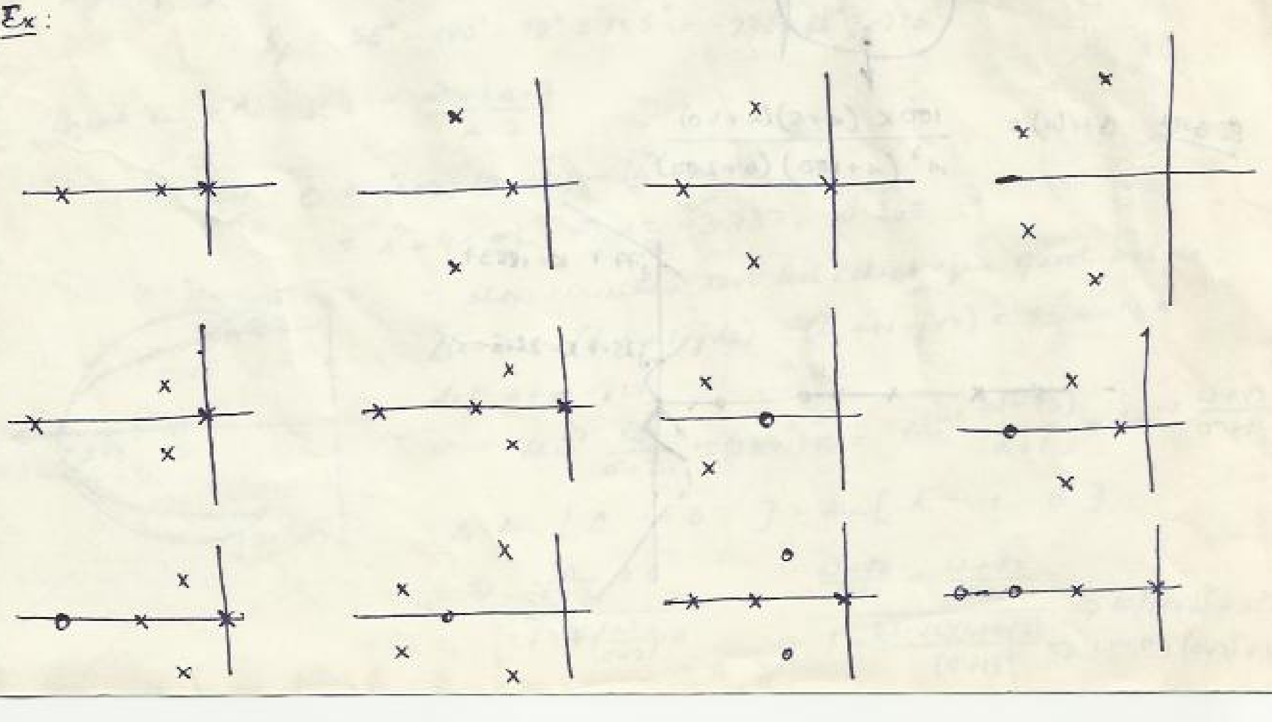
$GH(s) = \frac{K}{s}$ $\qquad \frac{K}{s^2}$ $\qquad \frac{K}{s+1}$ $\qquad \frac{K}{s-1}$



$\frac{K}{(s+2)^2}$ $\qquad \frac{K(s+1)}{(s+2)^2}$ $\qquad \frac{K(s+2)}{(s+1)^2}$ $\qquad \frac{K}{s^2+2s+2}$



Ex:



Assignment problems :

Ex 6.3 $G(s)H(s) = \dfrac{K}{s(s+4)(s+4\pm j4)}$

Ex 6.5. $G(s)H(s) = \dfrac{K}{s(s^2+6s+25)}$; G.M. at $K_1 = 15$, $K_2 = 1500$

Ex 6.6 $G(s)H(s) = \dfrac{K}{s(s+2)(s^2+6s+25)}$

Ex 6.7 $G(s)H(s) = \dfrac{K(s+3)}{s(s+2)}$

Ex 6.8 T.F. = $\dfrac{20(1+ks)}{s(s+1)(s+4)}$; find k s.t. $\zeta = 0.4$. [Let $20k = K$]

$1+GH = s^3+5s^2+4s+20+20ks \rightarrow GH(s) = \dfrac{Ks}{(s+5)(s\pm j2)} = \dfrac{Ks}{(s+5)(s^2+4)}$

for $\zeta = 0.4 = \cos\theta \Rightarrow \theta = \pm 66.42^\circ \rightarrow s = -1.05+j2.4$ at P

$K = 8.98 = 20k \therefore k = 0.449$. better response

$s = -2.15+j4.95$ at Q $\therefore K = 28.26 \Rightarrow k = 1.413$.

For $k = 0.449$, CL pole $s = -1.05\pm j2.4, -2.902$.

$\therefore \dfrac{C(s)}{R(s)} = \dfrac{20(1+9.04s)}{s^3+5s^2+12.98s+20}$

VERIFY. $\rightarrow c(t) = 1 - 0.747e^{-2.902t} - 0.253e^{-1.05t}\cos 2.4t - 1.01\times 3e^{-1.05t}\sin 2.4t$.

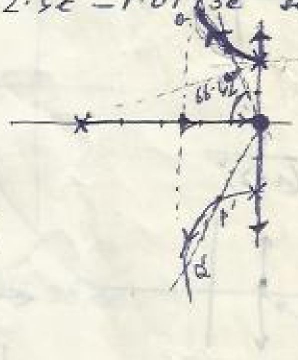
For $k = 1.413$, $s = -2.15\pm j4.95, -0.6823$.

$\therefore$ sluggish

Asymptote centroid $= \dfrac{-5-0}{3-1} = -2.5$

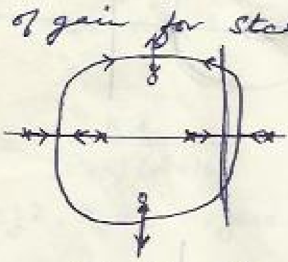
L of dep: $180^\circ = 90^\circ - 90^\circ - \tan^{-1}2/5 - \theta$ 21.77

$\Rightarrow \theta = -180^\circ - 21.77^\circ = -201.77^\circ$

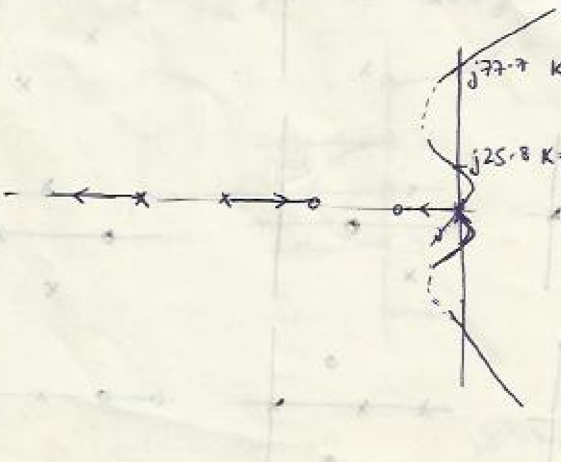


Ex 6.9 $GH(s) = \dfrac{K(s^2+1.5s+1.5625)}{(s-0.75)(s+0.25)(s+1.25)(s+2.25)}$

Values of gain for stable CL sys.



Ex 6.12 $GH(s) = \dfrac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}$



Ex 6.1

Characteristic eqn. of $\mathcal{E}$: $s(s+4)(s^2+4s+20) + K = 0$.

$\therefore 1 + K\, G_1(s)\, H_1(s) = 0$

$\Rightarrow G_1(s)H_1(s) = \dfrac{1}{s(s+4)(s^2+4s+20)}$

Poles at $0, -4, -2 \pm j4$ $\therefore P = 4,\ Z = 0,\ P - Z = 4$

$\therefore$ $\angle$s of asymptotes : $\pm\pi/4, \pm 3\pi/4$ centroid at -2

existence of root loci on real axis bet. 0 and -4.

R-H of $s^4 + 8s^3 + 36s^2 + 80s + K = 0$ yields $K = 260$ at crossover.

$\omega = \pm 3.16$ rad/s

$\angle$ of departure $= -90°\big|_{-2+j4}$

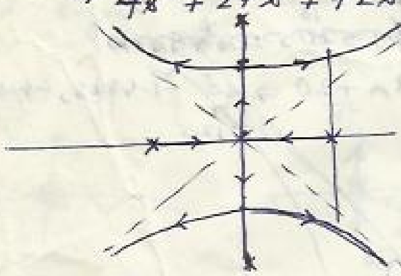
Break away pt : $K = -s(s+4)(s^2+4s+20)$

$\therefore \dfrac{dK}{ds} = \dfrac{d}{ds}\left[-(s^4+8s^3+36s^2+80s)\right] = 0$

$\Rightarrow 4s^3 + 24s^2 + 72s + 80 = 0 \Rightarrow s^3 + 6s^2 + 18s + 20 = 0 = (s+2)(s+2 \pm j2.45)$

$s^3 + 2s^2 + 4s^2 + 8s + 10s + 20$

$= (s+2)(s^2 + 4s + 10)$



Ex 6.2 $G(s)H(s) = \dfrac{K(s+2)}{s^2+2s+3} = \dfrac{K(s+2)}{(s+1 \pm j\sqrt{2})}$

a) value of K for repetitive roots to occur

b) range of K for UD $\mathcal{E}$ $(\zeta < 1)$

c) K for $\zeta = 0.7$.

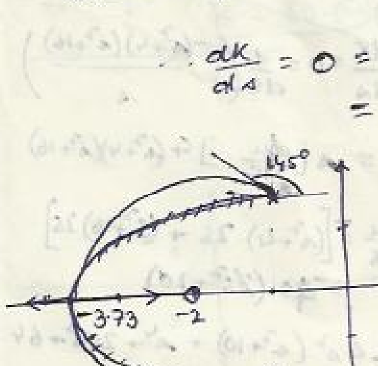
Angle of departure : $\theta_z - (\theta_p + \theta_p^*) = 180°$

$\therefore \theta_p = 55° - 180° - 90° = 145° = -270° + 55° = -215°$

Break in pt. $K = -\dfrac{s^2+2s+3}{s+2}$

$\therefore \dfrac{dK}{ds} = 0 = (s+2)(2s+2) - (s^2+2s+3)$

$= s^2 + 4s + 1 \Rightarrow s = -3.732, -0.268$



To show circular root loci, check eqns. of root loci as :

$\angle(s+2) - (\angle s+1-j\sqrt{2}) - (\angle s+1+j\sqrt{2}) = \pm(2k+1)\pi$

Let $s = \sigma + j\omega$

$\therefore \tan^{-1}\dfrac{\omega}{\sigma+2} \pm (2k+1)\pi = \tan^{-1}\dfrac{(\omega-\sqrt{2})}{\sigma+1} + \tan^{-1}\dfrac{\omega+\sqrt{2}}{\sigma+1}$

$\therefore \tan[\,C + D\,] = \tan[\,A + B\,]$

$\Rightarrow \dfrac{\frac{\omega}{\sigma+2} \pm 0}{1 \mp \left(\frac{\omega}{\sigma+2}\right)\cdot 0} = \dfrac{\frac{\omega-\sqrt{2}}{\sigma+1} + \frac{\omega+\sqrt{2}}{\sigma+1}}{1 - \frac{(\omega-\sqrt{2})(\omega+\sqrt{2})}{(\sigma+1)^2}} \Rightarrow \omega[(\sigma+2)^2 + \omega^2 - 3] =$

$\Rightarrow \omega = 0,\ (\sigma+2)^2 + \omega^2 = ($

a) K at repeated roots $\Rightarrow$ K at break in $\Rightarrow \zeta = 1$

$$\left| \frac{K(s+2)}{s^2+2s+3} \right|_{s=-3.732} = 1 \Rightarrow K = 5.4641$$

b) $\zeta < 1$ for $0 < K < 5.4641$

c) $\zeta = 0.7 \Rightarrow \cos\theta = 0.7 \Rightarrow \theta = 45.57^\circ$

$\frac{\omega_d}{\sigma} = \frac{|\omega_n\sqrt{1-\zeta^2}|}{|\zeta\omega_n|}$ $\therefore \tan\theta = 1.0202 \Rightarrow \omega_d = 1.0202\sigma$

$\therefore (\sigma+2)^2 + (1.0202\sigma)^2 = \sqrt{3}^2 \Rightarrow \sigma = -1.6659 = \zeta\omega_n$

$\therefore \omega_d = 1.6995 = \omega_n\sqrt{1-\zeta^2}$

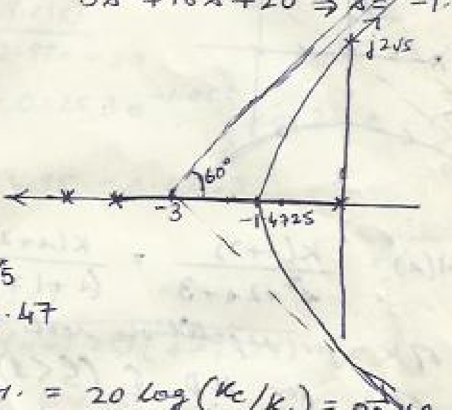
$\therefore K\Big|_{-1.6659+j1.6995} = 1.3318$

Ex 6.4 $G(s)H(s) = \frac{K}{s(s+4)(s+5)}$ $\Rightarrow s^3 + 9s^2 + 20s + K = 0$

Q poles $0, -4, -5$ $\quad P = 3$, $Z = 0$, $P - Z = 3$ asymptotes at $\frac{\pi}{3}, \pi, -\pi/3$.
centroid -3

$K = -(s^3 + 9s^2 + 20s)$ $\therefore \frac{dK}{ds} = 0 = 3s^2 + 18s + 20 \Rightarrow s = -1.4725, -4.5275$.

$K\Big|_{-1.4725} = 13.128$.



R-H array

s^3	1	20
s^2	9	K
s	$\frac{180-K}{9}$	
s^0	K	

$\therefore K_c = 180$

$A(s) = 9s^2 + 180 = 0$
$\Rightarrow s^2 + 20 = 0$
$\Rightarrow j\omega = \pm\sqrt{20} = \pm j2\sqrt{5}$
$= \pm j4.47$

Gain margin

$K_c = 180$ $\therefore$ For $K_1 = 18$, G.M. $= 20\log(K_c/K_1) = 20\,dB$.

Ex 6.10 $G(s)H(s) = \frac{Ks}{(s^2+4)(s^2+16)}$ $\quad 0 < K < \infty$

Q poles $\pm j2, \pm j4$. , zero at $s = 0$. $\quad P = 4$, $Z = 1$ $\quad P - Z = 3$
$\angle$s of asymptotes $\pi/3, \pi, -\pi/3$.

Intersection of asymptotes $= 0$, Break in at $s = -\infty$ ~~-9.03, 2.36~~ $\pm j3, \pm\sqrt{2.36}$ $s = -\sqrt{2.36} = -1.536$

$\angle$ of dep. for s at $j4 = 0^\circ$, s at $j2 = \pi$

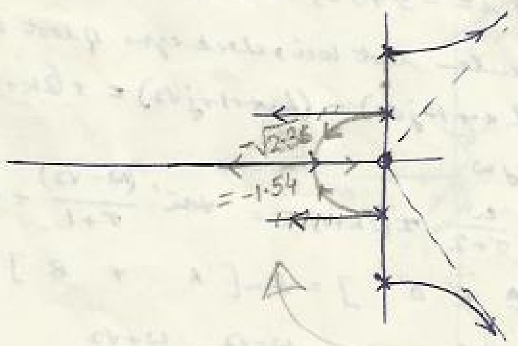
$\frac{dK}{ds} = 0 = \frac{d}{ds}\left(\frac{-(s^2+4)(s^2+16)}{s}\right)$

$= s\left[\frac{du}{ds}\right] + (s^2+4)(s^2+16)$

$\frac{du}{ds} = \left[(s^2+4).2s + (s^2+16)2s\right]$
$= 4s.(s^2+10)$

$\Rightarrow -4s^2(s^2+10) + s^4 + 20s^2 + 64$
$= 3s^4 + 20s^2 - 64 = 0$ $\therefore s^2 = \frac{-20 \pm\sqrt{400+768}}{6} = \frac{-20\pm\sqrt{1168}}{6}$

check $3s^4 + 20s^2 - 64 = 0$



$s^2 = -9.03, 2.36 \Rightarrow s^2 = \frac{-60 \pm\sqrt{3600 - 415.64}}{10}$
$\Rightarrow s = \pm j3, \pm\sqrt{2.36}$
$= \pm 1.54$

$= \frac{-60\pm\sqrt{3600-1280}}{10} = \frac{-6\pm\sqrt{2320}}{10}$ $= \frac{-20\pm3}{6}$ $= -9.03$

verify angle conditions are violated

Ex 6.8

$$T.F. = \frac{20(1+ks)}{s(s+1)(s+4)} \quad \text{find } k \text{ s.t. } \zeta = 0.4 \quad [\text{Let } 20k = K]$$

$$1 + GH(s) = s^3 + 5s^2 + 4s + 20 + \underbrace{20k}_{K}s \quad \therefore GH(s) = \frac{Ks}{(s+5)(s^2+4)} = \frac{Ks}{(s+5)(s\pm j2)} \quad \ldots (1)$$

For $\zeta = 0.4 = \cos\theta$, $\theta = \pm 66.42^\circ \Rightarrow \tan\theta = 2.29 = \frac{\omega_d}{\sigma}$

$$\therefore 1 + GH(s) = s^3 + 5s^2 + (4+K)s + 20 = (\sigma+j\omega)^3 \cancel{+ 5(\sigma+j\omega)^2 + (4+K)(\sigma+j\omega) + 20 = 0}$$

$\angle$ condn. in (1), put $s = \sigma + j\omega$.

$$\therefore \angle GH(s) = \frac{\angle \sigma+j\omega}{(\sigma+5+j\omega)(\sigma+j(\omega+2))(\sigma+j(\omega-2))} = \tan^{-1}\frac{\omega}{\sigma} - \tan^{-1}\frac{\omega}{\sigma+5} - \tan^{-1}\frac{\omega+2}{\sigma} - \tan^{-1}\ldots$$

$$= (2k+1)\pi \qquad \left[\tan(A+B) = \frac{\tan A \ldots}{1-\tan\ldots}\right.$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{\omega}{\sigma} - \frac{\omega}{\sigma+5}}{1 + \frac{\omega^2}{\sigma^2+5\sigma}}\right) - \tan^{-1}\left(\frac{\frac{2\omega}{\sigma}}{1 - \frac{\omega^2-4}{\sigma^2}}\right) = (2k+1)\pi$$

$$\Rightarrow \tan^{-1}\left(\frac{5\omega}{\sigma^2+5\sigma+\omega^2}\right) - \tan^{-1}\left(\frac{2\sigma\omega}{\sigma^2-\omega^2+4}\right) = \tan^{-1}\cancel{\frac{5\omega(\sigma^2-\omega^2+4) - 2\sigma\omega(\sigma^2+5\sigma+\omega^2)}{10\sigma\omega^2 + (\sigma^2+5\sigma+\omega^2)(\sigma^2-\omega^2+4)}} = (2k\ldots$$

$$\Rightarrow \cancel{\tan^{-1}} \frac{5\omega\sigma^2 - 5\omega^3 + 20\omega - 2\sigma^3\omega - \overset{5}{\cancel{10}}\sigma^2\omega - 2\sigma\omega^3}{\sigma^4 + 5\sigma^3 + \sigma^2\omega^2 - \sigma^2\omega^2 - 5\sigma\omega^2 + 4\omega^2 + 10\sigma\omega^2} = 0$$

$$\Rightarrow \omega\left[-2\sigma^3 - 5\sigma^2 - 2\sigma\omega^2 - 5\omega^2 + 20\right] = 0. \Rightarrow \omega = 0 \text{ or } 2\sigma^3 + 5\sigma^2 + (2\sigma+5)\omega^2 - 20 = 0.$$

Req: $\sigma = \zeta\omega_n = 0.4\omega_n$, $\omega = \omega_d = \omega_n\sqrt{1-(0.4)^2} = 0.916\omega_n$

$0.8\omega_n^3 + 5\omega_n^2 - 20 = 0.$

$$\Rightarrow 0.128\omega_n^3 + 0.8\omega_n^2 + (\underset{0.8\omega_n}{5+\cancel{0}})(0.84)\omega_n^2 - 20 = \cancel{0.128\,\omega_n^3}$$

$1.75 \to -0.5$

$$\Rightarrow \omega_n^3 + 6.25\omega_n^2 - 25 = 0 \cancel{\Rightarrow \omega_n^2(\omega_n + 6.25) = 25} \qquad \omega_n \in [1, 2] \quad +ve \ \ -ve$$

$$\Rightarrow \underbrace{(\omega_n - 1.775)}(\omega_n^2 + 8.025\omega_n + 14.245) = (\omega_n - 1.775)(\omega_n + 10.75)(\omega_n + \cancel{5.30})$$

$$\downarrow$$

$$\cancel{\frac{-8.025 \pm \sqrt{7.420}}{=}} = 2.724$$

$\omega_d = 1.626$

$\cancel{\sigma} = 0.71.$

$$\angle GH(s) = \cancel{\frac{20+Ks}{s(s+1)(s+4)}} \ldots = \cancel{\ldots} \cdot \quad 1 + GH = \frac{Ks}{s^3+5s^2+4s+20} = \frac{K \cdot s^{\to \sigma+j\omega}}{(s+5)(s^2+4)}.$$

$\sigma + j\omega$ $\quad \downarrow \ \sigma^2 - \omega^2 + 2\sigma\omega j$

$$\Rightarrow \tan^{-1}\frac{5\omega}{\sigma^2+5\sigma+\omega^2} = (2k+1)\pi + \tan^{-1}\frac{2\sigma\omega}{\sigma^2-\omega^2+4} =$$

$$\Rightarrow \frac{5\cancel{\omega}}{\sigma^2+5\sigma+\omega^2} = \frac{2\sigma\cancel{\omega}}{\sigma^2-\omega^2+4} \Rightarrow 5\sigma^2 - 5\omega^2 + 20 = 2\sigma^3 + 10\sigma^2 + 2\sigma\omega^2.$$

$$\Rightarrow 20 = 2\sigma^3 + 5\sigma^2 + \underbrace{2\sigma\omega^2}_{10.5\sigma^3} + \underbrace{5\omega^2}_{26.22\sigma^2}$$

$$\Rightarrow \omega = 2.29\sigma \quad \left[\because \tan\theta = \frac{\omega_d}{\sigma} = 2.29\right] \Rightarrow \cancel{12.5\sigma^3 + 31.22\sigma^2 - 20 = 0}.$$

$12.5\sigma^3 + 31.22\sigma^2 - 20 = 0.$ $\quad 0.7 \to -0.5$

$$= \sqrt{187.1} = 13.68$$

$$(\sigma - 0.708)(12.5\sigma^2 + 40.07\sigma + 28.37) \qquad 28.37 \to \sigma = \frac{-40.07 \pm \sqrt{40.07^2 - 4(28.37)(12.5)}}{25} = \underbrace{-1.0556\,;\,-2.15}$$

$$\omega = 2.417\,;\,4.92.$$

$$\left|\frac{Ks}{(s+5)(s^2+4)}\right| = 1 \Rightarrow K = \left|\frac{(s+5)(s^2+4)}{s}\right| = \left|\frac{\sqrt{(\sigma+5)^2+\omega^2}\cdot\sqrt{[(\sigma^2-\omega^2+4)^2+(2\sigma\omega)^2]}}{\sqrt{\sigma^2+\omega^2}}\right|$$

$s_1 = -1.0556 + j2.417$

$s_2 = -2.15 + j4.92$

$s^2+4 = (\sigma+j\omega)^2+4 = \underbrace{\sigma^2-\omega^2+4}+j2\sigma\omega$

$$① = \frac{(4.63)(5.15)}{2.64} = 9.04 = K \Rightarrow k = 0.452$$

$(s+j2) = \sigma+j(\omega+2) = \sqrt{\sigma^2+(\omega+2)^2}$

$$② = \frac{(5.685)(26.28)}{\cancel{28.83}\ 5.37} = 27.825 = K$$

$k = 1.39$

$$\mathcal{E} = \frac{G}{1+G} = \frac{20+Ks}{s^3+5s^2+(4+K)s+20}$$

$$① = \frac{20+9.04s}{s^3+5s^2+13.04s+20} \longrightarrow s = -1.0556 \pm j2.417,\ -2.9$$

Ex 6.9 $KG_1H_1(s) = \dfrac{K(s^2+1.5s+1.5625)}{[(s-0.75)(s+0.25)(s+1.25)(s+2.25)]} = \cancel{64s^2+96s+100}$

$= (s^2+1.5s-1.6875)(s^2+1.5s+0.3125)$

$= s^2(s^2+3s+2.25) - 1.375(s^2+1.5s) - 0.527$

$= \{s^4+3s^3+0.875s^2-2.0625s-0.527\} + K(s^2+1.5s+1.5625)$

s^4	1	$(0.875+K)$	$(-0.527+1.5625K)$
s^3	3	$(-2.0625+1.5K)$	
s^2	$\{0.875+K\} - (0.5K-0.6875)\} = 1.5625+0.5K$	$(1.5625K-0.527) = 0$	
s^1	$Ⓐ = -2.0625+1.5K$		
s^0	$1.5625K-0.527$		

verify.

$XA(s) = 3s^3+(-2.0625+1.5K)s = 0$

$\Rightarrow s[3s^2-2.0625+1.5K] = 0$

$\Rightarrow s^2 = \dfrac{2.0625-1.5K}{3}$, $s=0$

$= 0.6875-0.5K$

$\therefore$ ~~$1.5625+0.5K = 0 \Rightarrow K = -$~~ not valid

~~$1.5625K-0.527 = 0 \Rightarrow$~~

$(1.5625+0.5K)s^2+(1.5625K-0.527) = 0.$

$\Rightarrow s^2 = \dfrac{-1.5625K+0.527}{1.5625+0.5K}$

$\Rightarrow 1.5625K-0.527 = 0$

$\Rightarrow K_1 = \dfrac{0.527}{1.5625} = 0.337.$

$K_2 = \dfrac{2.0625}{1.5} = 1.375.$

$-2.0625+1.5K$

6.9
$$\frac{K(s^2+1.5s+1.5625)}{(s-0.75)(s+0.25)(s+1.25)(s+2.25)} = GH(s).$$

$$1+GH(s) = (s^2+1.5s-1.6875)(s^2+1.5s+0.3125) + K(s^2+1.5s+1.5625)$$

$$= s^4+3s^3+(0.875+K)s^2+(1.5K-2.0625)s+(1.5625K-0.527) = 0.$$

s^4	1	$(0.875+K)$	$(1.5625K-0.527)$
s^3	3	$(1.5K-2.0625)$	
s^2	$(1.5625+0.5K)$	$1.5625K-0.527$	
s^1	$\frac{(1.5K-2.0625)(1.5625+0.5K)-3(1.5625K-0.527)}{(1.5625+0.5K)}$		
s^0	$1.5625K-0.527.$		

$$s^0>0 \Rightarrow K \geqslant \frac{0.527}{1.5625} = 0.34$$

s^1 row $=0 \Rightarrow$ roots on imaginary axis

$$\Rightarrow 0.75K^2-3.375K-1.642=0 \quad \therefore K = \frac{+3.375}{1.5} \pm \frac{\sqrt{16.32}}{1.5} = 4.04$$

$$= 2.25 \pm 2.69 = 4.94, \cancel{-0.44}$$

$\therefore K = 4.94. \Rightarrow s^1$ row is 0.

$$\therefore A(s) = (1.5625+0.5K)s^2+(1.5625K-0.527)\Big|_{K=4.94} = 0$$

$$= 4.0325s^2+7.192 \Rightarrow s^2 = -\frac{7.192}{4.0325} = -1.7834 \Rightarrow s = \pm j1.335.$$

Ex 5 9(11)

$$1+p+2p^2+2p^3+3p^4=0.$$

p^4	3	2	1
p^3	2	1	
p^2	0.5	+2	
p^1	−3		
p^0	2		

2 sign changes

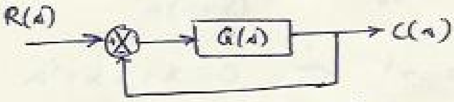
$$(s+1)(s^4+s^3+2s^2+2s+3) = s^5+2s^4+3s^3+4s^2+5s+3$$

s^5	1	3	5
s^4	2	4	3
s^3	1/2	7/2	
s^2	−10	3	
s^1	+7.3		
s^0	3		

2 sign changes

Root Loci with Compensator

(Pole placement)

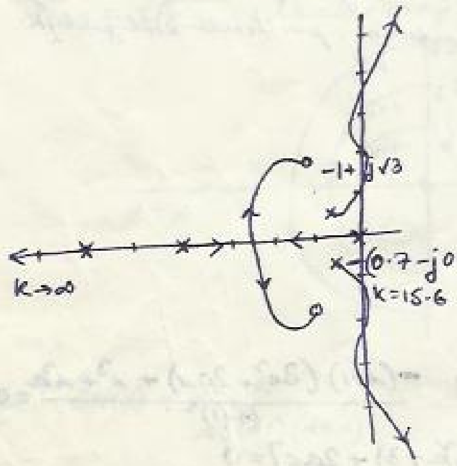


$$1 + GH(s) = 1 + \frac{K(s^2+2s+4)}{s(s+4)(s+6)(s^2+1.4s+1)} = \frac{s(s+4)(s+6)(s^2+1.4s+1) + K(s^2+2s+4)}{D(s)} = 0.$$

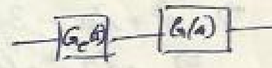
$-0.7 \pm j0.7$

Check: (i) System stable for $0 < K < 15.6$; $67.5 < K < 163.6$

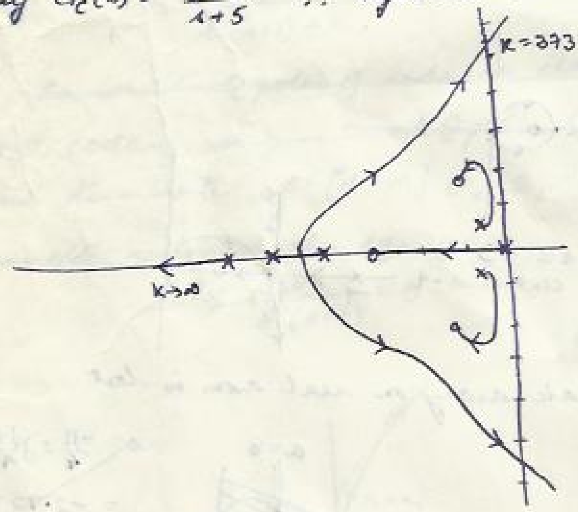
(ii) Unstable for $15.6 < K < 67.5$; $163.6 < K < \infty$.



Use compensating network $G_c(s)$ in fwd path with $G(s)$



Say $G_c(s) = \frac{s+3}{s+5}$ $\therefore$ system stable $\forall$ $0 < K < 373$.

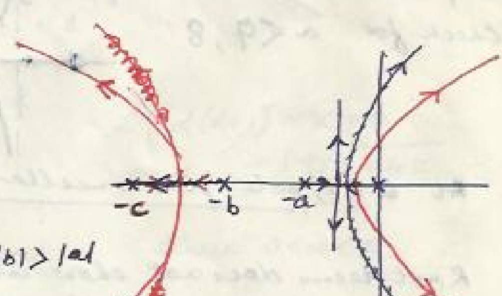


Effect of adding poles:

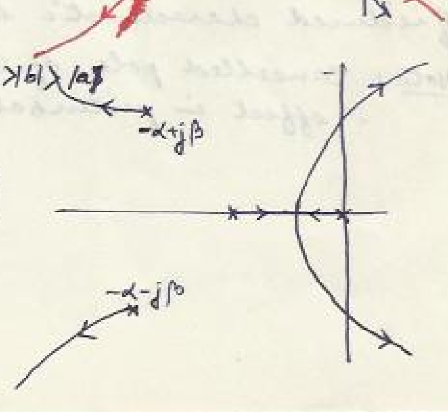
$GH(s) = \frac{K}{s(s+a)}$; $a > 0$

Add pole at $-b$ $\therefore$ $GH(s) = \frac{K}{s(s+a)(s+b)}$ $|b| > |a|$

Another pole at $-c$ $\therefore$ $GH(s) = \frac{K}{s(s+a)(s+b)(s+c)}$ $|c| > |b| > |a|$



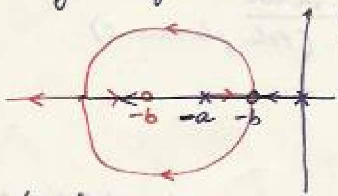
Pair of poles at $(s+\alpha \pm j\beta)$ $\therefore$ $GH(s) = \frac{K}{s(s+a)(s+\alpha \pm j\beta)}$



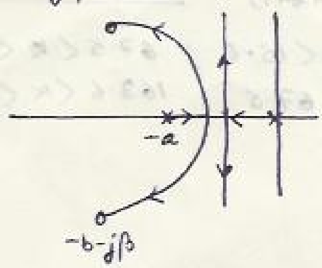
$\therefore$ Same as increasing ORDER of system:
makes it more UNSTABLE.

Addition of zeros: stabilizes system.

$$GH(s) = \frac{K(s+b)}{s(s+a)}$$



Two complex zeros at $-b \pm j\beta$



effect: more stable system for higher gains K.

limitation: practicable realizable high K

Effect of varying pole position:

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+a)} \quad \therefore K = \frac{-s^2(s+a)}{s+1}$$

For $\frac{dK}{ds} = 0 = \frac{-(s+1)\left[2s(s+a) + s^2\right] + s^2(s+a)}{(s+1)^2} \Rightarrow \frac{-(s+1)(3s^2+2as) + s^3 + s^2a}{(s+1)^2} = 0$

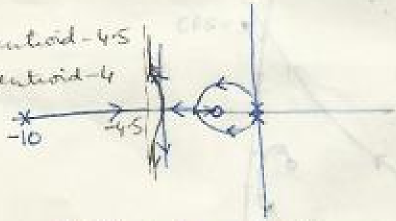
$\Rightarrow s^3 + s^2a - 3s^3 - 3s^2 - 2as^2 - 2as = -\left[2s^3 + s^2(a+3) + 2as\right] = 0$

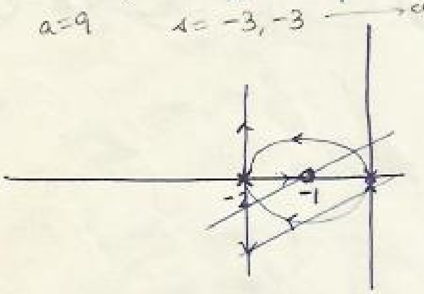
$\Rightarrow s\left[2s^2 + (a+3)s + 2a\right] = 0 \quad \therefore s = 0$ or $s = \frac{-(a+3) \pm\sqrt{(a+3)^2 - 16a}}{4} = \frac{-(a+3)}{4} \pm \frac{1}{4}\sqrt{a^2 - 10a + 9}$ check

$(a-9)(a-1)$

For $a = 10$; $s = -2.5, -4$ → centroid -4.5

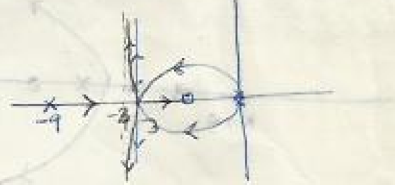
$a = 9$ $\quad s = -3, -3$ → centroid -4





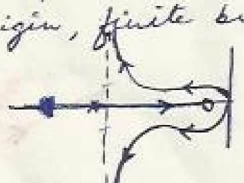
$a = 9$; $s = -3, -3$

centroid -4

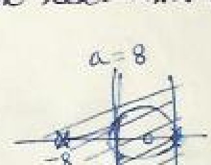


As pole shifts towards origin, finite break away on real axis is lost.

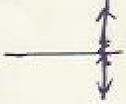
Check for $a < 9, 8$



$a = 8$ $\quad s = \frac{-11}{4} \pm j\frac{\sqrt{7}}{4} = -2.75 \pm j\frac{\sqrt{7}}{4}$



At $a = 1$, pole zero cancellation $\therefore$



Root locus does not show all roots of characteristic eqn., only those of reduced characteristic eqn.

* **Note**: Cancelled pole of $G(s)H(s)$ is a CL pole of original system $\therefore$ effect in perturbations!!

Root Contour

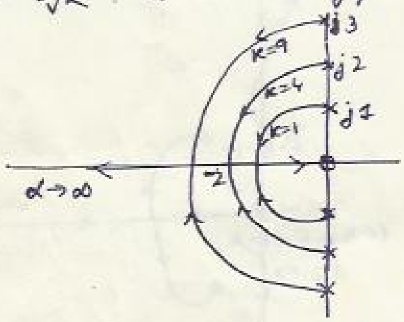
$G(s) = \dfrac{K}{s(s+\alpha)}$ $\therefore$ Effect of α also to be seen.

$s^2 + \alpha s + K = 0 \Rightarrow 1 + \alpha\left(\dfrac{s}{s^2+K}\right) = 0.$

Allow $0 < \alpha < \infty$ for different values of $K \rightarrow$ gives root contours.

$$\frac{d\alpha}{ds} = \frac{d}{ds}\left(-\frac{s^2+K}{s}\right) = \cancel{\frac{s(s^2+K)+s(2s^2K)}{s^2}} = \frac{-s(2s)+(s^2+K)}{s^2} = \frac{-(s^2-K)}{s^2} = 0$$

$\therefore s = \pm\sqrt{K}$. Break away pt.



Ex. $GH(s) = \dfrac{K}{s(s+1)(s+\alpha)}$

$1 + GH(s) = 1 + \dfrac{K}{s(s+1)(s+\alpha)} = 0$ or $s^2(s+1) + \alpha(s+1)s + K = 0$

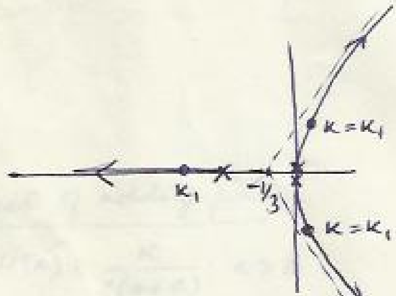
$\therefore 1 + \dfrac{\alpha s(s+1)}{s^2(s+1)+K} = 0$

~~For which Q poles of reduced charac. eqn. $s^2(s+1)+K=0$ are at poles~~

Root contours originate at Q poles of reduced charac. eqn. $s^2(s+1)+K=0$ $[\alpha=0]$ and terminate at $[\alpha=\infty]$ $0, -1, -\infty$.

Rewrite reduced ch. eqn. as

$1 + \dfrac{K}{s^2(s+1)} = 0.$



Use R-H array for α cross-over for par. K.

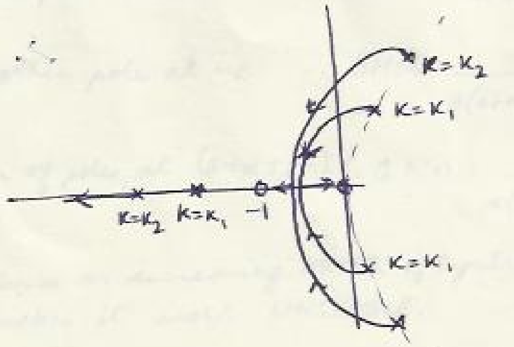
$s^3 + (\alpha+1)s^2 + \alpha s + K = 0.$

s^3	1	α
s^2	$\alpha+1$	K
s	$\dfrac{\alpha(\alpha+1)-K}{\alpha+1}$	
s^0	K	

$\therefore \alpha(\alpha+1) - K = 0$

$\Rightarrow \alpha = \dfrac{-1 \pm \sqrt{1+4K}}{2}$

Since $0 < \alpha < \infty$



Root locus for system with transportation lag.

$$GH(s) = \frac{Ke^{-sT}}{s(s+2)} \quad \text{say } T = 1s.$$

$$e^{-sT} \approx 1 - sT.$$

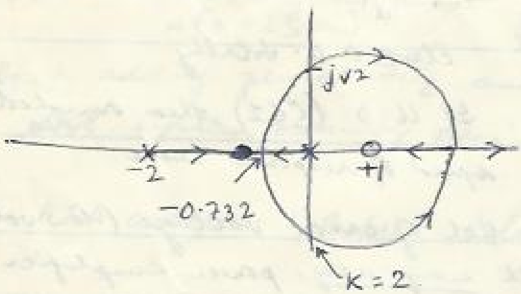
$$\therefore GH(s) = \frac{K(1-s)}{s(s+2)} = \frac{-K(s-1)}{s(s+2)}$$ → already (−)ve ∴ use +ve fb.

$\therefore$ ① $1 - GH = s(s+2) + K(s-1) = 0$

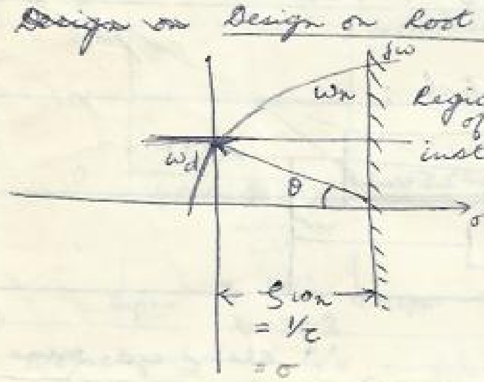
Also ② $\angle$ cond. becomes $\angle \frac{K(s-1)}{s(s+2)} = (2k)\pi$

③ Root loci exists for even number of poles & zeros to rt. side on real axis for Q TrF. $\frac{K(s-1)}{s(s+2)}$

Other rules same.



~~Design on~~ Design on Root locus



depends on rate of exponential envelope $(e^{-\zeta\omega_n})$

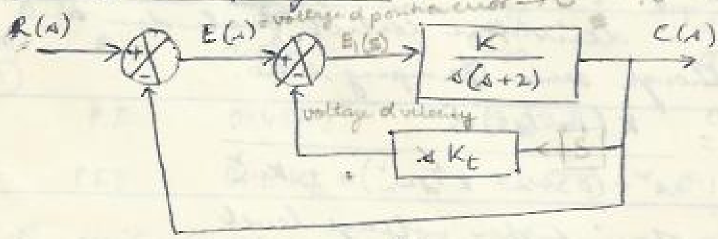
Speed of response of system given by largest time constant τ of system appearing in charac. eqn.

$\therefore (1+s\tau) \Rightarrow s = -\frac{1}{\tau} = -\zeta\omega_n$ (2nd order) $= -\sigma$

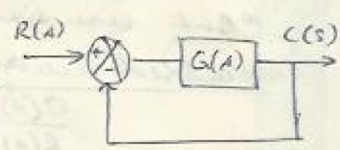
∴ lines ∥ to imaginary axis represent constant $1/\tau$ lines.

$\cos\theta = \zeta$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

(Rate fb) Multiple loop system: (HIERARCHY) ⟹ $K/s \cdot \frac{1}{s+2}$



≡

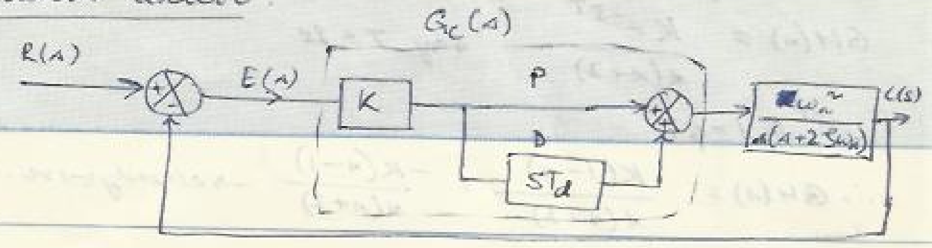


$$\therefore G(s) = \frac{C(s)}{E(s)} = \frac{K}{s(s+2) + sKK_t} \;;\; 1 + G(s) = 1 + \frac{sKK_t}{s(s+2)+K} = 1 + \frac{\alpha s}{s(s+2)+K} = 0$$

≈ to Rate fb loop.

$$G_F(s) = \frac{\frac{K}{s(s+2)}}{1 + \frac{KK_t}{s+2}} = \frac{K}{s(s+2)+sKK_t} = \frac{K}{s[s+2+KK_t]}$$

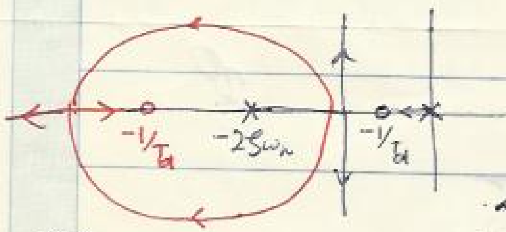
Derivative Control:



$\therefore$ Q w/o derivative control: $\dfrac{K\omega_n^2}{s(s+2\zeta\omega_n)}$

with derivative $\dfrac{K\omega_n^2(1+sT_d)}{s(s+2\zeta\omega_n)}$

$\rightarrow$ addn. of zero.



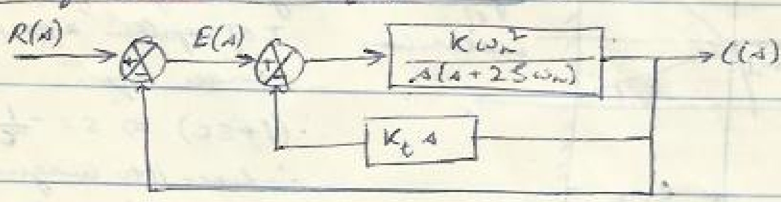
$|1/T_d| > |2\zeta\omega_n| \rightarrow$ then U.D. response

Else O.D. totally

If U.D. $(\zeta<1)$, then overshoot but speed of response more.

**<u>NOTE</u>: Derivative control of error voltage (low voltage signal) in f/w path <u>may req.</u> power amplifier $\Rightarrow K$ high $\Rightarrow$ system O.D. $\therefore$ slow.

Rate feedback (Tacho f/b) Control:



$$\frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2+(2\zeta\omega_n + KK_t\omega_n^2)s + K\omega_n^2} \quad \therefore \text{charac. eqn. same.}$$

just replace T_d by K_t $\therefore$ same change of damping ratio

*But rise time in derivative control faster due to added zero although same damping ratio.

$$\frac{C(s)}{R(s)} = \frac{K(1+T_d s)\omega_n^2}{s^2+(2\zeta\omega_n + KT_d\omega_n^2)s + K\omega_n^2}$$

But Rate f/b on o/p $\therefore$ higher voltage level

Derivative control on f/w path $\therefore$ error signal low

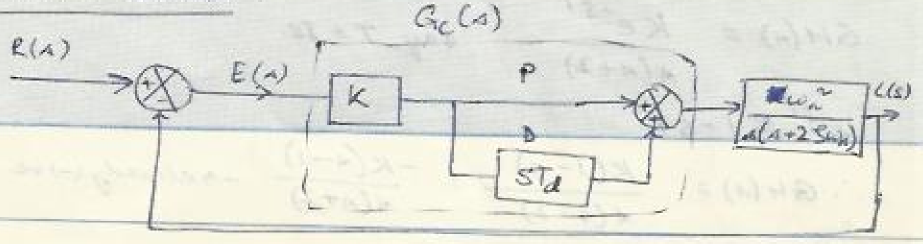
$\therefore$ Use high pass RC (else derivative of noise too!)

ex. o/p of potentiometer is discontinuous

$\therefore$ the jumps differentiated more.

+ power amplifier $\therefore K\uparrow$ $\therefore$ O.D.

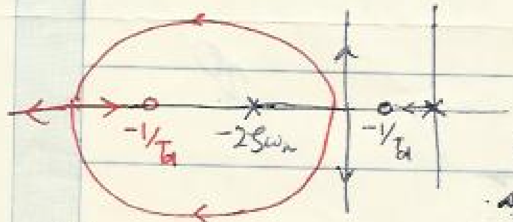
Derivative Control :



∴ Q w/o derivative control : $\dfrac{K\omega_n^2}{s(s+2\zeta\omega_n)}$

with derivative $\dfrac{K\omega_n^2(1+sT_d)}{s(s+2\zeta\omega_n)}$

→ addn. of zero.



$|1/T_d| > |2\zeta\omega_n|$ → then U.D. response

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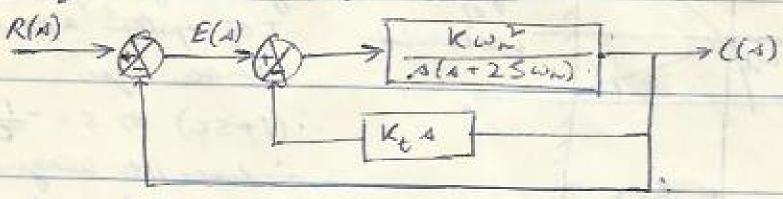
Rate feedback (Tacho f/b) Control :



$$\frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2+(2\zeta\omega_n+KK_t\omega_n^2)s+K\omega_n^2}$$ ∴ Charac. eqn. same.

just replace T_d by K_t ∴ same change of damping ratio

*But rise time in derivative control faster due to added zero although same damping ratio.

$$\frac{C(s)}{R(s)} = \frac{K(1+T_d s)\omega_n^2}{s^2+(2\zeta\omega_n+KT_d\omega_n^2)s+K\omega_n^2}$$

But Rate f/b on o/p ∴ higher voltage level

Derivative control on f/w path ∴ error signal low.

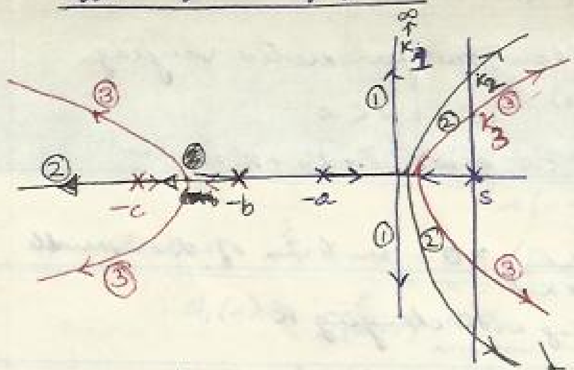
∴ use high pass RC (else derivative of noise too!)

ex. o/p of potentiometer is discontinuous

∴ the jumps differentiated more.

+ power amplifier ∴ K↑ ∴ O.D.

Effect of Adding poles.



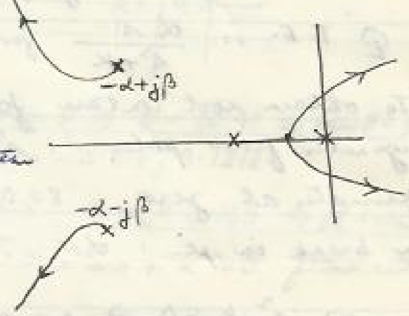
① $GH_1 = \dfrac{K}{s(s+a)}$ $a>0$

② $GH_2 = \dfrac{K}{s(s+a)(s+b)}$ $|b|>|a|$

③ $GH_3 = \dfrac{K}{s(s+a)(s+b)(s+c)}$ $|c|>|b|>|a|$

$\therefore K_1 = \infty > K_2 > K_3$

$GH_4 = \dfrac{K}{s(s+a)(s+\alpha\pm j\beta)}$



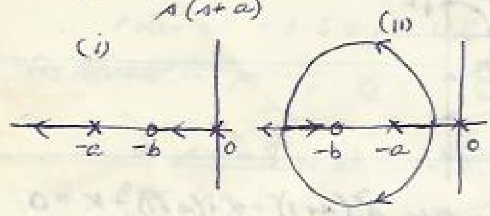
Addn. of pole $\equiv$ inc. ORDER of system
— decreases stability

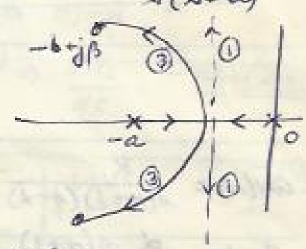
Effect of Adding Zeros:

$GH_1 = \dfrac{K}{s(s+a)}$ $a>0$

$GH_2 = \dfrac{K(s+b)}{s(s+a)}$

$GH_3 = \dfrac{K(s+b\pm j\beta)}{s(s+a)}$





Addn. of zero : stabilizes system
Q. – effect on transient ? (loss of causality)?

Effect of varying pole position:

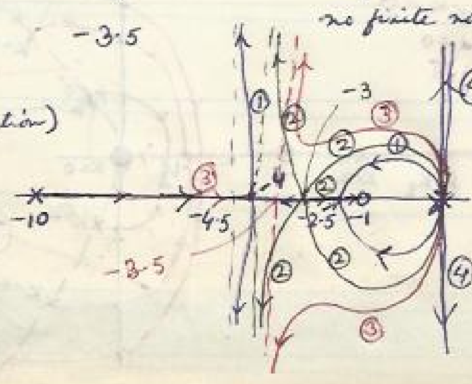
$GH = \dfrac{K(s+1)}{s^2(s+a)}$ $\therefore K = -\dfrac{s^2(s+a)}{s+1}$ for root loci.

$\therefore \dfrac{dK}{ds} = \dfrac{-(s+1)(3s^2+2as)+s^2(s+a)}{(s+1)^2} = 0 \Rightarrow s(2s^2+(a+3)s+2a)=0$

$\Rightarrow s=0$ or $\dfrac{-(a+3)}{4} \pm \dfrac{\sqrt{a^2-10a+9}}{4}$ are possible break-in pts. or breakaway

	a	L asymptote	intersection of asymptotes σ_A	Break-in/away pts.
①	10	$\pm 90°$	-4.5	-2.5, -4
②	9	$\pm 90°$	-4	-3
③	8	$\pm 90°$	-3.5	no finite non-zero pt.

** ④ 1 (pole zero cancellation)
NOT reflected in root locus.



MANAGEMENT REPORT

Root Contour :: More than one parameter varying.

$R(s) \to \bigotimes \to \boxed{\dfrac{K}{s(s+\alpha)}} \to C(s)$ $\qquad 0 < K, \alpha < \infty$

ch. eqn. $s^2 + \alpha s + K = 0$.

Rewrite as $1 + \alpha\left(\dfrac{s}{s^2+K}\right) = 0$ in terms of α as variable.

with the root loci changing with changing K.

$\therefore$ Q T.F. is $\dfrac{\alpha s}{s^2+K}$ for any particular K.

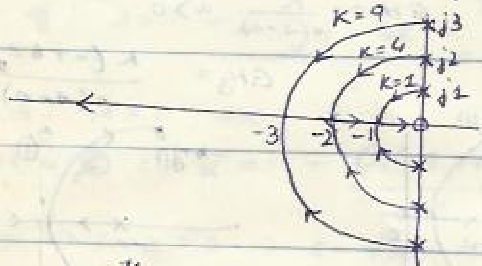
$\therefore$ To obtain root contour for various $K \longrightarrow$

originates from poles $\therefore s^2 + K = 0 \Rightarrow s = \pm j\sqrt{K}$

terminates at zeros $s = 0, -\infty$.

For break in pt. : $\alpha = -\dfrac{(s^2+K)}{s}$ $\therefore \dfrac{d\alpha}{ds} = \dfrac{-2s^2 + s^2 + K}{s^2} = 0$

$\Rightarrow s^2 - K = 0 \Rightarrow s = \pm\sqrt{K}$ possible $\therefore$ valid $s = -\sqrt{K}$.



Ex 6.15 $GH(s) = \dfrac{K}{s(s+1)(s+\alpha)}$ $\quad \therefore$ ch. eqn. $s^2(s+1) + \alpha s(s+1) + K = 0$.

$\therefore 1 + \dfrac{\alpha\, s(s+1)}{s^3+s^2+K} = 0$ $\quad \therefore$ Q T.F. $\dfrac{s(s+1)}{s^3+s^2+K}$ II

Root contours originate ($\alpha = 0$) at Q poles of reduced ch. eqn.

$s^3 + s^2 + K = 0$. Rewrite as $1 + \dfrac{K}{s^2(s+1)} = 0$

I $\therefore$ Root locus of $\dfrac{K}{s^2(s+1)}$: provide start of root contour

Root contours terminate at $s = 0, -1, -\infty$.

$j\omega$ axis crossings : R-H array : $s^3 + s^2(\alpha+1) + \alpha s + K = 0$

s^3	1	α
s^2	$\alpha+1$	K
s^1	$\dfrac{\alpha^2+\alpha-K}{\alpha+1}$	
s^0	K	

$\therefore K > 0$

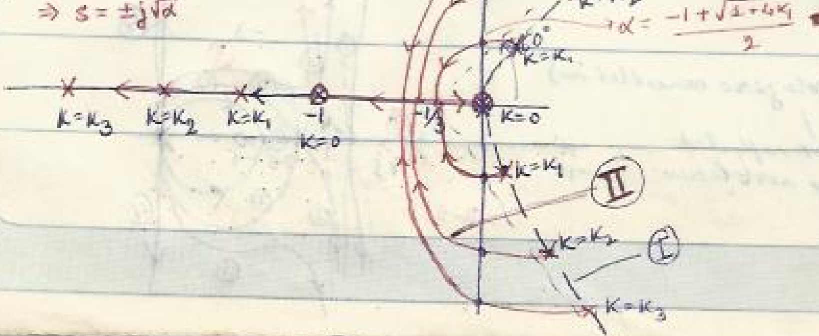
$\alpha(\alpha+1) - K = 0$ are cross over pts. $\Rightarrow (\alpha+1) = \dfrac{K}{\alpha}$

$\Rightarrow \alpha = \dfrac{-1 \pm \sqrt{1+4K}}{2}$ $\quad (\because \alpha > 0)$.

$(\alpha+1)s^2 + K = 0$

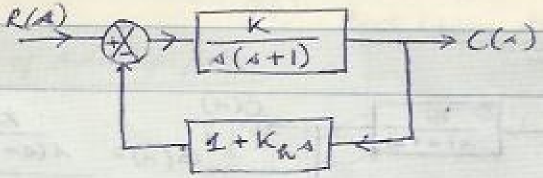
$\Rightarrow K s^2 + K\alpha = 0$

$\Rightarrow s = \pm j\sqrt{\alpha}$



$\alpha = \dfrac{-1+\sqrt{1+4K_1}}{2}$

Ex 6.16.



(i) Let $K_h = 0$ So $GH(s) = \frac{K(1+K_h s)}{s(s+1)} \Rightarrow$ ch. eqn. $s(s+1) + KK_h s + K = 0$.

becomes $s^2 + s + K = 0$ Reduced ch. eqn.

$\therefore G_1(s) = \frac{K}{s(s+1)}$ with root locus plot

(ii) value of K for $\zeta = 0.158 = \cos\theta \Rightarrow \theta = 80.9° \Rightarrow \tan\theta = 6.243 = \frac{p}{b}$

$\therefore b = -0.5$ (break away divides) $\Rightarrow p = 3.1216 \Rightarrow s_1 = -0.5 \pm j3.1216$

$\zeta\omega_n = 0.5 \Rightarrow \omega_n = 3.165 \Rightarrow K = \omega_n^2 = 10$

$\because s^2 + (2\zeta\omega_n)s + \omega_n^2 = s^2 + s + K = 0$ from reduced ch. eqn.

(iii) For $K = 10$, draw root loci with K_h as variable.

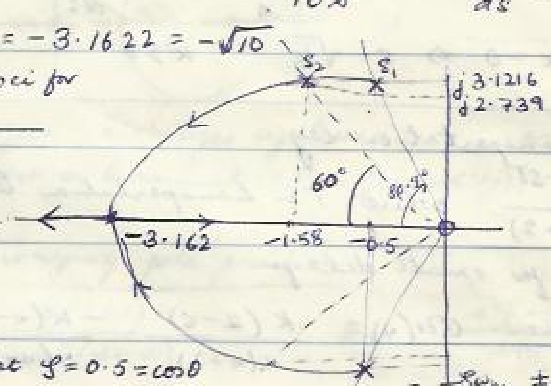
$\therefore s^2 + 10K_h s + s + 10 = 0 \Rightarrow 1 + K_h \frac{10s}{s^2+s+10} = 0$.

$\therefore$ Poles at $-0.5 \pm j3.1216$, zeros at $0, -\infty$.

For Break in pt.; $K_h = -\frac{(s^2+s+10)}{10s} \Rightarrow \frac{dK_h}{ds} = \frac{-10s(2s+1) + 10(s^2+s+10)}{100s^2} = 0$

$\Rightarrow s = -3.1622 = -\sqrt{10}$ $\qquad = \frac{-10(s^2-10)}{100s^2} = 0$

Root loci for $K = 10$

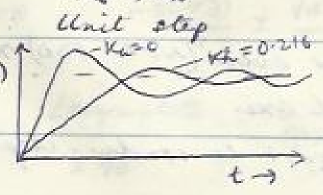


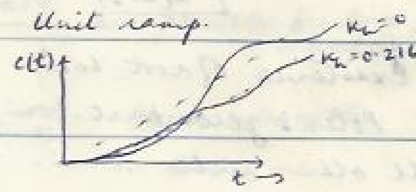
(iv) K_h so that $\zeta = 0.5 = \cos\theta$

$\Rightarrow \theta = 60°$ also $\omega_n = \sqrt{10} = 3.162$ $\therefore s_2 = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -1.581 \pm j2.739$

$\therefore 2\zeta\omega_n = 2(0.5)\omega_n = 3.1622 = 1 + 10K_h \Rightarrow K_h = 0.216$

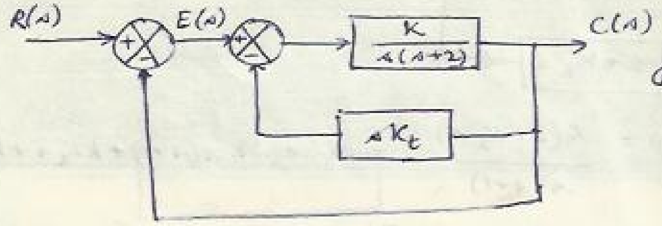
Responses: Unit step Unit ramp





Note: $\zeta\omega_n \triangleq \frac{1}{\tau}$ for 1st order
for 2nd order

Multiple loop system:

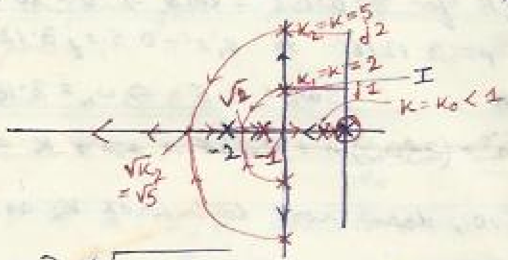


$$G(s) = \frac{\frac{K}{s(s+2)}}{1 + \frac{sKK_t}{s(s+2)}} = \frac{K}{s(s+2) + sKK_t}$$

$\therefore$ ch. eqn.: $s(s+2) + sKK_t + K = 0$.

$$\therefore\ 1 + \frac{sK \cdot K_t}{s(s+2)+K} = 1 + \frac{\alpha s}{s^2+2s+K} = 0 \quad \text{where } KK_t = \alpha. \quad \text{II}$$

Reduced ch. eqn.: $s(s+2) + K = 0 \Rightarrow 1 + \frac{K}{s(s+2)} = 0$ I



Root contour

Poles of: $s = \dfrac{-2 \pm \sqrt{4-4K}}{2} = -1 \pm \sqrt{1-K}$

Zeros: $s = 0, -\infty$.

For Break in pt.: $\alpha = -\dfrac{(s^2+2s+K)}{s} \quad \therefore \dfrac{d\alpha}{ds} = \dfrac{-s(2s+2)+s^2+2s+K}{s^2} = 0$

$\Rightarrow s^2 - K = 0 \Rightarrow s = \sqrt{K} \quad \because K > 0.$

System with Transportation lag:

$GH(s) = \dfrac{Ke^{-sT}}{s(s+2)}$ where T is transportation lag/delay

$e^{-sT} \approx 1 - sT$ for small delays.

Let $T = 1s$, then $GH(s) = \dfrac{K(1-s)}{s(s+2)} = \dfrac{-K(s-1)}{s(s+2)}$

$\therefore$ ~ to positive fb: $1 - GH(s) = 0$.

(i) $\angle$ condn. $\angle \dfrac{K(s-1)}{s(s+2)} = 2k\pi$

So

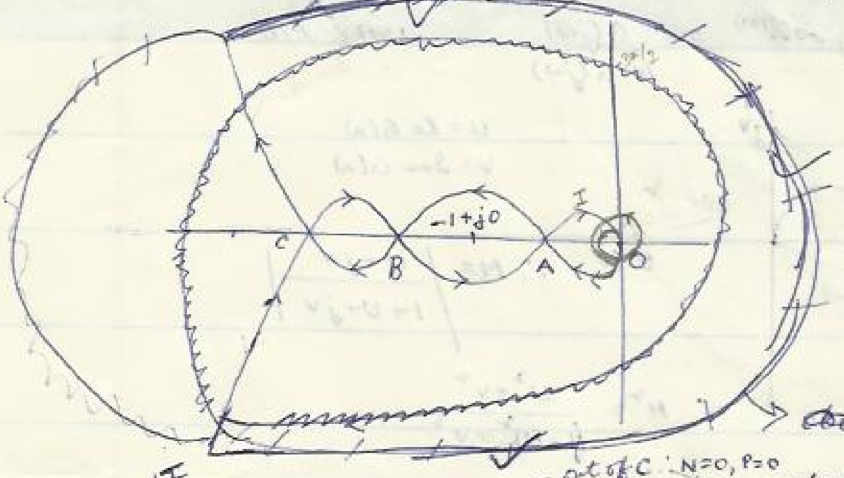
(ii) Existence of root loci: for even (NOT ODD) no. of Poles & zeros to rt. on real axis.

All other rules same: magnitude condn. dependent.

NYQUIST STABILITY Contd. 2 (21)

Conditionally stable systems:

① $GH(s) = \dfrac{K(s^2+2s+4)}{s(s+4)(s+6)(s^2+1.4s+1)}$ ∴ minimum phase (no poles in RH GH(s) plane)



II $s = \lim_{R\to\infty} Re^{j\phi}$ $\phi \in [\pi/2, -\pi/2]$ CW

$GH(s) = \dfrac{K}{R^3} e^{-3j\phi}$ $\phi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$ CCW

IV $s = \lim_{\epsilon\to 0} \epsilon e^{j\theta}$ $\theta = [-\frac{\pi}{2}, \frac{\pi}{2}]$ CCW

$GH(s) = \dfrac{4K}{\epsilon \cdot 24} = \frac{1}{\epsilon} e^{-j\theta}$ $\theta = [\frac{\pi}{2}, -\frac{\pi}{2}]$ CW

~~closed~~

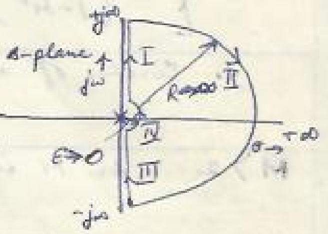
I For $0 < K < 16.36$, $(-1,0)$ bet. $\{B \& C$ (out of C $N=0, P=0$) change! $N=2$, $P=0$ ∴ $Z=2$

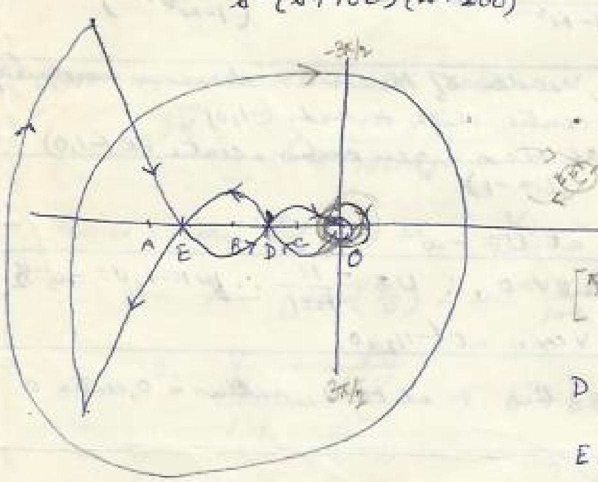
III $67.5 < K < \approx 160$ bet $A \& B$ — $O \& A$

II For $16.36 < K < 67.5$, $(-1,0)$ bet $A \& B$ $N = -1+1 = 0$, $P=0$ ∴ stable.

IV $\approx 160 < K < \infty$ bet $O \& A$

② $GH(s) = \dfrac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}$ ∴ $P=0$





IV $\left[+\frac{3\pi}{2}, -\frac{3\pi}{2}\right]$ CW ← $\left[-\pi/2, \pi/2\right]$

$\left[\pi/2, -\pi/2\right] \xrightarrow{II} \left[-\frac{3\pi}{2}, +\frac{3\pi}{2}\right]$ CCW

D → $\omega = 77.7$; $K=1, \sigma = -5.3\times 10^{-5}$

$K = 19000, \sigma = -1.007$

E → $\omega = 25.8$; $K=1, \sigma = -3.497\times 10^{-4}$

$K = 2900, \sigma = -1.014$.

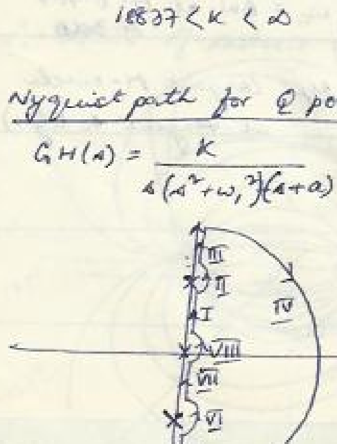
∴ For $0 < K < 2818$, $(-1,0)$ in A zone $N=2$

$2818 < K < 18837$ $(-1,0)$ in B zone : $N = -1+1 = 0$

$18837 < K < \infty$ $(-1,0)$ in C zone. $N=2$

Nyquist path for 2 poles on $j\omega$ axis

$GH(s) = \dfrac{K}{s(s^2+\omega_1^2)(s+a)}$ $a > 0$



I $s = j\omega$ $j0^+$ to $j\omega_1^-$

II $s = \lim_{\epsilon\to 0} j\omega_1 + \epsilon e^{j\theta}$ $\theta \in [-\pi/2, \pi/2]$

III $s = j\omega$ $j\omega_1^+$ to $j\infty$

IV $s = \lim_{R\to\infty} Re^{j\phi}$ $\phi \in [\pi/2, -\pi/2]$

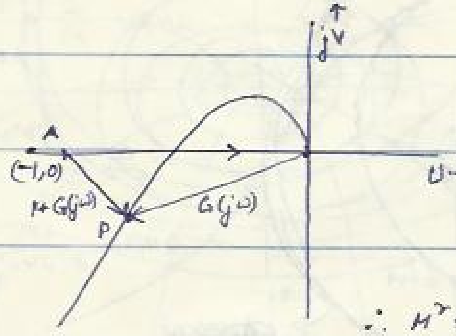
V to VII IPP of III to I

VIII $s = \epsilon e^{j\theta}$ $\theta \in [-\pi/2, \pi/2]$

CLOSED LOOP FREQUENCY RESPONSE:

Constant Magnitude Loci: M-circles

$$C.T.F. = T(j\omega) = M(\omega)\, e^{j\phi(\omega)} = \frac{G(j\omega)}{1+G(j\omega)} \qquad \text{UNITY F/B.}$$



$U = \mathrm{Re}\, G(s)$
$V = \mathrm{Im}\, G(s)$

$$\therefore M = \left| \frac{U+jV}{1+U+jV} \right|.$$

$$\therefore M^2 = \frac{U^2+V^2}{(1+U)^2+V^2}$$

$$\Rightarrow (1-M^2)U^2 + (1-M^2)V^2 - 2M^2 U = M^2$$

Divide by $(1-M^2)$ & add $\left(\frac{M^2}{1-M^2}\right)^2$ & complete squares

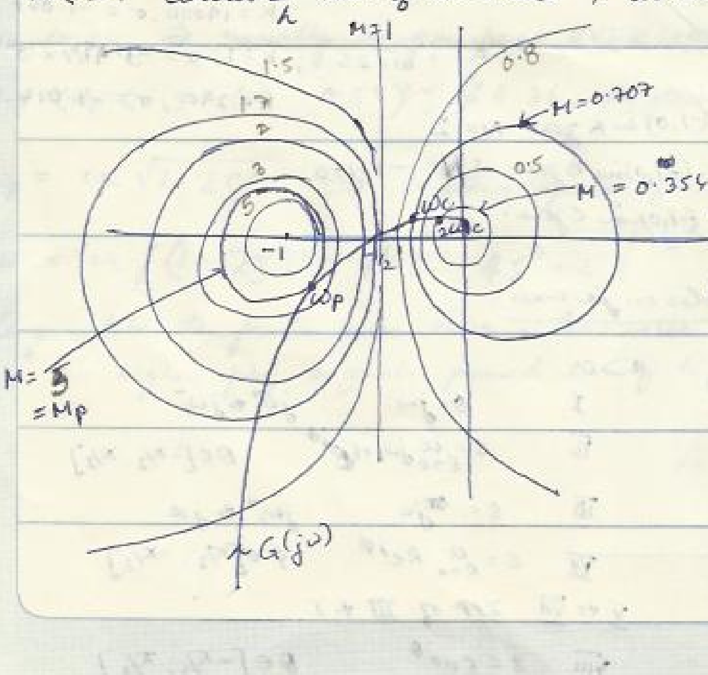
$$\therefore \left[U - \left(\frac{M^2}{1-M^2}\right)\right]^2 + V^2 = \left(\frac{M}{1-M^2}\right)^2 \quad \Rightarrow U = \pm\frac{M}{1-M^2} + \frac{M^2}{1-M^2} = \frac{M(1\pm M)}{(1+M)(1-M)} = \frac{-M}{1\pm M}$$

$\therefore$ Circles of radius $\left|\frac{M}{1-M^2}\right|$ with centre at $\left(\frac{M^2}{1-M^2}, 0\right)$

M > 1 : As M increases, radii of M circles decrease monotonically
centre shifts towards $(-1,0)$
at $M=\infty$, zero radius, centre at $(-1,0)$
$(U=-1)$

$M=1$: radius ∞, centre at $U=-\infty$
intercept at $-1/2$ [Put $V=0$, $\therefore U = \frac{-M}{M\pm1}$ $\therefore$ for $M=1$, $U=-\infty, -\frac{1}{2}$
$\therefore$ st. line $\parallel$ to V axis at $(-1/2, 0)$

$M<1$: circles of dec. radius & rt. of $M=1$ line; at $M=0$, radius is 0, centre 0.



ω_c = BW at $M=0.707$
or 3dB

M_p: largest M-circle at tangent to $G(j\omega)$.

CONSTANT PHASE ANGLE LOCI : N CIRCLES.

$$\angle e^{j\alpha} = \angle \frac{U+jV}{1+U+jV}$$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{V}{U}\right) - \tan^{-1}\frac{V}{1+U}$ $\quad$ let $\therefore \tan\alpha = N$

$$\therefore N = \tan\left[\tan^{-1}\left(\frac{V}{U}\right) - \tan^{-1}\frac{V}{1+U}\right]$$

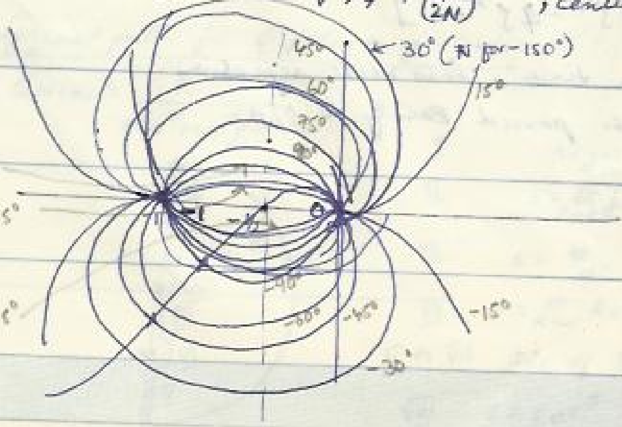
$$= \frac{\frac{V}{U} - \frac{V}{1+U}}{1 + \frac{V^2}{U(1+U)}} \Rightarrow U^2 + U + V^2 - \frac{1}{N}V = 0$$

Add $\left(\frac{1}{4} + \frac{1}{(2N)^2}\right)$ to both sides

for $V=0$, $U(1+U)=0$ $\Rightarrow U = 0, -1$ intersections $\forall N$

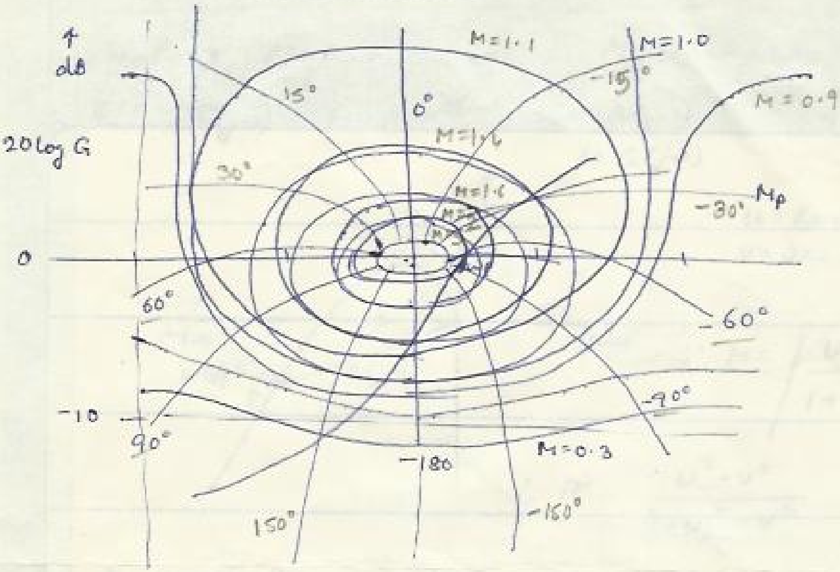
$$\Rightarrow \left(U+\frac{1}{2}\right)^2 + \left(V - \frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2$$

$\therefore$ circle of radius $\sqrt{1/4 + \left(\frac{1}{2N}\right)^2}$, center $\left(-\frac{1}{2}, \frac{1}{2N}\right)$



NICHOLS CHART :

Log magnitude vs phase plots



Non unity fb system

$$T(s) = \frac{G(s)}{1+GH(s)} = \frac{1}{H(s)} \; T_0(s)$$

$\therefore$ Plot for $T_0(s) \rightarrow$ use M & N circles for $G_0(s)$ to obtain $T_0(s)$

Then $T_0(s) \times \frac{1}{H(s)} \rightarrow$ use Bode plot $\rightarrow T(s)$

Freq. domain specs for design

2nd order prototype $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad 0<\zeta<0.707 \text{ (given } \zeta)$$

$$= 1 \quad \zeta \geq 0.707$$

$$\Rightarrow \zeta^2 = \frac{1}{2} - \frac{1}{2}\sqrt{\left(1-\frac{1}{M_p^2}\right)} \text{ for } M_p > 1 \text{ (Given } M_p)$$

M_p large $\Rightarrow$ usually large peak OVERSHOOT for STEP RESPONSE

$1.1 < M_p < 1.5 \Leftrightarrow 0.54 > \zeta > 0.36$ for stable system.

$\omega_p = \omega_n\sqrt{1-2\zeta^2} \Rightarrow$ system's speed of response

$$BW = \omega_n\left[(1-2\zeta^2) + \sqrt{4\zeta^4-4\zeta^2+2}\right]^{1/2}$$

larger BW $\Rightarrow$ faster rise time, less noise rejection.

$\because$ higher freq. signals passed easily to o/p.