Polar plot (NYQUIST PLOT) $\omega : 0$ to ∞

$\omega : -\infty$ to $+\infty$

plot of $|G(j\omega)|$ vs $\angle G(j\omega)$ in polar co-ord.

$\Rightarrow$ Re $G(j\omega)$ vs Im $G(j\omega)$ in Cartesian co-ord.

- $|G(j\omega)|$ and Re $G(j\omega)$ are even fns.
  $\angle G(j\omega)$ and Im $G(j\omega)$ are odd fns.

+ve phase angle is CCW.

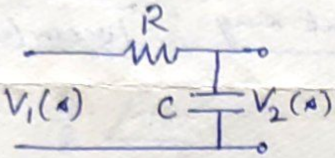
Disadv.: a) addn of poles & zeros to existing sys. req. recalculations

similar to root locus b) effects of addn. of individual poles/zeros not indicated

For cascaded T.F., use Bode plot & shift to polar plot.

Adv. freq. response over entire freq. range in one plot.

Ex

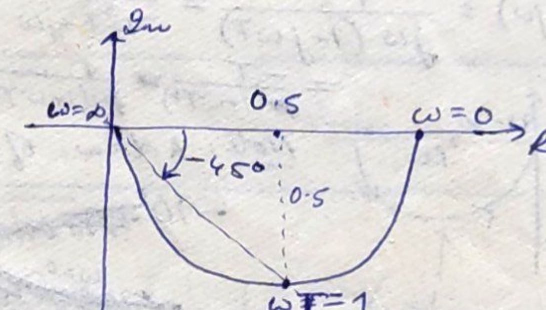


Low pass RC (LAG) ckt.

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1+sT}\ ; \quad T = RC$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0
$1/T$	0.707	$-45°$
∞	0	$-90°$



$\therefore$ Polar plot is a semicircle of rad. 0.5 and centre at 0.5.

$$(x - 1/2)^2 + y^2 = (1/2)^2 \quad \text{where } x = \text{Re } G(j\omega) = \frac{1}{1+\omega^2T^2},\ y = \text{Im } G(j\omega) = \frac{-\omega T}{1+\omega^2T^2}$$

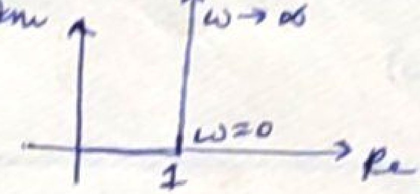
O/p lags behind i/p $\forall \omega$: indication is plot in ($-j\omega$) plane.

Ex

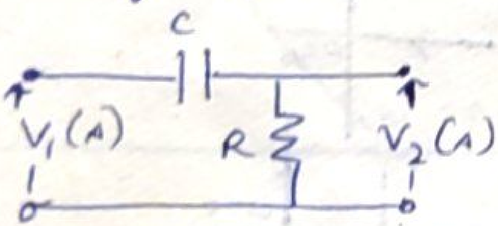
For $G(j\omega) = 1 + j\omega T \longrightarrow$ LEADS o/p.

ex. series RL ckt.

(NON CAUSAL)

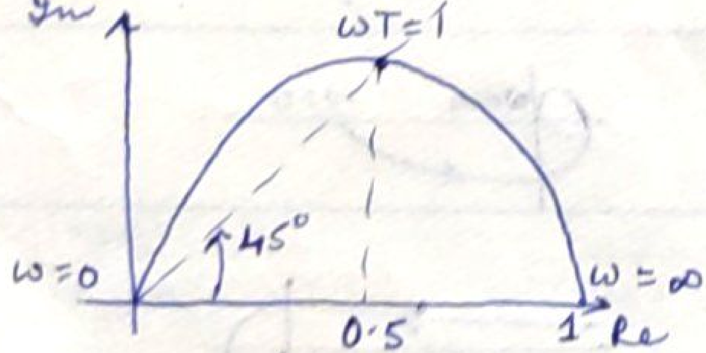


Ex High pass RC (LEAD) Network



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{s}{s + 1/T} \qquad T = RC.$$

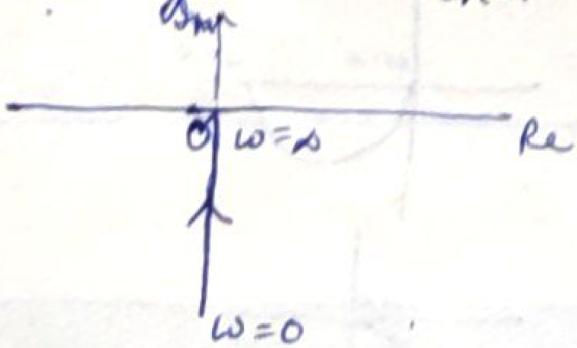
$$\Rightarrow G(j\omega) = \frac{j\omega}{j\omega + 1/T} = \left|\frac{\omega T}{\sqrt{1+\omega^2T^2}}\right| \angle 90^\circ - \tan^{-1}\omega T$$



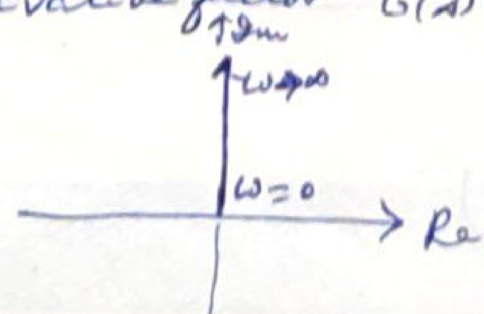
o/p voltage leads i/p at all times.

Ex Integrator $G(s) = 1/s$

$$\therefore G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} = \frac{1}{\omega} \angle -90^\circ$$



Ex Derivative factor $G(s) = s$

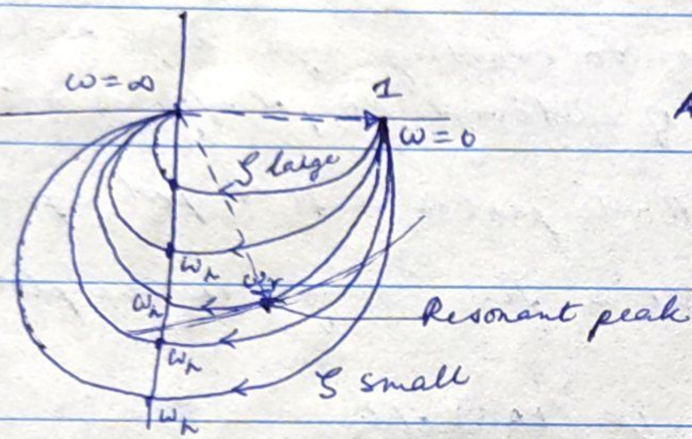


Ex. RLC series network with o/p across capacitor

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \omega_n = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{R}{2}\sqrt{\frac{L}{C}}$$

$$\therefore G(j\omega) = \frac{1}{1 + j2\zeta(\omega/\omega_n) - (\omega/\omega_n)^2}$$

ω	$G(j\omega)$
0	$1\angle 0^\circ$
∞	$0\angle -180^\circ \to -\tan^{-1}\left(\frac{2\zeta}{(-\omega/\omega_n)}\right) \to -\tan^{-1}(0)$



At $\omega = \omega_n$, $G(j\omega_n) = \frac{1}{j2\zeta} = -j\frac{1}{2\zeta}$

$\therefore$ on neg. real axis

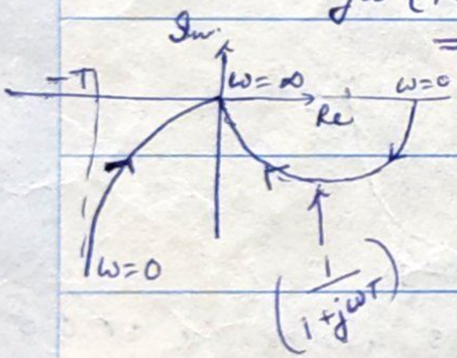
ω_r where distance from origin is max. $|G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ $\quad \omega_r = \omega_n\sqrt{1-2\zeta^2}$

$$\text{Resonant mag.} = \frac{|\text{vector}(\omega_r)|}{|\text{vector}(\omega=0)|}$$

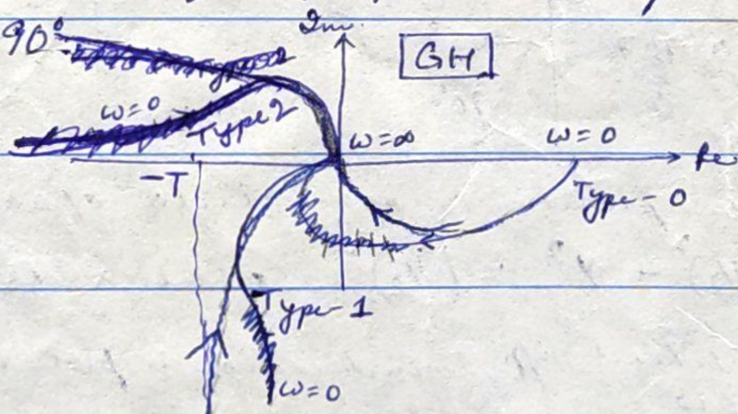
Ex $G(s) = \frac{1}{s(sT+1)}$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{-\omega^2 T + j\omega} = \frac{-T}{(1+\omega^2T^2)} - \frac{j}{\omega(1+\omega^2T^2)}$$

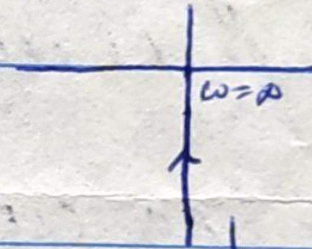
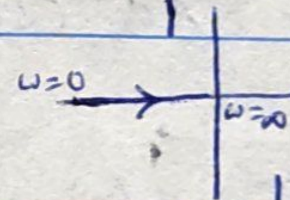
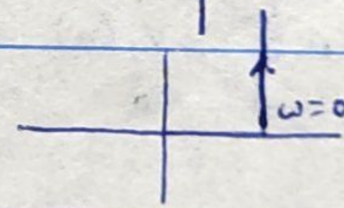
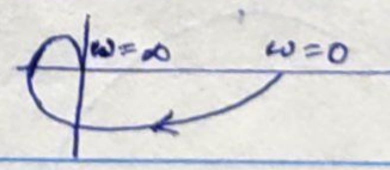
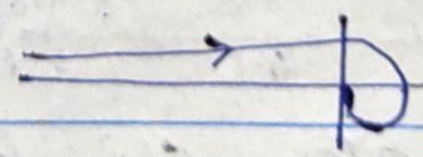
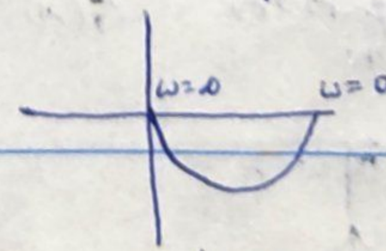
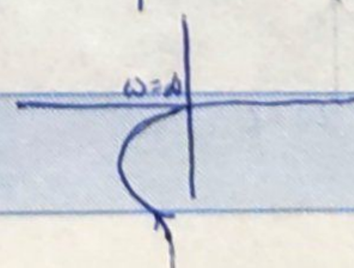
ω	$G(j\omega)$
0	$-T - j\infty = \infty\angle -90^\circ$
∞	$0\angle -180^\circ$



Note: Addn. of simple pole rotates polar plot by -90°

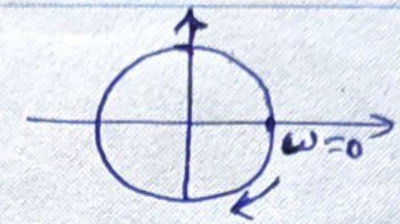


$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T)}$$
$$= \frac{-j}{\omega}\left[\frac{-T}{1+\omega^2T^2} - \frac{j}{\omega(1+\omega^2T^2)}\right]$$
$$= \frac{-1}{\omega^2(1+\omega^2T^2)} + j\frac{T}{\omega(1+\omega^2T^2)}$$
$$= -\infty + j\infty \text{ at } \omega = 0.$$

$G(j\omega)$	Pole-zero	Polar plot	Rem.
$\frac{1}{j\omega}$			
$\frac{1}{(j\omega)^2}$			
$1+j\omega T$			
$\frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$			
$\frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$			
$\frac{1}{1+j\omega T}$			
$\frac{1}{j\omega(1+j\omega T)}$			

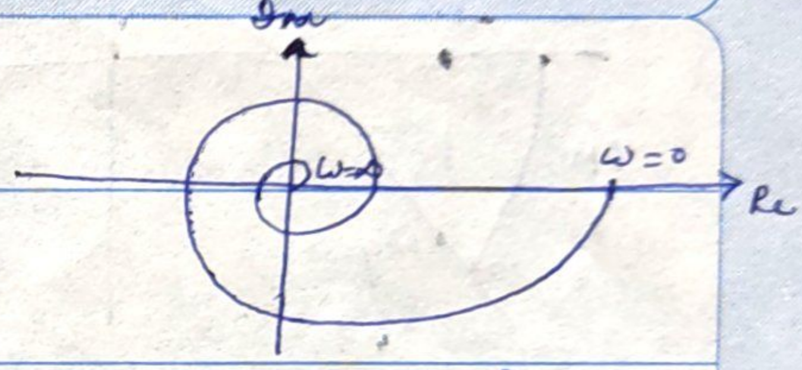
Ex Transportation lag $G(s) = e^{-sT}$

$G(j\omega) = 1 \angle -\omega T$



Ex. $G(s) = \dfrac{\exp(-sL)}{1+sT}$

$\therefore G(j\omega) = \dfrac{1}{\sqrt{1+\omega^2T^2}} \angle -\omega L - \tan^{-1}\omega T$



Ex $G(s) = \dfrac{10}{s(s+1)(s+2)}$ $\quad G(j\omega) = \dfrac{10(-j\omega)(1-j\omega)(2-j\omega)}{\omega^2(1+\omega^2)(2+\omega^2)}$

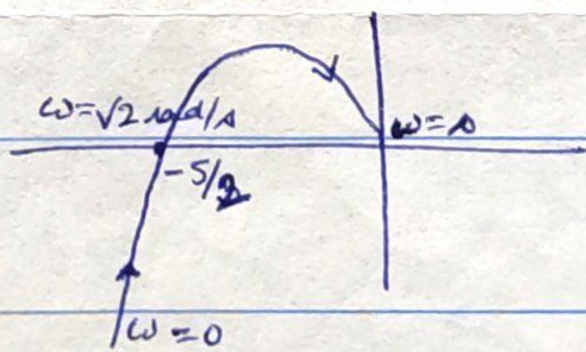
$$= \frac{10\left[-3\omega^2 - j\omega(2-\omega^2)\right]}{\omega^2(1+\omega^2)(2+\omega^2)}$$

ω	$G(j\omega)$
0	$\infty \angle -90^\circ$
∞	$0 \angle -270^\circ$
$\sqrt{2}$	$\frac{5}{2} \angle -180^\circ$

Re $G(j\omega) = 0$ at $\omega = \infty$

Im $G(j\omega) = 0$ at $\omega^2 = 2$.

$$\text{Re } G(j\omega)\Big|_{\omega=\sqrt{2}} = \frac{-30(2)}{2(3)(4)} = \frac{-5}{2}$$

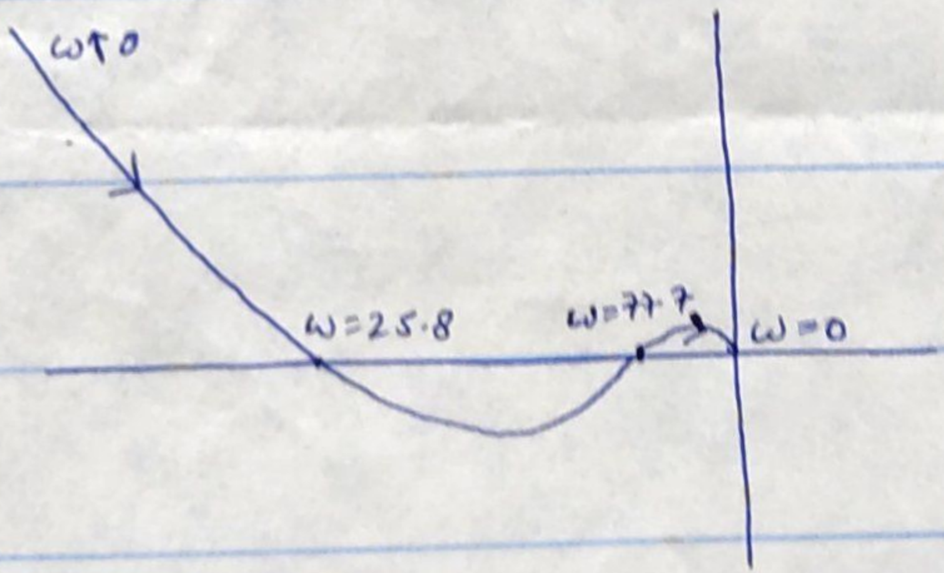


$\omega = 0$

Ex $\quad GH(s) = \dfrac{100k\,(s+5)(s+40)}{s^3\,(s+100)(s+200)}$

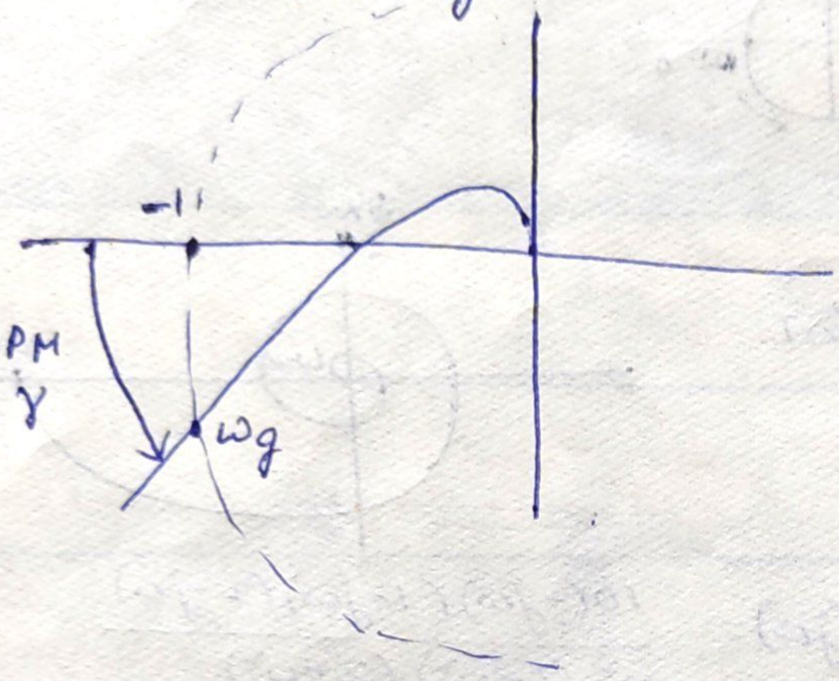
ω	$\lvert GH(j\omega) \rvert$	$\angle GH(j\omega)$
0	∞	-270°
$0 < \omega < 25.8$	finite	$-180^\circ < \theta < -270^\circ$
25.8	$(-3.497\times10^{-4})k$	-180°
	finite	$-180^\circ < \theta < -90^\circ$
77.7	$(-5.3\times10^{-5})k$	-180°
∞	0	-270°

Im $GH(j\omega) = 0$.
at $\omega = 25.8$ rad/s
& 77.7 rad/s

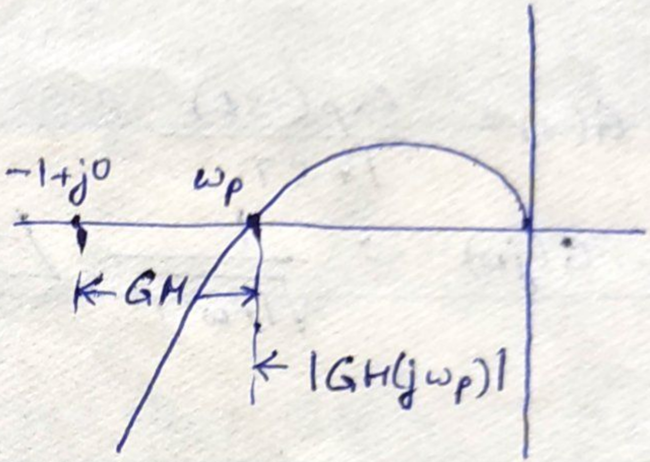


Relative stability : GM & PM.

Phase Margin



Gain Margin



$$\therefore \quad GM = 20 \log_{10} \frac{1}{|GH(j\omega_p)|} \text{ dB.}$$

For MP systems, both GM & PM +ve $\Rightarrow$ $\mathcal{E}$ stable SUFFICIENT CONDN.

NYQUIST STABILITY:

Basic philosophy:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$F(s) = 1 + G(s)H(s) = 0 \quad ; \quad GH(s) = \frac{N(s)}{D(s)}$$

$$= 1 + \frac{N(s)}{D(s)} = \frac{N'(s)}{D(s)}$$

∴ ① CL poles of system = Zeros of $F(s)$ = ROOTS of charac. eqn.

② OL T.F. $GH(s)$ poles = poles of $F(s)$

③ For asymptotic stability of CL sys.,

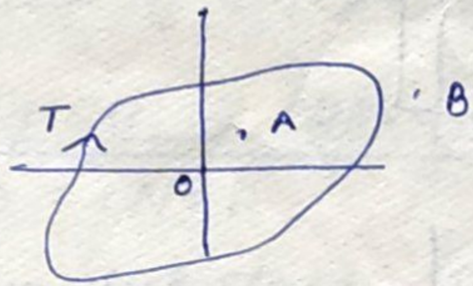
NO restriction on poles/zeros of OL T.F. $GH(s)$

BUT Poles of CL T.F. = Zeros of $F(s)$ in LHS.

Finding zeros of $F(s)$ is difficult but using $GH(s)$ roots, infer stability.

ANALYTIC FN $F(s)$ in s-plane if fn. & all its derivatives exist ⇒ no singular pts. ⇒ no POLES.

✓ENCIRCLED : A pt. is encircled by a Closed path if it is found inside the path. Usually CW (convention for +ve traversal)



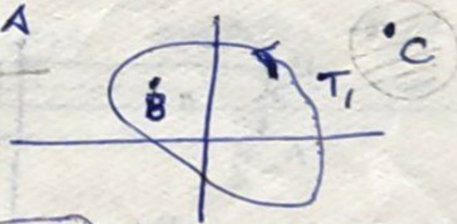
O, A encircled
B not.

ENCLOSED

A point or region is said to be enclosed by a closed path if it on the RT. of a path when the path is traversed in CW direction.

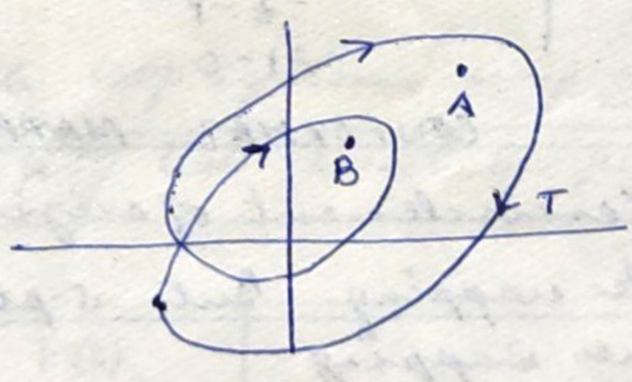
T encloses A

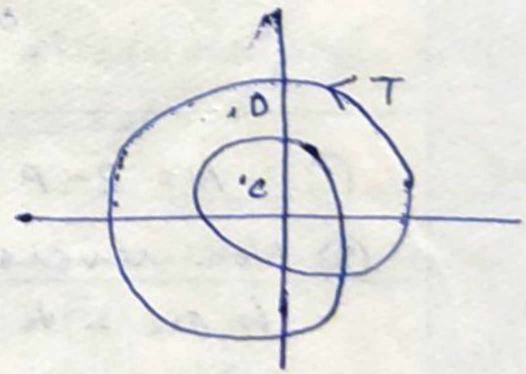
T_1 DOES NOT Enclose B
(While encloses C)



No. of encirclements

For B N = +2
A N = +1
C N = −2
D N = −1

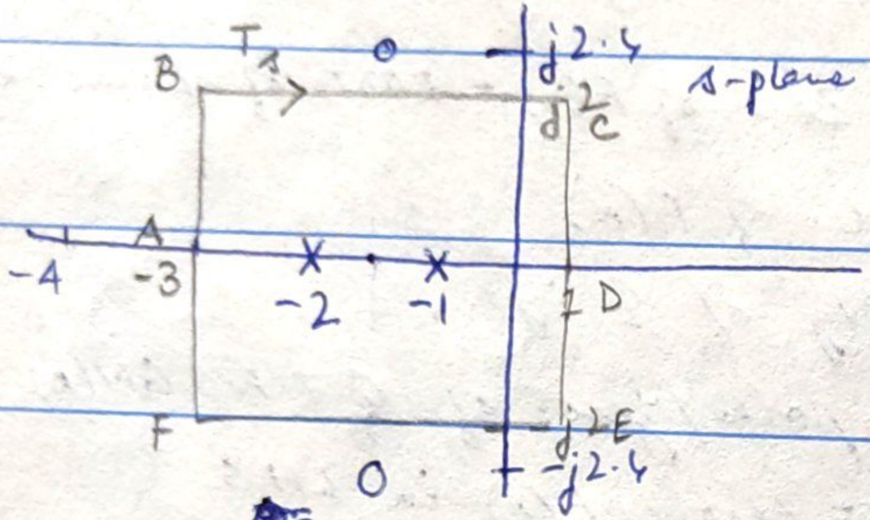


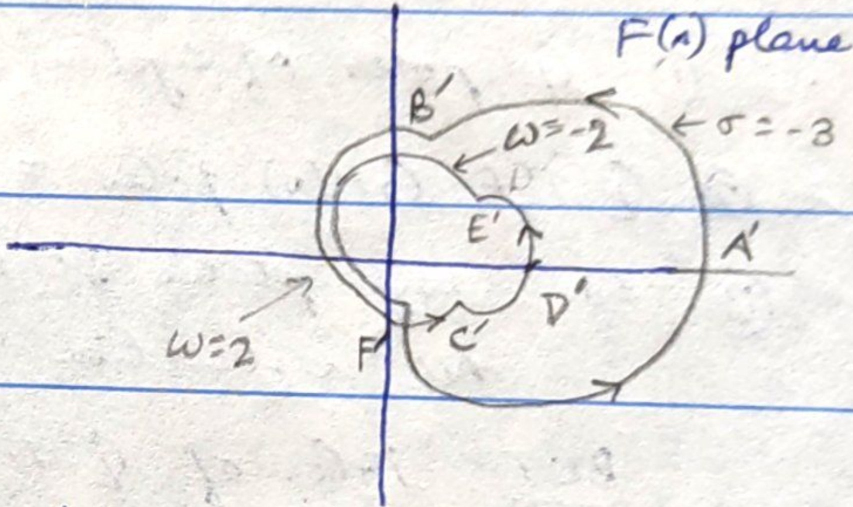


conformal mapping:

Say $GH(s) = \dfrac{6}{(s+1)(s+2)}$

$\therefore F(s) = 1 + GH(s) = \dfrac{(s+1.5+j2.4)(s+1.5-j2.4)}{(s+1)(s+2)} = 0$

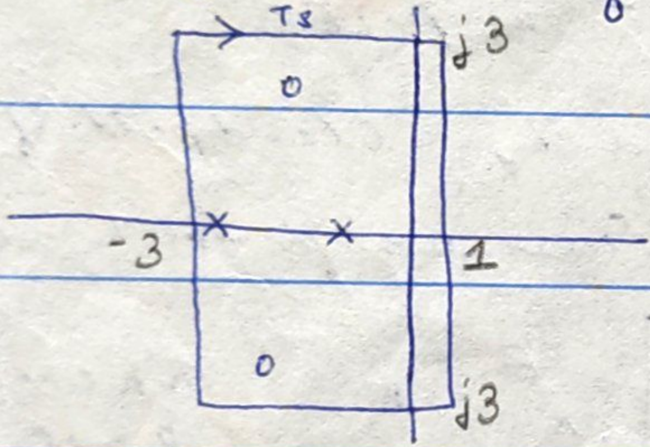


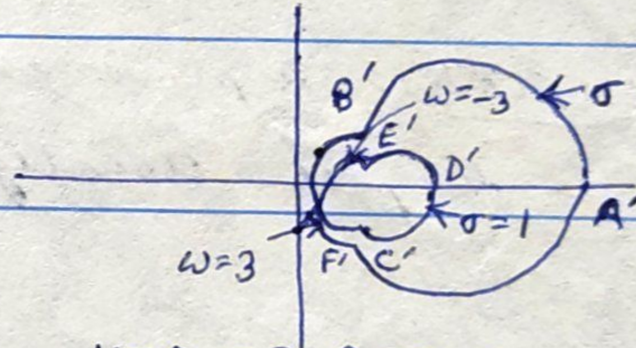


$s = \sigma + j\omega$	$A = -3 + j0$	$B = -3 + j2$	$C = 1 + j2$	$D = 1 + j0$	$E = 1 - j2$	$F = -3 - j2$
$F(s) = u + jv$	$4 + j0$	$0.7 + j0.9$	$1.11 - j0.577$	$2 + j0$	$1.11 + j0.57$	$0.7 - j0.9$

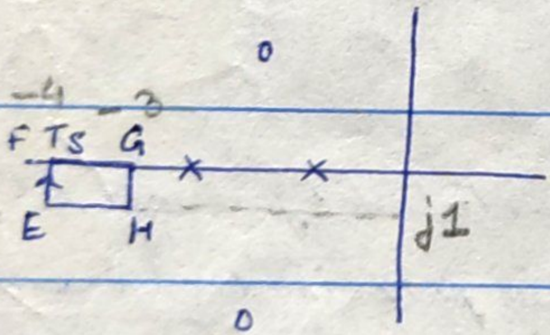
∴ No. of encirclements of origin $N = -2 = \boxed{Z-P}$ as in T_s → hence can be used as CHECK for ANALYTICAL FN. in specified region

$= 0 - 2$

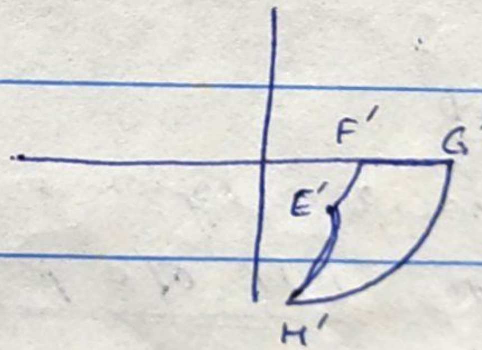


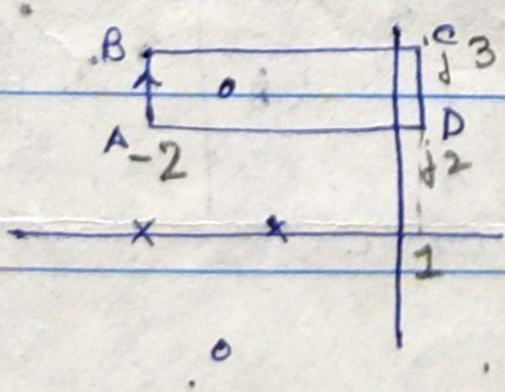


$N=0 = Z-P = 2-2$

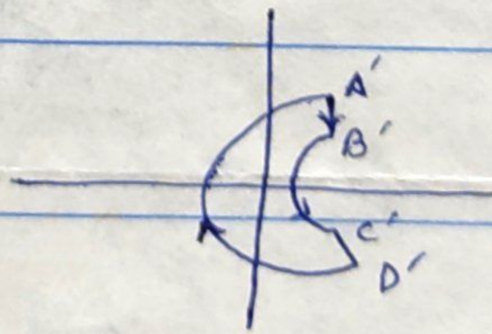


N=0





$N= +1$
$= Z-P$
$= 1-0$



CONFORMAL MAPPING.

II

s	F(s)
A = −3 + j0	4 + j0
B = −3 + j3	0·68 + j0·41
C = 1 + j3	0·92 − j0·38
D = 1 + j0	2 + j0
E = 1 − j3	0·92 + j0·38
F = −3 − j3	0·68 − j0·41

III

s	F(s)
A = −4 + j0	2 + j0
B = −3 + j0	4 + j0
C = −3 [illegible] j1	1·6 − j1·8
D = −4 − j1	1·6 − j0·6

IV

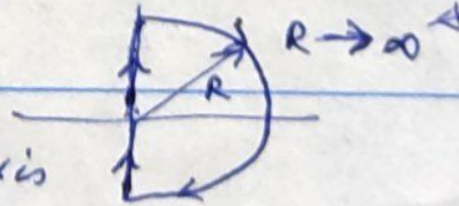
s	F(s)
A = −2 + j2	−0·2 + j0·6
B = −2 + j3	0·4 + j0·2
C = 1 + j3	0·92 − j0·38
D = 1 + j2	1·11 − j0·58

CONFORMAL MAPPING.

(1) $\therefore N = Z - P$ (encirclement of origin)

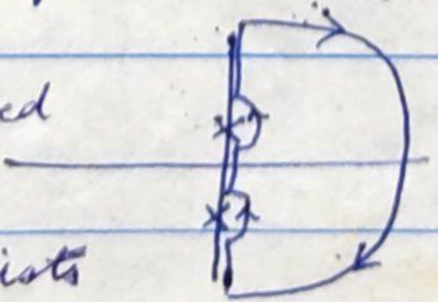
(2) Non reversible mapping But s-plane to $F(s)$ plane is one-to-one mapping

* Usually T_s as $j\omega$ axis & RHS s-plane



But if pt. of singularity or pole on $j\omega$ axis then small detour.

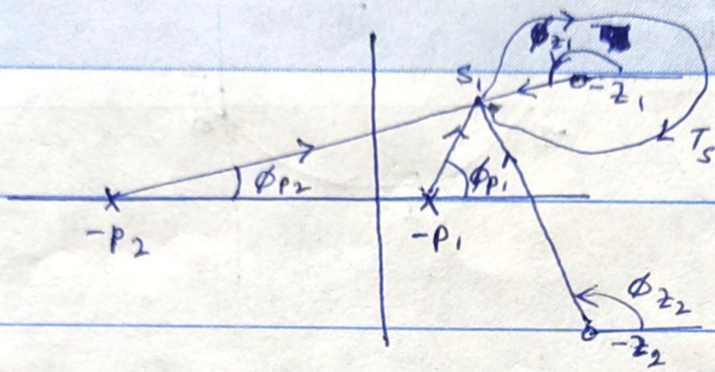
— to keep region enclosed in s-plane s.t. $F(s)$ is analytic — value exists



CAUCHY'S THEOREM:

Say $F(s) = \dfrac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} = |F(s)| \angle(\phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2})$

$\therefore$



$\therefore$ As s_1 traverses T_s, angles generated by $-p_1, -p_2, -z_2$ after one complete round are 0, while $\angle$ generated by $-z_1$ is 2π radians (clockwise)

$\therefore$ If Z zeros & P poles are encircled, then angle generated is $2\pi(Z) - 2\pi(P) = \phi_F = 2\pi N$ = net angle generated by T_F and hence net encirclement of origin of $F(s)$ plane in CW direction is $N = Z - P$.

Note $GH(s) + 1 = F(s) \Rightarrow GH(s)$ plane $(-1,0)$ pt. $\equiv F(s)$ plane origin

$\therefore$ Polar plot encirclement of $(-1,0)$ pt. $\equiv F(s)$ plane encirclement of origin.

$\therefore$ If there are P poles of GH in RHS and N encirclements of $(-1,0)$
$\Rightarrow$ Z of $F(s) = 1 + GH(s) = N + P$

Required to be zero for A stable $\mathcal{L}$ system $\Rightarrow N = -P$ required.

NYQUIST STABILITY CRITERION:

a) The Nyquist plot is obtained by mapping the fn. $GH(s)$ – polar plot.

b) The Nyquist contour is chosen to enclose entire RH of S-plane

Then $Z = N + P$

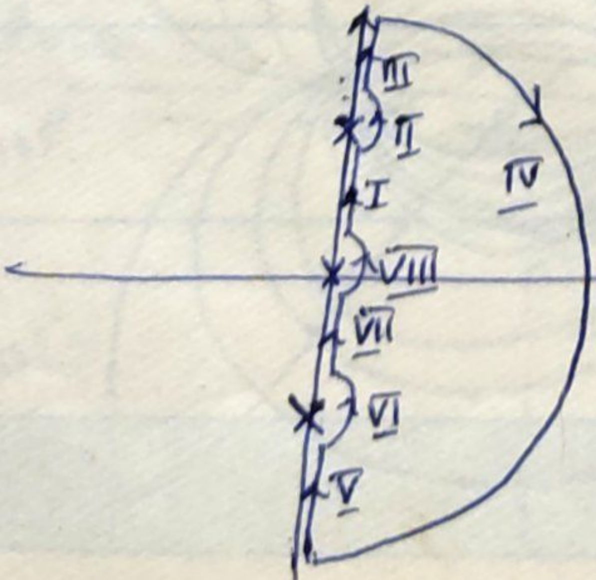
For $\mathcal{L}$ system to be stable, $Z = 0 \Rightarrow N = -P$.

<u>Note</u>: Since multiple loop systems may <u>NOT</u> include poles in RHS

$\therefore$ Routh's stability test must be performed.

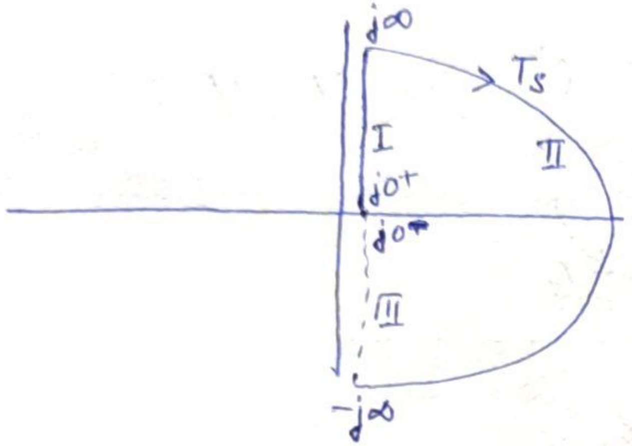
Nyquist path for 2 poles on $j\omega$ axis

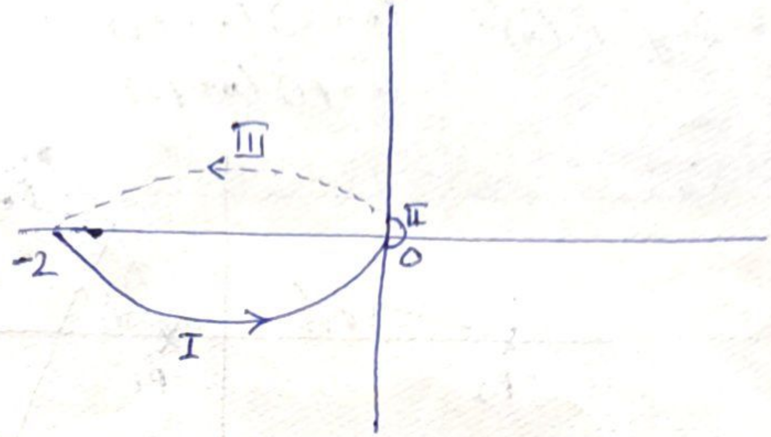
$$GH(s) = \frac{K}{s(s^2+\omega_1^2)(s+a)} \qquad a>0$$



I	$s = j\omega$	$j0^+$ to $j\omega_1^-$
II	$s = \lim_{\varepsilon\to 0} j\omega_1 + \varepsilon e^{j\theta}$	$\theta \in [-\pi/2, \pi/2]$
III	$s = j\omega$	$j\omega_1^+$ to $j\infty$
IV	$s = \lim_{R\to\infty} Re^{j\phi}$	$\phi \in [\pi/2, -\pi/2]$
V to VII	IPP of III to I	
VIII	$s = \varepsilon e^{j\theta}$	$\theta \in [-\pi/2, \pi/2]$

Ex 1 : $G(s)\,H(s) = \dfrac{K(s+2)}{(s+1)(s-1)}$ where $K = 1$.





I $\quad GH(s) = \dfrac{s+2}{(s+1)(s-1)}$

$\therefore |GH(j\omega)| = \left| \dfrac{j\omega+2}{(j\omega+1)(j\omega-1)} \right|$

Note : θ varies from $+\pi/2$ to $-\pi/2$ through 0° on T_s ~~complete~~ semicircle of infinite radius

$\therefore \angle GH = \angle \tan^{-1}\omega/2 - \angle \tan^{-1}\omega - \angle \pi - \tan^{-1}\omega = -\pi + \tan^{-1}\omega/2$

for $\omega = 0$ to ∞ $\longrightarrow \angle GH \in [-\pi, -\pi/2]$ ~~for ϕ~~

II $\quad s = \lim_{R\to\infty} R e^{j\phi} \quad \therefore \lim_{R\to\infty} GH(s)\Big|_{s=Re^{j\phi}} = \lim_{R\to\infty} \dfrac{1}{R} e^{-j\phi} = 0e^{-j\phi}$

$\therefore$ ε radius circle CCW $\quad \phi \in [\pi/2, -\pi/2]_{CW} \quad \angle GH \in [-\pi/2, \pi/2]_{CCW}$

III inverse plot of I.

$\therefore N = -1$, $P = +1$ $\quad \therefore Z = N + P = 0$ $\quad \therefore$ stable.

Ex 6 $GH(s) = \frac{K(s+3)}{s(s-1)}$ $K=10$.

I $GH(j\omega) = \frac{K(j\omega+3)}{j\omega(j\omega-1)} = \frac{K(j\omega+3)}{+j\omega(-1+j\omega)} = \frac{-4K}{1+\omega^2} - \frac{jK(\omega^2-3)}{\omega(1+\omega^2)}$

$\therefore \angle GH(j\omega) = \cancel{\pi} - \pi/2 + \tan^{-1}\frac{\omega}{3} - \cancel{\tan^{-1}(-\omega)}\ (\pi - \tan^{-1}\omega)$

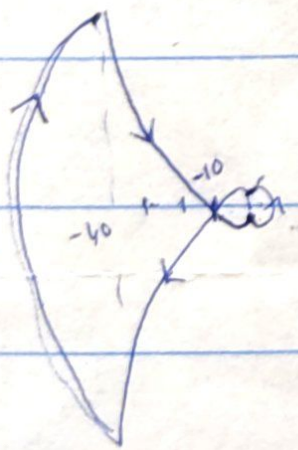
$= -3\pi/2 + \tan^{-1}\frac{\omega}{3} + \tan^{-1}\omega = \cancel{\pi/2 + \tan^{-1}\frac{\omega}{3} + \tan^{-1}\omega}$

$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$ CCW $\quad \cancel{= \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{CW}}$

$(\tan^{-1}A + \tan^{-1}B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\therefore$ For $\mathrm{Im}\, GH = 0$, $\pi = \pi/2 + \tan^{-1}\left(\frac{\frac{\omega}{3}+\omega}{1-\omega^2/3}\right) \Rightarrow \omega = \sqrt{3}$, $|GH|_{\sqrt{3}} = +10$

Ex 6



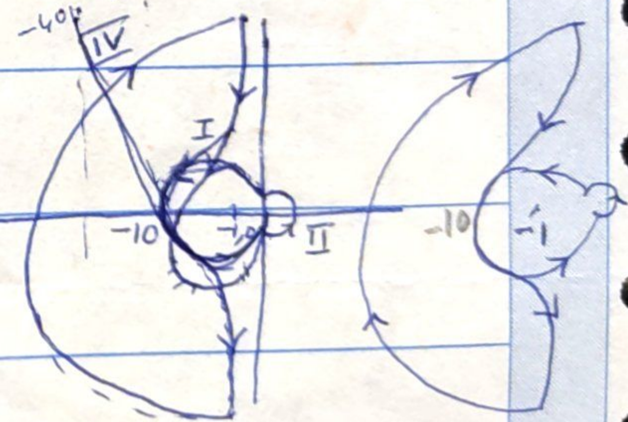
II $s = \lim_{R\to\infty} Re^{j\phi}$ $\quad \phi \in [\pi/2, -\pi/2]$ CW

$GH(s) = \lim_{R\to\infty} \frac{10}{R} e^{-j\phi}$ $\quad \therefore \angle GH \in [-\pi/2, \pi/2]$ CCW

III IPP $\left[\frac{+3\pi}{2}, \frac{+3\pi}{2}\right]$ CCW

IV $s = \lim_{\varepsilon\to 0} \varepsilon e^{j\theta}$ $\quad \theta \in [-\pi/2, \pi/2]$ CCW

$GH(s) = \left.\frac{30}{-s}\right|_{s=\lim_{\varepsilon\to0}\varepsilon e^{j\theta}} = (-1)\cdot\frac{30}{s} = \infty \angle{+\pi-\theta}$



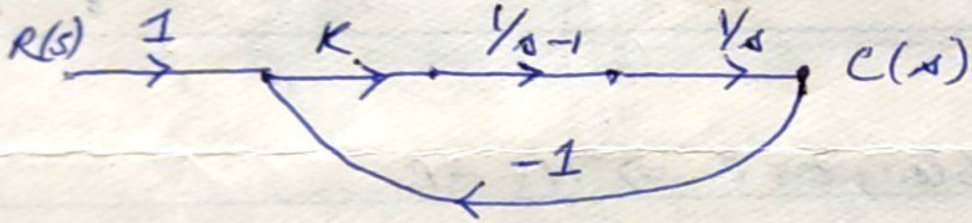
$\therefore \angle GH = [+3\pi/2, +\pi/2]$ CW

$\therefore P = 1$, $N = -1$ $\quad \therefore Z = 0$ STABLE.

Ex 2 $\quad GH(s) = \dfrac{K}{s(s+a)}$

Ex 3 $\quad GH(s) = \dfrac{K(T_2 s+1)}{s^2(T_1 s+1)} \quad T_1 > T_2$

Ex 4.



Ex 5. $\quad GH(s) = \dfrac{2(s+0.1)(s+0.6)(s^2+s+1)}{s^3(s-0.2)(s+1)}$

Ex 6. $\quad GH(s) = \dfrac{K(s+3)}{s(s-1)} \quad K = 10$

Ex 7. $\quad GH(s) = \dfrac{K(s-1)}{s(s+1)} \quad K > 0.$ check roots of $F(s)$

Ex 8. $\quad GH(s) = \dfrac{K}{s(T_1 s+1)(T_2 s+1)}$

Conditionally stable systems a) $GH(s) = \dfrac{K(s^2+2s+4)}{s(s+4)(s+6)(s^2+1.1s+1)}$

b) $GH(s) = \dfrac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}$