ROOT LOCUS TECHNIQUE — I

WR EVANS (1948)

— complete information of the poles from the pole-zero info w/o evaluating roots of the system when one or more parameters (say gain K) of the TF varied.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$ for negative f/b system where

$$G(s)H(s) = \frac{K\,P(s)}{Q(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = K\,G_1(s)H_1(s)$$

$m \leq n$ CAUSAL

Ch. eqn. $1 + G(s)H(s) = 0$ or $G(s)H(s) = -1$

(i) Magnitude condn. $|G(s)H(s)| = 1$

$$\therefore |G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{1}{|K|} \; ; \; 0 \leq K < \infty.$$

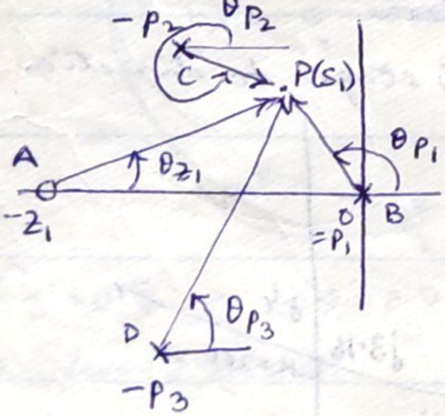
(ii) Phase condn. or $\angle$ condn. $\angle G(s)H(s) = (2k+1)\pi$; $k = 0, 1, 2 \dots$

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j) = (2k+1)\pi \quad \because \angle K = 0$$

∴ Any pt. satisfying both (i) & (ii) is a pt. on the root loci.

Let $G(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}$.

Say



∴ (i) Mag. condn. $|G_1(s_1)H_1(s_1)| = \frac{1}{|K|}$

$$= \frac{AP}{BP \cdot CP \cdot DP}$$

(ii) $\angle$ condn. : $\theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = (2k+1)\pi$.

Rules for root locus

1. Write characteristic eqn. in pole-zero form in terms of parameter of interest K as

$$\therefore 1 + K\, G_1(s)\, H_1(s) = 0$$

∴ Magnitude condn. $|G_1(s) H_1(s)| = \frac{1}{|K|}$

Phase condn. $\underline{/G_1(s) H_1(s)} = (2k+1)\pi$

2. Q poles and zeros of $G_1(s)\, H_1(s) = 0$
3. Locate segments of real axis where root loci exist.
4. Determine no. of separate root loci
5. Locate angles of asymptotes and intersection of asymptotes
6. Determine break away pt. on real axis and elsewhere.
7. Determine imaginary axis crossover pts. using R-H array.
8. Estimate angle of departure of root locus from complex poles and angle of arrival at complex zeros.

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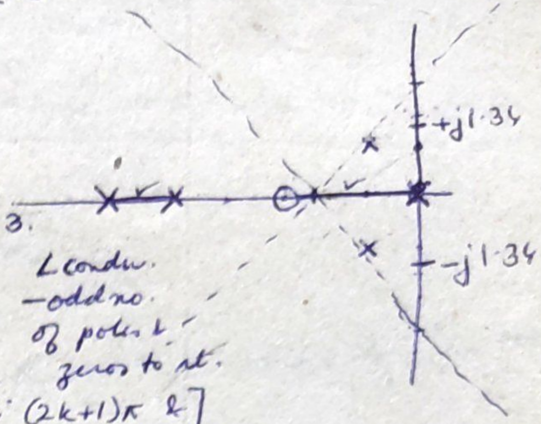
Q T.F. $G(s)H(s) = \dfrac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$ for $0 < K < \infty$.

1. Now, $G_1(s)H_1(s) = \dfrac{s+3}{s(s+5)(s+6)(s^2+2s+2)} = \dfrac{1}{K}$



Lcondn.
−odd no.
of poles &
zeros to rt.
[∵ (2k+1)π &
neg. f/b]

2. Q poles: $s = 0, -5, -6, -1 \pm j1$ $P = 5$
START
Q zeros: $s = -3$ real 4 zeros at infinity, $Z = 1$
END $= P - Z$

4. ∴ No. of root loci $N = P = 5$ ∵ $P > Z$.

5. $(P-Z)$ no. of infinite zeros ∴ $(P-Z)$ asymptotes intersecting at centroid σ_A on real axis s.t.

$$\sigma_A = \frac{\sum p_j - \sum z_i}{n-m} \quad \text{with angles } \phi_k = \frac{(2k+1)\pi}{P-Z}$$

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$$\therefore \sigma_A = \frac{(0-5-6-1-1)-(-3)}{4} = -2.5 \; ; \; \phi_k = \frac{(2k+1)\pi}{4} = \pm 45°, \pm 135°$$

6. $K(s+3) + s(s^2+11s+30)(s^2+2s+2) = s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$ ch. eqn

s^5	1	54	60+K
s^4	13	82	3K
s^3	47.7	60+0.769K	
s^2	65.60−0.212K	3K	
s^1	$\dfrac{3940-105K-0.163K^2}{65.65-0.212K}$		
s^0	3K		

∴ $K \geq 0$

$65.6 > 0.212K \Rightarrow K < 309$

$3940 - 105K - 0.163K^2 > 0$

$\Rightarrow K < 35.$

∴ $0 < K < 35$ for stability

Aux. eqn. : $58.2s^2 + 105 = 0$

$\Rightarrow s = \pm j1.34$ ∴ $\omega = \pm 1.34$ rad/s are the intersection pts. on imaginary axis.

7. At multiple roots. If r root loci branches meet at a pt., they break away at $\pm 180°/r$ w.r.t. angle at which they arrive. Usually on real axis.

I $F(s) = 1 + GH(s) = 0$ has multiple roots at $\frac{d}{ds}F(s) = 0$.

$\therefore \frac{d}{ds}GH(s) = \frac{d}{ds}(G_1 H_1)(s) = 0$ NECESSARY not sufficient

∴ all solns. not valid – need testing.

II $F(s) = 1 + K \cdot \frac{P(s)}{Q(s)} = 0 \Rightarrow Q(s) + KP(s) = 0$.

$\therefore \frac{dF(s)}{ds} = Q'(s) + KP'(s) = 0 \quad \therefore K = -\frac{Q'(s)}{P'(s)}$ & $F(s) = 0 \Rightarrow QP' - Q'P = 0$.

$\therefore K = -\frac{Q(s)}{P(s)}$ Note $\frac{dK}{ds} = -\frac{Q'P - QP'}{P^2} = 0$ is same as $\therefore \frac{dK}{ds} = 0$

$\downarrow$ get $s = s_1$

s.t. $\frac{d^2K}{ds^2} < 0$ for break away and $\frac{d^2K}{ds^2} > 0$ for break in.

and $K = \left.\frac{1}{G_1 H_1}\right|_{s=s_1}$

$$\frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} = -1 \Rightarrow K = \frac{-(s^5+13s^4+54s^3+82s^2+60s)}{s+3}$$

$$\therefore \frac{dK}{ds} = \frac{-(s+3)(5s^4+52s^3+162s^2+164s+60) + (s^5+13s^4+54s^3+82s^2+60s)}{(s+3)^2} = 0$$

$$\Rightarrow -4s^5 - 54s^4 - 264s^3 - 568s^2 - 492s - 180 = 0$$

$$\Rightarrow s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$$

From root locus plot, expected $-6 < s_1 < -5$ ∴ Bisection search gives

$$s_1 = -5.33.$$

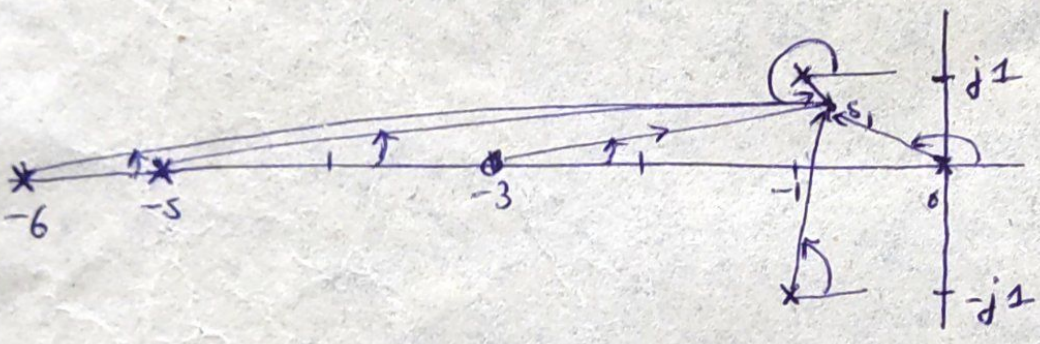
Other roots are $s = -0.656 \pm j0.468 ; -3.33 \pm j1.204$. not SUFFICIENT ∴ extra.

8. ∠ of departure from $-1 \pm j1$.

Say $s_1 = -1 \pm j1 + \delta$ (very close to pole) and on root locus leaving the pole.

$$\therefore \angle G_1H_1(s_1) = \angle(s_1+3) - [\angle s_1 + \angle s_1+1+j1 + \underset{}{\overset{=\theta}{\angle s_1+1-j1}} + \angle s_1+5 + \angle s_1+6]$$

$$= (2k+1)\pi$$



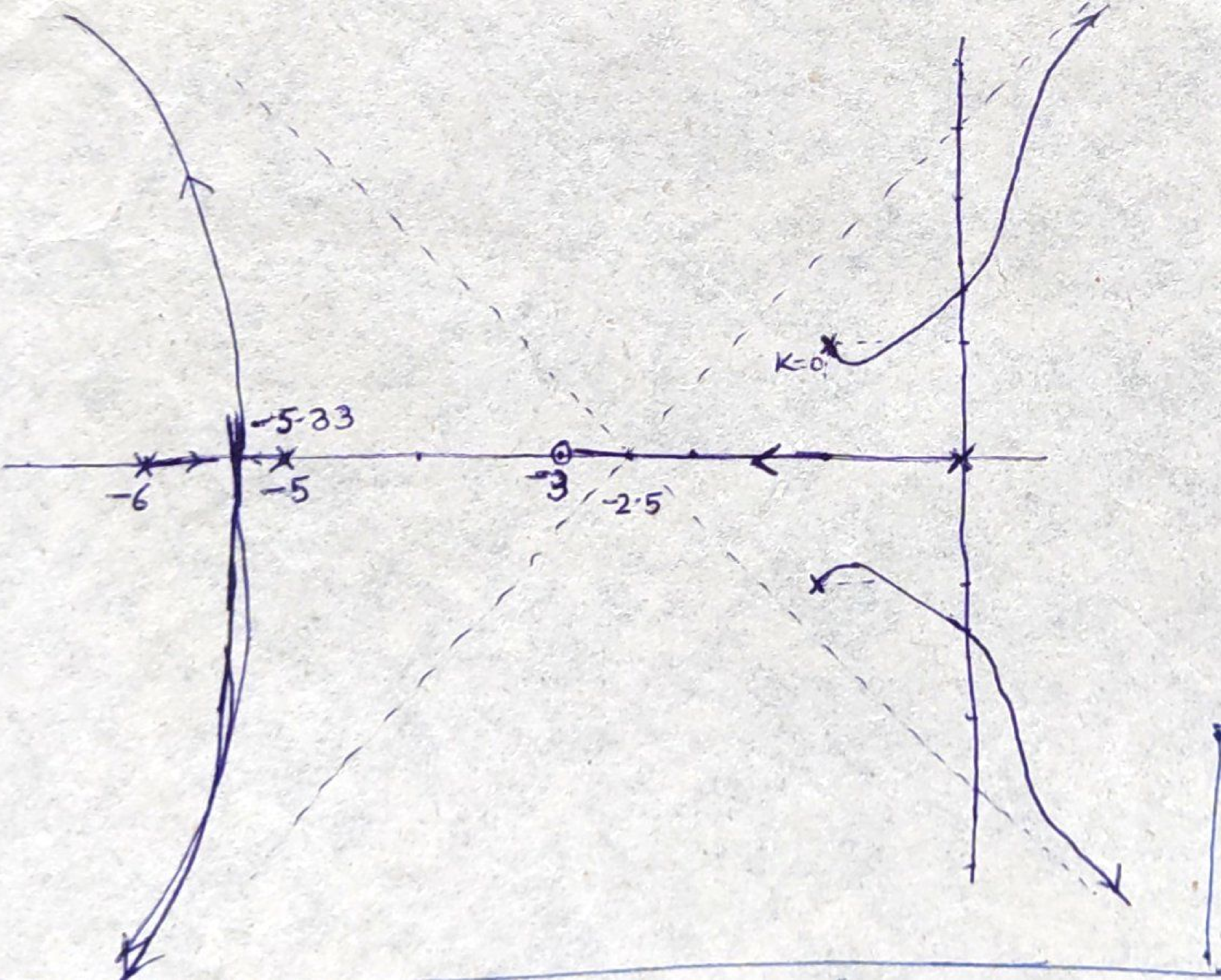
$$= (\tan^{-1}\tfrac{1}{2}) - (135° + 90° + \theta + \tan^{-1}\tfrac{1}{4} + \tan^{-1}\tfrac{1}{5})$$

$$= 26.56° - (225° + 14° + 11.3° + \theta)$$

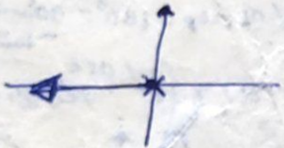
$$= -223.74° - \theta = -180°$$

$$\Rightarrow \theta = -43.74° = 316.26°$$

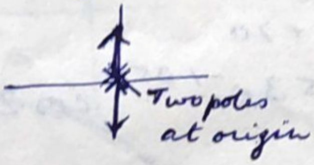
Note: Root loci symmetric about real axis ∴ ∠departure at $-1-j1 = -316.26°$


-5.33
-6
-5
-3
-2.5
K=0

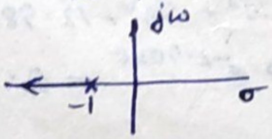
$GH(s) = \frac{K}{s}$



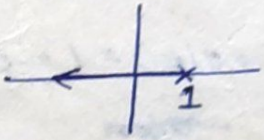
$\frac{K}{s^2}$



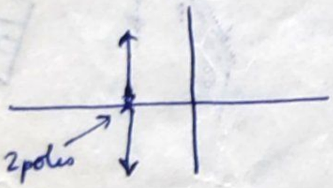
$\frac{K}{s+1}$



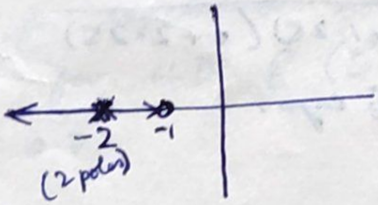
$\frac{K}{s-1}$



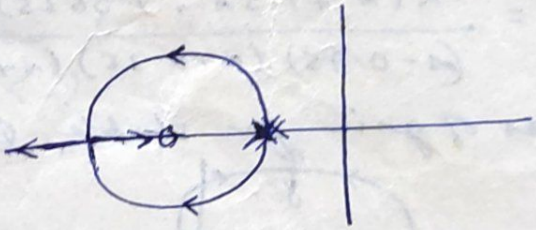
$\frac{K}{(s+2)^2}$



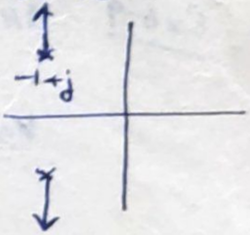
$\frac{K(s+1)}{(s+2)^2}$



$\frac{K(s+2)}{(s+1)^2}$



$\frac{K}{s^2+2s+2}$



Ex: