HURWITZ (1895) Germany

ROUTH- STABILITY ; ROBUST CONTROL

(1892) UK

STABLE LINEAR SYS iff all poles of system TF have (-)ve real parts.

—STRONG condn. since roots on $j\omega$ axis disallowed

Let $F(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$ be charac. polynomial of sys.

$\therefore$ (1) if all roots have (-) real parts, then NECESSARY – all coeffs of $F(s)$ have SAME SIGN

(2) No coeff. of $F(s) = 0$. Absence of a_0 or both a_0 & a_1 etc. $\Rightarrow$ one or more roots at origin or in right half.

Note : NECESSARY but NOT sufficient conditions

Hence R–H criterion

STABILITY : (i) Asymptotic : if impulse response $\to 0$ as $t \to \infty$.

(ii) BIBO

iii) Marginal if DISTINCT roots on $j\omega$ axis (NOT MULTIPLE !!)

Note 1: certain bounded i/ps like $\sin t$ produce unbounded o/p.

$s^2+1 = 0$ roots : $\pm j$ For $u(t) = \sin t$, $y(t) = t \sin t$ UNBOUNDED

Note 2 : Complex conjugate poles on $j\omega$ axis : $\dfrac{c}{s^2+\omega_0^2} \to \left(\dfrac{c}{\omega_0}\right) \sin \omega_0 t$ – NO EXP. DAMPING

$\therefore$ UNSTABLE for $\sin \omega_0 t$ i/p

~ly a) simple pole at origin driven by step $\to$ ramp o/p

b) repeated poles – UNBOUNDED

So, (CAUSAL) TI system with TF $G(s)$ is stable if

1. All poles of $G(s)$ are in left-half of s-plane.

2. $O(D(s)) \geq O(N(s))$ where $G(s) = N(s)/D(s) \to$ RATIONAL T.F.

$$D(s) = a_0 \prod_{i=1}^{p} (s+\alpha_i) \prod_{l=1}^{q} (s^2 + 2\beta_l s + \beta_l^2 + \phi_l^2)$$

ROUTH ARRAY for $F(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} \;\dots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} \;\dots \\
s^{n-2} & c_1 & c_2 & c_3 & c_4 \;\dots \\
s^{n-3} & d_1 & d_2 & d_3 & d_4 \;\dots \\
\vdots & \vdots & & & \\
s^3 & e_1 & e_2 & & \\
s^2 & f_1 & f_2 & & \\
s^1 & g_1 & & & \\
s^0 & h_1 & & &
\end{array}
$$

$$c_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$c_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$\vdots$

$$d_1 = \frac{c_1 a_{n-3} - c_2 a_{n-1}}{c_1}$$

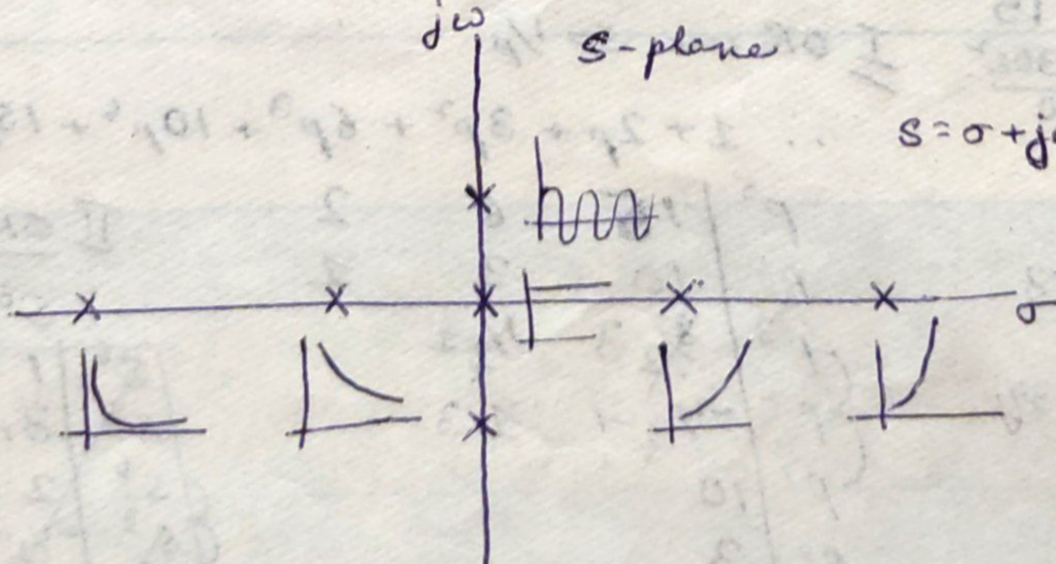
$$d_2 = \frac{c_1 a_{n-5} - c_3 a_{n-1}}{c_1}$$

$\vdots$

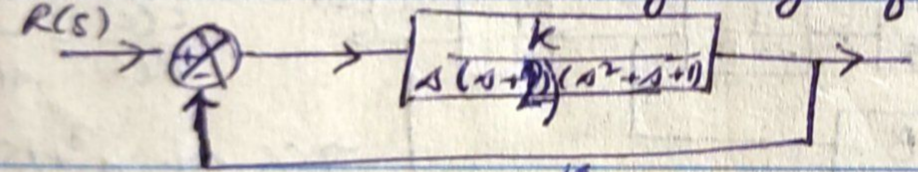
No. of sign changes in 1st column
= no. of roots of $F(s)$ with +ve real parts.

s-plane

$s = \sigma + j\omega$



Application in determining range of gain parameter K for stability.



∴ Ch. T.F. gives $F(s) = s^4 + 3s^3 + 3s^2 + 2s + K$

$G(s) = \dfrac{K}{s(s+2)(s^2+s+1)}$

$\dfrac{C(s)}{R(s)} = \dfrac{K}{s^4 + 3s^3 + 3s^2 + 2s + K}$

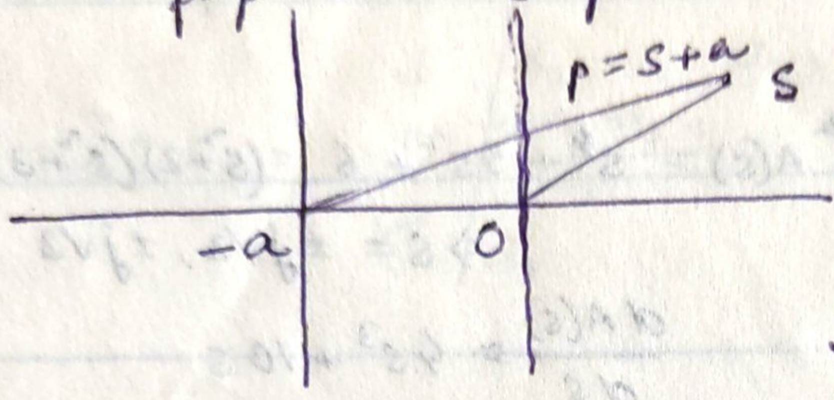
s^4	1	3	K
s^3	3	2	
s^2	$\cancel{7/3}$ 7	$\cancel{K}$ 3K	
s^1	$\frac{14-9K}{7}$		
s^0	K		

$\Rightarrow 14 - 9K > 0, \; K > 0$

$\Rightarrow \boxed{\frac{14}{9} > K > 0}$

** State-space representation : $|sI - A| = \Delta(s) = 0$.

Relative Stability:



$F(s) \to F(p)$ where $p = s+a$

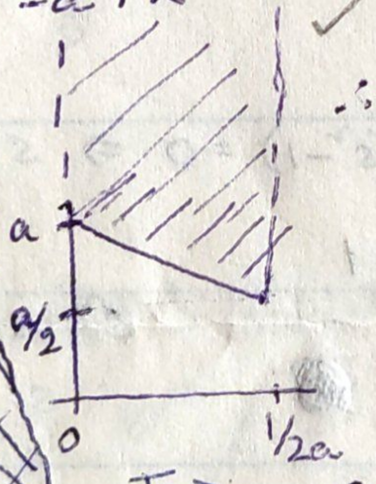
Ex 5.11 $G(s) = \dfrac{K}{s(sT+1)}$

Desired: All roots of Q eqn. to left of $s = -a$ for min. damping. Determine K, T.

$F(s) = 1 + G(s)H(s) = 0 = Ts^2 + s + K$

Let $s = p - a$ ∴ $T(p-a)^2 + (p-a) + K = Tp^2 + (1-2Ta)p + (a^2T - a + K)$
$= 0$.

p^2	T	$a^2T - a + K$
p^1	$1 - 2aT$	
p^0	$a^2T - a + K$	



∴ $T > 0$

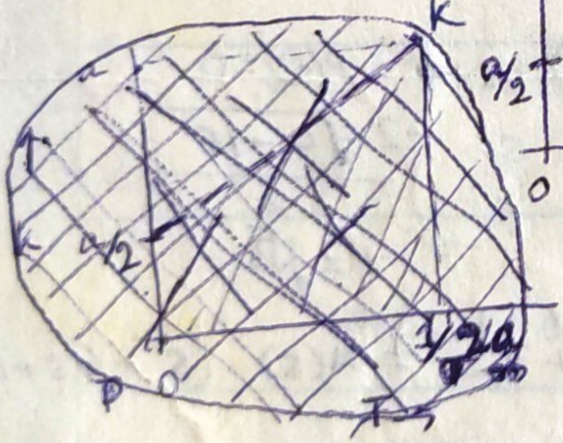
$1 - 2aT > 0 \Rightarrow \frac{1}{2a} > T > 0$

$a^2T - a + K > 0$

$\Rightarrow K > a(1 - aT) > \frac{a}{2}$

$a > a(1 - aT) > a\left(1 - a \cdot \frac{1}{2a}\right)$

$= \frac{a}{2}$



Ex 5.1 $F(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$

Routh array:

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & \frac{a_1 a_2 - a_3 a_0}{a_2} & \\ s^0 & a_0 & \end{array}$$

$\therefore$ Condn. for all roots to have −ve real parts

$a_1 a_2 - a_3 a_0 > 0$

$a_1, a_3, a_2, a_0 > 0$

$\therefore a_1 a_2 > a_3 a_0$

$\Rightarrow a_1 > \frac{a_3 a_0}{a_2} > 0$

No sign change in 1st column — SUFFICIENT CONDN. FOR STABILITY.

62

Ex 5.2 $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & \cancel{2}\,1 & \cancel{4}\,2 & \\ s^2 & \frac{3-2}{1} = 1 & 5 & \\ s^1 & 2-5 = -3 & & \\ s^0 & 5 & & \end{array}$$

2 sign changes $1 \to -3 \to 5$

$\therefore$ UNSTABLE with 2 roots in RHP.

poles at $0.29 \pm 1.42j$; $-1.29 \pm 0.86j$

Ex 5.3 $s^4 + (2\times10^3)s^3 + (3\times10^6)s^2 + (4\times10^9)s + (5\times10^{12}) = 0$

Let $s = 10^3 p \Rightarrow 10^{12}\,[p^4 + 2p^3 + 3p^2 + 4p + 5] = 0$

↳ same as in previous case

$\therefore$ UNSTABLE.

Ex 5.4 $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & \cancel{2}\,1 & \cancel{2}\,1 \\ s^1 & 0 \approx \varepsilon & \\ s^0 & 1 & \end{array}$$

If 1st column element zero while others nonzero/no remaining term, then replace by ε and evaluate rest of array assuming $\varepsilon \to 0^+$

ROW ZERO $\Rightarrow$ roots of EQUAL MAG. radially opp. in s-plane.

No poles in rt. half $\because$ no sign change but s^1 row 0.

$s^2 + 1 = 0 \Rightarrow s = \pm j$ are roots of the ch. eqn.

$\therefore s^3 + 2s^2 + s + 2 = (s^2+1)(s+2)$

Ex 5.5 $s^4 + s^3 + 2s^2 + 2s + 3 = 0$

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 3 \\ s^3 & 1 & 2 & \\ s^2 & 0 \approx \varepsilon & 3 & \\ s^1 & \frac{2\varepsilon - 3}{\varepsilon} & & \\ s^0 & 3 & & \end{array}$$

For $\varepsilon \to 0^+$, $2 - \frac{3}{\varepsilon}$ is −ve $\therefore$ 2 sign changes $\therefore$ 2 roots in RHP

poles at $0.406 \pm j\,1.29$; $-0.906 \pm 0.902j$

Ex 5.6 $s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15 = 0$

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 10 \\ s^4 & 2 & 6 & 15 \\ s^3 & 0 \approx \varepsilon & 5/2 & \\ s^2 & 6 - 5/\varepsilon & 15 & \\ s^1 & \frac{30\varepsilon - 25 - 30\varepsilon^2}{12\varepsilon - 10} & & \\ s^0 & 15 & & \end{array}$$

2 sign changes

$\therefore$ 2 roots in RHP

UNSTABLE

$$\frac{\frac{6\varepsilon-5}{\varepsilon}\cdot\frac{5}{2} - \frac{30\varepsilon^2}{2\varepsilon}}{\frac{(6\varepsilon-5)2}{2\varepsilon}}$$

I OR use $s = 1/p$

$\therefore 1 + 2p + 3p^2 + 6p^3 + 10p^4 + 15p^5 = 0$

$$\begin{array}{c|ccc} p^5 & 15 & 6 & 2 \\ p^4 & 10 & 3 & 1 \\ p^3 & \cancel{3/2}\,3 & \cancel{1/2}\,1 & \\ p^2 & \cancel{-1/3}\,-1 & \cancel{1}\,3 & \\ p^1 & 10 & & \\ p^0 & 3 & & \end{array}$$

poles: $0.83 \pm j\,1.53$

-1.84

$-0.91 \pm 1.37j$

II OR multiply by $(s+1)$ say

$\therefore s^6 + 3s^5 + 5s^4 + 9s^3 + \cancel{8}s^2 + \cancel{2}s + \cancel{1}$

16 25 15

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 8 & 1 \\ s^5 & \cancel{3}\,1 & \cancel{9}\,3 & \cancel{3}\,1 & \\ s^4 & 2 & 7 & 1 & \\ s^3 & \cancel{-1/2}\,-1 & 1/2 & & \\ s^2 & 9 & 1 & & \\ s^1 & 10/9 & & & \end{array}$$

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 16 & 15 \\ s^5 & 3 & 9 & 25 & \\ s^4 & \cancel{2}\,6 & \cancel{23/3}\,23 & \cancel{15}\,45 & \\ s^3 & \cancel{-5/2}\,-1 & \cancel{5/3}\,1 & & \end{array}$$

$$(s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15)(s+1) = s^6 + 3s^5 + 5s^4 + 9s^3 + 16s^2 + 25s + 15$$

$$
\begin{array}{c|cccc}
s^6 & 1 & 5 & 16 & 15 \\
s^5 & 3 & 9 & 25 & \\
s^4 & \cancel{2}\ 6 & \cancel{\frac{23}{3}}\ 23 & \cancel{15}\ 45 & \\
s^3 & \cancel{\frac{-5}{2}}\ -1 & \cancel{\frac{5}{2}}\ 1 & & \\
s^2 & +29 & 45 & & \\
s^1 & \frac{74}{29} & & & \\
s^0 & 45 & & &
\end{array}
$$

Ex 5.8 $F(s) = s^6 + 5s^5 + 11s^4 + 25s^3 + 36s^2 + 30s + 36$

s^6	1	11	36	36
s^5	$\cancel{5}$ 1	$\cancel{25}$ 5	$\cancel{30}$ 6	
s^4	$\cancel{6}$ 1	$\cancel{30}$ 5	$\cancel{36}$ 6	
s^3	$\cancel{0}\,\cancel{4}$ 2	$\cancel{0}\,\cancel{10}$ 5		
s^2	$\cancel{5/2}$ 5	$\cancel{6}$ 12		
s^1	1/5			
s^0	12			

$\longrightarrow A(s) = s^4 + 5s^2 + 6 = (s^2+2)(s^2+3)$

$\Rightarrow s = \pm j\sqrt{2}, \pm j\sqrt{3}$

$\frac{dA(s)}{ds} = 4s^3 + 10s$

$\therefore$ 4 poles on $j\omega$ axis
MARGINALLY STABLE

Note: $F(s) = A(s) \cdot (s+2)(s+3)$

Ex 5.8a $F(s) = s(s^2-1)(s^2+2s+4) = s^5 + 2s^4 + 3s^3 - 2s^2 - 4s = 0$

s^5	1	3	-4
s^4	$\cancel{2}$ 1	$\cancel{-2}$ -1	0
s^3	$\cancel{4}$ 1	$\cancel{-4}$ -1	
s^2	$\cancel{0}$ 3	$\cancel{0}$ -1	
s^1	-2/3		
s^0	-1		

$\therefore A(s) = s^3 - s = s(s^2-1) = 0 \Rightarrow s = 0, \pm 1$

$\frac{dA(s)}{ds} = 3s^2 - 1$

$\therefore$ 1 poles in RHP

$s^2 + 2s + 4 = P(s) = 0$

$\therefore s = \frac{-2 \pm \sqrt{4-16}}{2}$

$= -1 \pm j\sqrt{3}$

Ex 5.9 $F(s) = s^6 + 4s^5 + 12s^4 + 16s^3 + 41s^2 + 36s + 72 = A(s) \cdot (s^2+4s+8)$

s^6	1	12	41	72
s^5	$\cancel{4}$ 1	$\cancel{16}$ 4	$\cancel{36}$ 9	
s^4	$\cancel{8}$ 1	$\cancel{32}$ 4	$\cancel{72}$ 9	
s^3	$\cancel{0}\,\cancel{4}$ 1	$\cancel{0}\,\cancel{8}$ 2		
s^2	2	9		
s^1	-5			
s^0	9			

$A(s) = s^4 + 4s^2 + 9 = 0 \quad \therefore \frac{dA(s)}{ds} = 4s^3 + 8s$

$\Rightarrow s^2 = \frac{-4 \pm \sqrt{16-36}}{2}$

$= -2 \pm j\sqrt{5}$

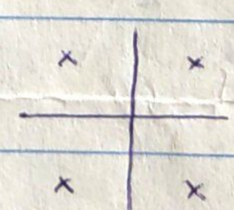
2 poles in RHP

** or $A(s) = s^4 + 6s^2 + 9 - 2s^2 = 0$

$= (s^2+3)^2 - (\sqrt{2}s)^2 = (s^2 - \sqrt{2}s + 3)(s^2 + \sqrt{2}s + 3)$

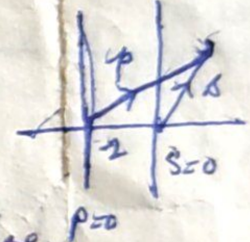
$\therefore s = \frac{\pm\sqrt{2} \pm \sqrt{2-12}}{2} = \pm\frac{\sqrt{2}}{2} \pm j\sqrt{\frac{5}{2}}$

$= \pm\frac{1}{\sqrt{2}} \pm j\sqrt{\frac{5}{2}}$



Ex. 5.12 For $F(s) = s^3 + 9s^2 + 26s + K$

Determine K s.t. dominant time constant ≤ 0.5. System should be slightly UD.



$T = 0.5 \Rightarrow s + \frac{1}{0.5} = 0 \Rightarrow s = -2$ is the relative stability line for $p = 0$

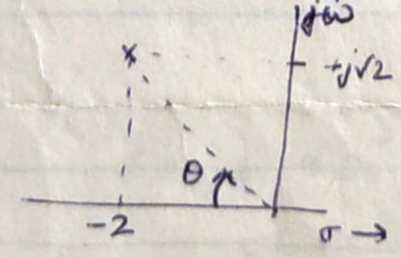
$\therefore p = s+2$

Lt $\Rightarrow s = p - 2$.

$F(p) = (p-2)^3 + 9(p-2)^2 + 26(p-2) + K = p^3 + 3p^2 + 2p + (K-24)$

p^3	1	2
p^2	3	$K-24$
p^1	$\frac{30-K}{3}$	
p^0	$K-24$	

$\therefore 24 \leq K \leq 30$



At $K = 24$, single pole at origin of p-plane $\Rightarrow$ at $s = -2$.

At $K = 30$,

$3p^2 + 6 = 0 \Rightarrow p^2 + 2 = 0$

$\Rightarrow p = s + 2 = \pm j\sqrt{2}$.

$\Rightarrow s = -2 \pm j\sqrt{2}$.

$\therefore \zeta = \cancel{\cos\left(\frac{2}{\sqrt{2}}\right)} \cos\left(\tan^{-1}\frac{\sqrt{2}}{2}\right) = 0.816$