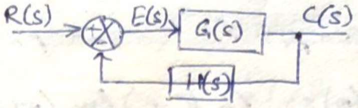
Steady state error:



$$E_S = \frac{R(s)}{1+G(s)H(s)} \rightarrow \text{FINAL VALUE THEOREM. } \lim_{s\to 0} sE(s)$$

Say $G(s)H(s) = \dfrac{K(T_a s+1)\dots(T_n s+1)}{s^N(T_1 s+1)\dots(T_p s+1)} = \dfrac{N(s)}{D(s)}$

s^N denotes poles at origin of multiplicity N : TYPE ~~ORDER~~ of system.

Type 0 : no pole at origin.

Unit Step Input: $r(t) = u(t)$ $R(s) = 1/s$.

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p}$$

$K_p = \lim_{s\to 0} G(s)H(s) = G(0)H(0) = G(0)$ for unity f/b.

$\triangleq$ static position error coefficient.

For Type 0, $K_p = K$ $\therefore e_{ss} = \frac{1}{1+K}$; Type 1 onwards : $K_p = \infty \Rightarrow e_{ss} = 0$ unless $N(s)$ has zeros at origin too! → III B

Unit Ramp Input: $r(t) = t$; $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s(1+G(s)H(s))} = \lim_{s\to 0} \frac{1}{sG(s)H(s)} = \frac{1}{K_v}$$

$K_v = \lim_{s\to 0} sG(s)H(s) \triangleq$ static velocity error coeff.

For Type 0, $e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$ UNABLE TO FOLLOW RAMP I/P

Type 1, $e_{ss} = \frac{1}{K_v} = \frac{1}{K}$; Type 2 onwards, $e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$

Follows with error — Follows w/o ss error.

Unit Parabolic (Accln.) I/p: $r(t) = t^2/2$ $\therefore R(s) = 1/s^3$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)H(s)} = \frac{1}{K_a}$$; $K_a \triangleq$ static accln. error coeff.

For Type 0 and 1, $e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$

Type 2, $e_{ss} = \frac{1}{K_a} = \frac{1}{K}$

Type 3 onwards $e_{ss} = 0$.

$\therefore$ Responses for these i/ps give information about multiple poles at origin.

Steady state Error in terms of CL TF : [Considering $H(s) = 1$]

$T(s) = \dfrac{C(s)}{R(s)}$ $E(s) = R(s) - C(s) = R(s)[1 - T(s)]$

$$\therefore e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)[1 - T(s)]$$

I Unit step i/p : $e_{ss} = \lim_{s\to 0} s \cdot \frac{1}{s}[1 - T(s)] = 1 - T(0)$.

$\therefore$ If $T(0) = 1$, then no ss error.

II Unit Ramp I/p $e_{ss} = \lim_{s\to 0} s \cdot \frac{1}{s^2}[1 - T(s)] = \lim_{s\to 0}\left[\frac{1-T(s)}{s}\right]$

So if $T(0) = 1$, then $e_{ss} = \lim_{s\to 0}\left[-\frac{dT(s)}{ds}\right]$,

else ∞

If $T(s) = K \frac{(s+z_1)\dots(s+z_m)}{(s+p_1)\dots(s+p_n)}$

Then $\ln T(s) = \ln K + \sum \ln(s+z_i) - \sum \ln(s+p_j)$

$$\frac{d}{ds}(\ln T(s)) = \frac{1}{T(s)}\frac{dT(s)}{d(s)} \Rightarrow -\frac{dT(s)}{d(s)} = -T(s)\frac{d}{ds}[\ln T(s)]$$

OR III A2

$$\Rightarrow -\frac{dT(s)}{d(s)} = -T(s)\left[\sum^{m} \frac{1}{s+z_i} - \sum^{n} \frac{1}{s+p_j}\right]$$

$$\therefore e_{ss} = -\left[\sum^{m} \frac{1}{z_i} - \sum^{n} \frac{1}{p_j}\right] \qquad \because T(0)=1$$

III Unit parabolic i/p $R(s) = 1/s^3$.

$$e_{ss} = \lim_{s\to 0} \frac{1-T(s)}{s^2}$$

$\therefore$ If $T(0)=1$ and $\lim_{s\to 0}\frac{d}{ds}[\ln T(s)] = 0$

$$\Rightarrow \sum\frac{1}{z_i} = \sum\frac{1}{p_j}$$

OR III A3

$$\therefore e_{ss} = \lim_{s\to 0}\left[-\left(\frac{1}{2}\right)\frac{d^2}{ds^2}(T(s))\right]$$ L'Hospital's rule twice.

$$= \frac{1}{K_a} = \frac{1}{2}\left[\sum^{m}\left(\frac{1}{z_i^2}\right) - \sum^{n}\left(\frac{1}{p_j^2}\right)\right]$$ $\longrightarrow$ CONT. (10)

Automatic Control Systems : Kuo 7th ed pg 377, pg 383. (10)(d)

Reln. bet. S.S. error and $\overset{\text{C.T.F.}}{T(s)}$: Unity & non-unity f/b sys.

A/ If $\boxed{\text{Let } \lim_{s\to 0} H(s) = H(0) = K_H = \text{constant}}$

f/b signal $= K_H\, y(t)$ at s.s.

$\therefore$ Let reference be modified to $\dfrac{r(t)}{K_H}$

$\therefore e(t) = \dfrac{r(t)}{K_H} - y(t) \Rightarrow E(s) = \dfrac{1}{K_H} R(s) - Y(s) = \dfrac{1}{K_H}\left[1 - K_H\, T(s)\right] R(s).$

where $T(s) = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$ Let $n > m$ [CAUSAL]

also $T(s)$ has all LHP poles (STABLE)

$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \dfrac{1}{K_H}\left[1 - K_H T(s)\right] s R(s) = \dfrac{1}{K_H} \lim_{s\to 0} \dfrac{s^n + \dots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} sR(s)$

I Step i/p $R(s) = R/s$.

$\therefore e_{ss} = \dfrac{1}{K_H}\left(1 - \dfrac{b_0 K_H}{a_0}\right) R$. $\therefore e_{ss} = 0$ only if $T(0) = \dfrac{b_0}{a_0} = \dfrac{1}{K_H}$ [TYPE 0 SYS]

... has all LHP poles (STABLE)

$$e_{SS} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{K_H}\left[1 - K_H T(s)\right] s R(s) = \frac{1}{K_H}\lim_{s\to 0} \frac{s^n + \dots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} s R(s)$$

I Step i/p $R(s) = R/s$.

$$\therefore e_{SS} = \frac{1}{K_H}\left(1 - \frac{b_0 K_H}{a_0}\right) R \quad \therefore e_{SS} = 0 \text{ only if } T(0) = \frac{b_0}{a_0} = \frac{1}{K_H} \left[\begin{matrix}\text{TYPE } 0\\ \text{SYS}\end{matrix}\right]$$

II Ramp i/p $R(s) = R/s^2$

$$e_{SS} = \frac{1}{K_H}\lim_{s\to 0} \frac{s^n + \dots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0)} R$$

$$\therefore e_{SS} = 0 \quad \text{if} \quad \underline{a_0 - b_0 K_H = 0} \text{ and } a_1 - b_1 K_H = 0$$

$$= \frac{a_1 - b_1 K_H}{a_0 K_H} \cdot R = \text{cons.} \quad \text{if } \underline{a_0 - b_0 K_H = 0} \text{ \& } a_1 - b_1 K_H \neq 0$$

$$= \infty \quad \text{else.}$$

$\rightarrow$ BOTH TYPE 0 SYSTEMS.

$\frac{b_0}{a_0} = \frac{1}{K_H}\{=1 \text{ if } K_H = 1\}$

$$\boxed{\text{Note: } \frac{\Sigma(\Pi \lambda_i)}{\Pi \lambda_i} = \frac{\text{coeff of } s^1}{\text{coeff of } s^0}}$$

III Parabolic i/p $R(s) = R/s^3$

$$e_{SS} = \frac{1}{K_H}\lim_{s\to 0} \frac{s^n + \dots + (a_2 - b_2 K_H)s^2 + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^2(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0)} R$$

$$\therefore -\frac{b_1 K_H}{a_0 K_H} + \frac{a_1}{a_0 K_H} = -\left[\frac{b_1 K_H}{b_0 K_H K_H} - \frac{a_1}{a_0 K_H}\right]$$

$$= 0 \quad \text{if } a_i - b_i K_H = 0 \text{ for } i = 0, 1 \text{ and } 2$$

$$= \frac{a_2 - b_2 K_H}{a_0 K_H} R = \text{cons} \quad \text{if } a_i - b_i K_H = 0 \text{ for } i = 0 \text{ and } 1, \text{ not } 2$$

$$= \infty \quad \text{if } a_i - b_i K_H \neq 0 \text{ for } i = 0 \text{ or } 1 \text{ and else.}$$

B/ Non Unity F/b : $H(s)$ has N^{th} order zero at $s = 0$.

Ref. signal modified to $R(s)/s^N K_H$ and $E(s) = \frac{1}{s^N K_H} R(s) - Y(s)$

where $K_H = \lim_{s\to 0} \frac{H(s)}{s^N}$

If $N = 1$, then $e_{SS} = \frac{1}{K_H}\lim_{s\to 0} \frac{s^{n-1} + \dots + (a_2 - b_1 K_H)s + (a_1 - b_0 K_H)}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s} sR(s)$

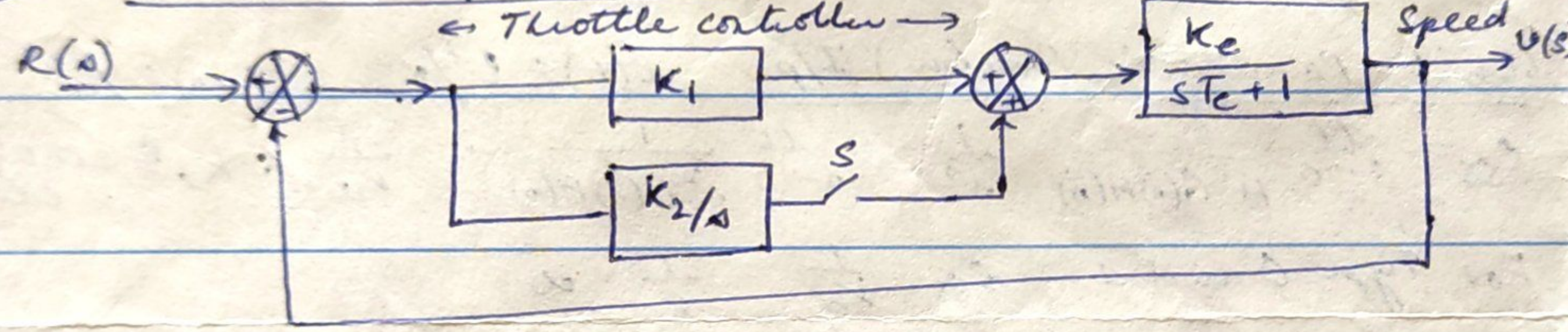
$\therefore$ For step i/p $R(s) = \frac{R}{s}$ and

$$e_{SS} = 0 \quad \text{if } a_2 - b_1 K_H = 0 \text{ and } a_1 - b_0 K_H = 0$$

$$= \frac{a_2 - b_1 K_H}{a_1 K_H} R = \text{cons.} \quad \text{if } \neq 0 \text{ and } = 0$$

$$= \infty. \quad \text{if } a_1 - b_0 K_H \neq 0.$$

Ex 2 Automobile Speed Control.



$G_1(s) = K_1 + \frac{K_2}{s} = \frac{K_1 s + K_2}{s}$ when S closed, else K_1

$G_2(s) = \frac{K_e}{sT_e+1}$.

I: S open : $K_p = \lim_{s\to 0} G_1(s)G_2(s) = K_1 K_e$

$e_{ss} = 1/(1+K_1K_e) = \frac{1}{1+K_1K_e}$

e_{ss} for step i/p and ramp i/p

Let K_p, K_v, e_{ss}

II S closed : $K_p = \lim_{s\to 0} \frac{(K_1 s + K_2)}{s} \cdot \frac{K_e}{sT_e+1} = \infty \quad \therefore e_{ss} = 0$.

$K_v = \lim_{s\to 0} s\,G_1(s)\,G_2(s) = K_2 K_e \quad \therefore e_{ss} = \frac{1}{K_v} = \frac{1}{K_2 K_e}$.

$\therefore$ To reduce e_{ss}, reqd. to increase $K_v \Rightarrow K_2 K_e \uparrow$ but then ζ decreases $\therefore M_p \uparrow$ $\therefore$ TRADE OFF.

INTEGRAL PERFORMANCE CRITERION : ⑩ (C)

Qualitative measure of system performance : Performance Index

COST FN.

① $J_1 = \int_0^T e^2(t)\,dt$ ISE criterion

② $J_2 = \int_0^T |e(t)|\,dt$ IAE

③ $J_3 = \int_0^T t\,e^2(t)\,dt$ ITSE

④ $J_4 = \int_0^T t\,|e(t)|\,dt$ ITAE