$\therefore \frac{1}{\zeta\omega_n} = \frac{1}{\sigma} = T$ of equivalent ~~second~~ first order system.
~~envelope~~

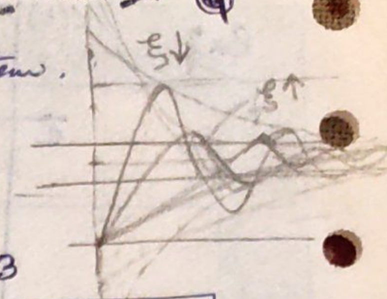
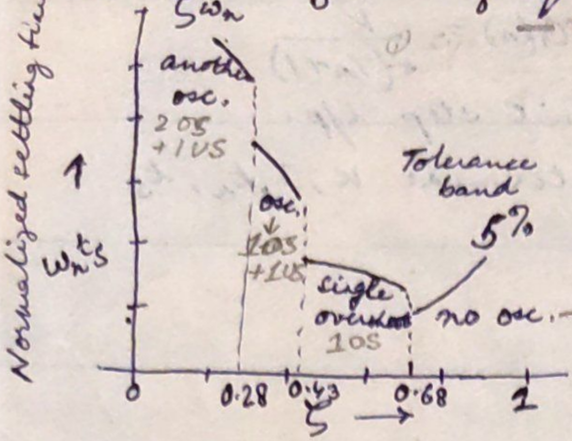


fig with 2OS major

$\frac{P}{100} = e^{-\zeta\omega_n t_s}$

I



$t_S = \frac{3}{\zeta\omega_n}$

$\therefore (\omega_n t_S) \cdot \zeta = 3$

DESIGN
$M_p \to 5-25\%$
$\zeta \to 0.4-0.7$

$\zeta = 1/\sqrt{2} = 0.707$
$M_P = 4.3\%$

$\zeta = 0.68$
$M_P = 5.4\%$

$\zeta = 0.7$
$M_P = 4.6\%$

$\zeta = 0.9$
$M_P = 0.15\%$

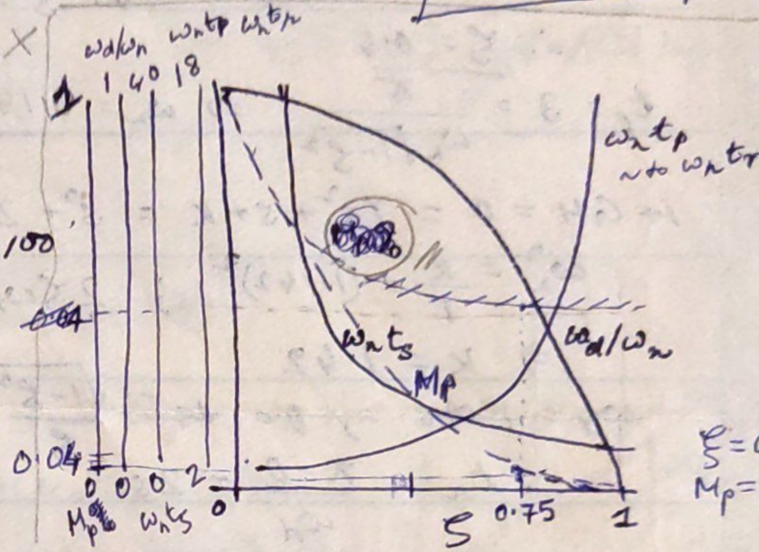
$\therefore \omega_n t_p = \frac{\pi}{\sqrt{1-\zeta^2}}$

$M_p(\%) = e^{(-\pi\zeta/\sqrt{1-\zeta^2})}$ ×100

$\omega_n t_S = \frac{3}{\zeta}$

$\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$

$\Rightarrow \left(\frac{\ln M_p}{\pi}\right)^2 (1-\zeta^2) = \zeta^2$



$\zeta = 0.75$
$M_p = 0.04$

Usually M_p, t_s specified ⟶ significance in relw. to freq. responses.

Note: (i) $t_d = 0.5\, t_{r_1} = 0.5\, t_r$ (t_r being for 0-100%)

(ii) $t_r = \frac{\pi - \theta}{\omega_d}$ $\qquad \theta = \cos^{-1}\zeta \Rightarrow \omega_n t_r = \frac{\pi-\theta}{\sqrt{1-\zeta^2}}$

I $\omega_n = c$

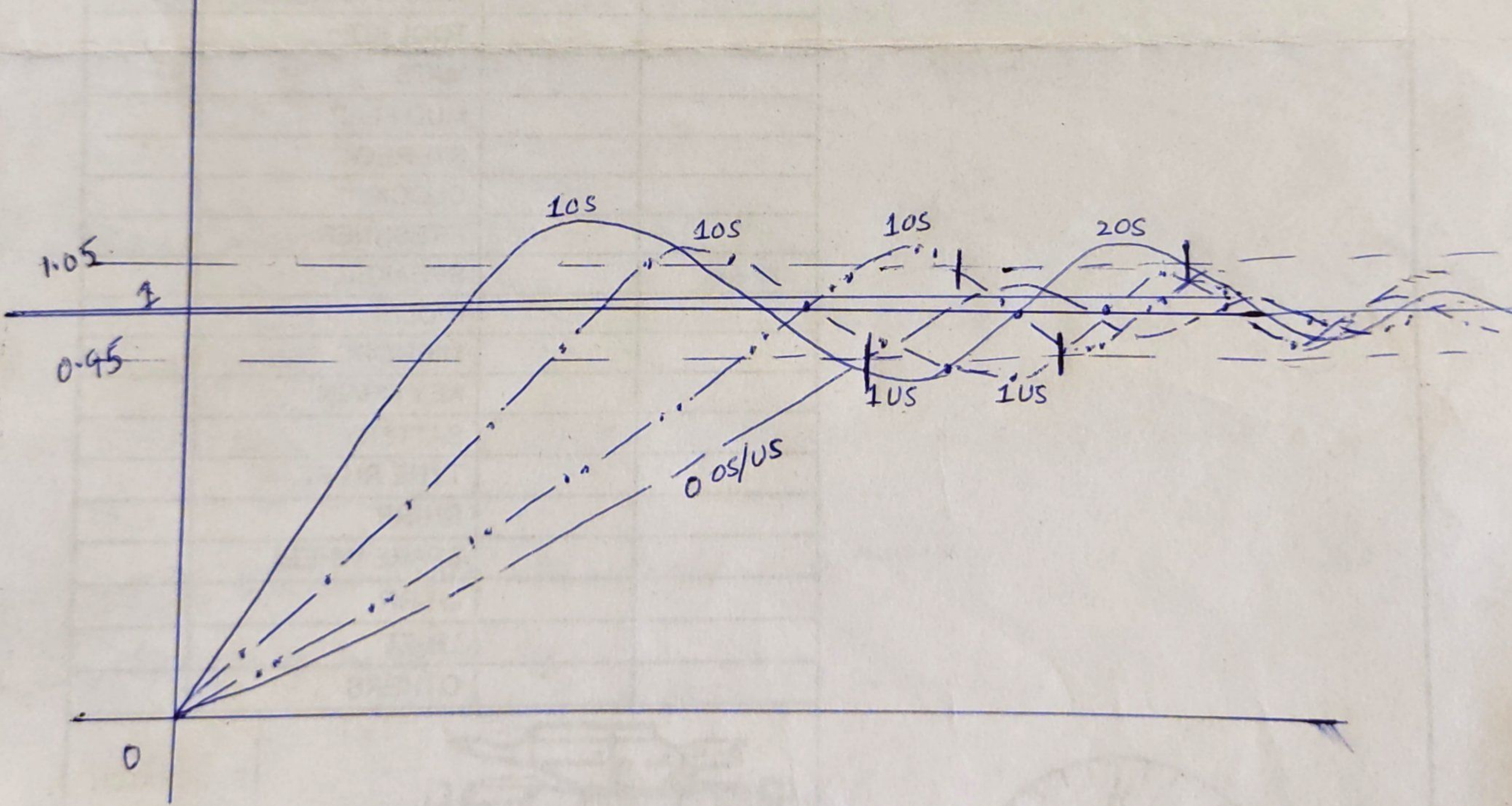
$\therefore \zeta \uparrow \Rightarrow t_d, t_r \uparrow$; t_s is large for small ζ & large ζ (beyond 0.707)

$M_p \downarrow$ $(0 < \zeta < 1)$

$\therefore$ optimum $\zeta = 0.707$: $M_p = 4.3\%$ while t_d, t_r, t_s reasonable.

II ζ fixed, $\omega_n \uparrow$, $M_p = c$, $t_d, t_r, t_s, t_p \downarrow$

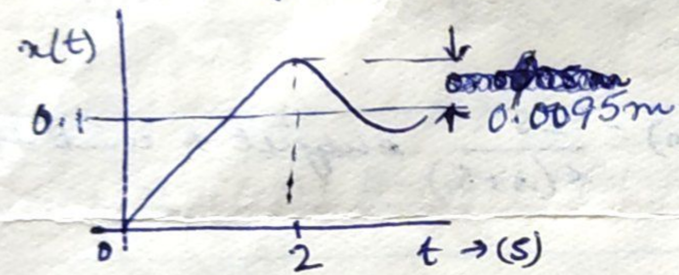
$\therefore$ desire high ω_n.



10S
10S
10S
20S
1.05
1
0.95
1US
1US
0 OS/US
0

$\therefore$ desire high ω_n.

Ex 1.



Spring-mass-damper system

Step force 2 N

s.s. displacement of M = 0.1m

Obtain M, f, K.

$$(Ms^2 + fs + K)X(s) = U(s) = \frac{2}{s}$$

$$\therefore X(s) = \frac{2}{s(Ms^2 + fs + K)}$$

FINAL VALUE THEOREM

$$\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \frac{2}{K} = 0.1 \Rightarrow K = 20\,N/m.$$

$$\frac{0.0095}{0.1} = M_p = 0.095 = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right) \Rightarrow \zeta = 0.6$$

$$(\ln 0.095)^2 = \frac{\zeta^2\pi^2}{1-\zeta^2}$$

$$\Rightarrow \sqrt{\frac{(\ln 0.095)^2}{(\pi^2 + (\ln 0.095)^2)}} = \zeta$$

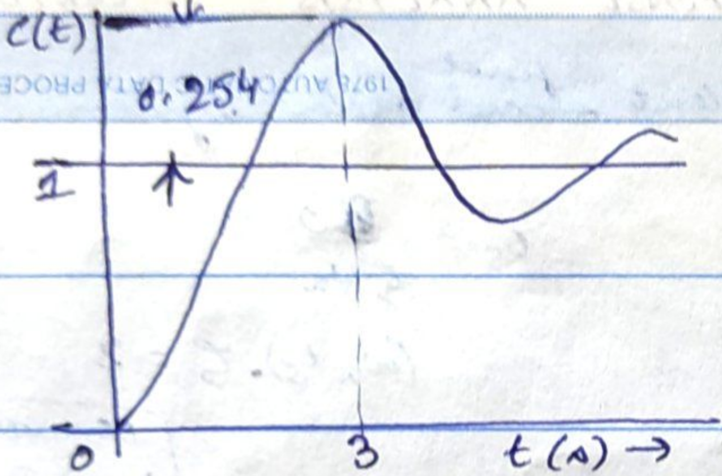
$$t_p = \frac{\pi}{\omega_d} = 2 \Rightarrow \omega_n\sqrt{1-0.36} = \frac{\pi}{2} \Rightarrow \omega_n = 1.96\ rad/s.$$

$$\omega_n = \sqrt{K/M} \Rightarrow M = \frac{20}{(1.96)^2} = 5.2\,kg$$

$$\zeta = \frac{1}{2}\sqrt{\frac{f}{KM}} \text{ or } 2\zeta\omega_n = \frac{f}{M} \Rightarrow f = M.\frac{2\zeta\omega_n}{(M)} = 12.2\ \frac{Nm}{rad/s}$$

$$\zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \Leftarrow \left(\frac{\ln 0.095}{\pi}\right)^2 = \left[1 + \left(\frac{\ln 0.095}{\pi}\right)^2\right]\zeta^2 \Leftarrow \left(\frac{\ln 0.095}{\pi}\right)^2(1-\zeta^2) = \zeta^2$$

Ex 2



Unity f/b system with

$$G(s) = \frac{K}{s(Ts+1)}$$

Unit step i/p.

Determine K, T, t_r, t_s

$M_p = 0.254 = \exp\left(-\pi\zeta / \sqrt{1-\zeta^2}\right)$

$\therefore \zeta = 0.4$

$t_p = 3 = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \omega_n = 1.142$ rad/s.

$1 + GH = 0 = Ts^2 + s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$

$\therefore \omega_n^2 = \dfrac{K}{T} = (1.142)^2$; $2\zeta\omega_n = \dfrac{1}{T} \Rightarrow T = 1.0946$

$\Rightarrow K = 1.42$

$\omega_d = 1.046$; $\theta = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta} = 66.4°$

$\therefore t_r = \dfrac{\pi - \theta}{\omega_d} = 2s$, $t_s = \dfrac{4}{\zeta\omega_n}$ for 2% $=$ ~~6.97 s~~ ?, 8.76 s ✓

$= 8T$

$\dfrac{3}{2\zeta\omega_n}$ for 5% → 6.57 s

Ex3. Unity f/b control system with unit step i/p & DESIGN

$G(s) = \frac{K}{s(Ts+1)}$ K, T : motor gain & time constants.

Fast transient desired with $M_p < 5\%$, $t_s < 4s$ (2%)

$\therefore t_s = \frac{4}{\zeta\omega_n} \leq 4s \Rightarrow \zeta\omega_n \geq 1$.

$\therefore \zeta\omega_n = 1$ and $\zeta = 0.707 = \frac{1}{\sqrt{2}}$ ($M_p = 4.3\%$) satisfies requirement.

$\therefore \omega_n^2 = 2$

$1 + GH = Ts^2 + s + K = 0 = s^2 + 2\zeta\omega_n s + \omega_n^2$ CHARACTERISTIC EQN.

$\therefore \omega_n^2 = \frac{K}{T}$, $2\zeta\omega_n = \frac{1}{T} \Rightarrow T = 0.5s$

$K = 0.5 \times 2 = 1$.

Ex4. Unity f/b with Q TF $G(s) = \frac{25}{s(s+5)}$ subject to unit step i/p

Find t_r, t_s, t_{p_1}, M_p

$s^2 + 5s + 25 = 0 = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\therefore \omega_n = 5$, $\zeta = 0.5$. $\therefore \omega_d = 4.33$, $\sigma = \zeta\omega_n = ~~1~~$ 2.5

$\cos^{-1}\frac{1}{2} = \theta = \tan^{-1}\frac{\omega_d}{\sigma} = ~~1.344 rad~~ = \pi/3 = 1.047$ rad $\therefore t_r = \frac{\pi - \theta}{\omega_d} = ~~0.416s~~$ 0.484

$t_s = \frac{4}{\zeta\omega_n}$ (2%) $= ~~4s~~$ 1.6s, $\frac{3}{\zeta\omega_n}$ (5%) $= ~~3s~~$ 1.2s

$t_{p_1} = \frac{\pi}{\omega_d} = 0.726s$

$M_p = \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right) \times 100\% = 16.3\%$

$\frac{\sqrt{1 - 1/4}}{1/2} = \frac{\sqrt{3/4}}{1/2} = \frac{\sqrt{3}/2}{1/2}$

$\theta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{\sqrt{0.75}}{0.5} = \tan^{-1}\frac{\sqrt{0.25 \times 3}}{0.5} = \tan^{-1}\sqrt{3}$

$= \cos^{-1}\zeta$

Impulse Response of 2nd order system:

$R(s) = 1$

$\therefore C(t) = \mathcal{L}^{-1} C(s) = \mathcal{L}^{-1} \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

1. For $0 < \zeta < 1$ UD

$c(t) = \dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$

2. $\zeta = 1$ CD $c(t) = \omega_n^2 t e^{-\omega_n t}$

3. $\zeta > 1$ OD $c(t) = \dfrac{\omega_n}{2\sqrt{\zeta^2-1}} \left[e^{-(\zeta - \sqrt{\zeta^2-1})\omega_n t} - e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} \right]$

Ramp Response

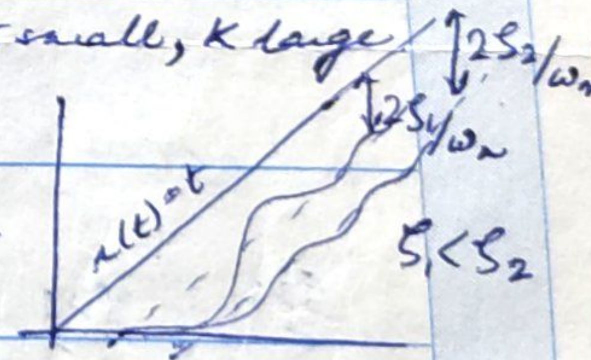
$R(s) = 1/s^2$

$c(t) = t - \dfrac{2\zeta}{\omega_n} + \dfrac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$

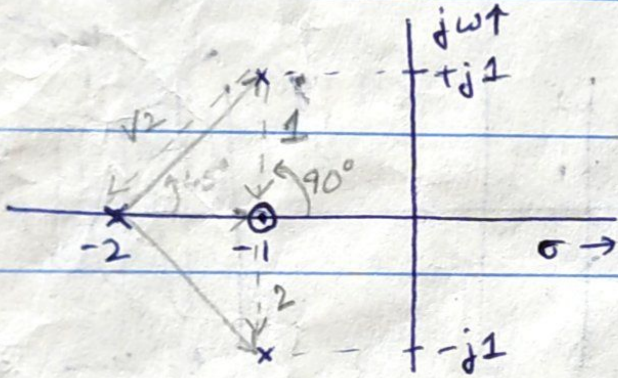
$e_{ss} = \dfrac{2\zeta}{\omega_n} = \lim_{s\to 0} sE(s) \Rightarrow \zeta$ not large, ω_n not small, K large

$\dfrac{2\zeta\omega_n}{\omega_n^2} = f/K = \dfrac{f}{M} \cdot \dfrac{M}{K}$ $\because 1 + GH = (Ms^2 + fs + K) = 0.$

For lower ζ, ramp response is more oscillatory.



GRAPHICAL INTERPRETATION OF LAPLACE-HEAVISIDE EXPANSION:



$$G(s) = \frac{4(s+1)}{(s+2)(s^2+2s+2)}$$

$$= \frac{C_1}{s+2} + \frac{C_2}{s+1-j} + \frac{C_3}{s+1+j}$$

① towards origin ~~posit~~ θ = 0
② CCW from σ axis tow. +jω → θ +ve
from Z/P to test pt.

$$C_1 = \frac{4(-2-(-1))}{\sqrt{2}\cdot\sqrt{2}} = -2 = \frac{4\,1e^{j180^\circ}}{\sqrt{2}e^{-j135}\sqrt{2}e^{j135}}$$

$$C_2 = \frac{4\{(-1+j1)-(-1)\}}{\{(-1+j1)-(-2)\}\{(-1+j1)-(-1-j1)\}} = \frac{4j}{(1+j1)(2j)} = 1-j1$$

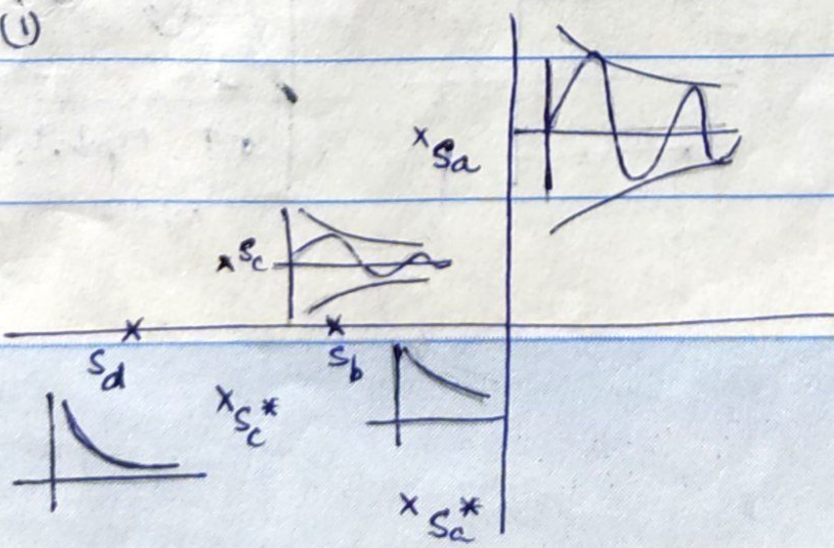
① tow. j0 +ve θ = 0

$$= \frac{4.\,1e^{j90^\circ}}{e^{j90^\circ}\,2\sqrt{2}(e^{j45^\circ})}$$

|ly $C_3 = 1+j1$

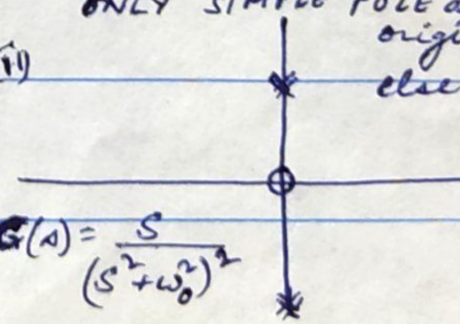
TIME DOMAIN BEHAVIOUR FROM POLE-ZERO PLOT

(i)



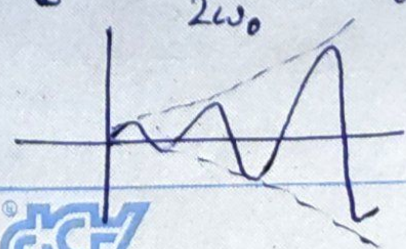
(ii)

critical stable when ONLY SIMPLE POLE at origin else

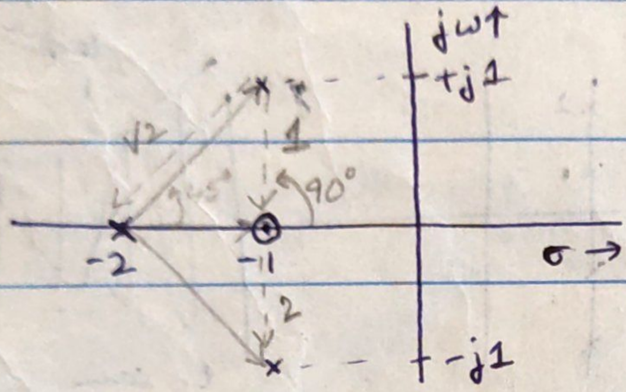


$$G(s) = \frac{s}{(s^2+\omega_0^2)^2}$$

$$\therefore g(t) = \frac{t}{2\omega_0}\sin\omega_0 t$$



GRAPHICAL INTERPRETATION OF LAPLACE-HEAVISIDE EXPANSION:



$$G(s) = \frac{4(s+1)}{(s+2)(s^2+2s+2)}$$

$$= \frac{C_1}{s+2} + \frac{C_2}{s+1-j} + \frac{C_3}{s+1+j}$$

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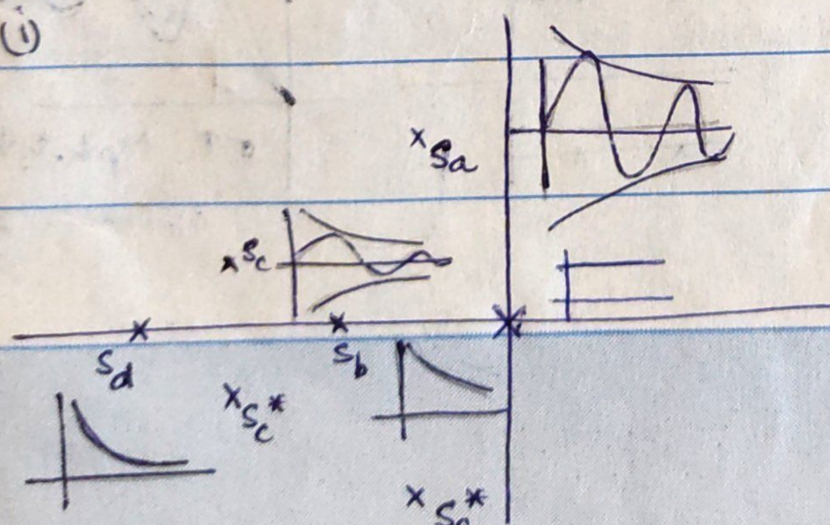
① tow. j0 +ve θ=0

$$= \frac{4\cdot 1e^{j90^\circ}}{e^{j90}\,2\sqrt{2}(e^{j45^\circ})}$$

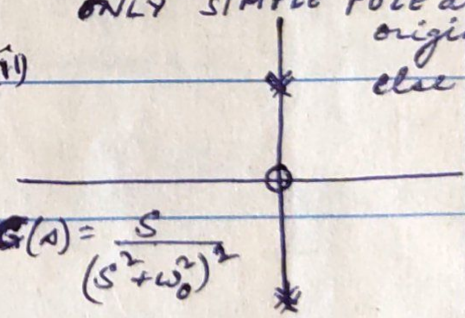
~ly $C_3 = 1+j1$

TIME DOMAIN BEHAVIOUR FROM POLE-ZERO PLOT

(i)

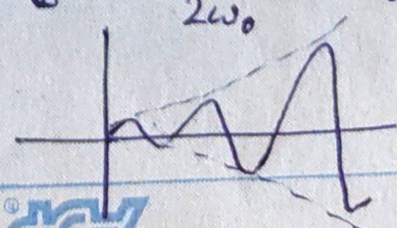


(ii) critical stable when ONLY SIMPLE POLE at origin else

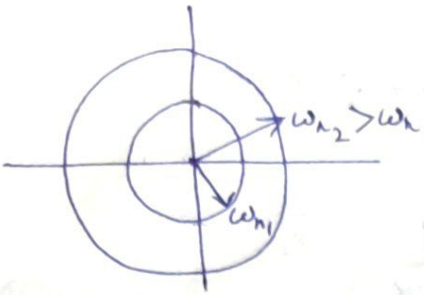


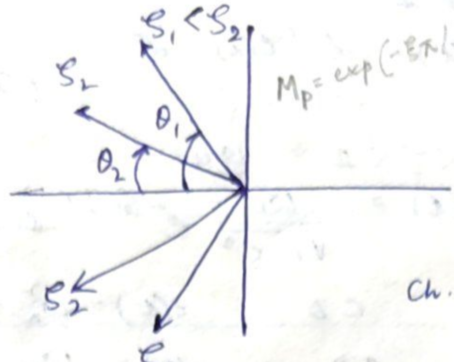
$$G(s) = \frac{s}{(s^2+\omega_0^2)^2}$$

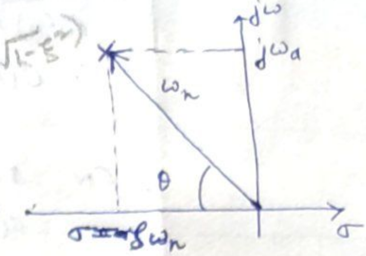
$$\therefore e(t) = \frac{t}{2\omega_0}\sin\omega_0 t$$



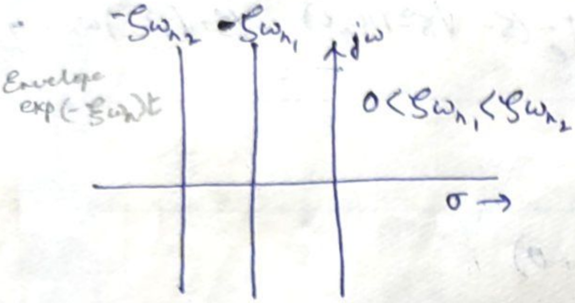
PERFORMANCE LINES

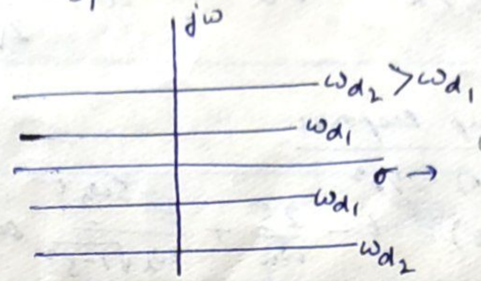




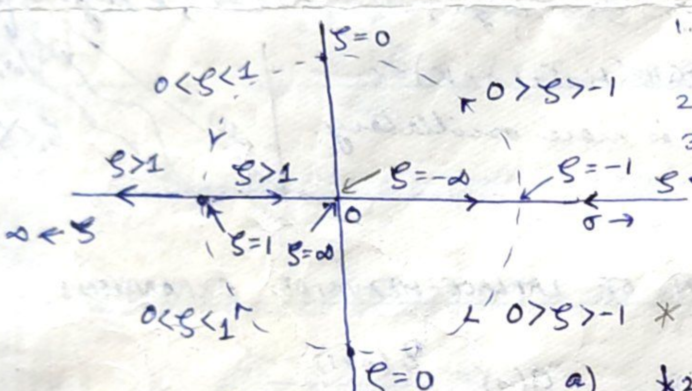


Ch. eqn: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

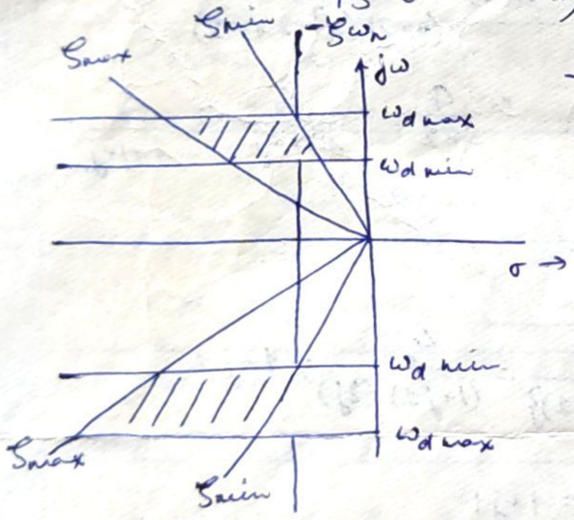


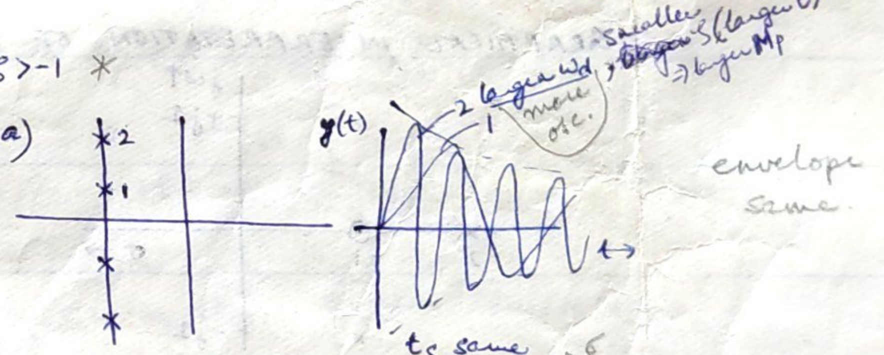


$\omega_d = \omega_n\sqrt{1-\zeta^2}$

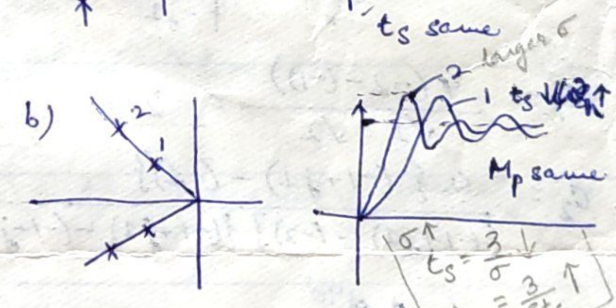


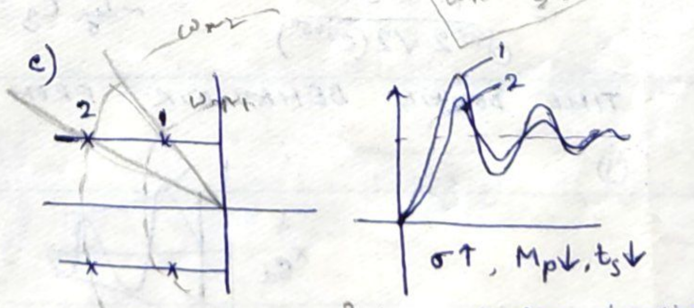
1. $\zeta < 0$ $\quad s_{1,2} = +\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ $\quad \mathrm{Re}\, s_{1,2} > 0$ DIVERGING RESPONSE
2. $\zeta = 0$ $\quad \pm j\omega_n$ $\quad$ OSCILLATORY
3. $0 < \zeta < 1$ $\quad -\zeta\omega_n \pm j\omega_d$ $\quad$ UNDERDAMPED
4. $\zeta = 1$ $\quad -\zeta\omega_n$ $\quad$ C. DAMPED
5. $\zeta > 1$ $\quad -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ $\quad$ O.D. $\quad \mathrm{Re}\, s_{1,2} < 0$





envelope same.





σ more for 2, $t_s = \frac{3}{\sigma} \downarrow$ $\quad$ ω_d same $\therefore$ period same.

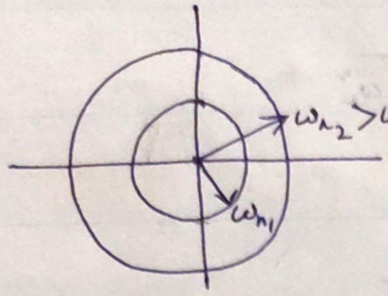
also $\zeta_2 > \zeta_1 \Rightarrow M_{p_2} \downarrow M_{p_1}$

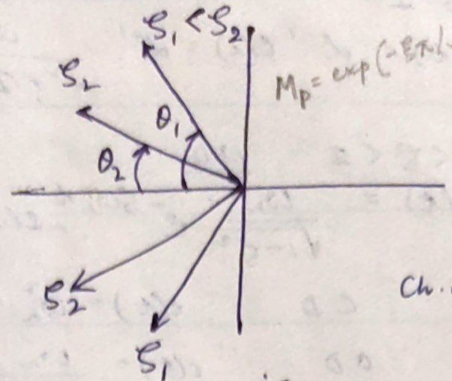
$\zeta \downarrow \Rightarrow \omega_d \uparrow \Rightarrow T \downarrow \Rightarrow$ faster osc.
or $\omega_n \uparrow$

$\theta \downarrow \Rightarrow \zeta \uparrow \Rightarrow M_p \downarrow$

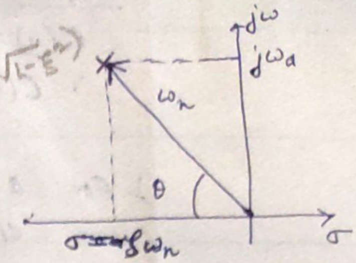
$|\sigma| \uparrow \Rightarrow$ transient faster, envelope sharper

PERFORMANCE LINES

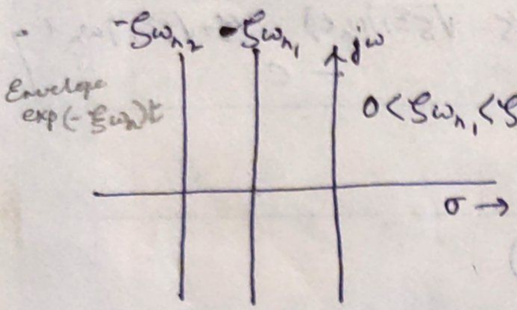


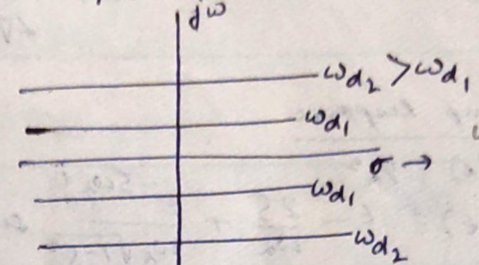


$M_p = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)$

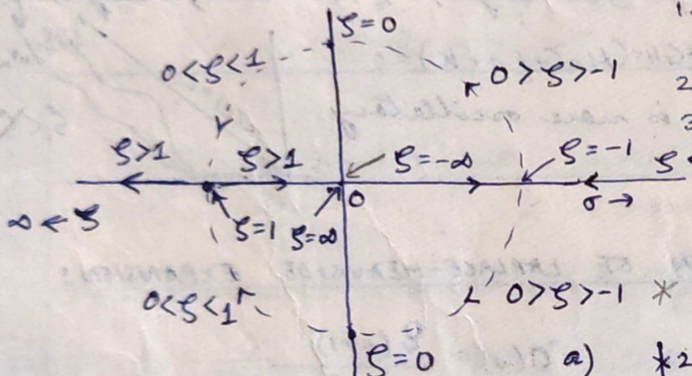


Ch. equ. $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$





$\omega_d = \omega_n\sqrt{1-\zeta^2}$



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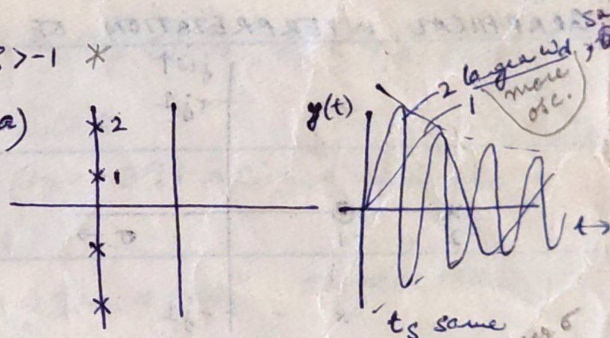
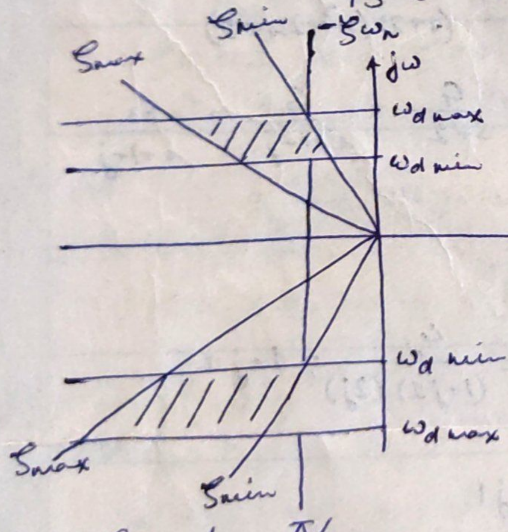
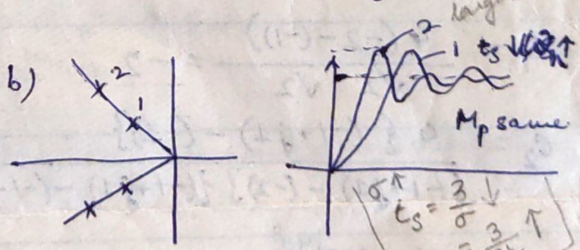
a)



envelope same.

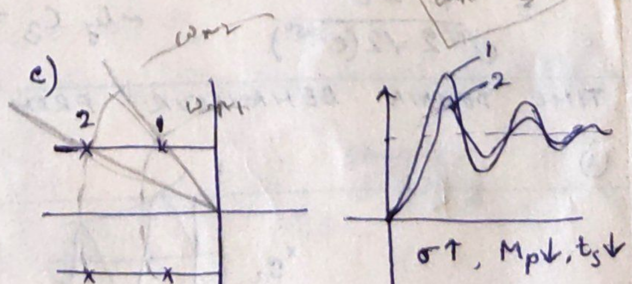


b)



$M_p \rightarrow \zeta$; $t_p = \pi/\omega_d$

c)



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or $\omega_n \uparrow$

$\theta \downarrow \Rightarrow \zeta \uparrow \Rightarrow M_p \downarrow$

$|\sigma| \uparrow \Rightarrow$ transient faster, envelope sharper

ω_d same $\therefore$ period same.

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