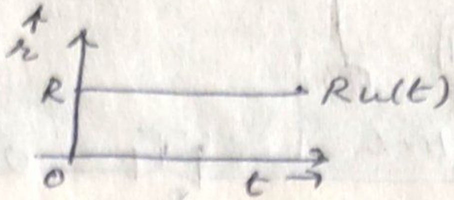
TRANSIENT RESPONSE ANALYSIS

$$c(t) = c_n(t) + c_f(t) = c_t(t) + c_{ss}(t)$$

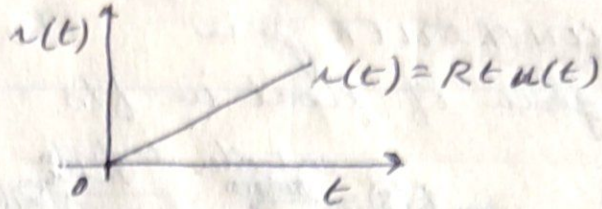
Typical Test Signals

(i) Step fn. $r(t) = R\,u(t)$
Displacement i/p



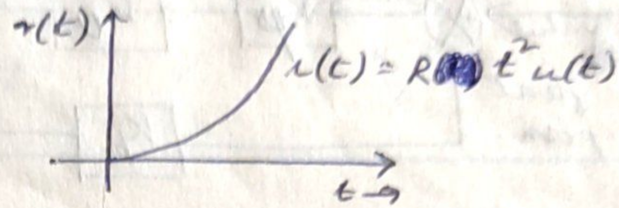
(ii) Velocity i/p : Ramp fn.

$r(t) = Rt\,u(t)$



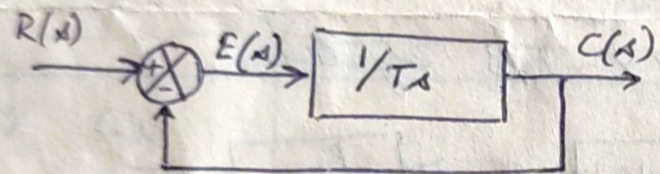
(iii) Acceleration i/p : Parabolic fn.

$r(t) = Rt^2 u(t)$



Note: $u(t)$: unit step fn.

1st order systems



$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

a) UNIT STEP RESPONSE

$R(s) = \frac{1}{s}$ $\therefore\ C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+(1/T)}$

$\therefore\ c(t) = 1 - e^{-t/T}$ $\quad t \geq 0$

Error $e(t) = r(t) - c(t) = e^{-t/T}$ $\quad$ T: time constant

$\therefore$ S.S. error $e_{ss}(t) = 0 = \lim_{t \to \infty} e(t)$

Properties of T (time constant):

(I) $c(T) = 0.632$

(II) smaller T, faster response

(III) $\left.\frac{dc}{dt}\right|_{t=0} = 1/T$, slope decreases MONOTONICALLY to zero. $\frac{dc}{dt} = +\frac{1}{T}e^{-t/T} \to 0$ as $t \to \infty$

(IV) $c(2T) = 0.865$, $c(3T) = 0.95$ w/i 5% of final $c(4T) = 0.982$ <2% $c(5T) = 0.993$ <1%.

→ ~~Tolerance band~~

DELAY TIME t_d: $c(t_d) = 0.5\, c_{ss}(t)$ (CONTROL)

RISE TIME t_r: t_r is time taken for response to reach from 10% to 90% of SS value.

For underdamped usually 0 to 100% of SS.

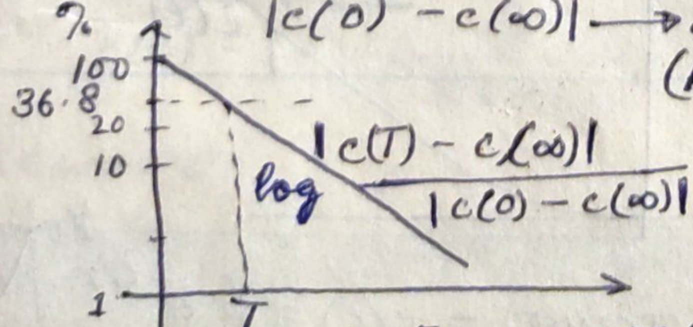
Note: $c(t) - c(\infty) = 1 - e^{-t/T} - 1 = -e^{-t/T}$ $c(0) - c(\infty) = 0 - 1 = -1$

$\therefore \log \frac{(c(t) - c(\infty))}{c(0) - c(\infty)} = -t/T$ line with slope $= -1/T$

$\therefore$ To experimentally determine if system is 1st order:

plot $\log \frac{|c(t) - c(\infty)|}{|c(0) - c(\infty)|}$ vs t OR SEMILOG: $\frac{|c(t) - c(\infty)|}{|c(0) - c(\infty)|}$ vs t

→ ~~origin setting, i.e. = 0~~ (NORMALIZE)

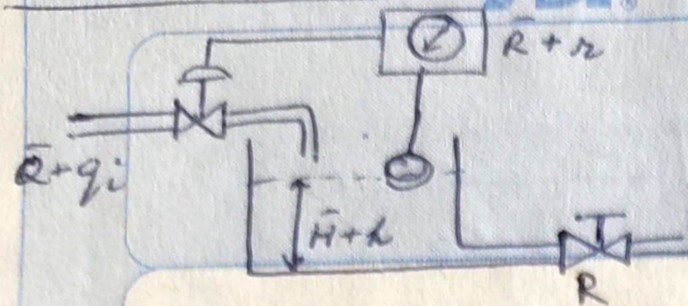


→ if st. line drop all then $t|_{36.8\%} = T$.

Note: As $T\uparrow$, t_d, t_r, $t_s \uparrow$

LIQUID LEVEL CONTROL

MANAGEMENT REPORT



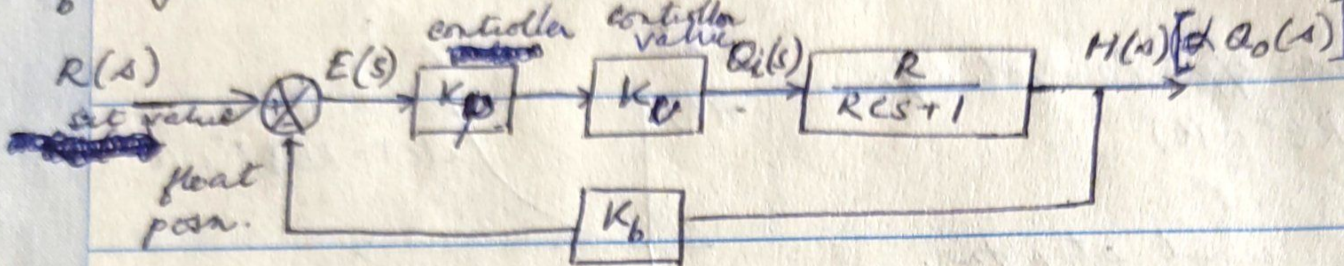
Let $r, q_i, h, q_0 \to$ small $\therefore$ linear model

$$q_i = K_v K_p e$$

overhead tanks

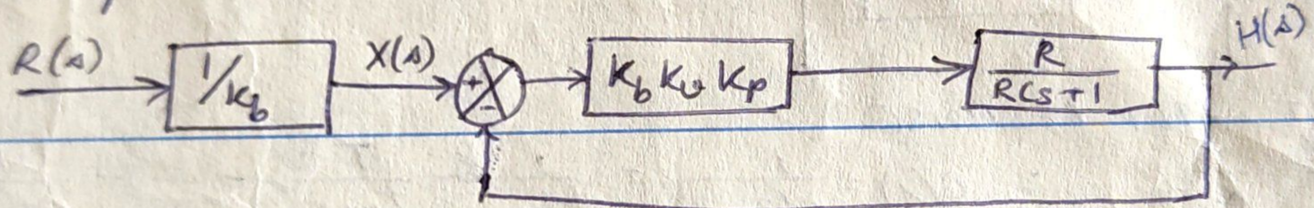
K_p : CONTROLLER gain $\quad K_v$: Control valve gain

K_b : gain of float in f/b



$$\frac{H(s)}{R(s)} = \frac{K_p K_v R}{(RCs+1) + K_b K_p K_v R}$$

For representation as UNITY F/b system,



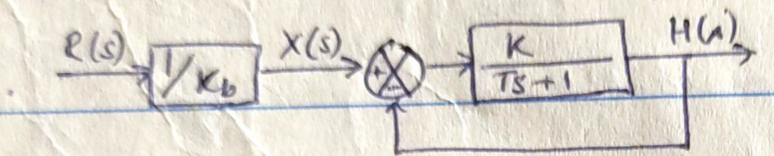
or $$\frac{H(s)}{R(s)} = \frac{X(s)}{R(s)} \times \frac{H(s)}{X(s)} = \frac{1}{K_b} \cdot \frac{K_b K_v K_p R}{RCs + 1 + K_b K_v K_p R}$$

$$= \frac{1}{K_b} \cdot \frac{K}{Ts + 1 + K}$$

ASSIGNMENT

or $\dfrac{H(s)}{R(s)} = \dfrac{X(s)}{R(s)} \times \dfrac{H(s)}{X(s)} = \dfrac{1}{K_b} \cdot \dfrac{K_b K_v K_p R}{RCs + 1 + K_b K_v K_p R}$

$$= \frac{1}{K_b} \cdot \frac{K}{Ts + 1 + K}$$

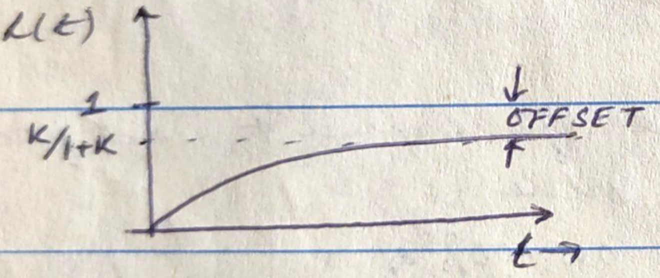
$\therefore$ 

Say unit step change in scaled down i/p $x(t) = \frac{1}{K_b} r(t)$

$\therefore h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{K}{Ts+1+K} \times \frac{1}{s}\right] = \frac{K}{1+K}\left(1 - e^{-t/T_1}\right) \; \forall t \geq 0$

where $T_1 = \dfrac{T}{1+K}$

So $h(\infty) = \dfrac{K}{1+K}$ $\quad \therefore e_{ss} = 1 - \dfrac{K}{1+K} = \dfrac{1}{1+K}$ <u>OFFSET</u>.



~~Note: As T ↑, t_d, t_r, t_s ↑~~

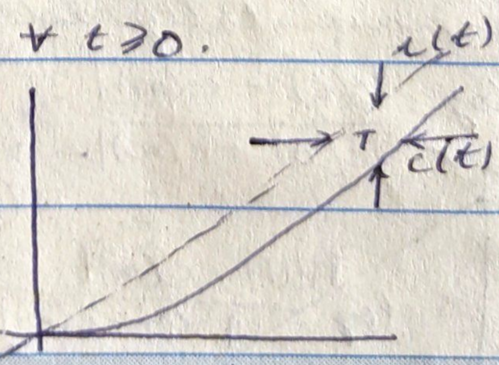
b) UNIT RAMP RESPONSE : $R(s) = \frac{1}{s^2}$; $r(t) = t$

$C(s) = \dfrac{1}{Ts+1} \cdot \dfrac{1}{s^2} = \dfrac{1}{s^2} - \dfrac{T}{s} + \dfrac{T^2}{Ts+1}$

$\therefore c(t) = t - T + Te^{-t/T} \quad \forall t \geq 0.$

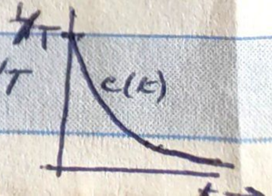
$e(t) = T(1 - e^{-t/T})$

$\therefore e_{ss}(t) = T = e(\infty)$

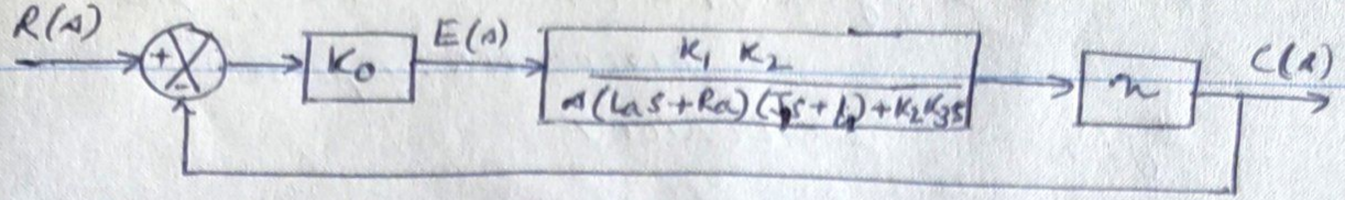


c) UNIT IMPULSE RESPONSE

$C(s) = \dfrac{1}{Ts+1}$ $\therefore$ $\dfrac{d}{dt}$(UNIT STEP RESPONSE) $= c(t) = \dfrac{1}{T} e^{-t/T}$



SECOND ORDER SYSTEM



Assume $L_a \to 0$

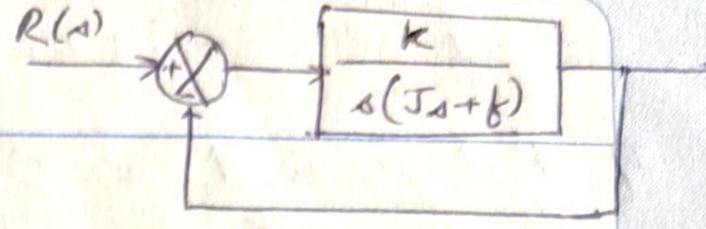
K_0 : potentiometric ~~gain~~ constant   n : gear turn ratio < 1

K_1 : amplifier gain

K_2 : motor torque constant

K_3 : back emf constant

≡



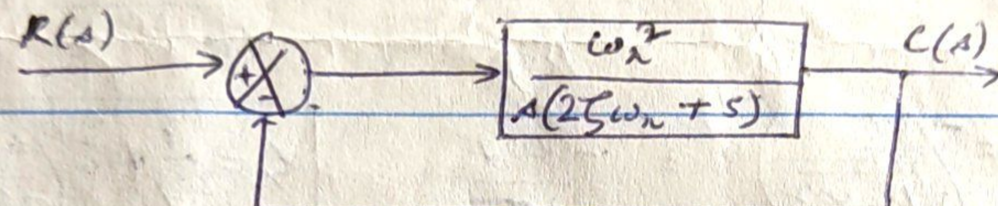
$J = J_1/n^2$; $f = \left(\frac{K_2K_3}{R_a} + f_1\right)/n^2$ referred to o/p shaft.

J_1 , f_1 referred to motor shaft $K = \frac{K_0 K_1 K_2 n}{R_a}$

At motor $J = J_m + n^2 J_L$ on load side $J_1 = J/n^2$

a) UNIT STEP RESPONSE :
UNDERDAMPED

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+fs+K} = \frac{K/J}{s^2+(f/J)s+(K/J)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

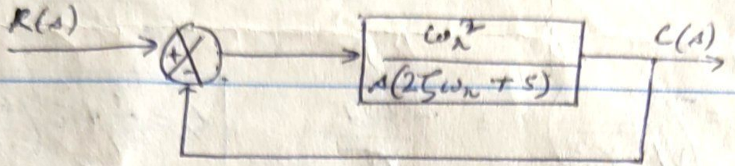


$$= \frac{\omega_n^2}{(s+\zeta\omega_n \pm j\omega_d)}$$

$\therefore \omega_d = \omega_n\sqrt{1-\zeta^2}$ DAMPED NATURAL FREQ. [at which oscillations occurs]

For $R(s) = 1/s$, $C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+fs+K} = \frac{K/J}{s^2+(f/J)s+(K/J)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



$$= \frac{\omega_n^2}{(s+\zeta\omega_n \pm j\omega_d)}$$

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For $R(s) = 1/s$, $C(s) = \dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \dfrac{1}{s} - \dfrac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$

$$\boxed{-\frac{(\zeta\omega_n/\omega_d)\cdot\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}}$$

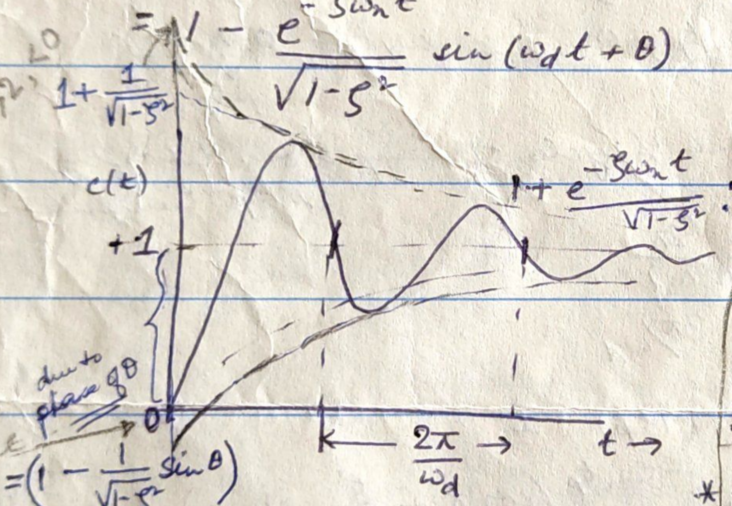
$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \dfrac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$

$= 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$ where $\theta = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) = \cos^{-1}\zeta = \sin^{-1}\sqrt{1-\zeta^2}$

$\zeta = 0$

$1-\zeta^2 = 1$

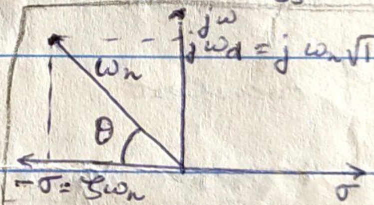
$1 \pm \frac{1}{\sqrt{1-\zeta^2}} = (2, 0)$



$\therefore \boxed{\tau = \frac{1}{\zeta\omega_n}}$

$\therefore c_{ss}(t) = 1$

$\therefore e_{ss}(t) = 0.$



LATER

** So $\cos\theta = \zeta$ as $\cos\theta = 1$, real roots OVERDAMPED

$\cos\theta = 0$, oscillatory : no damping.

$\zeta\downarrow$ $\omega_d\uparrow$ for $0<\zeta<1$.

$c(t)$ more oscillatory

$e(t) = r(t) - c(t) = \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$ DAMPED SINUSOID.

For $\zeta = 0$, $c(t) = 1 - \cos\omega_n t$ UNDAMPED.

Derive ASSIGNMENT

CRITICALLY DAMPED:

$$c(t) = \mathcal{L}^{-1}\left(\frac{\omega_n^2}{s(s+\omega_n)^2}\right) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

OVERDAMPED:

$$c(t) = \mathcal{L}^{-1}\left[\frac{\omega_n^2}{s(s+\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1})}\right]$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

$s_{1,2} = \omega_n(\zeta \pm \sqrt{\zeta^2-1})$

$\therefore\ \zeta < 0 \qquad s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad Re\, s_{1,2} > 0$

DIVERGING response.

$\zeta = 0$	$\pm j\omega_n$	OSCILLATORY
$0 < \zeta < 1$	$-\zeta\omega_n \pm j\omega_d$	UNDERDAMPED
$\zeta = 1$	$-\zeta\omega_n$	C. DAMPED
$\zeta > 1$	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$	OVERDAMPED $\quad Re\, s_{1,2} < 0$

If $K \uparrow \Rightarrow \omega_n \uparrow \Rightarrow \zeta \downarrow \Rightarrow \omega_d \uparrow \therefore$ faster response

motor gain constant $\quad = \sqrt{K/J} \qquad \propto \frac{1}{\sqrt{K}} \quad \frac{b}{\sqrt{KJ}} = \zeta \qquad$ stability reduces

Performance indices:

(i) Rise time t_r : 0 to 100%

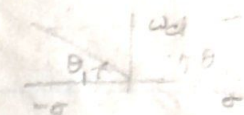
$$\therefore c(t_r) = 1 - 0 = 1 = 1 - \underbrace{e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]}_{=0.}$$

since $e^{-\zeta\omega_n t_r} \neq 0$

$$\Rightarrow \tan\omega_d t_r = -\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^{-1} = -\frac{\omega_d}{\sigma} \qquad \therefore \boxed{t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\theta}{\omega_d}}$$

$$\sigma = \zeta\omega_n$$

poles $s_{1,2}$



$\tan\theta_1$

$= \tan(\pi-\theta)$

(ii) Peak time $t_p = +\zeta\omega_n e^{-\zeta\omega_n t}[a+b] - \omega_d e^{-\zeta\omega_n t}\left[-\sin\omega_d t_r + \cancel{\frac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_d t_r}\right]$

$$\left.\frac{dc}{dt}\right|_{t=t_p} = 0 = \sin\omega_d t_p \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0 \qquad \left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \frac{\omega_n(1-\zeta^2)}{\sqrt{1-\zeta^2}}\right] \cdot e^{-\zeta\omega_n t} \sin\omega_d t$$

$$\Rightarrow \sin\omega_d t_p = 0 \qquad \therefore \omega_d t_p = 0, \underset{1^{st}\text{ peak}}{\pi}, 2\pi$$

$$\therefore \boxed{t_p = \frac{\pi}{\omega_d}} \rightarrow \text{one } \tfrac{1}{2} \text{ cycle of damped oscillation freq.}$$

(p - p : time period)

$\therefore$ 1st undershoot at $\frac{2\pi}{\omega_d}$

2nd overshoot $3\pi/\omega_d$ ....

(iii) Max. overshoot $M_p = c(t_p) - 1$

$$= e^{-\zeta\omega_n t_p}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$e^{-\pi/0} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$

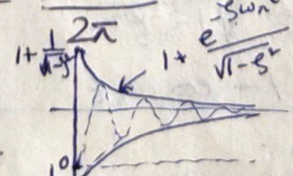
* (Sometimes in %), then $(\times 100)$% $\quad \Rightarrow \in [1, e^{-\pi}, 0] \ \forall \zeta \in [0, \frac{1}{\sqrt{2}}, 1]$

$\zeta = 1/\sqrt{2} \Rightarrow 0.043 \Rightarrow 4.3\%$

Design
M_p : 5–25%
ζ : 0.4–0.7

(iv) SUBSIDENCE RATIO (ratio of two subsequent OVERshoots)
DECAY RATIO
— separated by $2\pi\,\omega_d$

$$\frac{M_{p_1}}{M_{p_2}} = R:1 \Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\ln R}{2\pi}$$



(v) Settling time t_s

Envelopes are $1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ Rate of decay : $1/T = \zeta\omega_n$

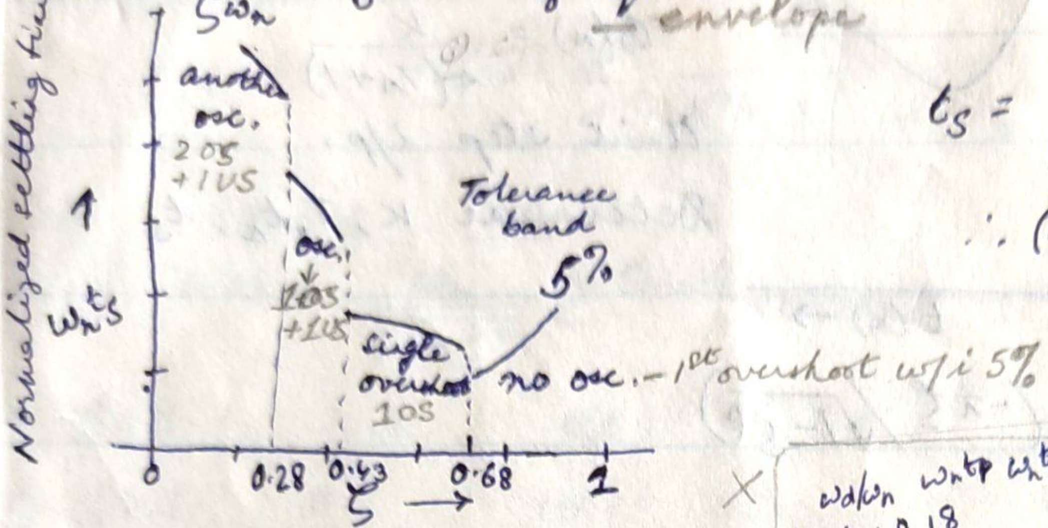
$$\therefore t_s \text{ for } p\% \text{ settling time} : \boxed{\frac{p}{100} = e^{-\zeta\omega_n t_s}}$$

$$\therefore t_s = \frac{4}{\zeta\omega_n} \text{ for } 2\%\ t_s \quad , \text{ for } 5\%\ t_s = \frac{3}{\zeta\omega_n} \ \ldots \text{ as for 1st order systems}$$

$e^{-4} \approx 0.02 \qquad e^{-3} \approx 0.05$

$\therefore \frac{1}{\zeta\omega_n} = \frac{1}{\sigma} = T$ of equivalent ~~second~~ first order system.

— envelope

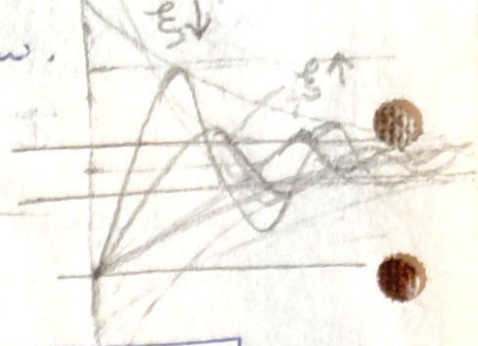


big with 2OS major

$\frac{\rho}{\rho_0} = e^{-\zeta\omega_n t}$

$t_s = \frac{3}{\zeta\omega_n}$

$\therefore (\omega_n t_s)\cdot\zeta = 3$



$\boxed{\begin{array}{l}\text{DESIGN} \\ M_p \to 5\text{–}25\% \\ \zeta \to 0.4\text{–}0.7\end{array}}$

ω_d/ω_n $\omega_n t_p$ $\omega_n t_r$

0.18