MATHEMATICAL MODELLING OF SYSTEMS:

1. MECHANICAL TRANSLATIONAL SYSTEMS:

(i) inertial force f_M due to MASS

$$f_M = Ma = M\frac{du}{dt} = M\frac{d^2x}{dt^2}$$

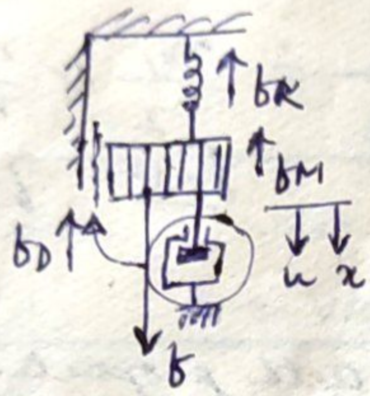
(ii) viscous damping force f_D

$$f_D = Du = D\frac{dx}{dt}$$

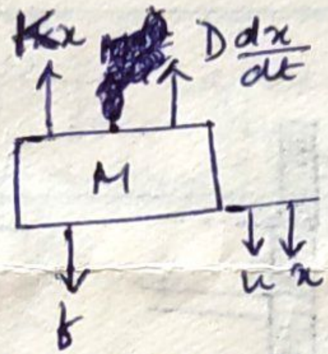
(iii) Spring force f_K [K_c = compliance = $1/K$ stiffness]

$f_K = Kx$ where K = stiffness or spring constant

∴



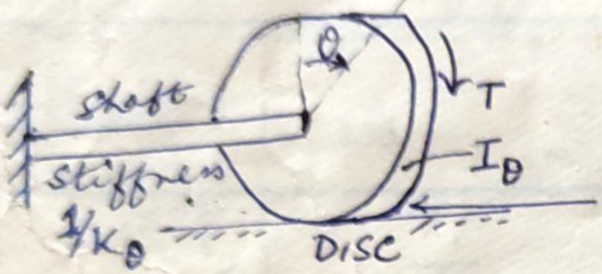
$$\therefore f + f_M + f_D + f_K = 0$$

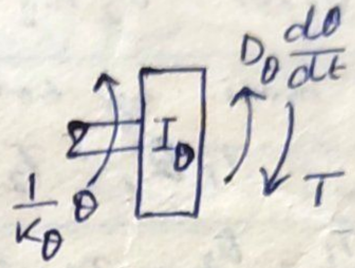


FREE BODY DIAGRAM

$$\therefore f = M\ddot{x} + D\dot{x} + Kx$$

2. MECHANICAL | ROTATIONAL SYSTEM





- 3 types of torques resisting rotational motion

(i) Inertial torque $T_I = I_\theta \alpha = I_\theta \dot{\omega} = I_\theta \ddot{\theta}$

(ii) Damping torque $T_D = D_\theta \omega = D_\theta \dot{\theta}$

(iii) Spring torque $T_K = \left(\frac{1}{K_\theta}\right)\theta$ where K_θ : torsional compliance

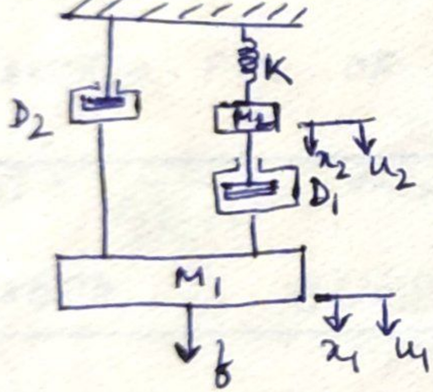
$$\therefore\ T = I_\theta \ddot{\theta} + D_\theta \dot{\theta} + \left(\frac{1}{K_\theta}\right)\theta$$

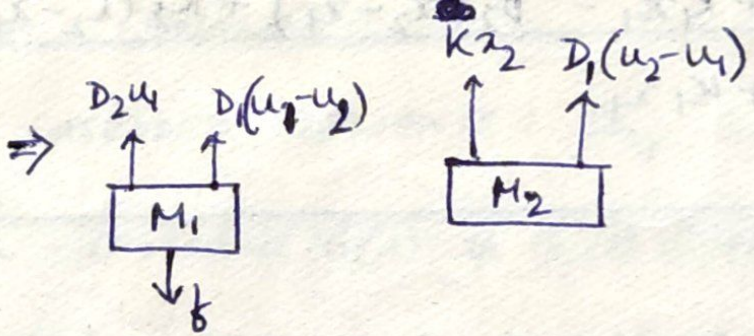
ANALOGY:

(ease of transformation)

Mechanical system		Electrical System	
Translational	Rotational	F - i / T - i	F - v / T - v ✓✓
Force f (N)	Torque T (N-m)	Current i	Voltage v
Velocity u (m/s)	Ang. vel. ω (rad/s)	Voltage v	Current i
Displacement x (m)	Ang. dis. θ (rad)	Flux linkage ϕ	Charge q
Mass M (kg)	Moment of Inertia I_θ (kg m²)	Capacitance C	Inductance L
V.D. coeff. D (N/s)	Rotational Damping Coeff. D_θ (N-m/rad/s)	Conductance G	Resistance R
Compliance K	Torsional Compliance K_θ	Inductance L	Capacitance C (Sense of direction retained)

Ex I





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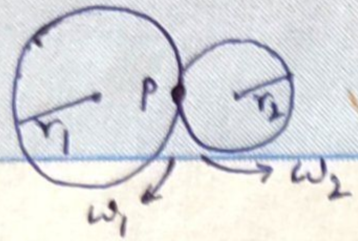
$$\therefore\ f = M_1\ddot{x}_1 + (D_1 + D_2)\dot{x}_1 - D_1\dot{x}_2$$

$$0 = M_2\ddot{x}_2 + D_1\dot{x}_2 + Kx_2 - D_1\dot{x}_1$$

*SHARE same displacement : SERIES
force : PARALLEL

MIT opencourseware : 07lecture08 : JOEL VOLDMAN

3. Friction wheels (analogous to transformers)



— same linear velocity at point of contact P

$\therefore r_1\omega_1 = r_2\omega_2$

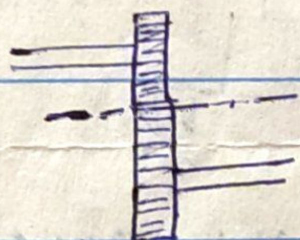
— Move together so experience equal and opposite forces

$$\therefore \frac{T_1}{r_1} = \frac{T_2}{r_2}$$

$$\therefore \frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}$$

Note $\frac{r_1}{r_2} \doteq \frac{n_1}{n_2}$ or $\frac{n_2}{n_1}$
<u>f-v</u> <u>f-i</u>

4. Gear train



— no. of teeth proportional to radii

$r_1/r_2 = n_1/n_2$

— distance travelled along each gear is same

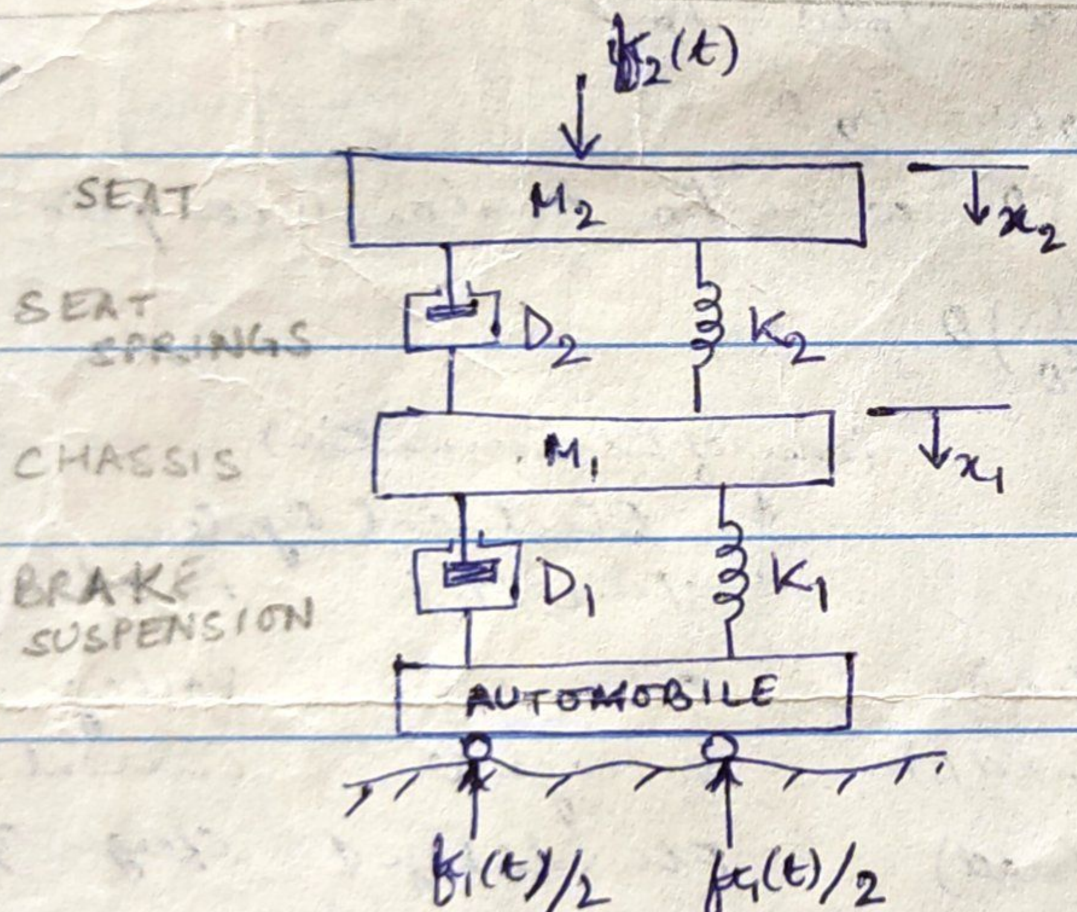
$\therefore \theta_1 r_1 = \theta_2 r_2$

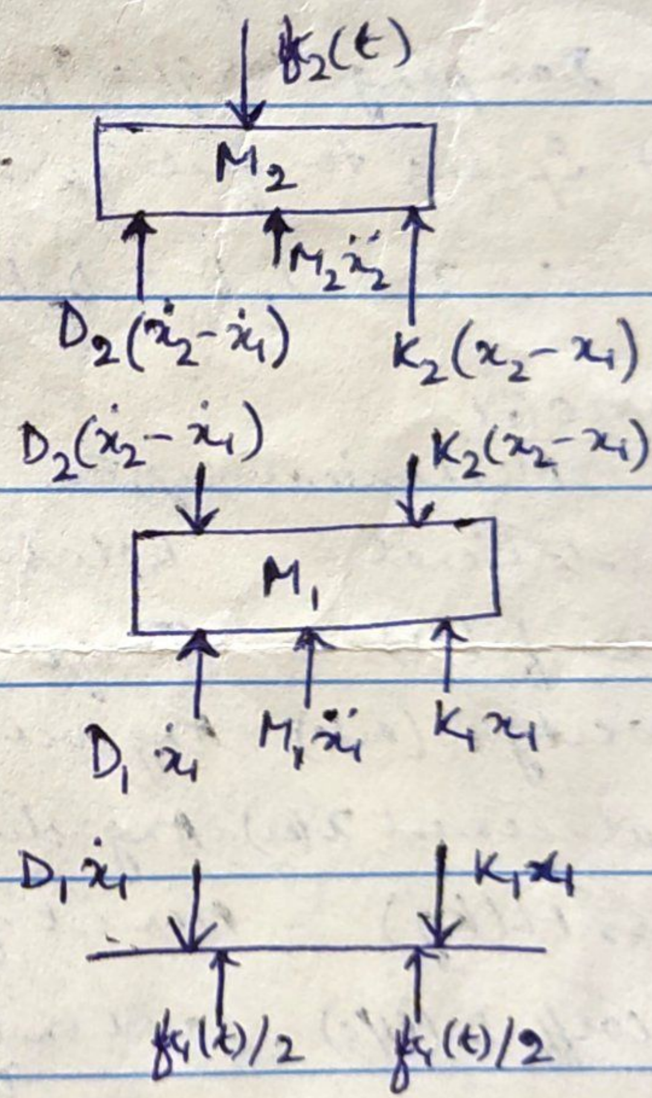
— work done by both is same

$\therefore T_1\theta_1 = T_2\theta_2$

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{n_1}{n_2} = \frac{r_1}{r_2}$$

II





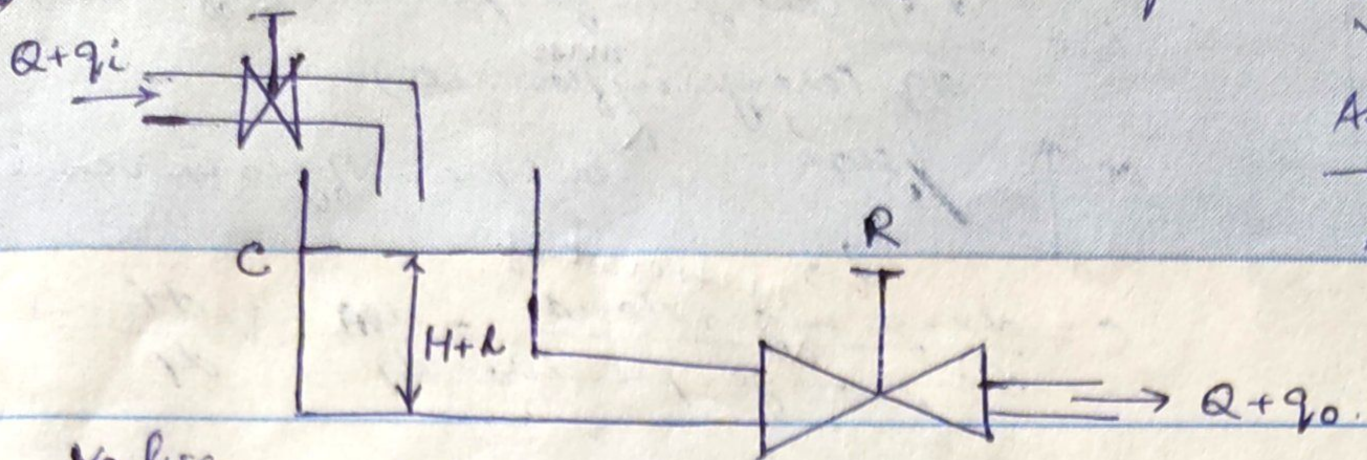
$$f_2 = M_2\ddot{x}_2 + D_2(\dot{x}_2-\dot{x}_1) + K_2(x_2-x_1)$$

$$M_1\ddot{x}_1 + D_1\dot{x}_1 + K_1x_1 = D_2(\dot{x}_2-\dot{x}_1) + K_2(x_2-x_1)$$

$$f_1 = D_1\dot{x}_1 + K_1x_1$$

LIQUID LEVEL SYSTEMS (Incompressible)

a)



Assume ρ constant
→ volumetric flow rates
ELSE MASS FLOW RATES
$\boxed{m = \rho V}$

Valve

Resistance R to liquid flowing in tank $= \dfrac{dH}{dQ} = \dfrac{\text{change in liquid level (head)}}{\text{change in flow rate}}$ (volumetric) $= \dfrac{\Delta v}{\Delta i}$

$$\frac{dV}{dt} = q_i - q_o = C\frac{dh}{dt} \rightarrow \text{TANK}$$

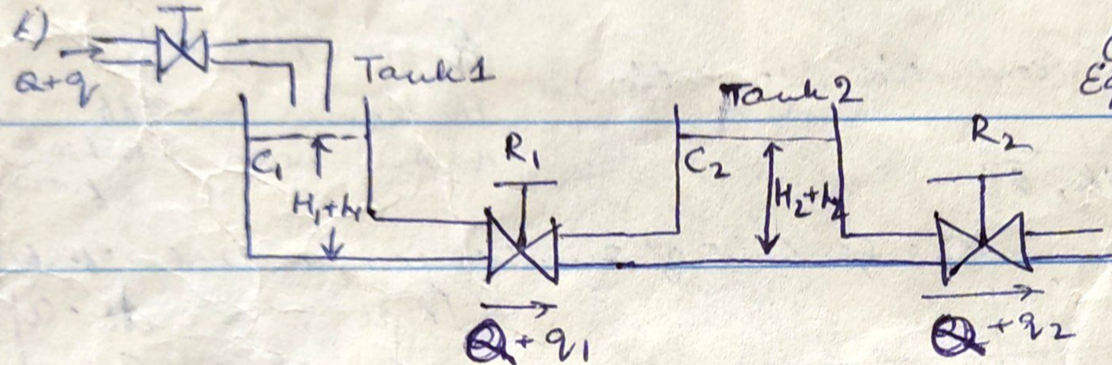
$dV = C\,dh$ where $C =$ fluid capacitance of tank $= \dfrac{\text{charge (flow volume)}}{\text{voltage (head)}}$
(Effective area of c.s!!)

$q_o = h/R$ → VALVE characteristics

$$\therefore C\frac{dh}{dt} = q_i - \frac{h}{R} \Rightarrow RC\frac{dh}{dt} + h = Rq_i$$

$$\therefore \frac{H(s)}{Q_i(s)} = \frac{R}{RCs+1}$$

neglecting ~~compressure~~ and inertia (causes sloshing)
→ inertance of pipe – high Q / large A_{cs} of pipe / high ρ_{liq}



(Nominal) Equilibrium values + Instantaneous values

VALVES

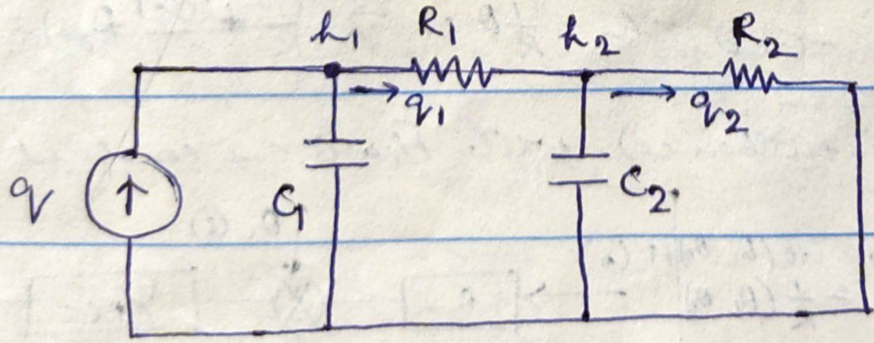
② $\dfrac{h_1-h_2}{R_1} = q_1$; ④ $\dfrac{h_2}{R_2} = q_2$

TANKS

① $C_1 \dfrac{dh_1}{dt} = q - q_1$

③ $C_2 \dfrac{dh_2}{dt} = q_1 - q_2$

$$\therefore \frac{Q_2(s)}{Q(s)} = \frac{1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1}$$



→ BLOCK DIAG.

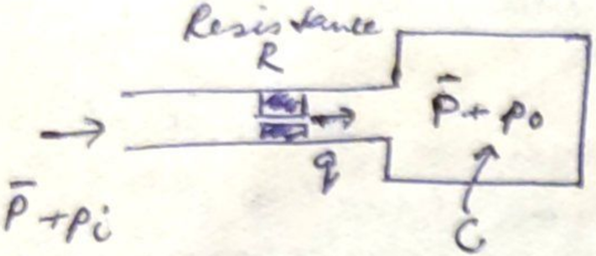
ELECTRICAL ANALOG.

** $\therefore$ TRANSFER FN. OF CASCADED ELEMENTS : Say $G_1(s) = \dfrac{1}{R_1C_1s+1}$

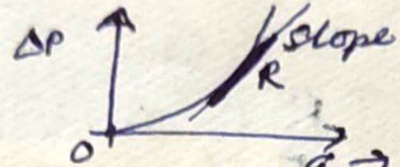
$G_2(s) = \dfrac{1}{R_2C_2s+1}$ then cascaded $G(s) \neq G_1(s)\,G_2(s)$ UNLESS o/p is

UNLOADED $\therefore$ BUFFERS / ISOLATING AMPLIFIERS used.

PRESSURE SYSTEM: flow (air) through pipes



$\therefore\; R = \dfrac{d\Delta P \text{ (change in gas pressure difference)}}{dq \text{ (change in mass flow rate)}}$



m: mass of gas in vessel

$C = \dfrac{\text{change in mass or wt. of gas stored}}{\text{change in gas pressure}} = \dfrac{dm}{dp} = V\dfrac{d\rho}{dp}$ d:

For polytropic expansion of gas → bet. isotopic & ADIABATIC $p\left(\dfrac{V}{m}\right)^n = \dfrac{p}{\rho^n} = \text{constant}.$ $\Rightarrow \dfrac{dp}{d\rho} = \text{cons.}$

X $C = V\dfrac{d\rho}{dP} = \boxed{\dfrac{k}{R_{gas}T}\cdot V}$? where $pV = R_{gas}T = \dfrac{R_u}{M}T$ ISOTHERMAL

M: molecular wt., m = wt., R_u = universal gas constant

$R = \dfrac{p_i - p_o}{q}$; $C\,dp_o = dm = q\,dt \Rightarrow \dfrac{P_o(s)}{P_i(s)} = \dfrac{1}{RCs+1}$

$\Rightarrow \frac{dt}{dp} = \frac{k}{p} + \frac{1}{v} \ldots$

$\Rightarrow \frac{dt}{dp} = \frac{1}{n}\frac{dt}{dp} + \frac{1}{v} \ldots$

$\Rightarrow \left(\frac{n-1}{n}\right)\frac{dt}{dp} = \frac{1}{v} \ldots$

THERMAL SYSTEMS: typically DISTRIBUTED SYSTEM PARAMETERS

CONDUCTION or Convection : $h = K\,d\theta$; $K = HA$ (for convection) — h: Heat flow rate

Radiation : $h = K_r(\theta_1^4 - \theta_2^4)$

$d\theta$: change in temp. $\sim v$

h : heat flow rate $\sim i$

H: convection coeff.

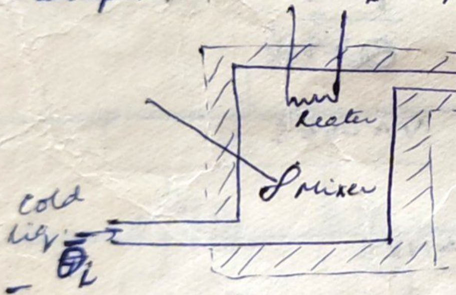
A: Area normal to heat flow

$$R \triangleq \frac{d\,d\theta}{dh} = \frac{1}{K} \text{ (conduction, convection)}$$

$$= \frac{1}{4K_r\bar{\theta}^3}$$

$C = mc$; c = sp. heat of substance, m: mass of substance

$$\boxed{\int h\,dt = mc\,d\theta = Q \Rightarrow q = Cv} \Rightarrow h = Gc\theta$$



Hot liquid — Outlet

$\bar{H} + h_o$, $\bar{\theta}_o + \theta$

Note : $\theta_o = f(h_i, \theta_i)$

Inlet — cold liq. $\bar{\theta}_i$

$\bar{H} + h_i$

Let $\theta_o = f(h_i)$

$h_o = Gc\theta$

$C = Mc$

$R = \frac{\theta}{h_o} = \frac{1}{Gc}$

G: steady state liq. flow rate/°C

c: liq. sp. heat

M: mass of liquid in tank

h: heat flow rate kcal/sec (same as q)

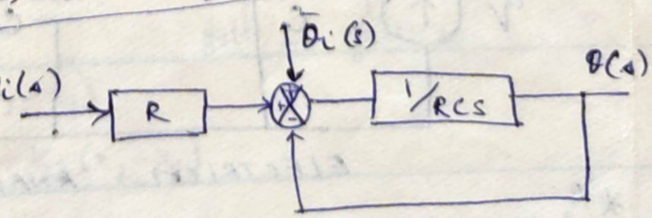
$$C\frac{d\theta}{dt} = h_i - h_o = h_i - Gc\theta = h_i - \frac{1}{R}\theta \;;\; \Rightarrow \frac{(1+RCs)}{R}\theta_o(s) = h_i(s)$$

If $\bar{\theta}_i \rightarrow \bar{\theta}_i + \theta_i$ (disturbance) while $\bar{H}_i$ & G are constant

Let $\theta_o = f(\theta_i)$

$$C\frac{d\theta_o}{dt} = Gc\,\theta_i - h_o = Gc(\theta_i - \theta_o) = \frac{1}{R}(\theta_i - \theta_o)$$

$$\therefore \frac{\theta_o(s)}{\theta_i(s)} = \frac{1}{RCs + 1}$$



∴ ASSUMING LINEARITY

$$RC\frac{d\theta_o}{dt} + \theta_o = \theta_i$$

$(1/R) \propto$ steady state liq. mass flow rate and sp. heat of liq.

Through & across variable analogies

Electrical system $\frac{v}{i} = R$; $\frac{v}{q} = \frac{1}{C}$; $\frac{\phi}{i} = L$

Electrical	Mechanical: Translational	Mechanical: Rotational	Liq level (Assume P const.)	Pressure	Thermal
Voltage v	Force f	Torque T	Level h	pressure diff. ΔP	Temp diff ΔT
Current i	Velocity u	ω	(volumetric) flow rate q	Mass flow rate q	heat flow rate h
Charge q	Displacement x	θ	Volume V	Mass stored dm	Heat transferred $Q = mc\theta$
Flux linkage ϕ					
Resistance R	Viscous damping coeff. D	D_θ	$\frac{h}{q} = R$ (valve)	$\frac{d\Delta P}{dq}$	$\frac{d\theta}{dh} = \frac{1}{k}$ or $\frac{1}{4K_r\theta^3}$
Inductance L	Mass M	I (Moment of Inertia)	—	—	
Capacitance C	Compliance K	K_θ	$\frac{dV}{dh} = C$ (effective Area of c.s.)	$\frac{dm}{dp} = V\frac{d\rho}{dp} = C$	$mc = C$