Laplace transform techniques:

$$\underline{V}(\underline{s}) = \int_{-\infty}^{\infty} e^{-\underline{s}t} v(t)\,dt = \int_{0^-}^{\infty} e^{-\underline{s}t} v(t)\,dt$$

One sided Laplace trans.
since most fns. do not
exist forever in time.
So consider that initiated
at $t=0$.

$t=0^-$ taken to take care of initial conditions.

$$\therefore v(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\underline{s}t}\,\underline{V}(\underline{s})\,d\underline{s}$$

$$\therefore \mathcal{L}[v(t)] = \underline{V}(\underline{s}) \quad \text{or} \quad v(t) = \mathcal{L}^{-1}[\underline{V}(\underline{s})]$$

Some basic formulae.

$$\mathcal{L}[u(t)] = \frac{1}{\underline{s}}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[t\,u(t)] = \frac{1}{\underline{s}^2}$$

$$\mathcal{L}[e^{-\alpha t} u(t)] = \frac{1}{\underline{s}+\alpha}$$

$$\mathcal{L}[\sin\omega t\, u(t)] = \frac{\omega}{\underline{s}^2+\omega^2}$$

$$\mathcal{L}[\cos\omega t\, u(t)] = \frac{\underline{s}}{\underline{s}^2+\omega^2}$$

$$\mathcal{L}[f_1(t) + f_2(t)] = \underline{F}_1(\underline{s}) + \underline{F}_2(\underline{s})$$

$$\mathcal{L}[k\,v(t)] = k\,\underline{V}(\underline{s})$$

$$\mathcal{L}\left[\frac{dv}{dt}\right] = \underline{s}\,\underline{V}(\underline{s}) - v(0^-)$$

$$\mathcal{L}\left[\frac{d^2v}{dt^2}\right] = \underline{s}^2\,\underline{V}(\underline{s}) - \underline{s}v^{\bullet}(0^-) - v^{1}(0^-)$$

$$\mathcal{L}\left[\int_{0^-}^{t} v(x)\,dx = \frac{\underline{V}(\underline{s})}{\underline{s}}\right]$$

$$*\mathcal{L}\left[\int_{-\infty}^{t} v(x)\,dx = \frac{1}{\underline{s}}\underline{V}(\underline{s}) + \frac{1}{\underline{s}}\int_{-\infty}^{0^-} v(x)\,dx\right.$$

$$\mathcal{L}[f(t-a)\,u(t-a)] = e^{-a\underline{s}}\,\underline{F}(\underline{s}) \quad a \geq 0$$

Convolution : $\int_{-\infty}^{\infty} f_1(\lambda)\, f_2(t-\lambda)\, d\lambda = f_1(t) * f_2(t)$

Let $F_1(s) = \mathcal{L}[f_1(t)]$, $F_2(s) = \mathcal{L}[f_2(t)]$

then $\mathcal{L}[f_1(t) * f_2(t)] = F_1(s) \cdot F_2(s)$

$$\mathcal{L}\left[\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t) \; , n = 1, 2, \ldots \right] = \frac{1}{(s+\alpha)^n}$$

$$\mathcal{L}\left[\frac{1}{\beta-\alpha} \left(e^{-\alpha t} - e^{-\beta t}\right) u(t) \right] = \frac{1}{(s+\alpha)(s+\beta)}$$

$$\mathcal{L}\left[e^{-\alpha t} \sin \omega t \; u(t) \right] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

$$\mathcal{L}\left[e^{-\alpha t} \cos(\omega t)\, u(t) \right] = \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

$$\mathcal{L}\left[\sin(\omega t + \theta)\, u(t) \right] = \mathcal{L}\left[(\sin \omega t \cos\theta + \cos \omega t \sin\theta)\, u(t) \right]$$

$$= \frac{\omega \cos\theta + s \sin\theta}{s^2 + \omega^2}$$

$\mathcal{L}[f(t)\, e^{-at}] = F(s$ [illegible]

$$\mathcal{L}[-t f(t)] = \frac{dF(s)}{ds}$$

$$\mathcal{L}\left[\frac{f(t)}{t} \right] = \int_s^{\infty} F(s)\, ds$$

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\mathcal{L}[f(0^+)] = \lim_{s \to \infty} s F(s)$$

$$\mathcal{L}[f(\infty)] = \lim_{s \to 0^+} s F(s)$$

(provided all poles of $F(s)$ in LHP)

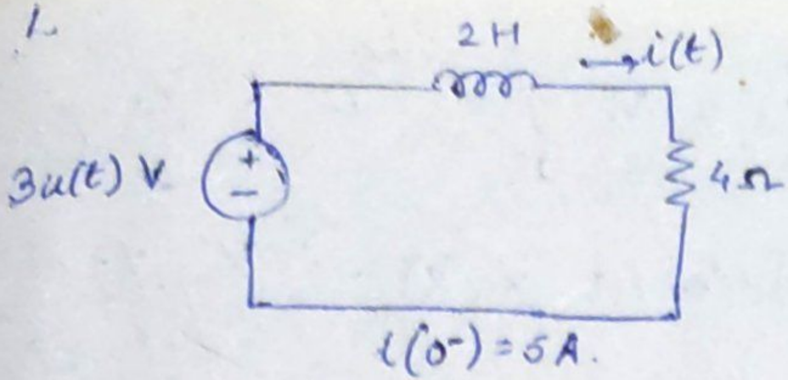
* For a periodic fn.

$f(t) = f(t + nT) \quad n = 1, 2, \ldots$

Say $F_1(s) = \int_{0^-}^{T} f(t)\, e^{-st}\, dt$

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-Ts}} F_1(s)$$

Ex. 1.



$$\therefore \quad 2\frac{di}{dt} + 4i = 3u(t)$$

Taking $\mathcal{L}$-transform of both sides.

or $$2[sI(s) - i(0^-)] + 4I(s) = \frac{3}{s}$$

$$\therefore \quad (2s+4)\,I(s) = \frac{3}{s} + 10$$

$$\therefore \quad I(s) = \frac{3}{2s(s+2)} + \frac{5}{s+2} = \frac{A}{s} + \frac{B}{s+2}$$

At $s=0$, $$sI(s) = \frac{1.5}{2} + \cancelto{0}{\frac{5s}{2}} = A + \cancelto{0}{\frac{Bs}{s+2}}$$

$$\therefore \quad A = 0.75.$$

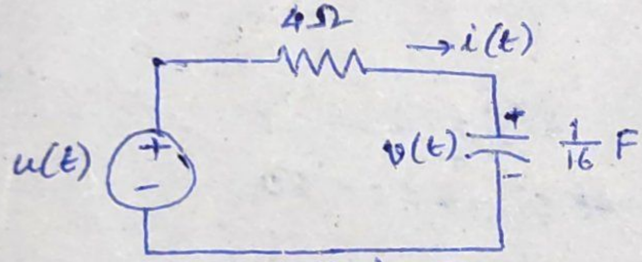
At $s=-2$, $$(s+2)\,I(s) = \frac{3}{2s} + 5 = \cancelto{0}{\frac{A(s+2)}{s}} + B$$

$$= \frac{-3}{4} + 5 = 4.25.$$

$$\therefore \quad I(s) = \frac{0.75}{s} + \frac{4.25}{s+2}$$

$$\therefore \quad i(t) = 0.75u(t) + 4.25e^{-2t}u(t).$$

Ex 2.



$v(0^-) = 9V$

$$u(t) = 4i(t) + 16\int_{-\infty}^{t} i(t)\,dt$$

$$= 4i(t) + v(0^-) + 16\int_{0^-}^{t} i(t)\,dt$$

Note $\mathcal{L}[v(0^-)]$
$= \mathcal{L}[v(0^-)u(t)]$ since one-sided $\mathcal{L}$-transform

Taking Laplace transform of both sides.

$$\frac{1}{s} = 4I(s) + \frac{9}{s} + \frac{16}{s}I(s)$$

$$\Rightarrow -8 = 4(s+4)I(s)$$

$$\Rightarrow I(s) = \frac{-2}{s+4}$$

$$\Rightarrow i(t) = -2e^{-4t}u(t).$$

If $v(t)$ had been desired, write nodal eqn.

$$\frac{v(t)-u(t)}{4} + \frac{1}{16}\frac{dv}{dt} = 0$$

Take Laplace.

$$\frac{V(s)}{4} - \frac{1}{4s} + \frac{1}{16}\left[sV(s) - v(0^-)\right] = 0.$$

$$\Rightarrow V(s)\left[1+\frac{s}{4}\right] = \frac{1}{s} + \frac{9}{4}$$

$$\Rightarrow V(s) = \frac{4}{s(s+4)} + \frac{9}{s+4} = \frac{1}{s} - \frac{1}{s+4} + \frac{9}{s+4} = \frac{1}{s} + \frac{8}{s+4}$$

$$\frac{4}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}.$$

$A(s+4) + Bs = 4$

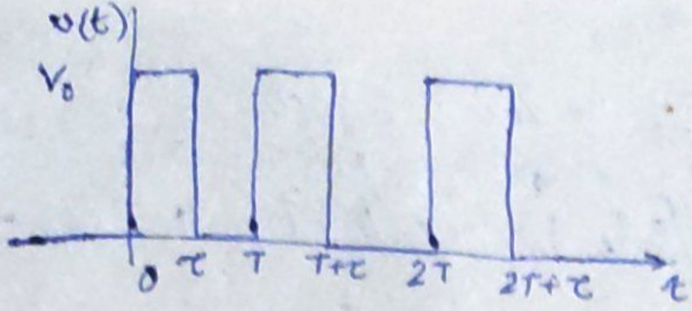
$\Rightarrow A + B = 0$

& $4A = 4 \Rightarrow A = 1$ $\therefore B = -1$

$$\Rightarrow v(t) = u(t) + 8e^{-4t}u(t) = \left(1 + 8e^{-4t}\right)u(t).$$

Check $\frac{1}{16}\frac{dv}{dt} = \frac{1}{16}\left[-32e^{-4t}\right] = -2e^{-4t} = \frac{-(v(t)-u(t))}{4}$ $\therefore$ correct.

Ex 3.



* $\therefore \underline{V}_1(\underline{s}) = V_0 \int_{0^-}^{\tau} e^{-\underline{s}t}\,dt = \frac{V_0}{\underline{s}}\left(1 - e^{-\underline{s}\tau}\right)$

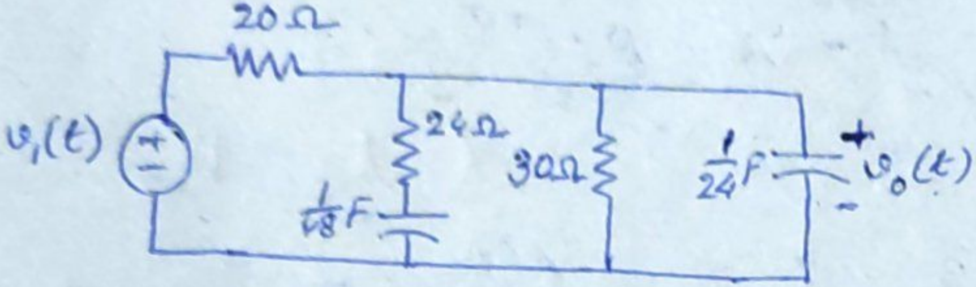
$\therefore \mathcal{L}[v(t)] = \underline{V}(\underline{s}) = \frac{V_0}{\underline{s}} \frac{1 - e^{-\underline{s}\tau}}{1 - e^{-\underline{s}T}}$

denominator : due to periodicity.

$e^{-\underline{s}\tau}$: due to time delay of negative eq. wave that turns off pulse.

$\frac{V_0}{\underline{s}}$: $\mathcal{L}[V_0 u(t)]$

Ex 4.



Assume no initial energy storage in network.

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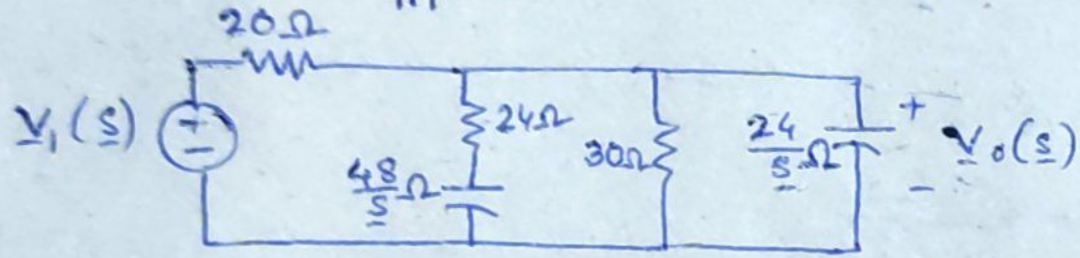


: $\frac{V_o(s)}{V_1(s)}$ required.

impedance of 3 parallel branches

$$Z_i(s) = \frac{1}{\frac{s}{24} + \frac{1}{30} + \frac{1}{24 + 48/s}} = \frac{120(s+2)}{5s^2 + 19s + 8}.$$

$$\therefore \frac{V_o(s)}{V_1(s)} = \frac{Z_i(s)}{20 + Z_i(s)} = \frac{6(s+2)}{5s^2 + 25s + 20}$$

$$\therefore H(s) = \frac{1.2(s+2)}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4} = \frac{0.4}{s+1} + \frac{0.8}{s+4}.$$

$$\therefore h(t) = (0.4e^{-t} + 0.8e^{-4t})u(t).$$

So, for $v_i(t) = \delta(t)$. $\quad v_o(t) = h(t)$

Say $v_i(t) = 50\cos 2t\, u(t)$ V.

$$\therefore \underline{V}_o(\underline{s}) = \underline{H}(\underline{s}) \cdot \underline{V}_i(\underline{s}) = \underline{H}(\underline{s}) \cdot \mathcal{L}[50\cos 2t\, u(t)]$$

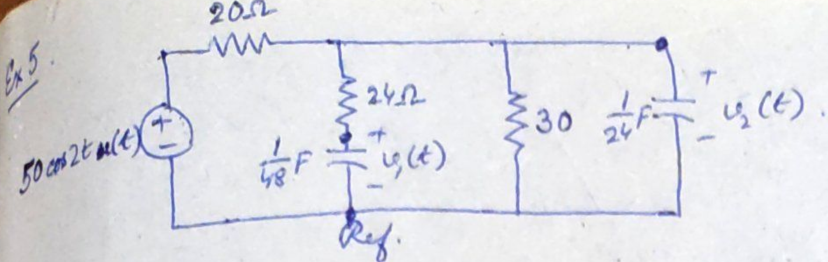
$$= \underline{H}(\underline{s}) \frac{50\underline{s}}{\underline{s}^2+4} = \frac{1.2(\underline{s}+2)}{(\underline{s}+1)(\underline{s}+4)} \cdot \frac{50s}{s^2+4}$$

$$= \frac{A}{\underline{s}+1} + \frac{B}{\underline{s}+4} + \underbrace{\frac{C}{s+j2} + \frac{D}{s-j2}}_{\frac{Cs+D}{s^2+4}}$$

$$= \frac{-4}{s+1} - \frac{8}{s+4} + \frac{12s+24}{s^2+4}$$

$$\therefore v_o(t) = (-4e^{-t} - 8e^{-4t} + 12\cos 2t + 12\sin 2t)\, u(t)$$

Ex 5.



Let $v_1(0^-) = 10\,V$
$v_2(0^-) = 25\,V$.

At ①, $\frac{1}{48}\frac{dv_1}{dt} + \frac{v_1 - v_2}{24} = 0$.

$$\Rightarrow \quad 2v_2 = 2v_1 + \frac{dv_1}{dt} \quad \ldots \text{①}$$

At ②, $\frac{1}{24}\frac{dv_2}{dt} + \frac{v_2}{30} + \frac{v_2 - v_1}{24} + \frac{v_2 - 50\cos 2t\, u(t)}{20} = 0$.

$$\Rightarrow \quad v_1 = \frac{dv_2}{dt} + 3v_2 - 60\cos 2t\, u(t) \quad \ldots \text{②}$$

Eliminate v_1 and $\frac{dv_1}{dt}$ by taking derivative of ② [Note $\frac{du(t)}{dt} = \delta(t)$].

$$\therefore \; v_1' = v_2'' + 3v_2' + 120\sin 2t\, u(t) - 60\,\delta(t) \quad \ldots \text{③}$$

Substituting ② & ③ in ①.

$$v_2'' + 5v_2' + 4v_2 = (120\cos 2t - 120\sin 2t)\,u(t) + 60\,\delta(t)$$

Taking Laplace transform.

$$\left[s^2 V(s) - s\,v_2(0^-) - v_2'(0^-)\right] + \left[5\,s\,V_2(s) - 5v_2(0^-)\right] + 4\,V(s)$$

$$= \frac{120s - 240}{s^2 + 4} + 60$$

$$\Rightarrow (s^2 + 5s + 4)\,V(s) = s\,v_2(0^-) + v_2'(0^-) + 5v_2(0^-) + \frac{120s - 240}{s^2 + 4} + 60.$$

~~let $v_2(0^-) = 25$.~~

To evaluate $v_2'(0^-)$, use (2),

$$v_1(0^-) = v_2'(0^-) + 3v_2(0^-) - 0.$$

$$\Rightarrow v_2'(0^-) = -65. \quad \text{while } v_2(0^-) = 25.$$

$$\therefore V_2(s) = \frac{25s^3 + 120s^2 + 220s + 240}{(s+1)(s+4)(s^2+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s+4} + \frac{Cs+D}{s^2+4}.$$

$$= \frac{23/3}{s+1} + \frac{16/3}{s+4} + \frac{12s+24}{s^2+4}.$$

$$\therefore v_2(t) = \left(\frac{23}{3}e^{-t} + \frac{16}{3}e^{-4t} + 12\cos 2t + 12\sin 2t\right)u(t).$$