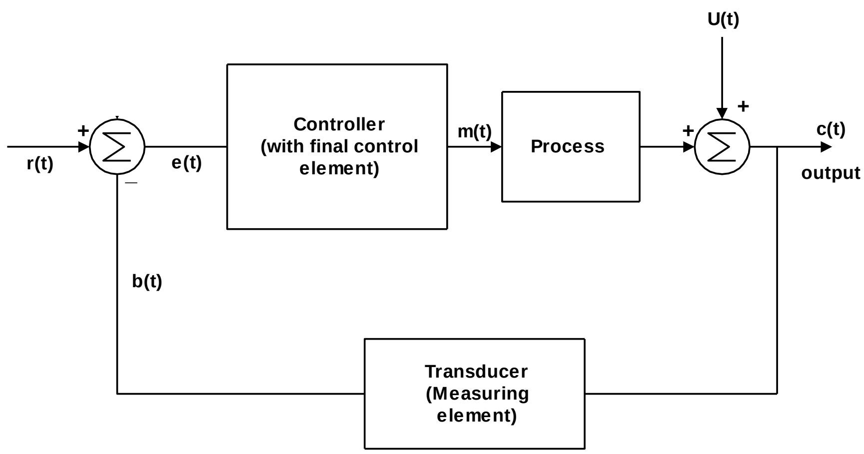
Digital Control of Continuous Time Process

A simple process control system may be represented as shown below.



r(t) = Reference input or Demand or Set point

c(t) = Output (Controlled variable)

e(t) = Error signal or deviation

m(t) = Manipulated variable

U(t) = Load disturbance variable

b(t) = Feedback variable.

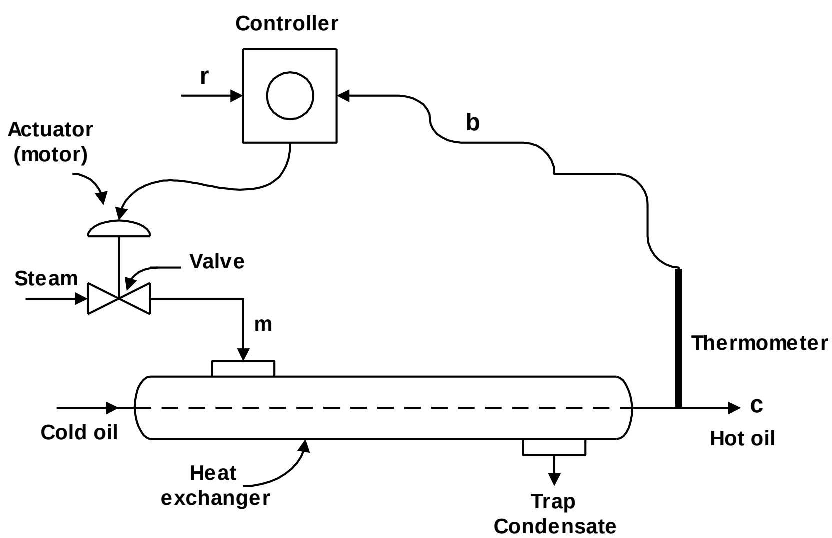
In an automatic process control system, the demand is usually fixed, and the purpose of the control system is to minimize the effect of load disturbances on the value of the controlled variable. It is also known as *regulator control*.

The alternative control problem is the control of a system to follow a changing reference input. The load variables are constant, or changes are of little significance. Control of this kind (tracking control) is generally a problem of s*ervomechanism* and is not as often encountered in process control.

Example of an Automatic Process Control System:-

A Simple heat-transfer system

The process element is a double pipe heat exchanger in which cold oil flowing through the inner pipe is heated by steam condensing in the jacket surrounding the inner pipe.



(Temperature of cold oil may act as a variable.)

Purpose of controller is to maintain a constant outlet oil temperature, as set by reference input.

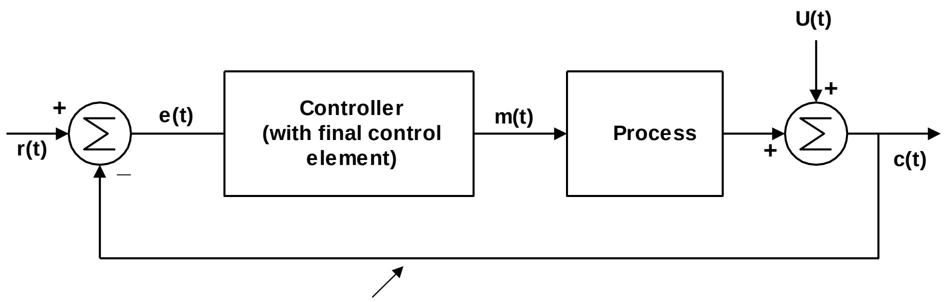
c = Outlet temperature of oil

m = Steam pressure

U = Inlet temperature of oil (Load disturbance).

The motor (actuator) driven valve is the final control element

If the time constant of the measuring element is small compared to the time constant of the process, the block diagram representation of the automatic process control system shown at the beginning, can be represented by the unity feedback system shown below.

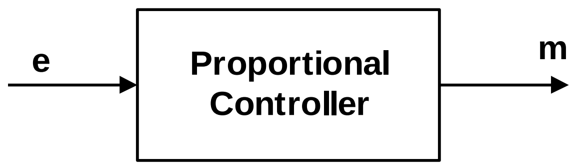


(Response time of the measuring element is small w.r.t. smallest time constant of the process)

Conventional Analog Controllers

A conventional analog controller can be a single term (proportional), or a two-term (Proportional-Integral or Proportional derivative) or a three-term (Proportional-Integral-Derivative) controller.

Proportional Controller (P)



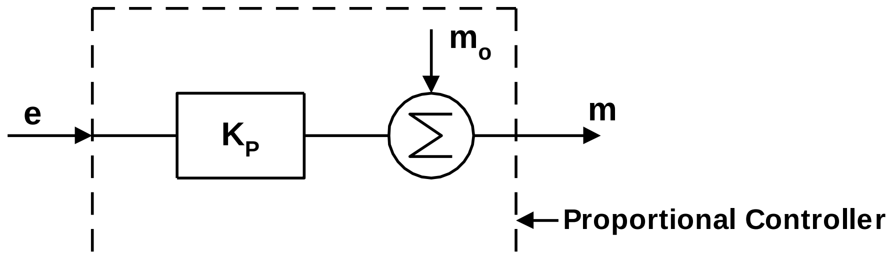
Proportional control follows the law

$m = K_P e + m_o$

m_o = Manual-reset constant (bias).

On most proportional controllers there is an adjusting knob or other mechanism for selecting the value of m_o.

K_P = Proportional sensitivity i.e. change of manipulated variable caused by unit change of deviation. K_P is also known as proportional gain.



Proportional Integral (PI) Controller

PI control follows the law

$$m = \underbrace{\frac{K_P}{T_i}\int e\,dt}_{\text{Integral action}} + \underbrace{K_P e + m_o}_{\text{Proportional action}}$$

$T_i \equiv$ Integral time or Integral time constant

$\frac{1}{T_i} \equiv$ Reset rate.

Proportional-Derivative (PD) Controller

The control law for the PD controller is given by

$$m = K_P T_d \frac{de}{dt} + K_P e + m_o$$

$T_d =$ Derivative time or derivative time constant.

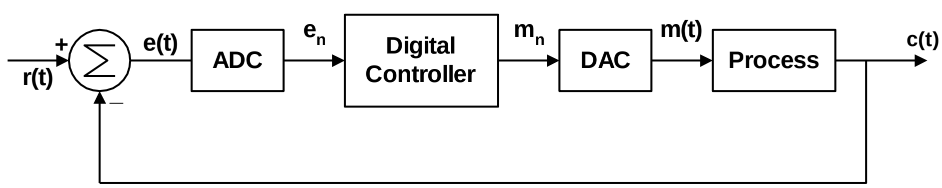
Proportional-Integral-Derivative (PID) Controller

PID control action is the additive combination of proportional action, integral action and derivative action. It is defined by the equation,

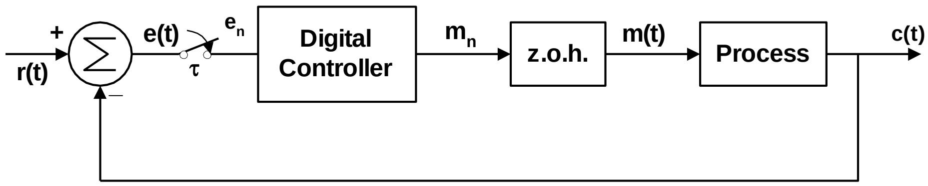
$$m = \frac{K_P}{T_i}\int e\,dt + K_P T_d \frac{de}{dt} + K_P e + m_o$$

Digital Control System

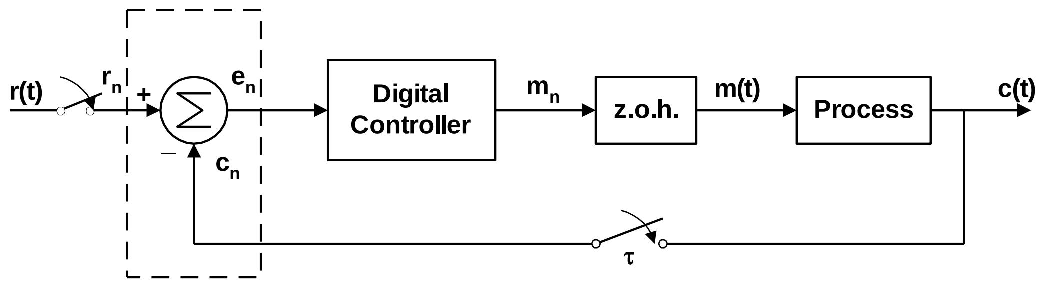
- A digital controller receives the error signal in the form of a sequence of digital words.
- The controller processes the data, and corresponding to each input word, it generates a control strategy in the form of a number (a digital word).
- A DAC converts the numerical control strategy from a digital word to an analog actuating signal for the actuator of final control element.
- ***There should however be a power amplifier stage in between, to drive the actuator of the final control element.***
- The digital controller can be a **computer program** or can also be in the form of a **sequential digital circuit** specially designed for this purpose.



- It is assumed that DAC stage contains necessary power amplifier and final control element.
- Since the ADC can be represented by an impulse sampler and DAC by a zero-order-hold, the above block diagram representation can be simplified to the following.



- Alternatively, ADC can also be placed in feedback path as shown below.



Let, $G_P(s) =$ Transfer function of C.T. process.

$G_{ho}(s) =$ Transfer function of zero-order hold (z.o.h)

$= \dfrac{1-e^{-s\tau}}{s}$, where τ is the sampling period.

Then the transfer function of the z.o.h along with the process is

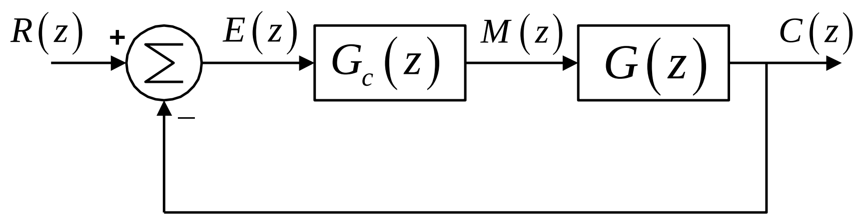
$$G(s) = G_{ho}(s)G_P(s) = \left[\frac{1-e^{-s\tau}}{s}\right]G_P(s)$$

∴ The z-transfer function of z.o.h along with the process is

$$G(z) = Z\left[G(s)\right] = Z\left[\left\{\frac{1-e^{-s\tau}}{s}\right\}G_P(s)\right]$$

Let $G_c(z)$ be the z-transfer function of the digital controller.

Then the block diagram representation of the control system becomes (in z-domain),



The open loop z-transfer function of the control system is $G_c(z)$ $G(z)$.

The closed loop z-transfer function of the control system is

$$H(z)=\frac{G_c(z)G(z)}{1+G_c(z)G(z)}$$

Transfer function of z.o.h.

$$G_{ho}(s)=\frac{1-e^{-s\tau}}{s}=\frac{1-\left[1-s\tau+\frac{s^2\tau^2}{2}-.....\right]}{s}\approx\tau\left[1-\frac{s\tau}{2}+.....\right]$$

If τ is small ≈ $\tau e^{-s\tau/2}$

∴A z.o.h introduces a time delay of approximately half the sampling period in a system.

DESIGN OF DIGITAL CONTROLLERS BY TRANSLATION OF ANALOG DESIGN:

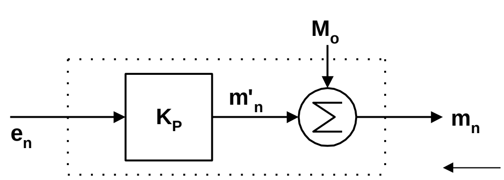
- **As tuning rules for analog controllers are simple and known, we go for this approach.**

In accordance with some performance specification the analog controller is first designed. Transform the analog controller to its digital counterpart by z-transform or directly by using difference equations.

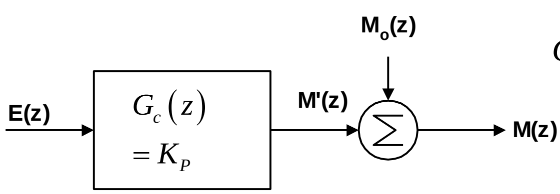
Proportional (P) Digital Controller

$m = K_P e + M_o \rightarrow$ bias or zero-error value of controller output or steady state value of controller output.

At nth instant, $m_n = K_P e_n + M_o$



Proportional control algorithm



$$G_c(z) = K_P = \frac{M'(z)}{E(z)}$$

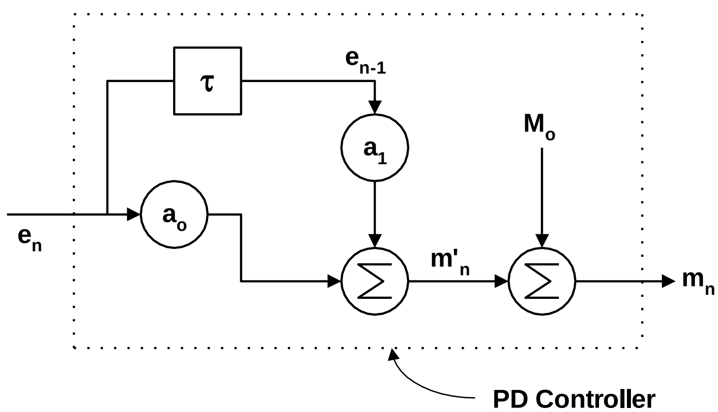
Proportional-Derivative (PD) Controller

$$m = K_P\left[e + T_d\,\frac{de}{dt}\right] + M_o$$

At the nth instant, $m_n = \underbrace{K_P e_n + K_P T_d\left(\frac{e_n - e_{n-1}}{\tau}\right)}_{m'_n} + M_o$

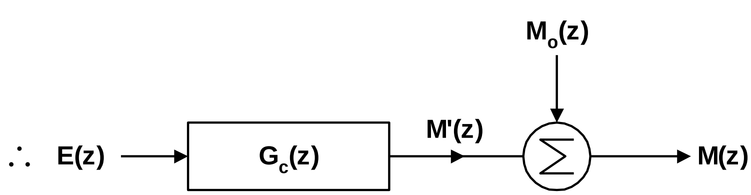
$$m_n = K_P\left(1+\frac{T_d}{\tau}\right)e_n - \frac{K_P T_d}{\tau}e_{n-1} + M_o$$

$$\therefore m_n = a_o e_n + a_1 e_{n-1} + M_o\left[a_o = K_P\left(1+\frac{T_d}{\tau}\right), a_1 = -\frac{K_P T_d}{\tau}\right]$$



$$\therefore M'(z) = a_o E(z) + a_1 z^{-1} E(z)$$

or, $G_c(z) = M'(z) = a_o + a_1 z^{-1}$



Proportional-Integral (PI) Controller

$$m = K_P\left[e + \frac{1}{T_i}\int_o^t edt\right] + M_o$$

or, $m_n = K_P e_n + \frac{K_P}{T_i} I_n + M_o = m_n' + M_o$

$$\therefore m_n' = K_p\left[e_n + \frac{\tau}{T_i}e_n + \frac{I_{n-1}}{T_i}\right]$$

$\left(\because I_n = I_{n-1} + \tau e_n,\right.$ using rectangular integration$\left.\right)$

$\left(m_n = \right.$ controller output at the nth instant$\left.\right)$

$$m_{n-1}' = K_p\left[e_{n-1} + \frac{I_{n-1}}{T_i}\right]$$

$$\therefore m_n' - m_{n-1}' = K_P\left[\left(1 + \frac{\tau}{T_i}\right)e_n - e_{n-1}\right]$$

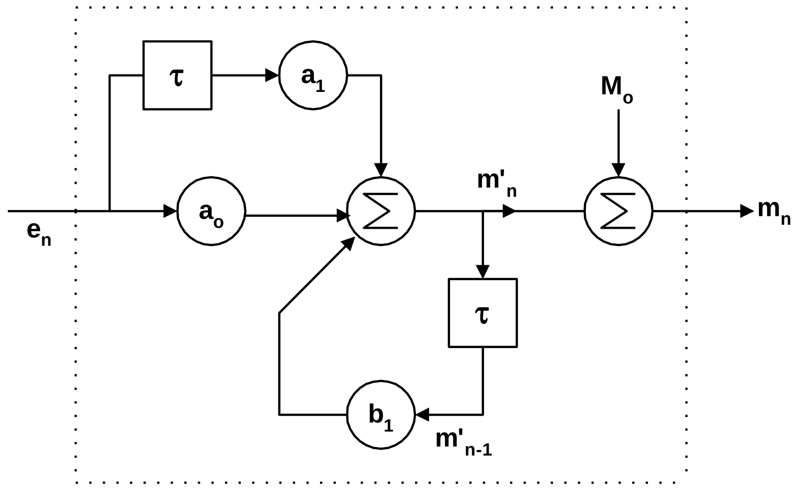
or, $m_n' = m_{n-1}' + K_P\left(1 + \frac{\tau}{T_i}\right)e_n - K_P e_{n-1}$

or, $m_n' = a_o e_n + a_1 e_{n-1} + b_1 m_{n-1}'$

$$m_n = a_o e_n + a_1 e_{n-1} + b_1 m_{n-1}' + M_o$$

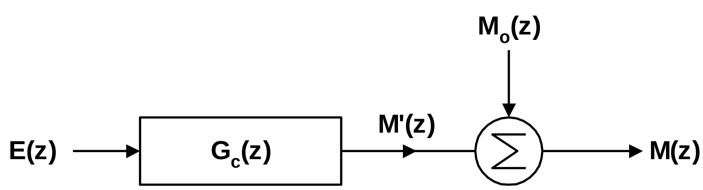
The pictorial representation of the algorithm will be :

(Structure using rectangular integration)



$\therefore$ z-transfer function will be $M'(z) = a_o E(z) + a_1 z^{-1} E(z) + b_1 M'(z) z^{-1}$

$\therefore$ Controller z-transfer function $G_c(z) = \dfrac{M'(z)}{E(z)} = \dfrac{a_o + a_1 z^{-1}}{1 - b_1 z^{-1}}$



Using Trapezoidal Integration

$I_n = I_{n-1} + \frac{\tau}{2}\left[e_n + e_{n-1}\right]$

$$\therefore m'_n = K_P e_n + \frac{K_P}{T_i} I_n = K_p e_n + \frac{K_P}{T_i} I_{n-1} + \frac{K_P \tau}{2T_i}\left(e_n + e_{n-1}\right)$$

$$= K_P\left(1 + \frac{\tau}{2T_i}\right) e_n + \frac{K_P \tau}{2T_i} e_{n-1} + \frac{K_P}{T_i} I_{n-1}$$

$$m'_{n-1} = K_P e_{n-1} + K_P \frac{I_{n-1}}{T_i}$$

$$m'_n - m'_{n-1} = K_P\left(1+\frac{\tau}{2T_i}\right)e_n + K_P\left(\frac{\tau}{2T_i}-1\right)e_{n-1}$$

$$\text{or, } m'_n = K_P\left(1+\frac{\tau}{2T_i}\right)e_n + K_P\left(\frac{\tau}{2T_i}-1\right)e_{n-1} + m'_{n-1}$$

$$= a_o e_n + a_1 e_{n-1} + b_1 m'_{n-1}$$

$$\therefore m_n = a_o e_n + a_1 e_{n-1} + b_1 m'_{n-1} + M_o \qquad \ldots(1)$$

(structure representation remains same, but the coefficients are different)

This form of algorithm shown in eqn.(1) is called *position form of PI control algorithm.*

There is another form of control algorithm, called velocity form of PI control or incremental form of PI control algorithm. This is given by:-

$$\Delta m_n = m_n - m_{n-1} = m'_n - m'_{n-1} = a_o e_n + a_1 e_{n-1}$$

Since it gives the change in position w.r.t. the position in the previous sampling instant, it is called velocity form of algorithm. Velocity form can be used with actuators which have *integral action behavior*. One example can be stepper motor whose angular position changes in discrete steps since it acts on pulse.

$$n = 0, \Delta m_o \rightarrow \Delta\theta_o$$

$$n = 1, \Delta m_1 \rightarrow \Delta\theta_1 \rightarrow \text{ actual position } = \Delta\theta_o + \Delta\theta_1$$

$$n = 2, \Delta m_2 \rightarrow \Delta\theta_2 \rightarrow \text{actual position } = \Delta\theta_o + \Delta\theta_1 + \Delta\theta_2$$

Proportional-integral-derivative (PID) controller

$$m = K_P \left[e + \frac{1}{T_i} \int_o^t e dt + T_d \frac{de}{dt} \right] + M_o$$

At the nth instant, , $m'_n = K_P \left[e_n + \frac{I_n}{T_i} + T_d \left(\frac{e_n}{\tau} - \frac{e_{n-1}}{\tau} \right) \right]$

$$m'_{n-1} = K_P \left[e_{n-1} + \frac{I_{n-1}}{T_i} + T_d \left(\frac{e_{n-1}}{\tau} - \frac{e_{n-2}}{\tau} \right) \right]$$

$$m'_n - m'_{n-1} = K_P \left[\left(e_n - e_{n-1} \right) + \frac{1}{T_i} \left(I_n - I_{n-1} \right) + \frac{T_d}{\tau} \left(e_n - 2e_{n-1} + e_{n-2} \right) \right]$$

Using rectangular integration

$$I_n - I_{n-1} = \tau e_n$$

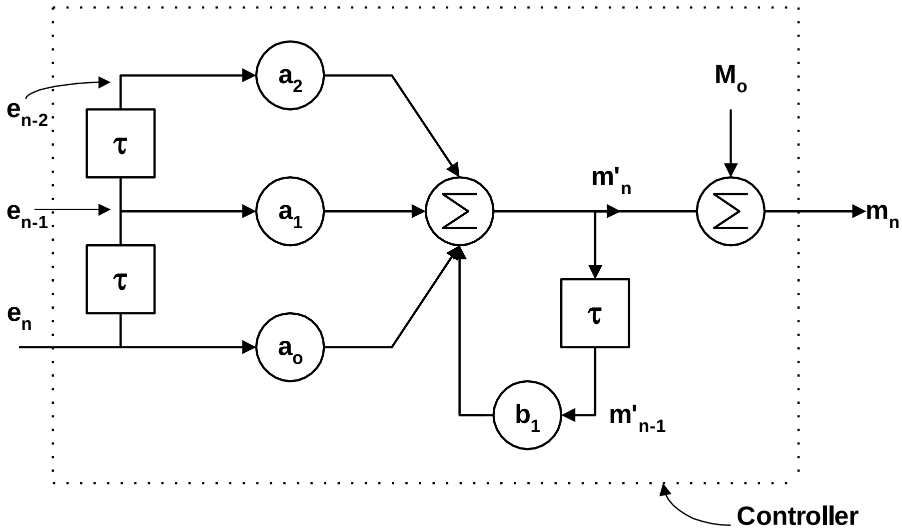
$$m'_n - m'_{n-1} + K_P \left[\left(e_n - e_{n-1} \right) + \frac{\tau}{T_i} e_n + \frac{T_d}{\tau} \left(e_n - 2e_{n-1} + e_{n-2} \right) \right]$$

or, $m'_n = K_P \left(1 + \frac{\tau}{T_i} + \frac{T_d}{\tau} \right) e_n - K_P \left(1 + \frac{2T_d}{\tau} \right) e_{n-1} + \frac{K_P T_d}{\tau} e_{n-2} + m'_{n-1}$

or, $m'_n = a_o e_n + a_1 e_{n-1} + a_2 e_{n-2} + b_1 m'_{n-1}$

$\therefore m_n = a_o e_n + a_1 e_{n-1} + a_2 e_{n-2} + b_1 m'_{n-1} + M_o$

Pictorial representation of the structure is



Controller Z.T.F. (ignoring bias) →

$$G_c(z) = \frac{M'(z)}{E(z)} = \frac{a_o + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1}}$$

Using trapezoidal integration

$$I_n - I_{n-1} = \frac{\tau}{2} e_n + \frac{\tau}{2} e_{n-1}$$

$$\therefore\ m'_n - m'_{n-1} = K_P \left[\left(e_n - e_{n-1}\right) + \frac{\tau}{2T_i}\left(e_n + e_{n-1}\right) + \frac{T_d}{\tau}\left(e_n - 2e_{n-1} + e_{n-2}\right) \right]$$

$$\text{or, } m'_n = K_P \left(1 + \frac{\tau}{2T_i} + \frac{T_d}{\tau}\right) e_n - K_P \left(1 - \frac{\tau}{2T_i} + \frac{2T_d}{\tau}\right) e_{n-1} + \frac{K_P T_d}{\tau} e_{n-2} + m'_{n-1}$$

$$= a_o e_n + a_1 e_{n-1} + a_2 e_{n-2} + b_1 m'_{n-1}$$

$$\text{and } m_n = m'_n + M_o \qquad \text{...(2)}$$

($\therefore$two integration methods give same structure, only coefficients are different).

This form of algorithm gives the position form of *PID control algorithm.*

Velocity form of PID control algorithm will be

$$\Delta m_n = m_n - m_{n-1} = a_o e_n + a_1 e_{n-1} + a_2 e_{n-2}$$

Observation made during PI controller can also be used in PID controller. These velocity or position form of algorithms are called ***ideal PID algorithm or non-interactive PID algorithm***

Modified form of ideal PID algorithm

The modified form can be employed for prevention of **derivative kick**. Here, set point r is considered as constant. In the ideal velocity form of algorithm e_n is replaced by $r - c_n$. Then, the modified form is obtained as:

$$\Delta m_n = K_P \left[\left(c_{n-1} - c_n \right) + \frac{\tau}{T_i} \left(r - c_n \right) + \frac{T_d}{\tau} \left(2c_{n-1} - c_{n-2} - c_n \right) \right],$$

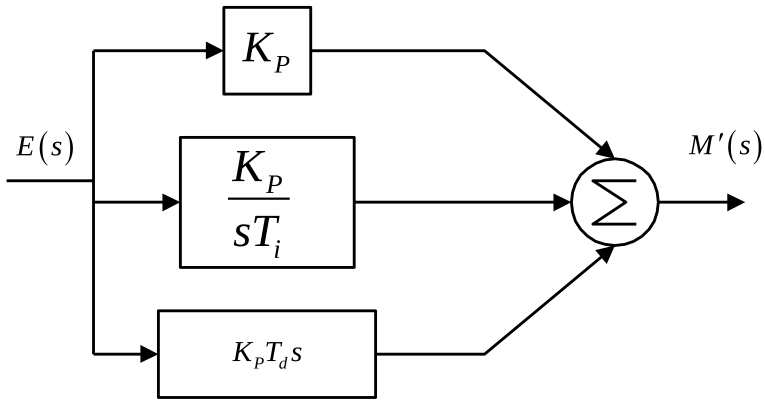
employing rectangular integration.

Employing trapezoidal integration,

$$\Delta m_n = K_P \left[\left(c_{n-1} - c_n \right) + \frac{\tau}{2T_i} \left(2r - c_n - c_{n-1} \right) + \frac{T_d}{\tau} \left(2c_{n-1} - c_{n-2} - c_n \right) \right]$$

Real PID algorithm

In continuous time form, the block diagram representation of a PID controller will be :-

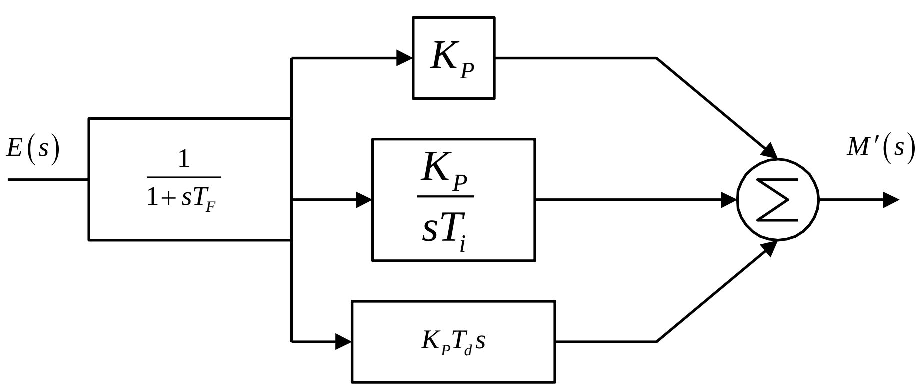


Ideal continuous – time PID controller

This is also known as non-interactive or separated-mode PID controller.

$$\frac{M'(s)}{E(s)} = K_P\left(1 + \frac{1}{sT_i} + sT_d\right)$$

For a practical controller using analog components, this structure should be preceded by a low-pass filter (in form of a first-order lag).



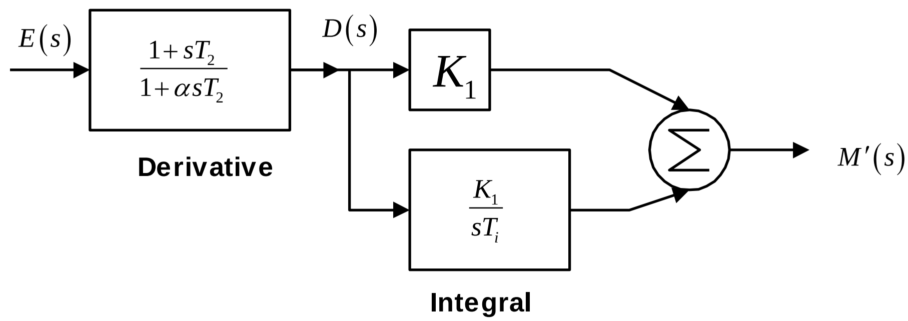
Real continuous-time PID controller

T_F = time lag of the filter.

The filter is required to cut off high frequency noises.

$$\therefore \frac{M'(s)}{E(s)} = \frac{K_P}{1+sT_F}\left(1+\frac{1}{sT_i}+sT_d\right)$$

$$\therefore \frac{M'(s)}{E(s)} = \left(\frac{1+sT_2}{1+\alpha sT_2}\right)K_1\left(1+\frac{1}{sT_1}\right)$$



Real or interactive PID controller

T_1 = Real integral time constant

T_2 = Real derivative time constant

α = Rate amplitude constant.

Let us first consider the derivative block. For the derivative block, in Laplace domain, $D(s)(1+\alpha sT_2)=(1+sT_2)E(s)$

In time domain, $d+\alpha T_2\frac{d}{dt}(d)=e+T_2\frac{de}{dt}$

In discrete form, $d_n+\alpha T_2\left(\frac{d_n-d_{n-1}}{\tau}\right)=e_n+\frac{T_2(e_n-e_{n-1})}{\tau}$

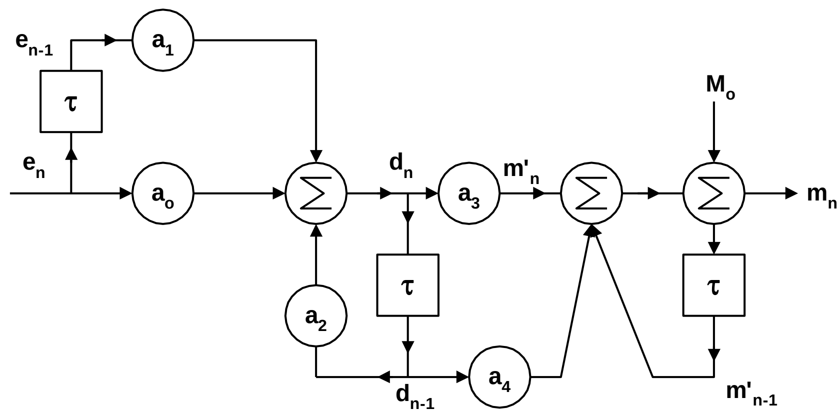
or, $d_n=\frac{\alpha T_2}{\alpha T_2+\tau}d_{n-1}+\frac{T_2+\tau}{\alpha T_2+\tau}e_n-\frac{T_2}{\alpha T_2+\tau}e_{n-1}$ ...(1)

Now, considering the PI block, using trapezoidal integration, we can finally write:-

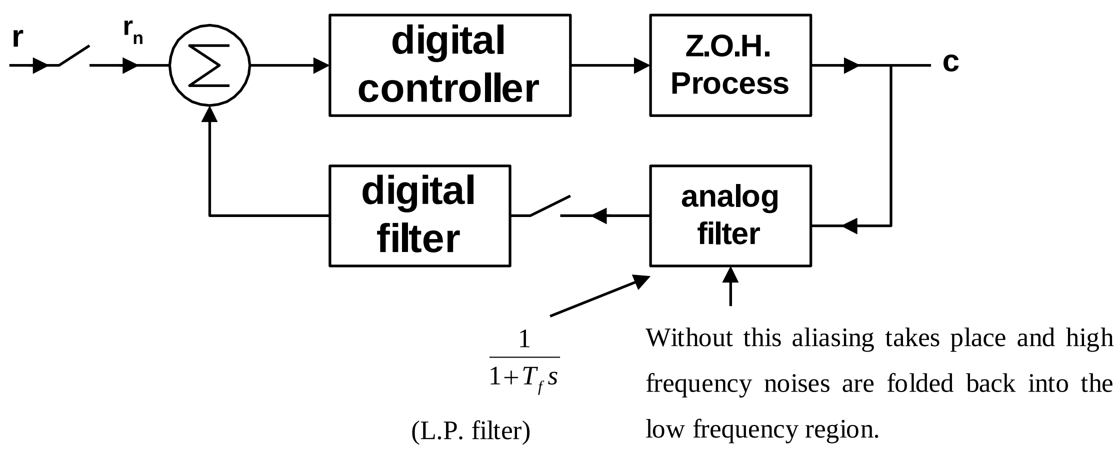
$$m_n' = K_1\left(1+\frac{\tau}{2T_1}\right)d_n + K_1\left(\frac{\tau}{2T_1}-1\right)d_{n-1} + m_{n-1}' \qquad ...(2)$$

The two coupled equations (1) and (2) describe the real PID controller algorithm.

Real PID algorithm



To reduce high frequency noise, we use a low-pass filter. This can be implemented as a first-order lag or cascade combination of several such lags.

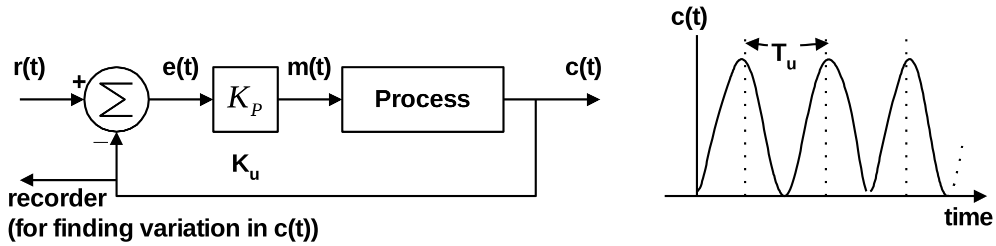


Tuning of Single term, two term and three term controllers:

Zieglar – Nichols (Z – N) tuning rules

There are 2 methods for tuning the controllers (at site).

1. Loop tuning method or ultimate-cycle method (or limit cycle method or ultimate sensitivity method).

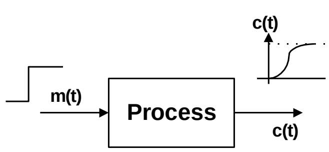


The controller is set on proportional action only. Integral and derivative actions (if any) are rendered inoperative (T_i as large as possible and T_d as small as possible). Starting with a small value of proportional gain, the gain is progressively increased in stages while creating small set point changes until a continuous oscillation of the controlled variable with fixed amplitude is produced. The proportional gain at this condition is the ultimate proportional gain K_u and corresponds to the maximum value for limiting stability. The period of continuous oscillation is also observed as the ultimate period T_u. The controller settings suggested by Zieglar and Nichols, based on these experimentally determined ultimate values, are given as:

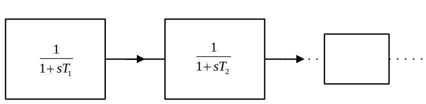
Control action	K_P	T_i	T_d
P	$0.5K_u$	--	--
PI	$0.45K_u$	$0.83T_u$	--
PID	$0.6K_u$	$0.5T_u$	$0.125T_u$

If the output does not exhibit sustained oscillations, for a values of K_P, this method can not be applied. This method suffers from several disadvantages including the possibility of damage to the plant and also the practical difficulty of determining when exactly sustained oscillation is obtained.

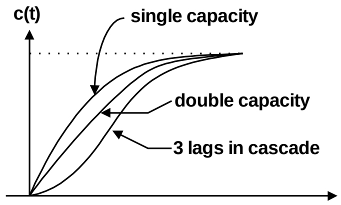
2. The second method of Z-N tuning rules is known as **Process Reaction curve method** (also known as transient response method)



Process reaction curve is the response of the process to a step input. This is an S-shaped curve, also called sigmoidal curve. This method is only applicable in cases where process reaction curves are S-shaped curves.



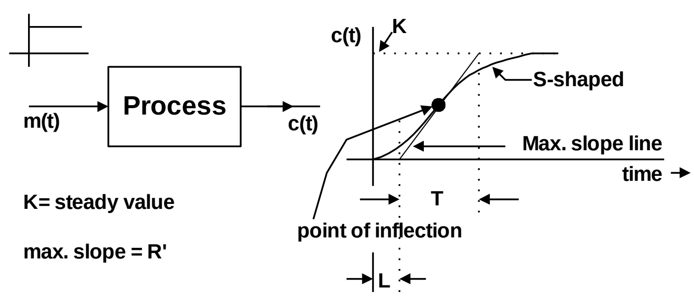
Normal industrial processes are cascade connections of 1st order lags. Then the process reaction curve can be obtained as an S-curve.

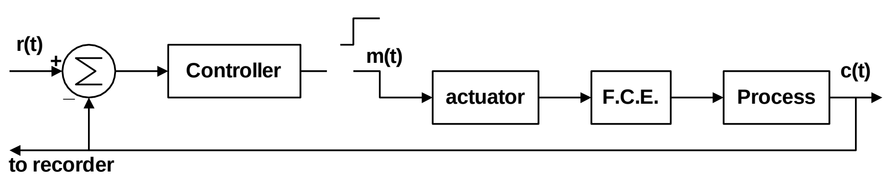


If there is only one 1st order lag, the curve is an exponential curve. This method uses the step response of the open-loop system (excluding the controller) to characterize the system dynamics.

Step response is often referred to as the reaction curve of the process.

With the controller disconnected, the input to the process is subjected to a unit step change and the monitoring system of the controller is used to record the reaction curve. If the process involves neither integrator nor complex conjugate poles, the reaction curve may look like an S-shaped curve. If the response does not exhibit an S-shaped feature, this method can not be applied.





L = apparent dead time or apparent delay.

We can approximate the process by the first order system T.F.

$$\rightarrow \frac{Ke^{-Ls}}{1+sT}$$

Then the controller settings suggested by Z-N rules are:

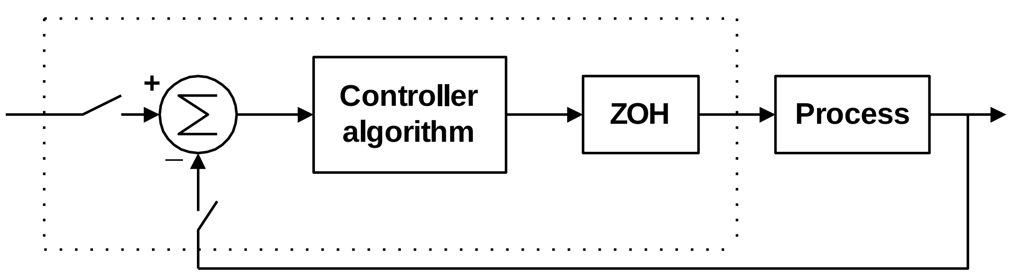
Control action	K_P	T_i	T_d
P	$\frac{1}{R'L}$	--	--
PI	$\frac{0.9}{R'L}$	3.32L	--
PID	$\frac{1.2}{R'L}$	2L	$\frac{L}{2}$

Application of Z-N tuning rules to digital controllers.

Choice of sampling period

For data obtained from process reaction curve :- $\tau \approx 0.05L - 0.25L$

For data obtained from limit cycling :- $\tau \approx 0.01T_u - 0.05T_u$



Application of process reaction curve method to digital controller with ideal algorithm

To take into account the time delay of approximately $\tau/2$ introduced by Z.O.H., the modified tuning rules for the digital controller are obtained by replacing L by $L+\frac{\tau}{2}$ in the standard Z-N rules.

Control action	K_P	T_i	T_d
P	$\frac{1}{R'\left(L+\frac{\tau}{2}\right)}$	--	--
PI	$\frac{0.9}{R'\left(L+\frac{\tau}{2}\right)}$	$3.32\left(L+\frac{\tau}{2}\right)$	--
PID	$\frac{1.2}{R'\left(L+\frac{\tau}{2}\right)}$	$2\left(L+\frac{\tau}{2}\right)$	$\frac{L+\frac{\tau}{2}}{2}$

For modified ideal algorithm (used for preventing set-point kick), with rectangular integration:-

$$\Delta m_n = K_P\left[\left(c_{n-1}-c_n\right)+\frac{\tau}{T_i}\left(r-c_n\right)+\frac{T_d}{\tau}\left(2c_{n-1}-c_{n-2}-c_n\right)\right].$$

Takahashi carried out a number of experiments with different sampling period and proposed the following rules (for preventing set-point kick):

Loop tuning method

$$K_P = 0.6K_u - 0.6K_u \frac{\tau}{T_u}$$

$$T_i = \frac{K_P T_u}{1.2K_u}; T_d = \frac{3K_u T_u}{40K_P}$$

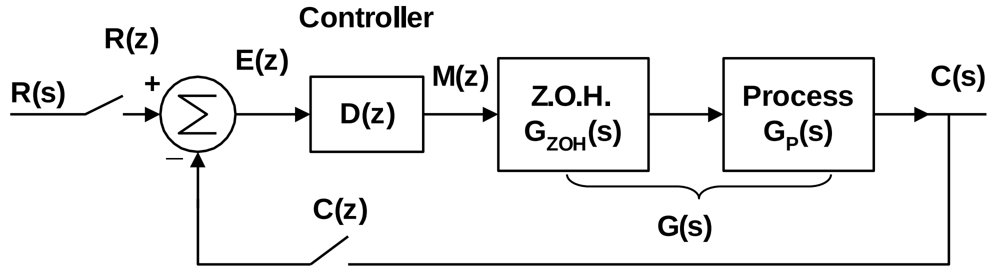
Process reaction curve method

$$K_P = \frac{1.2}{R'(L+\tau)} - \frac{0.3\tau}{R'\left(L+\frac{\tau}{2}\right)^2}$$

$$T_i = 1.67K_P\left(L+\frac{\tau}{2}\right)^2 R'; T_d = \frac{1}{2R'K_P}$$

For $\tau = 0$, the settings converge to those prescribed by standard Z-N rules.

Designing digital controllers by direct synthesis method



$$G_{ZOH}(s)=\frac{1-e^{-s\tau_s}}{s}$$

Designing a digital controller by direct synthesis method consists of determing the controller T.F., D(z), that is required to produce a specified closed loop response. The overall CLTF H(z) is so chosen that the system has a desirable transient response for a specific input with a time constant which is appropriate to the response time of the plant.

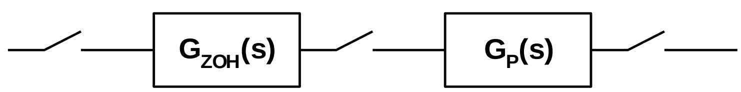
The direct synthesis approach assumes that the process can be represented by low order models.

$$G(z)=Z\left[G(s)\right]=Z\left[\frac{1-e^{-s\tau_s}}{s}G(s)\right] \qquad \left(\begin{array}{l} Z(G(s))\neq G_{ZOH}(z)G_P(z) \\ \because z-\text{T.F. are properties of} \\ \text{discrete time systems}\end{array}\right)$$

$$\Downarrow$$

$$G_{ZOH}(s)G_P(s)$$

For the case shown below:-



Then $Z\left[G_{ZOH}(s)G_p(s)\right]=G_{ZOH}(z)G_p(z)$

The overall C.L.T.F. $= H(z) = z\left[\frac{C(s)}{R(s)}\right] = \frac{D(z)G(z)}{1+D(z)G(z)}$

Hence D(z) can be found as : $D(z) = \frac{1}{G(z)} \cdot \frac{H(z)}{\left[1-H(z)\right]}$

The equation can be readily implemented but care must be taken in choosing H(z). The design calls for a D(z) which will cancel the process effects and add whatever is necessary to produce the desired H(z).

First order lag plus dead-time:-

$G_P(s) = \frac{Ke^{-Ls}}{1+s\tau}$ ≡> L = apparent dead time, τ = time constant, K= steady state gain

Cascaded lag (or 2nd order lag) plus dead time:-

$$G_P(s) = \frac{Ke^{-Ls}}{(1+s\tau_1)(1+s\tau_2)}$$

T.F. G(z) for different process models:

For first order lag plus dead-time:-

$$G(z) = Z\left[G_{ZOH}(s)G_P(s)\right] = Z\left[\frac{1-e^{-sT_s}}{s} \cdot \frac{Ke^{-Ls}}{1+s\tau}\right] \quad (T_s = \text{ sampling period})$$

$$= KZ\left[\frac{\left(1-e^{-sT_s}\right)e^{-Ls}}{s(1+s\tau)}\right] = KZ\left[\left(1-e^{-sT_s}\right)e^{-Ls}\left(\frac{1}{s} - \frac{\tau}{1+s\tau}\right)\right]$$

$$= KZ\left[\left(\frac{e^{-Ls}}{s} - \frac{e^{-Ls}}{s+\frac{1}{\tau}}\right) - \left\{\frac{e^{-s(L+T_s)}}{s} - \frac{e^{-s(L+T_s)}}{s+\frac{1}{\tau}}\right\}\right]$$

Let dead time L be an integral multiple of sampling period T_s, i.e., say, $L = NT_s$, where N is a positive integer. If L is not an integral multiple of T_s, then N is the nearest integer number of sampling periods in L.

$$G(z) = K\left[\frac{z^{-N}}{1-z^{-1}} - \frac{z^{-N}}{\left(1-e^{-T_s/\tau}z^{-1}\right)} - \left\{\frac{z^{-(N+1)}}{1-z^{-1}} - \frac{z^{-(N+1)}}{1-e^{-T_s/\tau}z^{-1}}\right\}\right]$$

$$= K\left\{z^{-N} - z^{-(N+1)}\right\}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T_s/\tau}z^{-1}}\right)$$

$$\vdots$$

$$= \frac{K\left(1-e^{-T_s/\tau}\right)z^{-(N+1)}}{\left(1-e^{-T_s/\tau}z^{-1}\right)}$$

Process described by second order lag plus dead time:-

$$G(z) = Z\left[\frac{1-e^{-sT_s}}{s}\frac{Ke^{-Ls}}{(1+s\tau_1)(1+s\tau_2)}\right] = KZ\left[\frac{\left(1-e^{-sT_s}\right)}{s}\frac{e^{-sNT_s}}{(1+s\tau_1)(1+s\tau_2)}\right] \quad \left(\text{Substituting } L \approx NT_s\right)$$

$$\vdots$$

$$= \frac{K\left(b_1 + b_2 z^{-1}\right)z^{-(N+1)}}{\left(1-e^{-sT_s/\tau_1}z^{-1}\right)\left(1-e^{-T_s/\tau_2}z^{-1}\right)}$$

where, $b_1 = \dfrac{\tau_1 e^{-T_s/\tau_1} - \tau_2 e^{-T_s/\tau_2}}{\tau_2 - \tau_1} + 1$

and $b_2 = e^{-T_s\left(\frac{1}{\tau_2}+\frac{1}{\tau_1}\right)} + \dfrac{\tau_1 e^{\frac{-T_s}{\tau_2}}\, \tau_2 e^{\frac{-T_s}{\tau_1}}}{\tau_2 - \tau_1}$

Constraint of Causality

For the digital controller to be physically realizable, causality should be ensured. That is, the controller output m_n at any instant n should depend only on the present and the past values of the input error sequence i.e., e_n, e_{n-1}, e_{n-2} etc. and not on future values of input i.e., e_{n+1}, e_{n+2} etc.

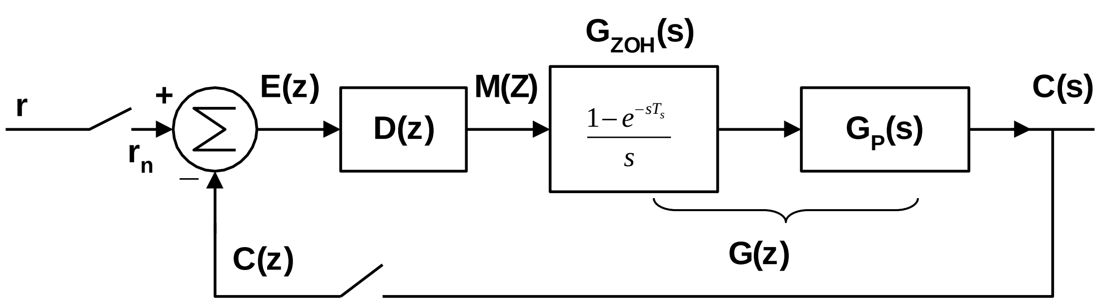
If we get the Z.T.F of a controller of the form

$$D(z)=\frac{M(z)}{E(z)}=\frac{b_o+b_1z+b_2z^2+....+b_kz^k}{a_o+a_1z+a_2z^2+....+a_jz^j} \qquad (\text{j and k are positive integers})$$

For causality, the necessary condition is $j \geq k$.

This means that, for a process which has a dead time, represented by z^{-N}, the desired C.L.T.F., H(z), must also include the same deal time.

Dahlin's algorithm



The desired C.L.T.F. is considered to be that of a first order lag with dead time.

$$H(s)=\frac{\lambda e^{-Ls}}{s+\lambda}=\frac{e^{-Ls}}{1+s/\lambda} \qquad \begin{pmatrix}\text{dead time} = L \\ \text{time const} = \frac{1}{\lambda}\end{pmatrix}$$

In the z-domain, the C.L.T.F. (with Z.O.H. included to ensure physical realizability of the controller) is given by

$$H(z) = \frac{\left(1 - e^{-\lambda T_s}\right) z^{-N-1}}{1 - e^{-\lambda T_s} z^{-1}}$$

Here λ is used as a tuning parameter. By adjusting λ, we can tune the controller. $\lambda = \frac{1}{\tau}$. For large λ, response of C.L. system to a particular set point change, will be faster. For small λ, we will have a sluggish response for the overall system. Hence, for large λ, we have tight control.

The closed loop time constant $\left(1/\lambda\right)$ is normally chosen as 2 to 3 times as fast as the open loop value (i.e., the process time constant)

$\therefore$ z-transfer function of the controller is :

$$D(z) = \frac{1}{G(z)} \frac{H(z)}{1 - H(z)} \left[G(z) = Z\left[G(s)\right] \right]$$

$$\therefore D(z) = \frac{1}{G(z)} \frac{\left(1 - e^{-\lambda T_s}\right) z^{-N-1}}{\left[1 - e^{-\lambda T_s} z^{-1} - \left(1 - e^{-\lambda T_s}\right) z^{-N-1}\right]} = \frac{M(z)}{E(z)}$$

(putting L = NT_s)

[Note: $G(z) = \frac{B(z)}{A(z)} . z^{-N} \Rightarrow$ representing dead time. To nullify it we have included dead time in H(z) in terms of z^{-N-1} Z.O.H also introduces a dead time. This will go up in the numerator and have positive powers of z. It must be compensated for physical realizability. Now, with the process dead time will be = the L in

$H(z)=\lambda\frac{e^{-Ls}}{\lambda+s}$. This L≥ dead time of the process, otherwise physical realizability is lost.]

Problem:

A certain thermal process can be approximated by a first order lag + dead time model. The steady state gain is 0.49. The time constant is 414 sec. and the dead time is 16 sec. Design a digital controller for this process following Dahlin's algorithm. Assume the sampling period of 8 secs.

$$G_P(s)=\frac{Ke^{-Ls}}{1+s\tau}; K=0.49, \tau=414\,\text{sec. and } L=16\,\text{sec.}$$

$$G(s)=G_{ZOH}(s)G_P(s)=\frac{1-e^{-ST_s}}{s}\frac{Ke^{-Ls}}{1+s\tau}; T_s=8\,\text{sec.}$$

$$G(z)=Z\left[G_{ZOH}(s)G_P(s)\right]=\frac{K\left(1-e^{-T_s/\tau}\right)z^{-N-1}}{\left(1-e^{-T_s/\tau}z^{-1}\right)}$$

$$L=NT_s \text{ or } N=\frac{L}{T_s}=\frac{16}{8}=2$$

$$\therefore G(z)=\frac{0.49\left(1-e^{-0.019}\right)z^{-3}}{\left(1-e^{-0.019}z^{-1}\right)}=\frac{0.009z^{-3}}{1-0.98z^{-1}}$$

$$\therefore \text{ C.L.T.F. }\left(\text{with Z.O.H}\right)=H(z)=\frac{\left(1-e^{-\lambda T_s}\right)z^{-N-1}}{\left(1-e^{-\lambda T_s}z^{-1}\right)}$$

$$\frac{1}{\lambda}=\frac{\tau}{2.5}\left(2.5\text{ assumed; should be between 2 and 3}\right)$$

$$=\frac{414}{2.5}=165.6\,\text{sec.};\lambda=0.006/\text{sec}.$$

$$\therefore H(z)=\frac{\left(1-e^{-0.048}\right)z^{-3}}{\left(1-e^{-0.048}z^{-1}\right)}=\frac{0.047z^{-3}}{1-0.9532z^{-1}}$$

$$D(z)=\frac{1}{G(Z)}\frac{H(Z)}{\left[1-H(Z)\right]}=\frac{1-0.98z^{-1}}{0.009z^{-3}}.\frac{0.047z^{-3}}{1-.0953z^{-1}-0.047z^{-3}}$$

$$=\frac{M(z)}{E(z)}=\frac{5.22\left(1-0.98z^{-1}\right)}{1-0.953z^{-1}-0.047z^{-3}};$$

$$\therefore M(z)=0.953z^{-1}M(z)+0.047z^{-3}M(z)+5.22\left[E(z)-0.98z^{-1}E(z)\right]$$

Controller output at the nth instant

$$: m_n'=0.953m_{n-1}+0.047m_{n-3}+5.22\left(e_n-0.98e_{n-1}\right)$$

$$\#1. K_u=1.2,\frac{1}{T_u}=\frac{0.2}{60}$$

PI controller: $m_n'=K_P\left(1+\frac{T_s}{2T_i}\right)e_n+K_P\left(\frac{T_s}{2T_i}-1\right)e_{n-1}+m_{n-1}'$

or $m_n'=0.54\left(1+\frac{6}{2\times 249}\right)e_n+0.54\left(\frac{6}{2\times 249}-1\right)e_{n-1}+m_{n-1}'$

$$\left[T_s=0.02T_u=0.02\times\frac{60}{0.2}=6\,\text{sec.};T_i=0.83T_u=0.83\times\frac{60}{0.2}=249\,\text{sec.},K_P=0.45\times1.2=.054\right]$$

$$\therefore m_n' = 0.546e_n - 0.5335e_{n-1} + m_{n-1}'$$
$$\therefore M(z) = 0.546E(z) - 0.5335z^{-1}E(z) + z^{-1}M(z)$$
$$\therefore D(z) = \frac{M(z)}{E(z)} = \frac{0.546 - 0.5335z^{-1}}{1 - z^{-1}}$$

PID controller

$$0.6K_u = K_P = 0.72; T_i = 0.5T_u = 150; T_d = 0.125T_u = 0.125 \times 300 = 37.5$$
$$m_n' = K_P\left(1 + \frac{T_s}{2T_i} + \frac{T_d}{T_s}\right)e_n - K_P\left(1 - \frac{T_s}{2T_i} + \frac{2T_d}{T_s}\right)e_{n-1} + \frac{K_P T_d}{T_s}e_{n-2} + m_{n-1}'$$
$$a_o = 0.72\left(1 + \frac{6}{2 \times 150} + \frac{37.5}{6}\right) = 5.2344$$
$$a_1 = -0.72\left(1 - \frac{6}{2 \times 150} + \frac{2 \times 37.5}{6}\right) = -9.7056$$
$$a_2 = \frac{0.72 \times 37.5}{6} = 4.5$$

Kalman

$$G(z)=\frac{K\left(P_1+P_2z^{-1}\right)z^{-(N+1)}}{\left(1-e^{-\frac{T_s}{\tau_1}}z^{-1}\right)\left(1-e^{-\frac{T_s}{\tau_2}}z^{-1}\right)}$$

$$P_1=\frac{1+\tau_1e^{-\frac{T_s}{\tau_1}}-\tau_2e^{-\frac{T_s}{\tau_2}}}{\tau_2-\tau_1}$$

$$P_2=e^{-T_s\left(\frac{1}{\tau_2}+\frac{1}{\tau_1}\right)}+\frac{\tau_1e^{-\frac{T_s}{\tau_2}}-\tau_2e^{-\frac{T_s}{\tau_1}}}{\tau_2-\tau_1}$$

$$\tau_1=0.132,\tau_2=0.377,K=1.24,L=NT_s\quad\therefore N=\frac{0.3}{0.3}=1$$

$$P_1=\frac{1+0.132e^{-\frac{0.3}{0.132}}-0.377e^{-\frac{0.3}{0.377}}}{0.377-0.132}=3.4428;$$

$$P_2=e^{-0.3\left(\frac{1}{.377}+\frac{1}{.132}\right)}+\frac{.132e^{-\frac{.3}{.377}}-.377e^{-\frac{0.3}{.132}}}{.377-.132}$$

$$=0.13107$$

$$H(z)=\frac{P_1}{P_1+P_2}z^{-(N+1)}+\frac{P_2}{P_1+P_2}z^{-(N+1)}=0.9633z^{-2}+0.0366z^{-3}$$

$$G(z)=\frac{1.24\left(3.4428+0.13107z^{-1}\right)z^{-2}}{\left(1-e^{-\frac{0.3}{.132}}z^{-1}\right)\left(1-e^{-\frac{0.3}{.377}}z^{-1}\right)}=\frac{4.269z^{-2}+0.1625z^{-3}}{\left(1-0.1032z^{-1}\right)\left(1-0.4512z^{-1}\right)}$$

$$\therefore D(z)=\frac{1}{G(z)}.\frac{H(z)}{1-H(z)}=\frac{\left(1-0.103z^{-1}\right)\left(1-0.4512z^{-1}\right)\left(0.9633z^{-2}+0.0366z^{-3}\right)}{\left(4.269z^{-2}+0.1625z^{-3}\right)\left(1-0.9633z^{-2}-0.03366z^{-3}\right)}$$

$$=\frac{z^2\left(1-0.103z^{-1}\right)\left(1-.4512z^{-1}\right)\left(0.9633z^{-2}+0.0366z^{-3}\right)}{\left(4.269+0.1625z^{-1}\right)\left(1-0.9633z^{-2}-0.03366z^{-3}\right)}$$

$$=\underset{z\to1}{Lt}\frac{z^2\left(1-0.103z^{-1}\right)\left(1-.4512z^{-1}\right)\left(0.9633z^{-2}+0.0366z^{-3}\right)}{4.269\left(1+0.038Z^{-1}\right)\left(1-0.9633Z^{-2}-0.03366Z^{-3}\right)}$$

$$=\frac{\left(1-0.103z^{-1}\right)\left(1-.4512z^{-1}\right)\left(0.9633+0.0366z^{-1}\right)0.225}{\left(1+0.038z^{-1}\right)\left(1-0.9633z^{-2}-0.03366z^{-3}\right)}$$