Time -Frequency Domain Analysis

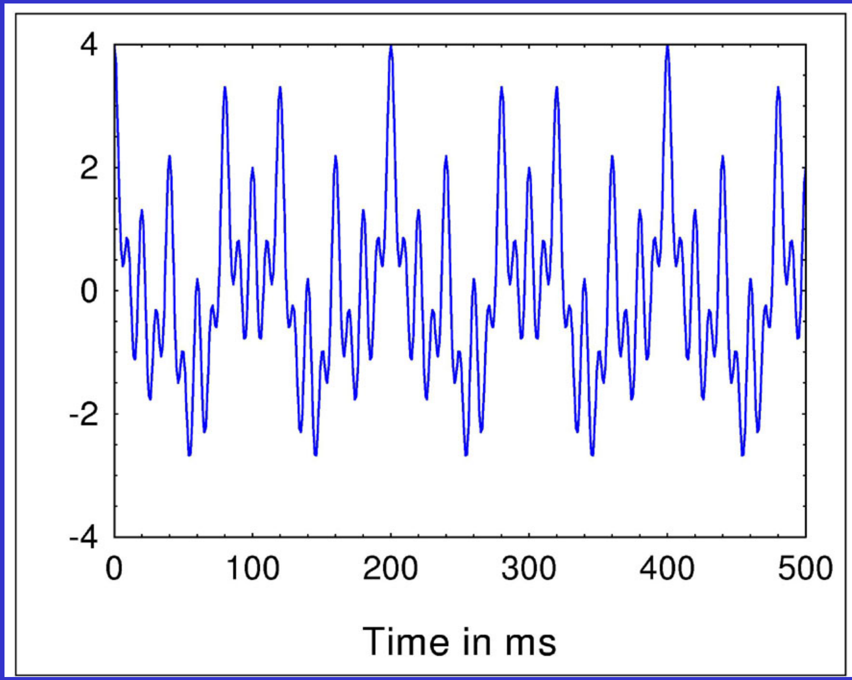
Topics of Discussion

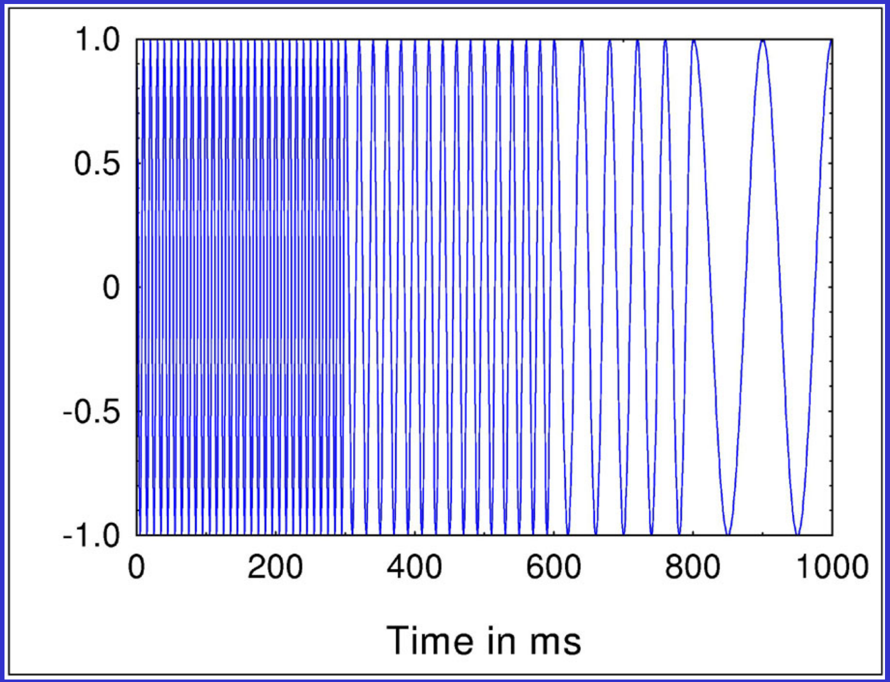
- Transform – What and why ?
- Stationary and Non-Stationary Signals
- Fourier Transform
- Short Term Fourier Transform
- Wavelet Transform
- Some Applications

Why Transform ?

- ☆ Raw Signal
- ☆ Time domain: Time-Amplitude Representation
- ☆ Frequency Domain: Frequency-Amplitude Spectrum
- ☆ Transforms are generally reversible
- ☆ Different domains are usually specific
- ☆ For special applications, two domain information is sought at the same time
- ☆ There lies the importance of Wavelet Transform

Classification of signals on the basis of frequency components





1. Stationary Signal:

x(t) = cos(2*pi*10*t) +
cos(2*pi*25*t) +
cos(2*pi*50*t) +
cos(2*pi*100*t)

2. Non-Stationary Signal:

0 to 300 ms → 100 Hz sinusoid,
300 to 600 ms → 50 Hz sinusoid,
600 to 800 ms → 25 Hz sinusoid &
800 to 1000 ms → 10 Hz sinusoid.

FOURIER TRANSFORM

***We are going to discuss FT mainly for three reasons*:**

1. It is a necessary background to understand how WT works.

2. It has been by far the most important signal processing tool for many years.

3.This will help us to get the drawbacks of FT & hence realize the importance of STFT and WT.

The Frequency Domain representation:

Amplitudes of the various frequency components of the signal.

Continuous Fourier Transform - the Basic Formulae:

Fourier transform of x(t) → $X(f) = \int_{-\infty}^{\infty} x(t) \bullet e^{-2j\pi ft} dt ....(1)$

Inverse Fourier transform of X(f) → $x(t) = \int_{-\infty}^{\infty} X(f) \bullet e^{2j\pi ft} df ....(2)$

Equation 1:

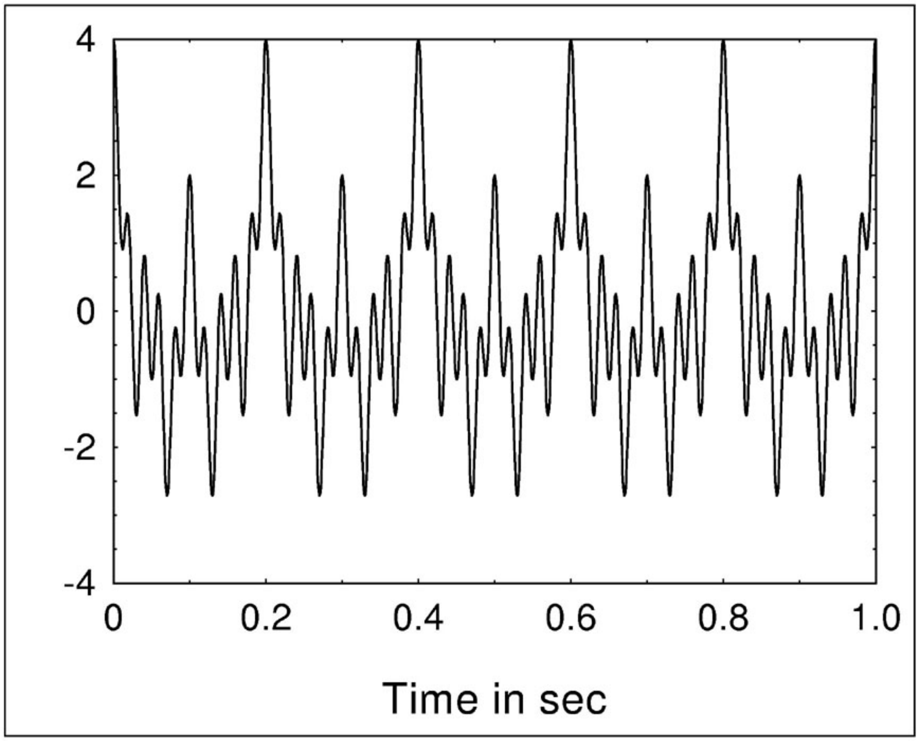
- **x(t) is multiplied with an exponential term, at certain frequency 'f', then integrated over the entire time domain.**
- **Integration result large →dominant spectral component at frequency 'f'.**

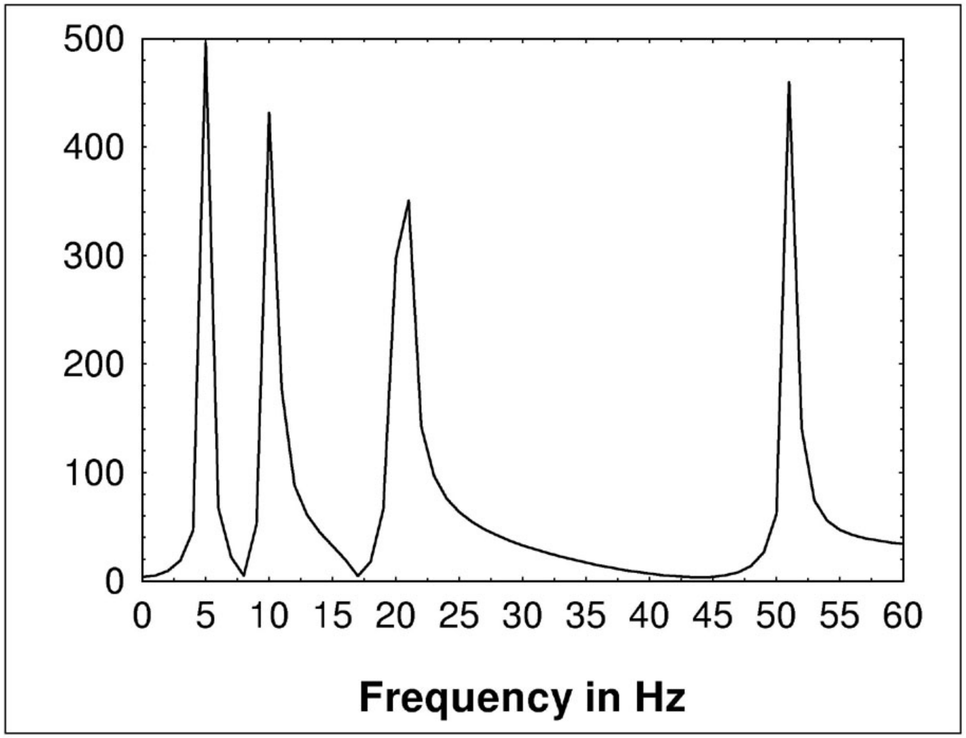
Example:

x(t)=cos(2*pi*5*t)+cos(2*pi*10*t)+cos(2*pi*20*t)+cos(2*pi*50*t)

x(t)

X(f)





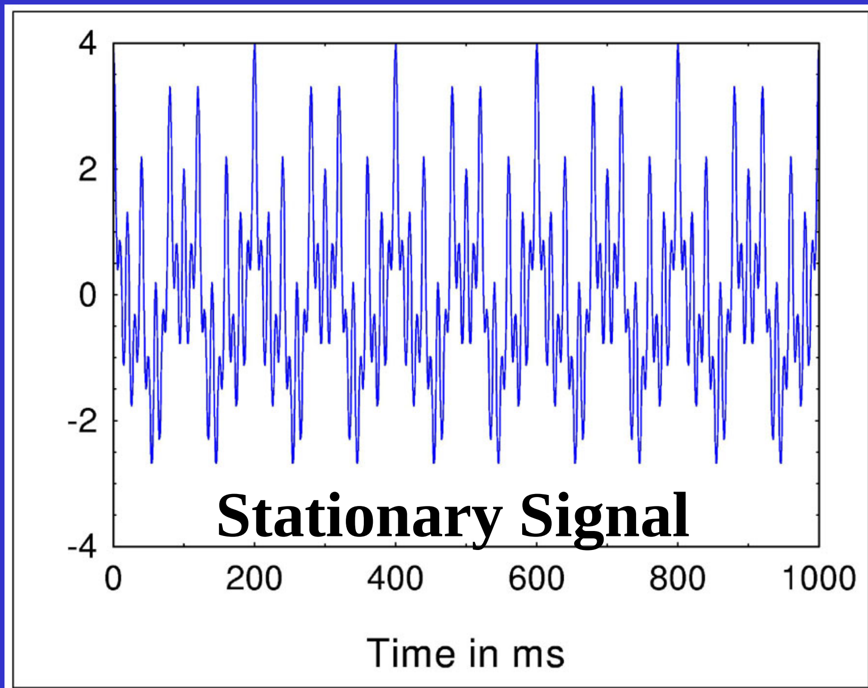
Drawbacks of FT:

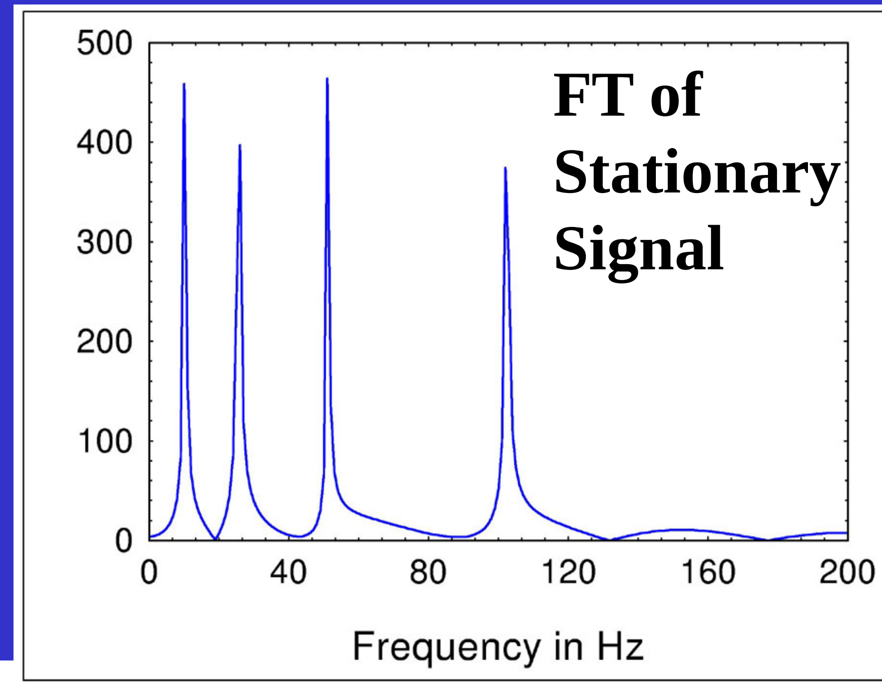
Frequency component appears irrespective of the time instant of its occurrence in the original signal.

In other words, FT tells only whether the frequency component exists or not.

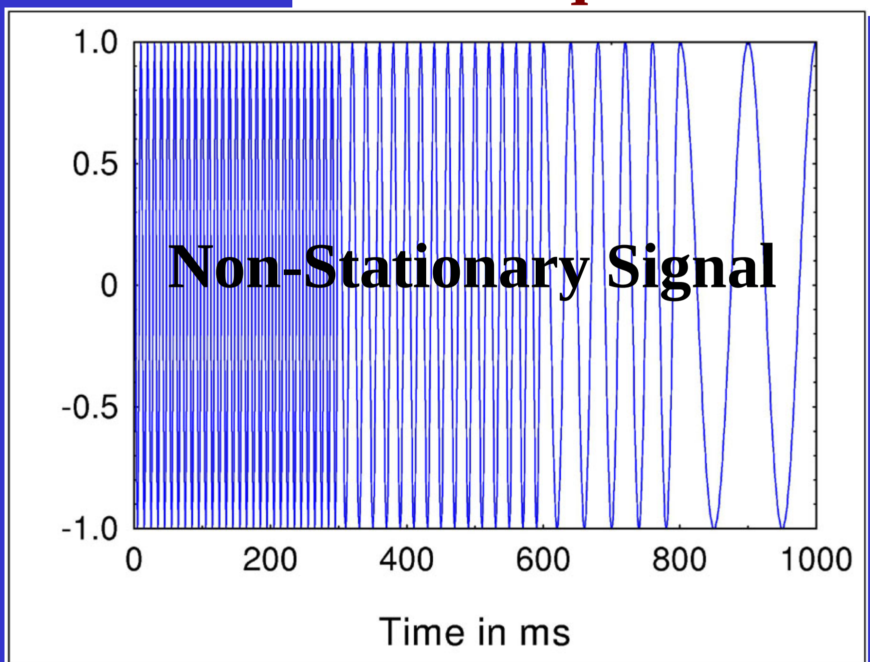
FT gives no information on the time instant at which a particular frequency exists.

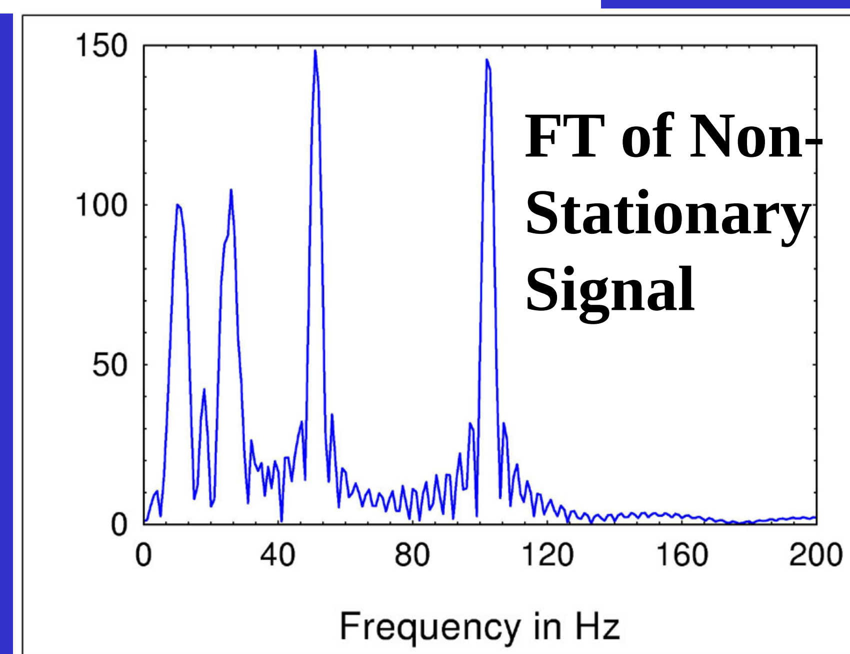
Hence, FT is unsuitable for signals with time varying frequencies, i.e. non-stationary signals.





Same frequencies at different time instants

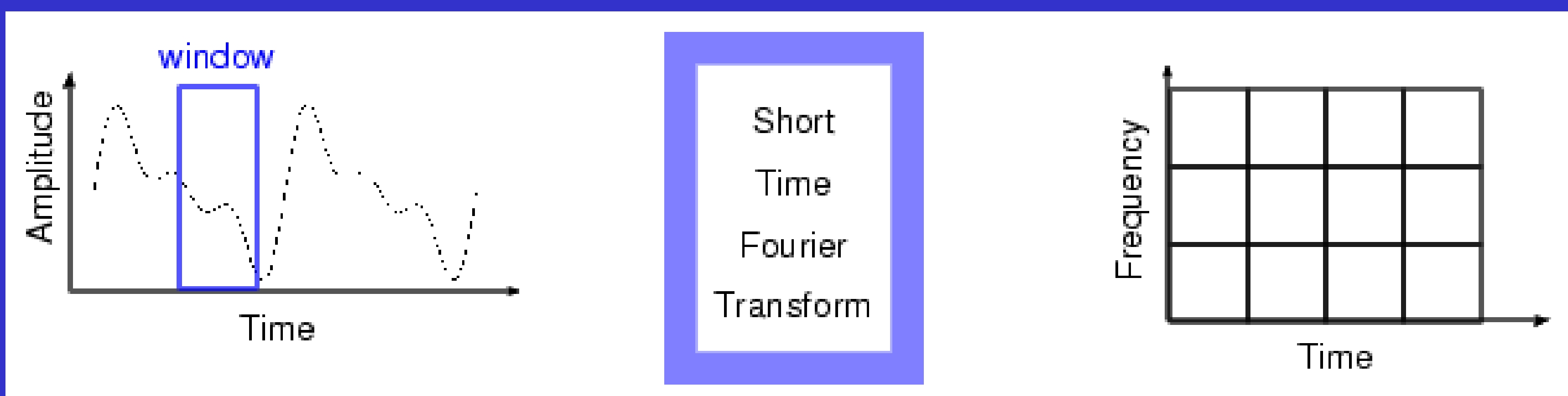




Short Time Fourier Transform

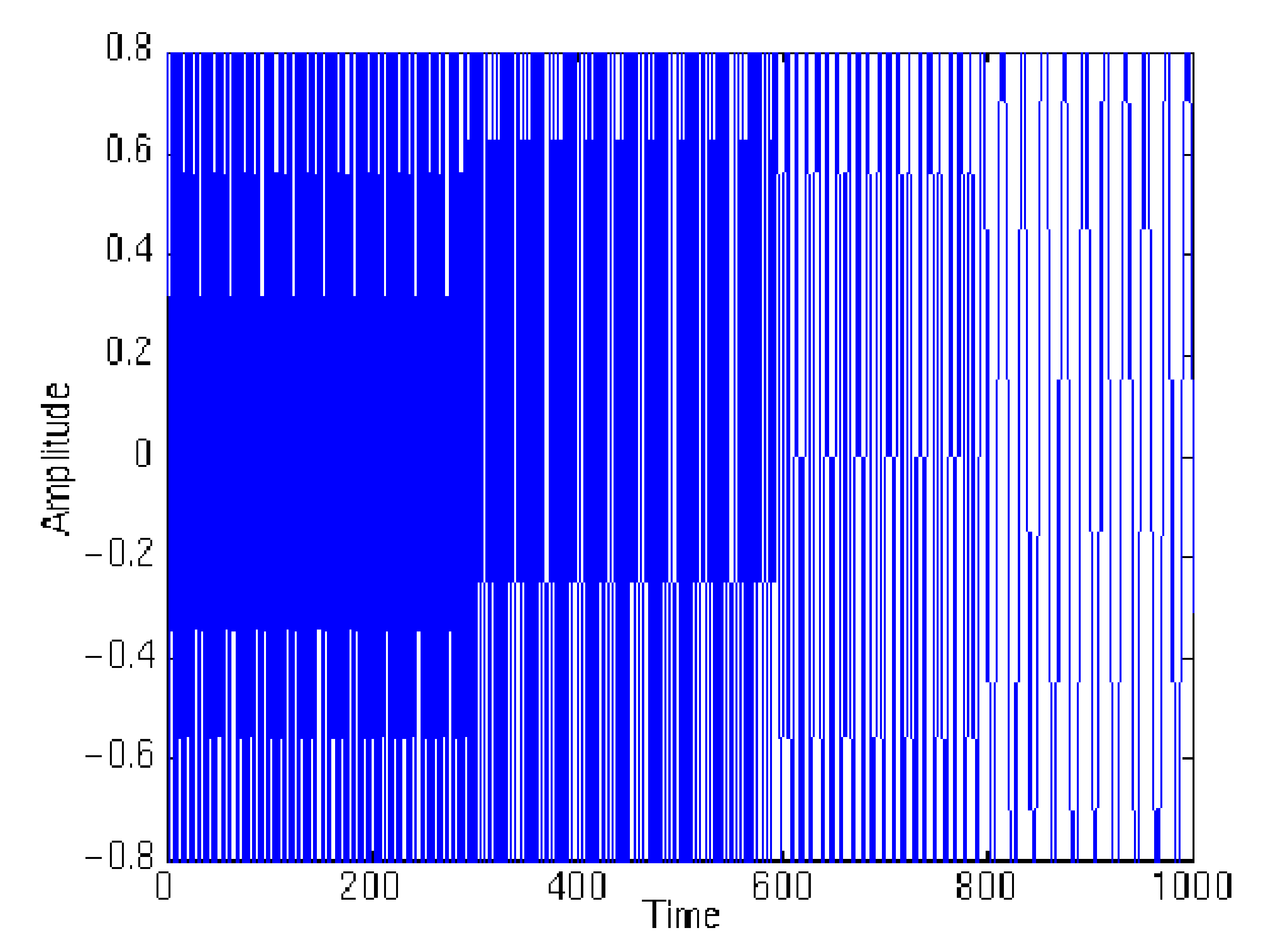
- A bridge between FT and WT
- Attempts to overcome the drawbacks of FT
- Assumption → signals is segment-wise stationary
- Window function (*w*) chosen
- The basic formula for the transform is

$$STFT_X^{(\omega)}(t,f) = \int [x(t) \bullet \omega^*(t - t')] \bullet e^{-j2\pi ft} dt$$



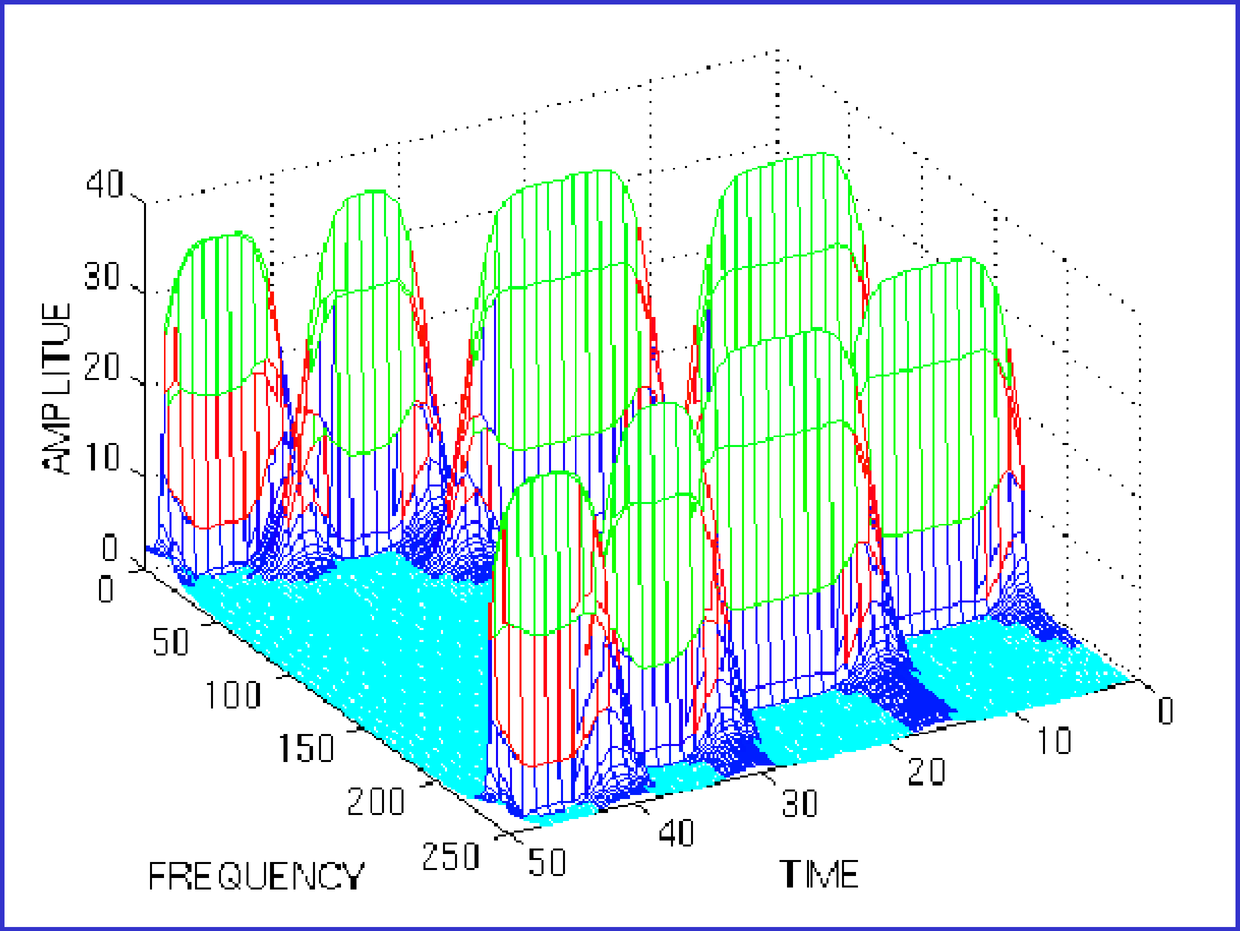
An example of STFT

A non – stationary signal



A signal with multiple frequency components at different time intervals.

STFT of the signal



- ***Window function:*** **w(t)=exp(-a*(t^2)/2) [a=0.001]**
- ***STFT is a 3-D plot (as expected). Multiple peaks at different time intervals (unlike FT) representing the original signal with multiple frequencies.***
- ***Thus time-frequency representation is obtained.***

Has the problem been solved ?

? As STFT gives the time-freq model , why go for WT

☹ Only band of frequencies available at certain time intervals. Consequently, resolution problem arises.

☹ Problem with the size of window function

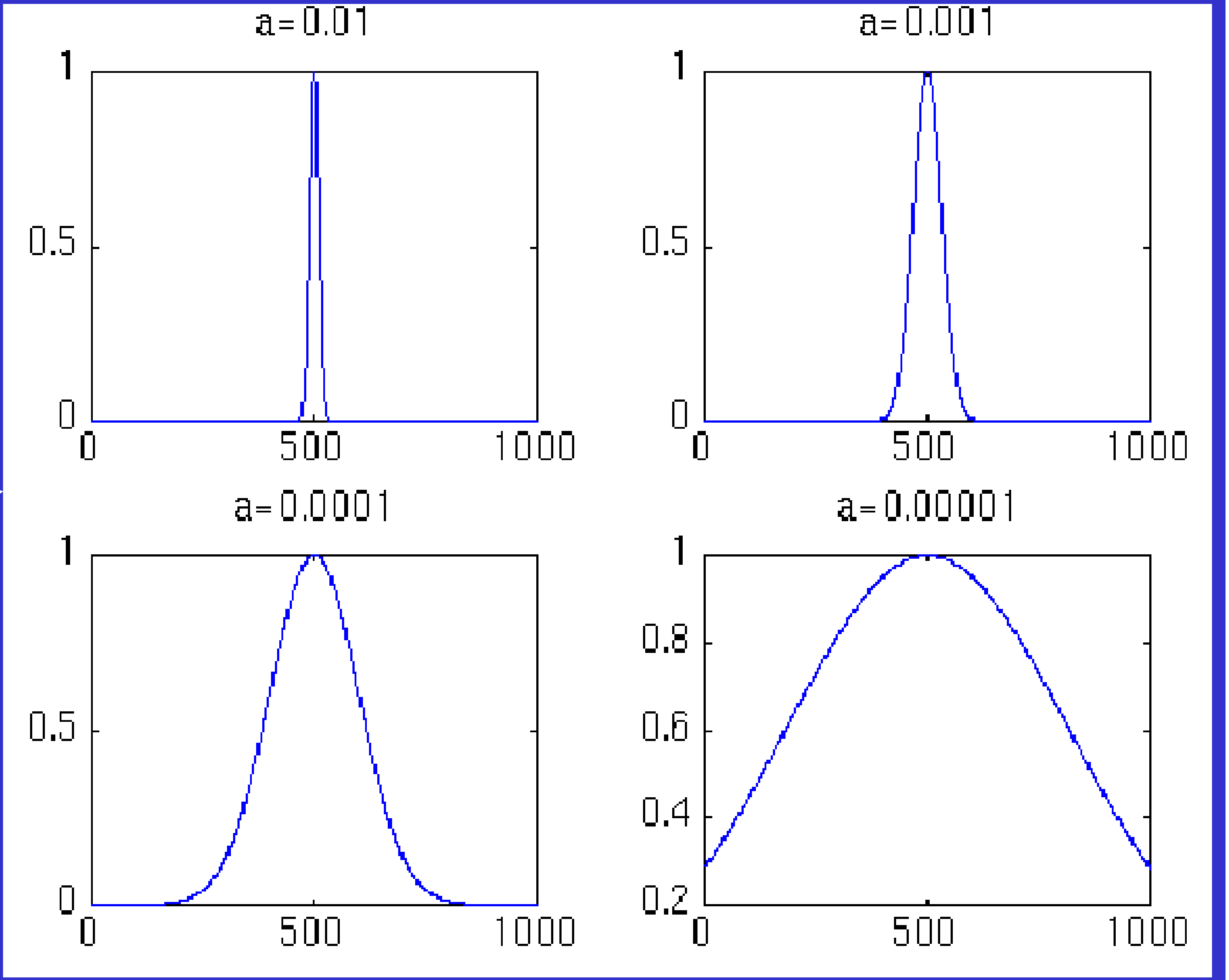
Wide window →good frequency resolution, poor time resolution.

Narrow window →good time resolution, poor frequency resolution.(for better stationary assumption)

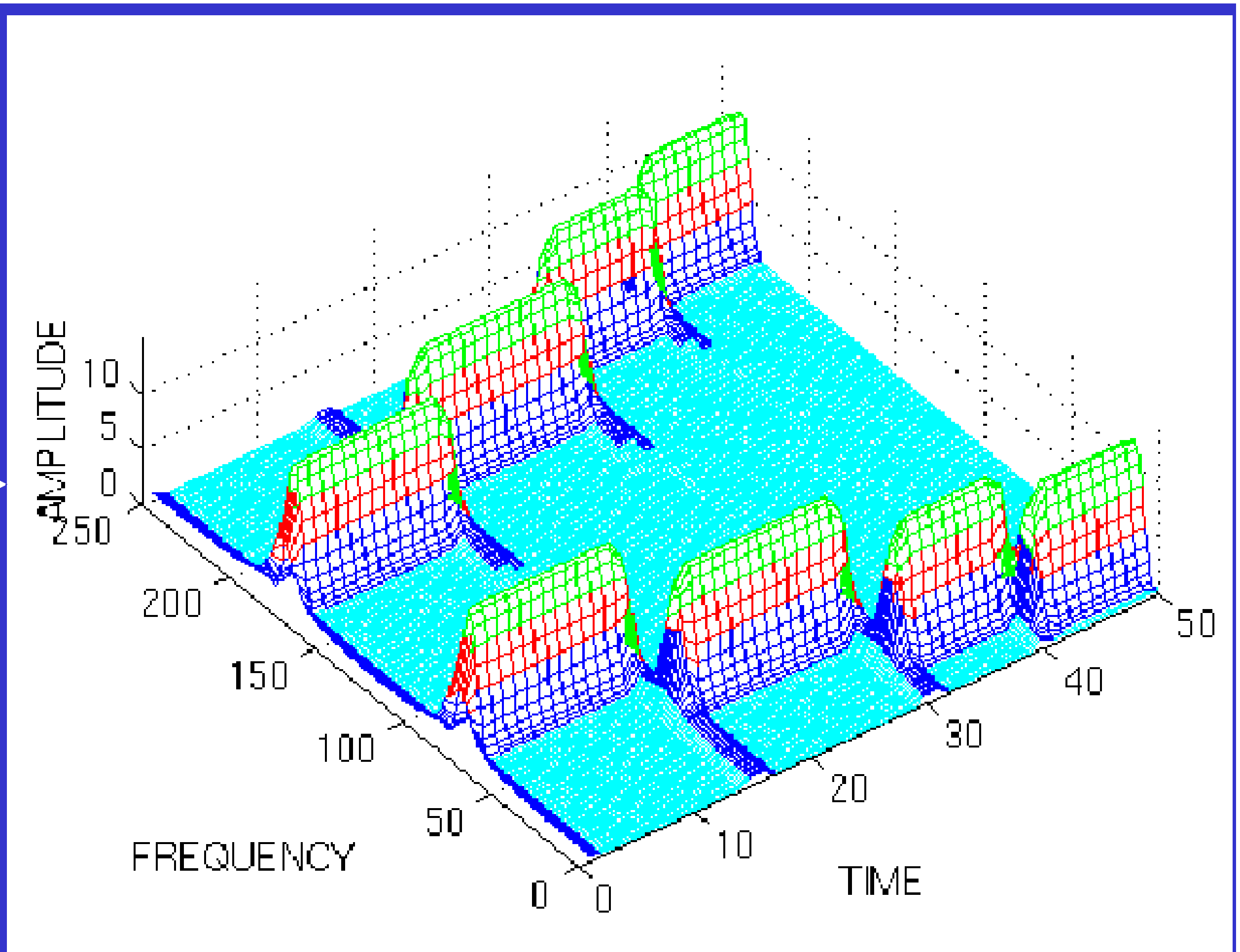
✓ To illustrate this let us take a window function → w(t)=exp(-a*(t^2)/2) and show STFT results on the previous signal.

Window function = exp(-a*(t^2)/2)

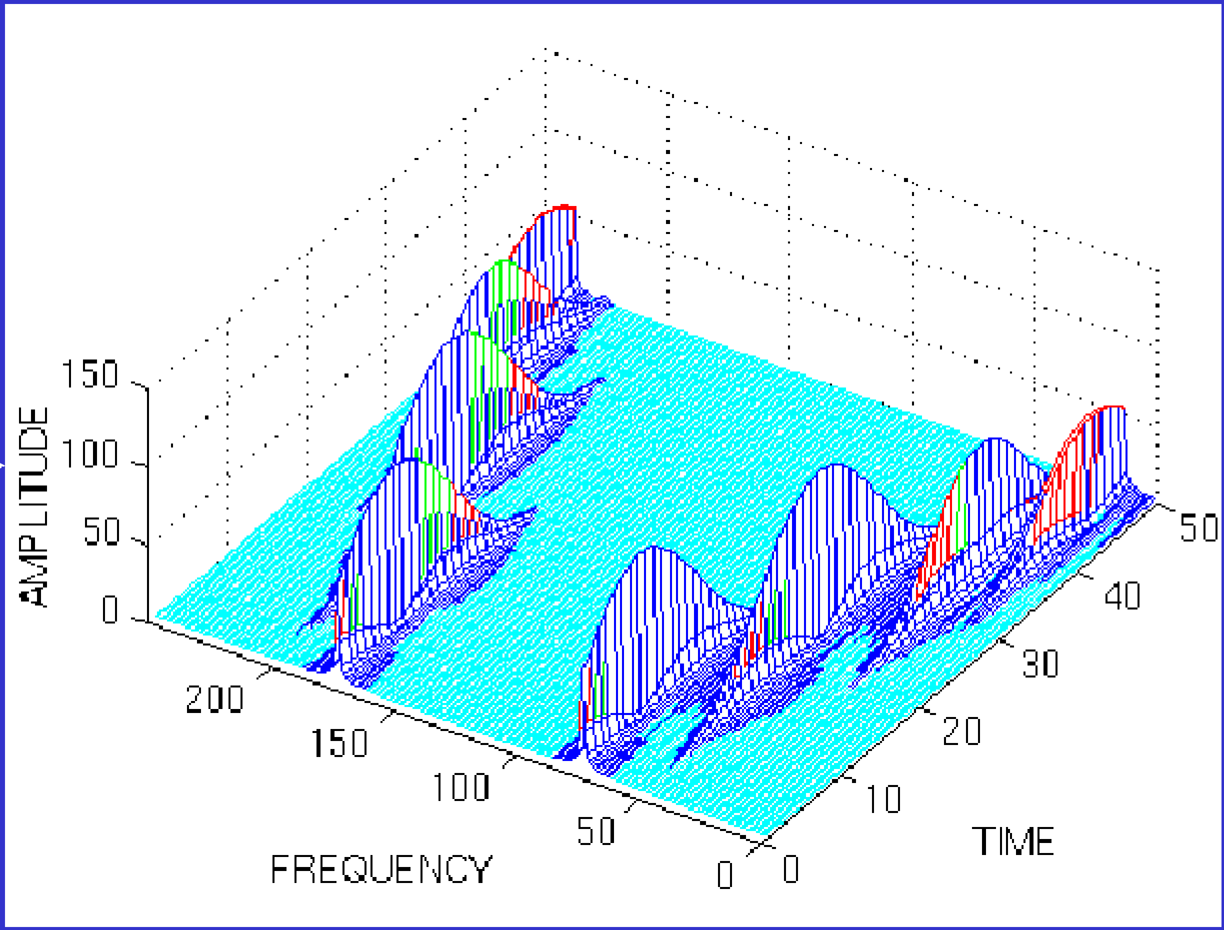
Window Function for 4 values of ‘a’_



STFT using the first window i.e. a= 0.01



- **Very good time resolution, but relatively poor frequency resolution.**
- **Four peaks are well separated from each other in time.**
- **In frequency domain, every peak covers a range of frequencies, instead of a single frequency value.**



STFT using the third window i.e. a = 0.0001

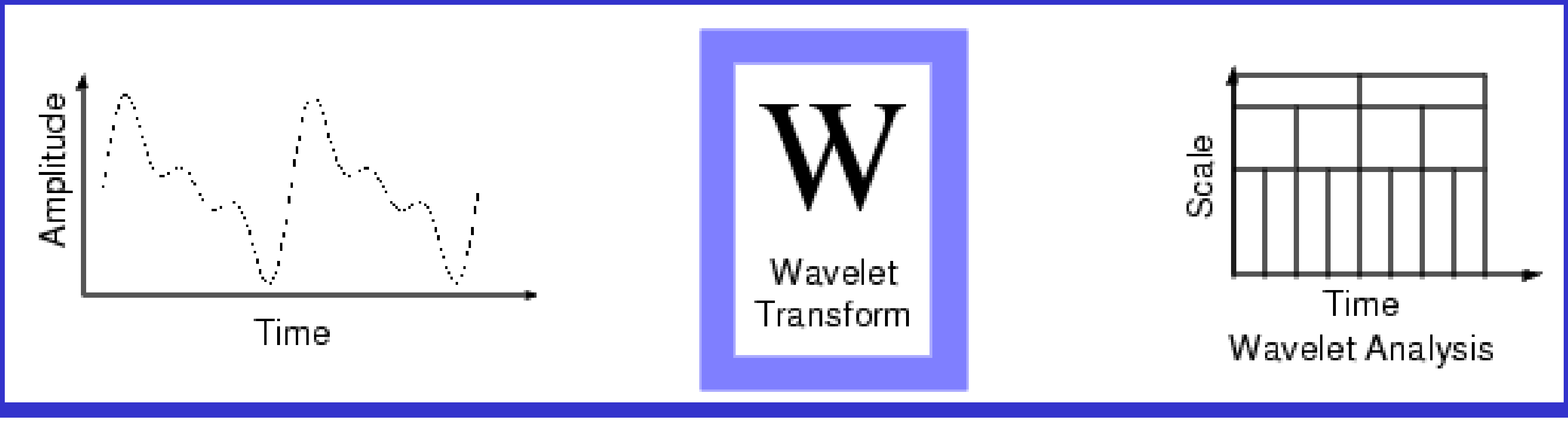
- **Peaks are not well separated from each other in time, unlike the previous case.**
- **In frequency domain the resolution is much better.**

Multi Resolution Analysis

- **Analysis with different resolutions at different frequencies.**
- **Every spectral component is not resolved equally, as was the case in the STFT.**
- ***Good time resolution with poor frequency resolution at high frequencies and good frequency resolution with poor time resolution at low frequencies.***

Multi Resolution Analysis

- **Analysis with different resolutions at different frequencies.**
- **Every spectral component is not resolved equally, as was the case in the STFT.**
- ***Good time resolution with poor frequency resolution at high frequencies and good frequency resolution with poor time resolution at low frequencies.***



WAVELET TRANSFORM

- **The signal is multiplied with a function (called wavelet here), similar to the window function in STFT.**
- **The transform is computed separately for different segments of the time-domain signal.**
- **The width of the window is changed as the transform is computed for every single spectral component, which is probably the most significant characteristic of the wavelet transform**
- ***THE BASIC FORMULA:***

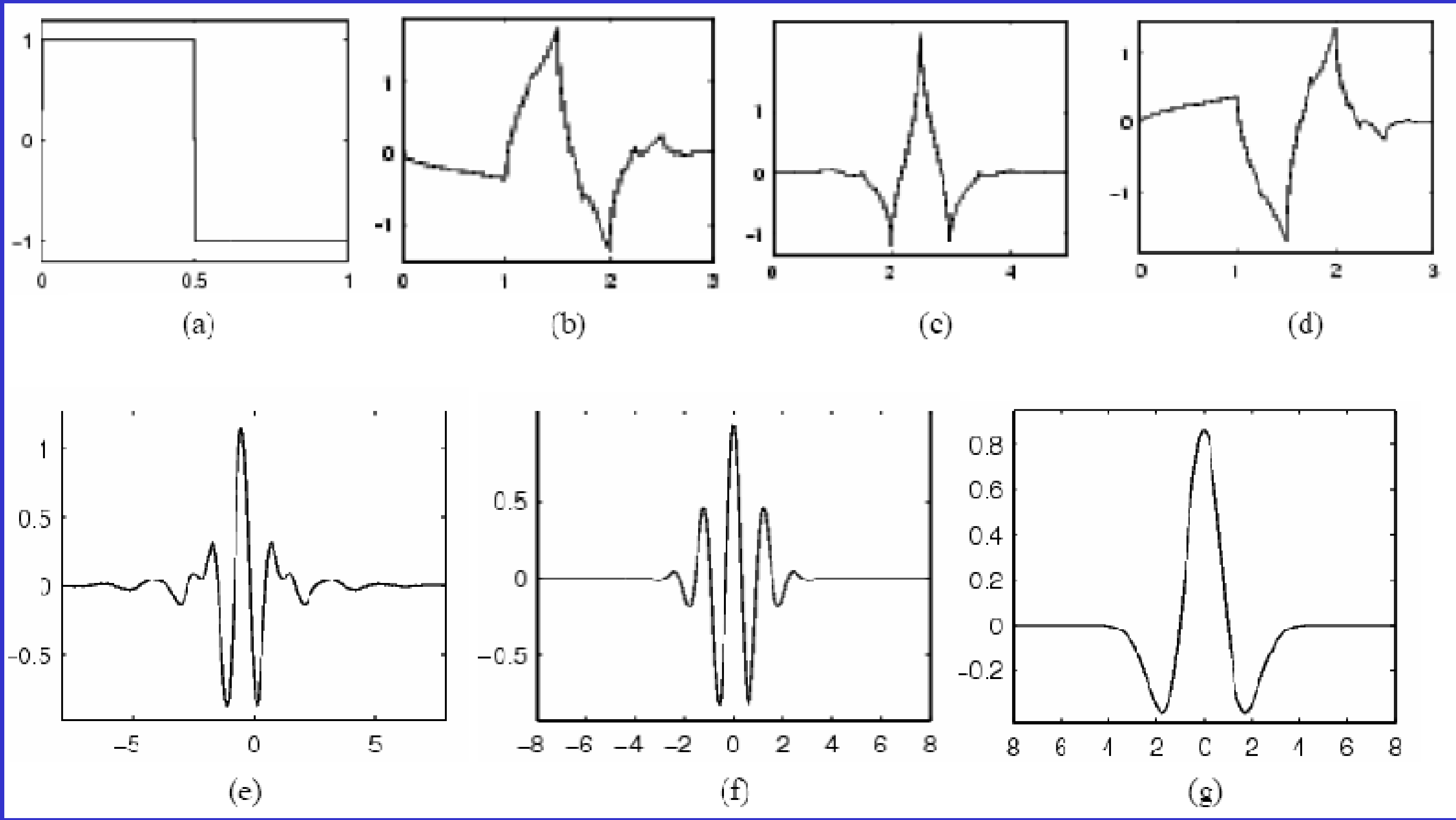
$$W(s,\tau) \equiv \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{|s|}} \psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

A glance into the past ……

- The beginning of WT can be traced back to the works of Grossman and Morlet (1984).
- Early results were related to what is now known as Continuous Wavelet Transform.
- Their motivation was provided by the fact that certain seismic signals could be modelled by combining translations and dilations of a simple, oscillatory function.
- However, ‘wavelet transformlike’ expressions can be found in earlier works done in several fields such as function estimation, quantum mechanics etc.
- The interest in WT was sustained and further nurtured when connections were established between Wavelet representation and developments in other fields [Meyer(1993) and Daubechies(1988, 1996)].

✓ **Wavelet : A simple oscillatory function of finite duration.**

✓ **Mother Wavelet: It is a wavelet that integrates to zero and is square integrable, i.e. has finite energy, and satisfies the admissibility condition. It is a prototype for generating other window functions. Examples- Morlet, Mexican Hat etc.**



Various Mother Wavelet families (a) Haar (b) Daubechies4 (c) Coiflet1 (d) Symlet2 (e) Meyer (f) Morlet (g) Mexican Hat

Properties of Mother Wavelet:

The most important properties of wavelets are the admissibility and the regularity conditions and these are the properties which gave wavelets their name. It can be shown [She96] that square integrable functions $\psi(t)$ satisfying the *admissibility condition*,

$$\int \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < +\infty ,$$

can be used to first analyze and then reconstruct a signal without loss of information. $\Psi(\omega)$ stands for the Fourier transform of $\psi(t)$. The admissibility condition implies that the Fourier transform of $\psi(t)$ vanishes at the zero frequency, i.e.

$$|\Psi(\omega)|^2 \Big|_{\omega=0} = 0 .$$

This means that wavelets must have a band-pass like spectrum. This is a very important observation, which we will use later on to build an efficient wavelet transform.

A zero at the zero frequency also means that the average value of the wavelet in the time domain must be zero,

$$\int \psi(t) dt = 0 ,$$

and therefore it must be oscillatory. In other words, $\psi(t)$ must be a *wave*.

Properties of Mother Wavelet:

As can be seen the wavelet transform of a one-dimensional function is two-dimensional; the wavelet transform of a two-dimensional function is four-dimensional. The time-bandwidth product of the wavelet transform is the square of the input signal and for most practical applications this is not a desirable property.

Therefore one imposes some additional conditions on the wavelet functions in order to make the wavelet transform decrease quickly with decreasing scale *s*. These are the *regularity conditions* and they state that the wavelet function should have some smoothness and concentration in both time and frequency domains.

Summarizing, the admissibility condition gave us the wave, regularity and vanishing moments gave us the fast decay or the *let*, and put together they give us the wavelet.

Mother Wavelet:

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{|s|}} \psi\left(\frac{t-\tau}{s}\right)$$

<u>Translation:</u>

For any given value of s, the function $\psi_{s,\tau}(t)$ is a shift of $\psi_{s,0}(t)$ by an amount of τ along the time axis. Thus the variable τ represents the time-shift or Translation.

<u>Scale:</u>

$$\psi_{s,0}(t) - \frac{1}{\sqrt{|s|}} \psi\left(\frac{t}{s}\right)$$

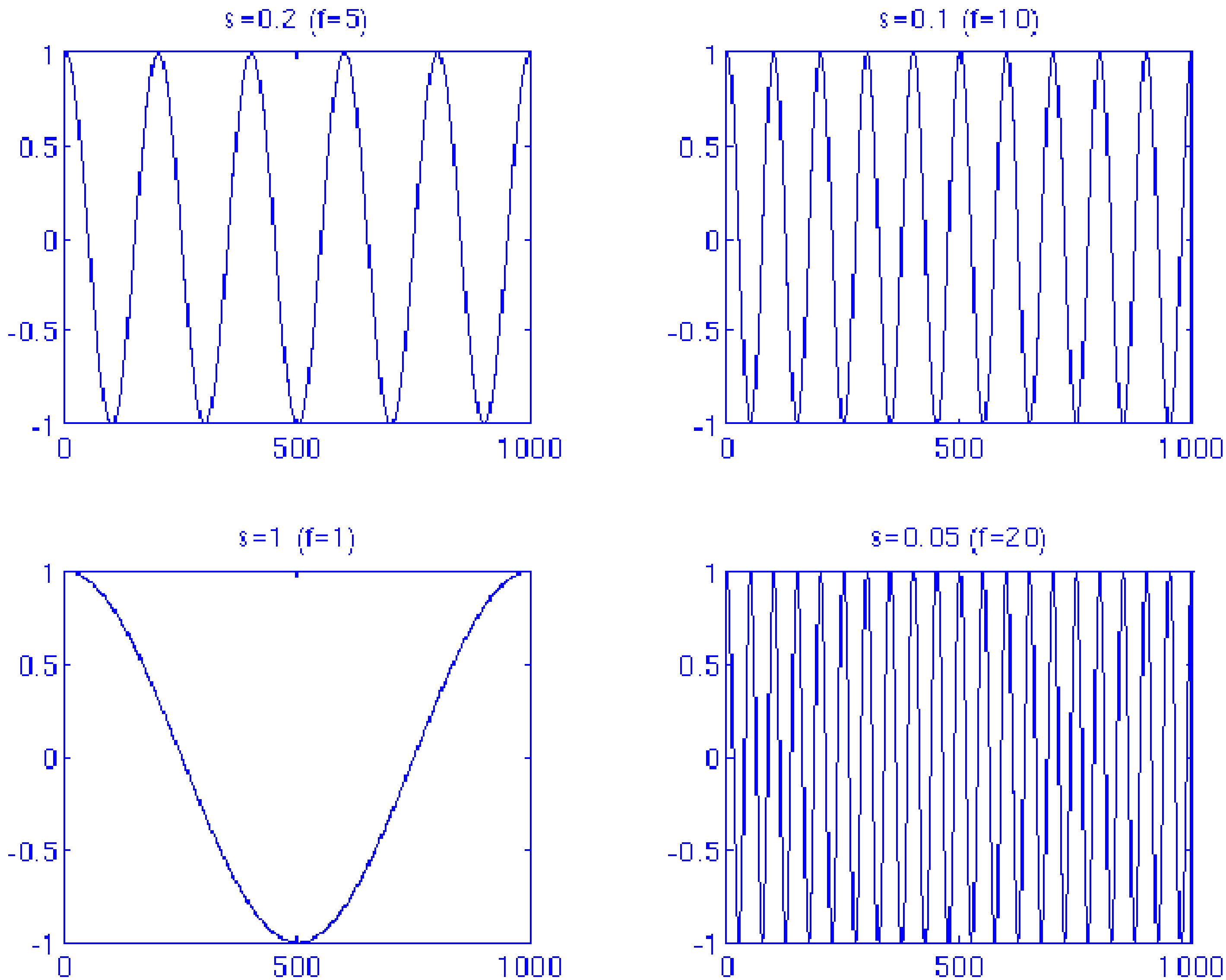
It follows that $\psi_{s,0}(t)$ is a time-scaled and amplitude –scaled version of $\psi(t)$. Since 's' determines the amount of time-scaling or dilation, it is referred to as the dilation variable or Scale.

Scale

- ✓ Similar to the scale used in maps (low scale - detailed view …)
- ✓ Low frequency → High scale → Global information → Usually the entire signal.
- ✓ High frequency → Low scale → Detailed information about the hidden pattern of the signal.
- ✓ Scaling, as a mathematical operation, either dilates or compresses a signal.
- ✓ Larger scales correspond to dilated (or stretched out) wavelets and small scales correspond to compressed wavelets.
- ✓ scale >1dilates the wavelets & scale < 1,compresses the wavelets.
- ✓ Actual frequency,

$$f = \frac{f_c}{s \cdot \Delta}$$

f_c = centre frequency of the mother wavelet and
Δ = sampling period of the signal



Dilated and compressed versions of the same cosine signal

If f(t) is a given function, then f(st) is *dilated* version of f(t) if s > 1 and *compressed* version of f(t) if s < 1 .

Time-Frequency Resolution of CWT

The CWT offers time and frequency selectivity, i.e. it is able to localize events both in time and frequency.

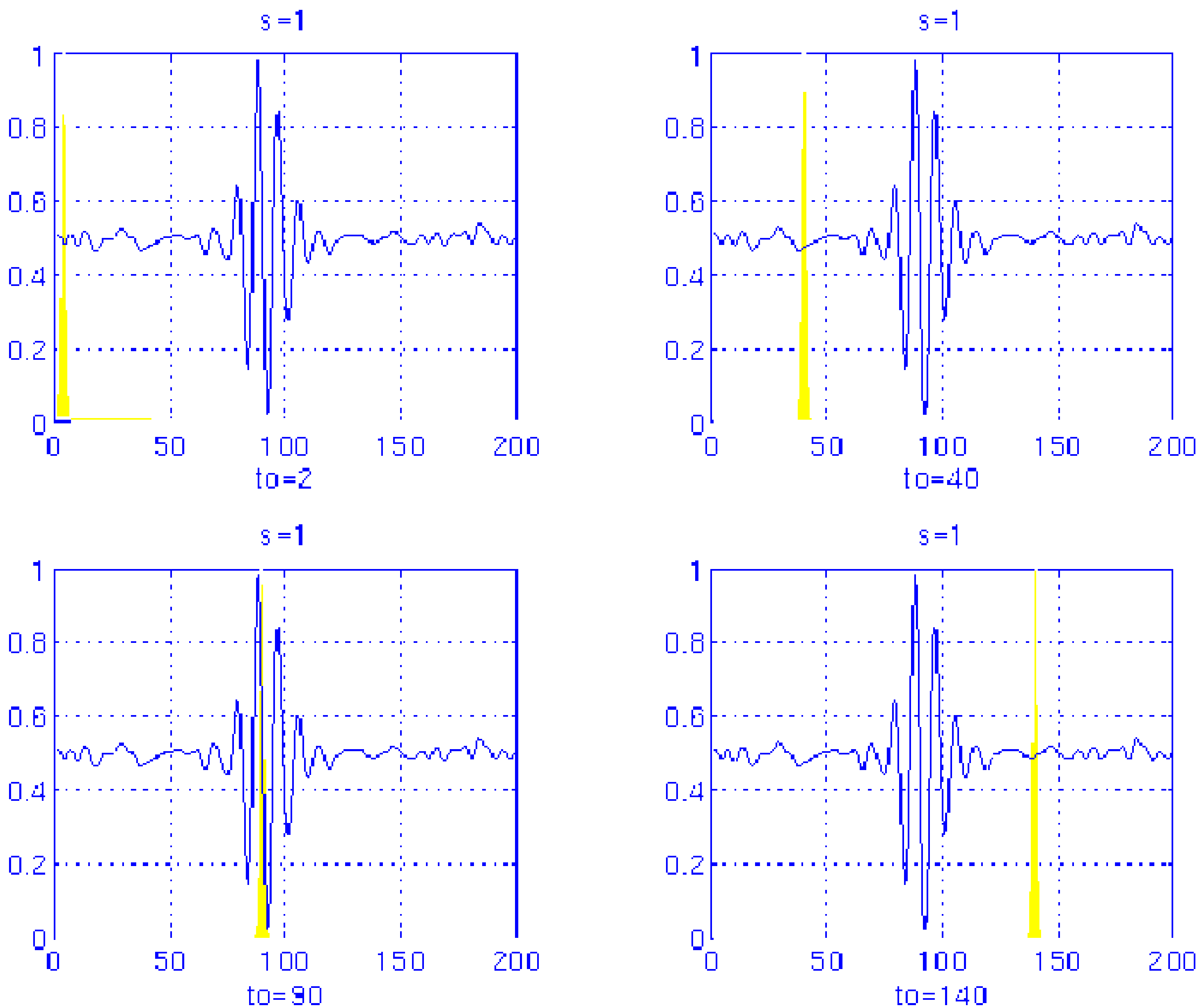
Time Selectivity: **The segment of f(t) that influences the value of W(s,τ) for any value of (s, τ) is that stretch of f(t) that coincides with the interval over which $\psi_{s,\tau}(t)$ has the bulk of its energy. This windowing effect results in the time selectivity of the CWT.**

Frequency Selectivity: **It can be explained using the interpretation of CWT as a collection of linear, time-invariant filters. From the plots of the wavelet and the Fourier Transform of the wavelet, it may be seen that the wavelet is essentially a bandpass function. It is this bandpass nature that gives rise to frequency selectivity of the CWT.**

Computation Of Continuous Wavelet Transform(CWT)

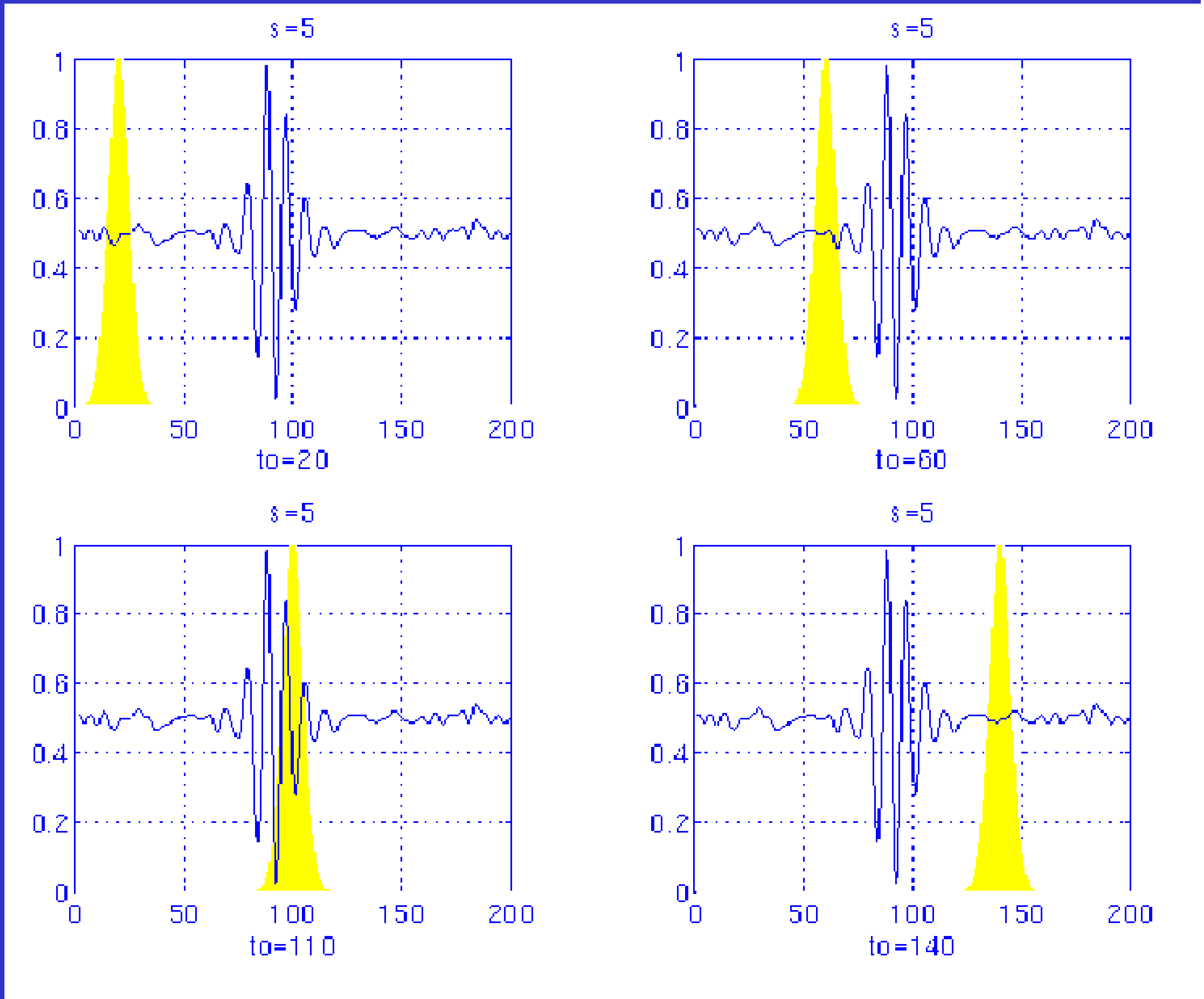
a. **x(t) is the signal to be analyzed**
b. **A mother wavelet is chosen to serve as a prototype of windows in the process**
c. **Other windows are dilated/compressed and shifted versions of the mother wavelet**
d. **To begin with, a scale s=1, is chosen (most compressed wavelet)**
e. **Wavelet placed at the beginning of the signal, i.e. at t=0**
f. **CWT is computed using the required expression for s=1 & t=0**
g. **Wavelet shifted to t=τ and CWT calculated for s=1 & t= τ**
h. **Process repeated till the end of signal is reached**
i. **THUS 1 ROW OF POINTS ON THE TIME SCALE FOR s=1 IS OBTAINED**
j. **The above steps are repeated now, for different values of 's'**
k. **Increment in the value of 's' depends on the mode of computation**
l. **This corresponds to the sampling of the time-scale plane**
m. **When the above process is completed for all desired values of 's', the CWT of the signal is said to be calculated**
n. **If the signal *has a spectral component* that corresponds to the current value of s, the product of the wavelet with the signal at that location gives a relatively large value. If *not* the product will have a small value (may even be zero).**

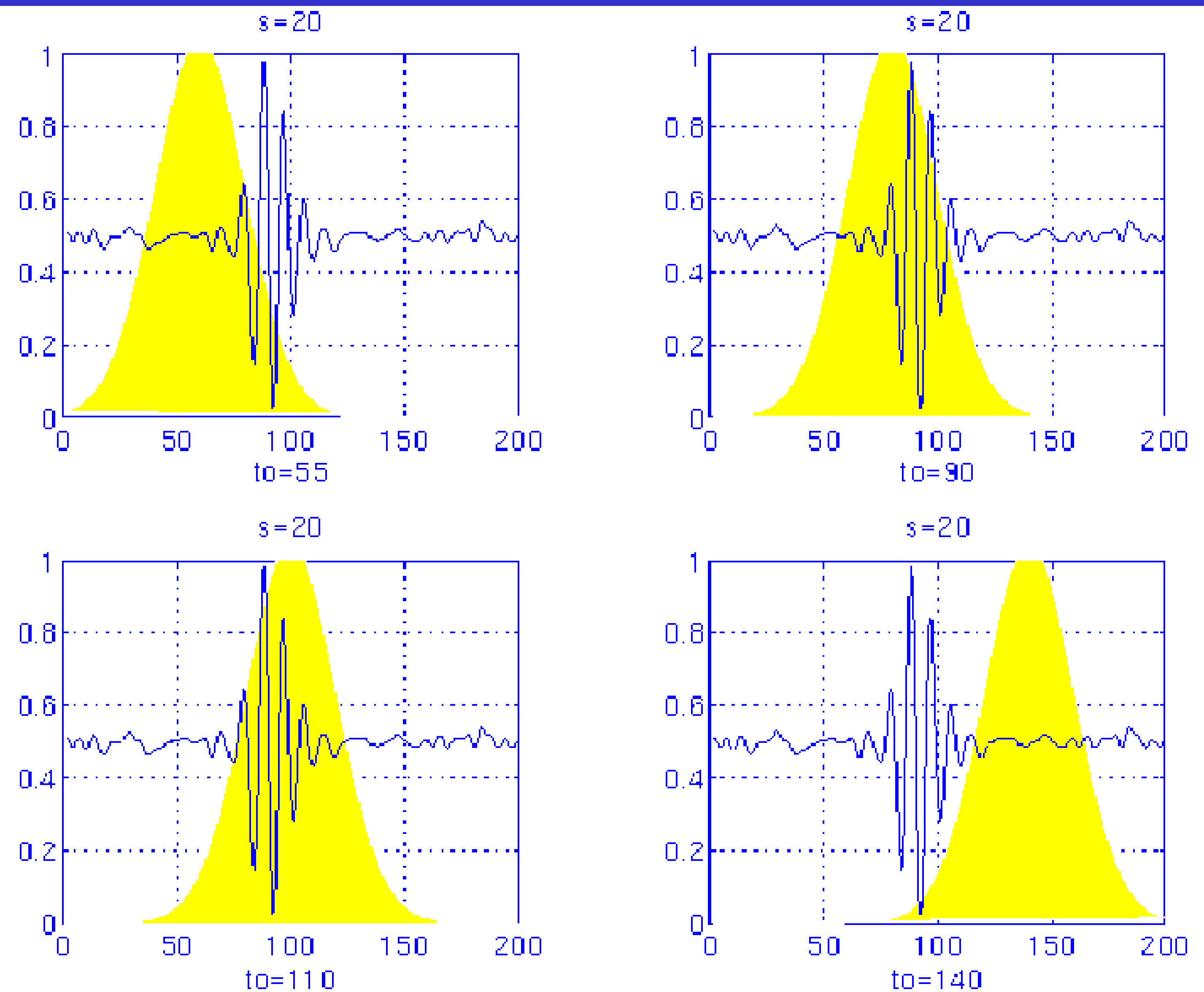
Signal and wavelet function for 4 different values of τ for ‘s=1’



Spectral components comparable to the window's width at s=1 around t=100 ms.

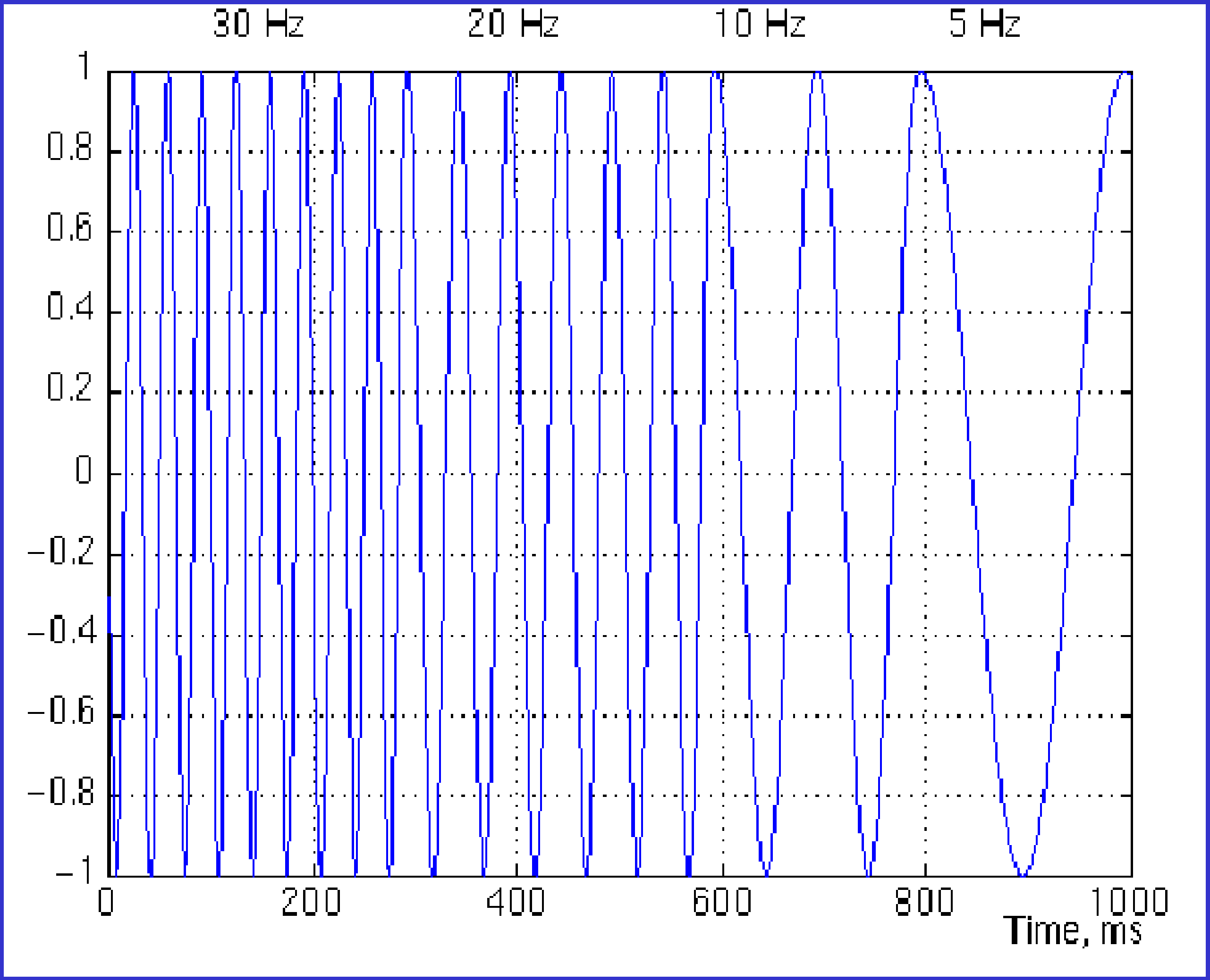
Signal and wavelet function for 4 values of τ at ‘s=5’





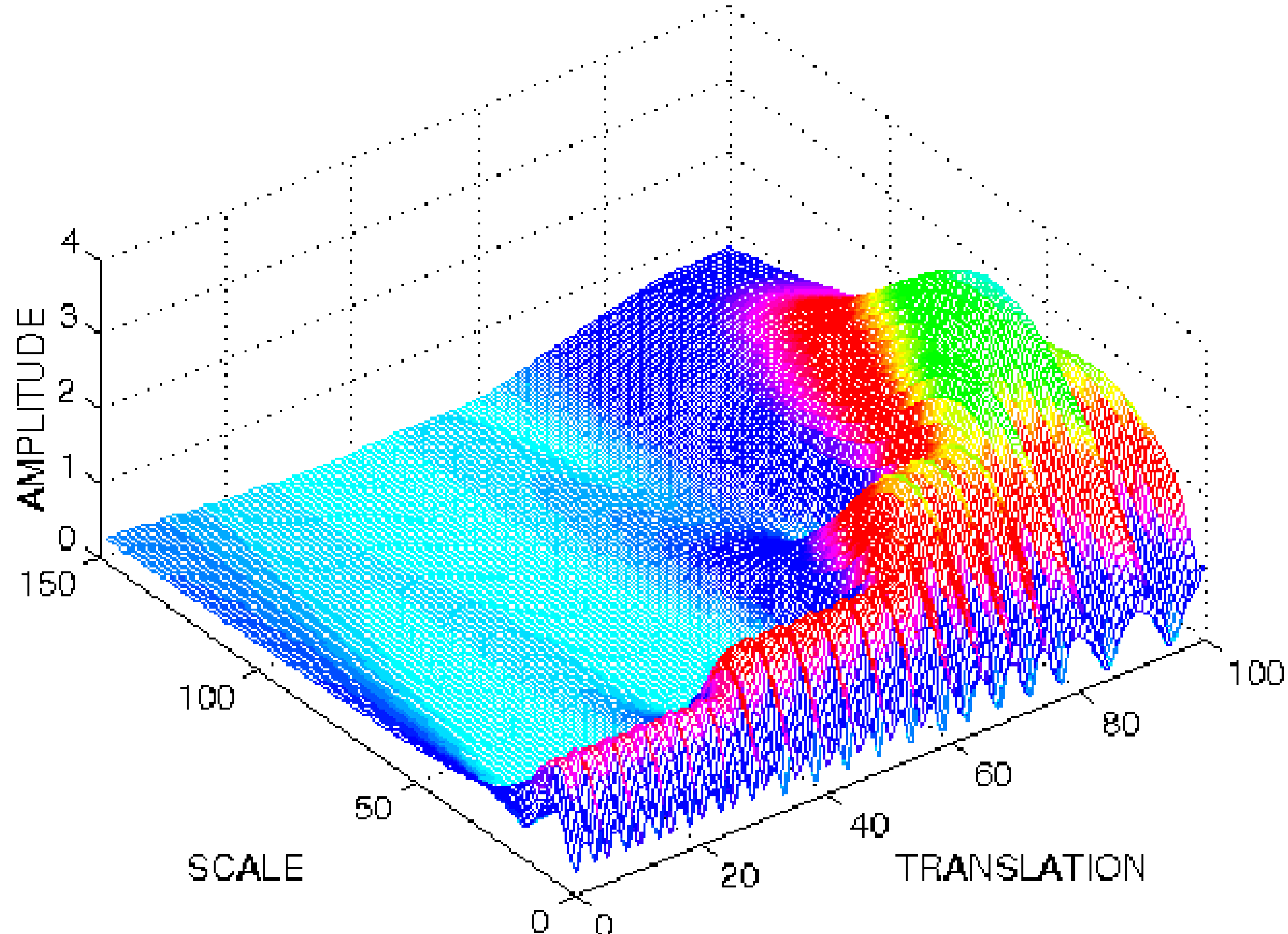
Signal and wavelet function for 4 values of τ at s=20

An Example of Wavelet Transform



Non stationary signal similar to that used in STFT

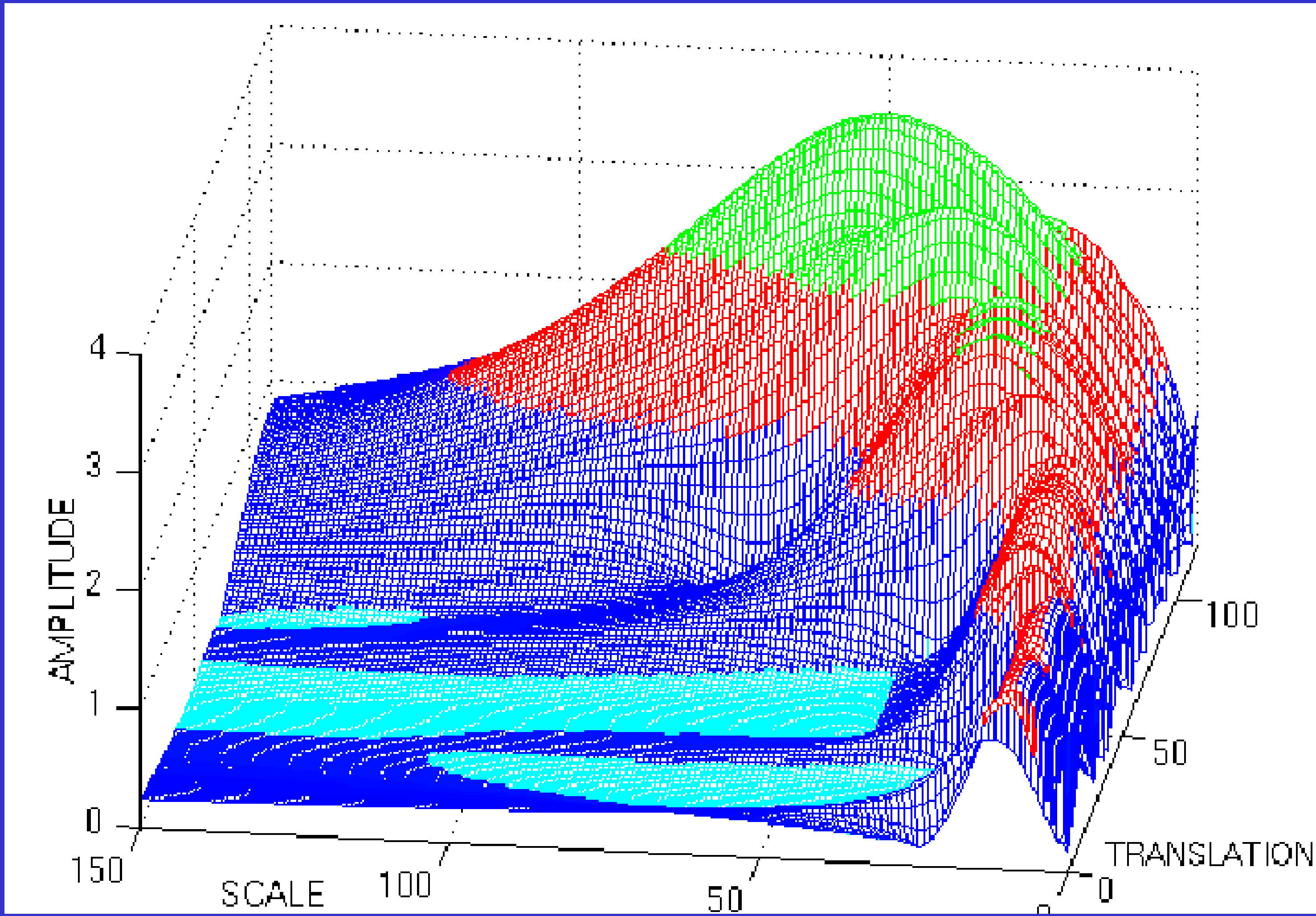
Continuous Wavelet Transform of the signal



Axes are *translation* and *scale*

- **Translation related to time**
  (Location of mother wavelet)
- **Scale(s) → inverse of frequency.**

Another view of the same Continuous Wavelet Transform



- Smaller scales correspond to higher frequencies
- Part of the plot with scales around zero ➔ highest frequencies in the analysis.

Inverse CWT

Inverse Wavelet Transform is possible if the mother wavelet satisfies the (sufficient) following condition, known as 'Admissibility Condition'.

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(\xi)|^2}{|\xi|} d\xi$$

If $0 < C_\psi < \infty$, then the Inverse CWT is given by

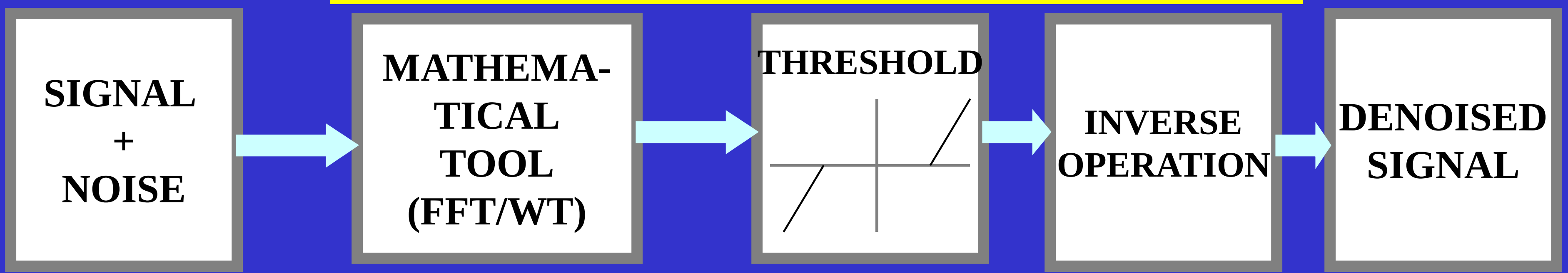
$$f(t) = \frac{1}{C_\psi} \int_{s=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} \frac{1}{|s|^2} W(s,\tau)\,\psi\left(\frac{t-\tau}{s}\right) d\tau\, ds$$

It shows that $\{W(s,\tau)/|s|^2\}$ provides the weighting function for constructing f(t) from $\psi_{s,\tau}(t)$ – the translates and dilates of the mother wavelet.

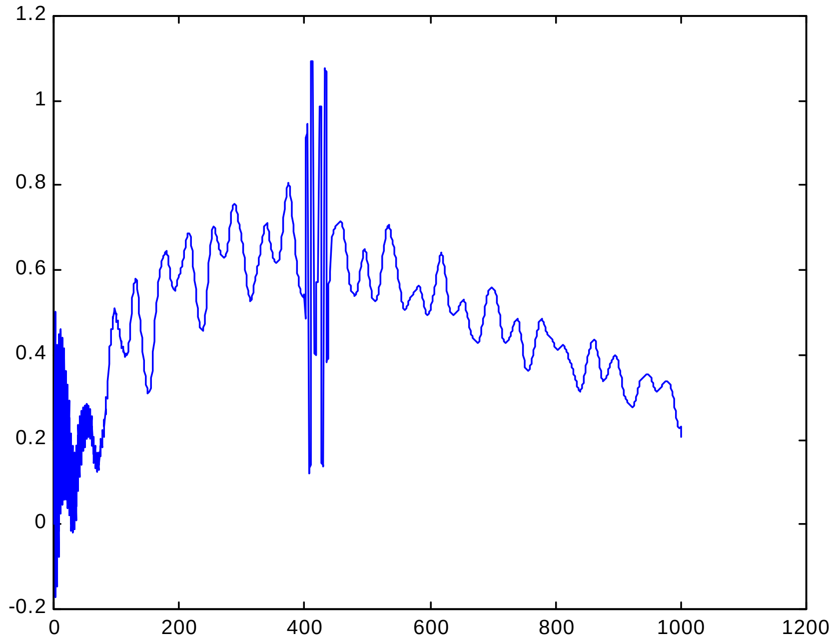
Wavelet Denoising

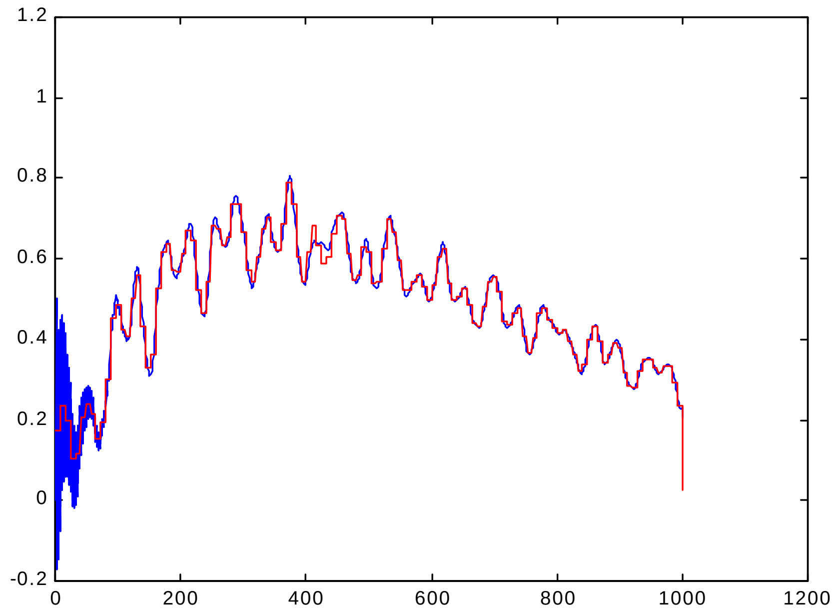
- ✓ WT can be applied for noise reduction.
- ✓ WT of real world signals tend to create low value coefficients at finer scales.
- ✓ The noise tend to be represented by wavelet coefficients at these finer scales.

BLOCK DIAGRAM REPRESENTATION



Examples of Wavelet Denoising



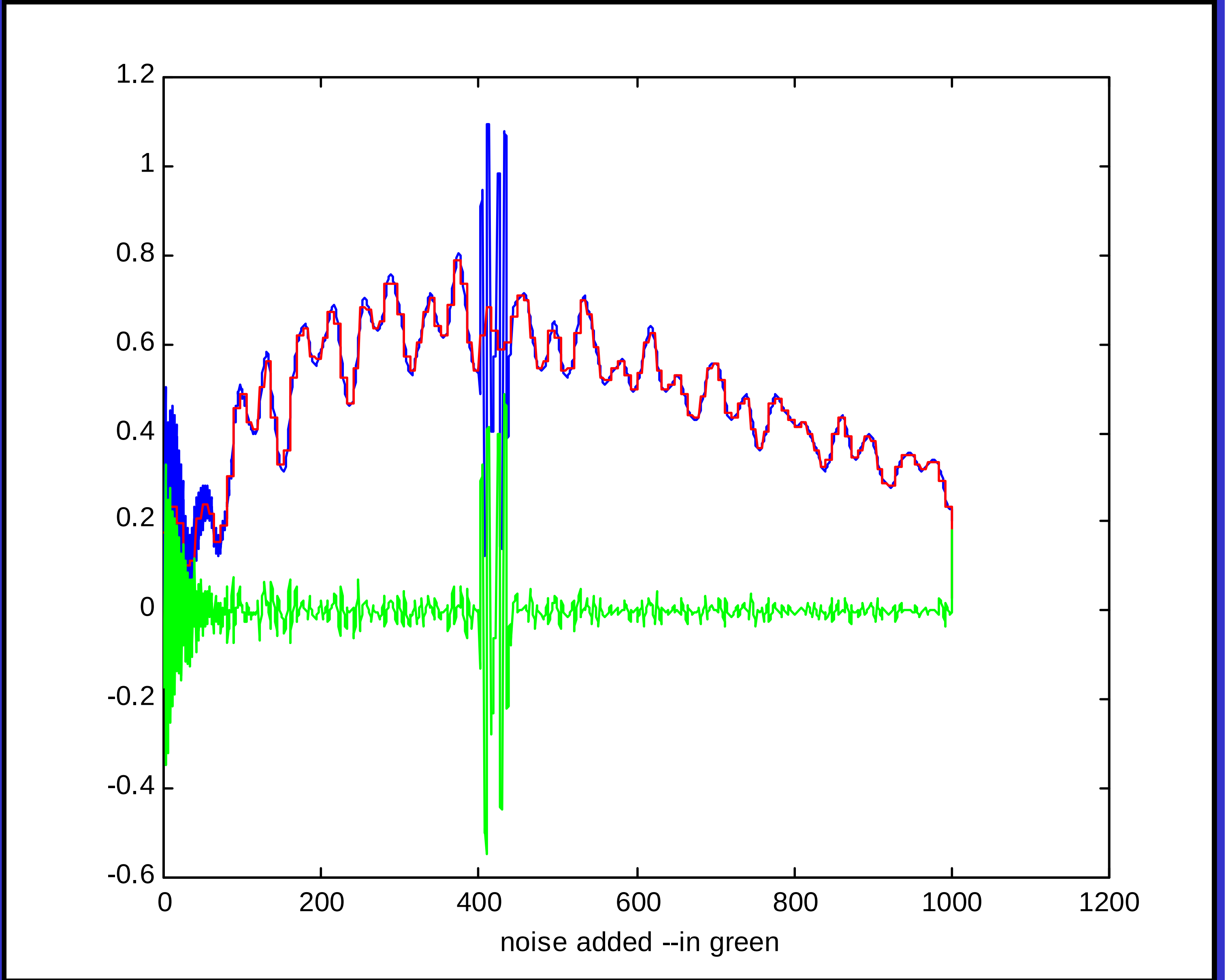


Noisy signal, Denoised signal and Noise

De-Noised signal

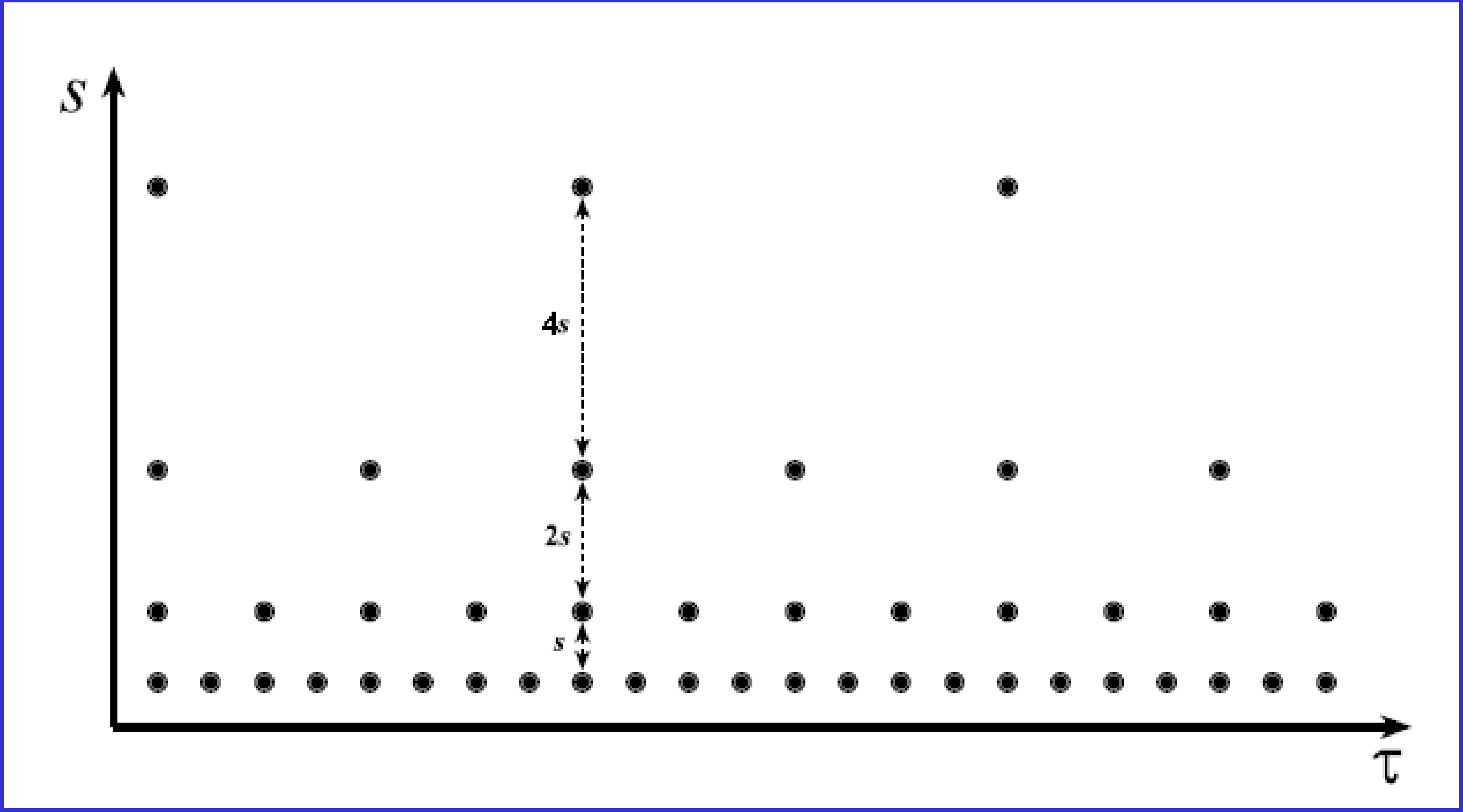
Noisy signal

Noise

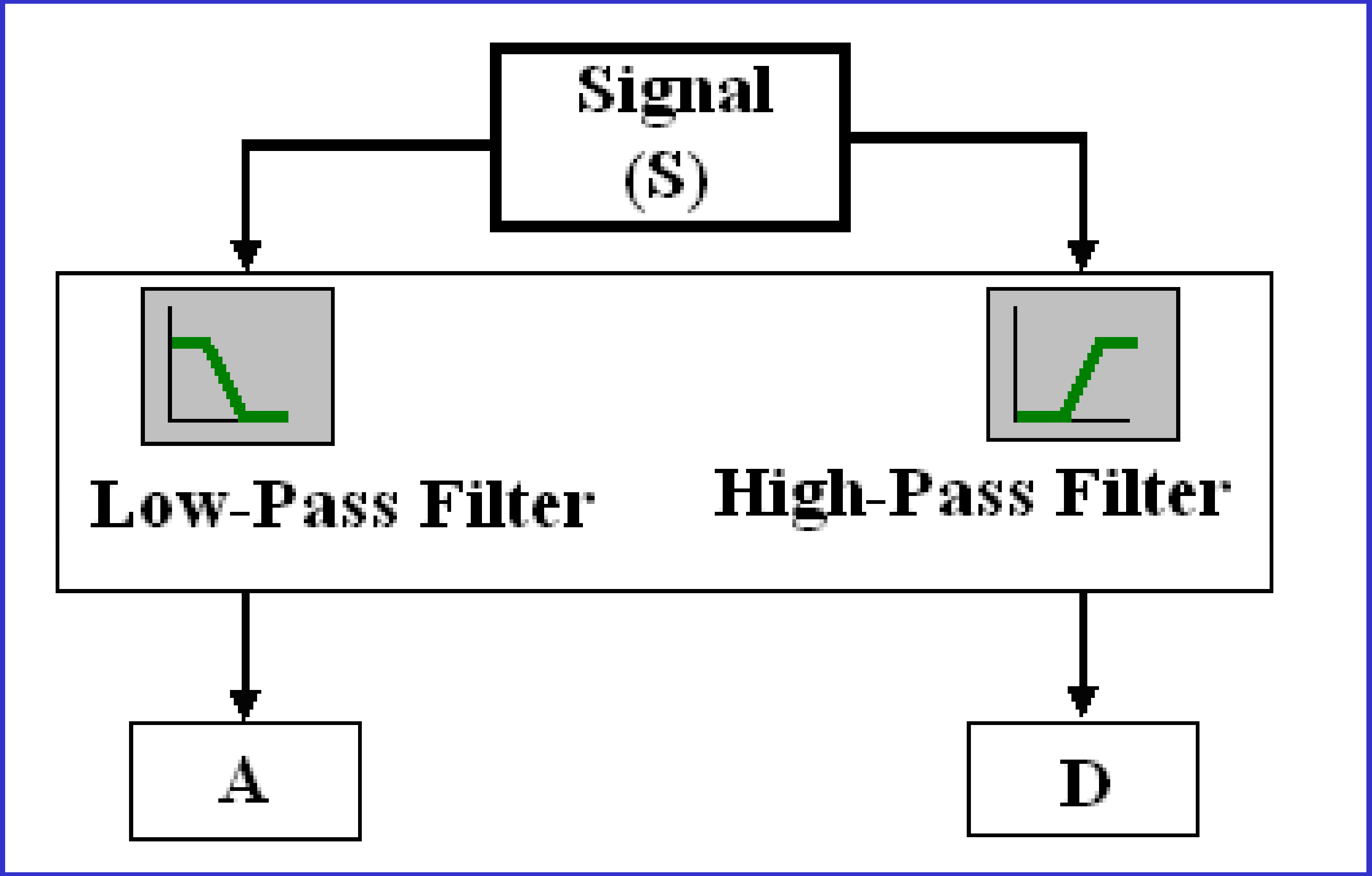


Discrete Wavelet Transform (DWT)

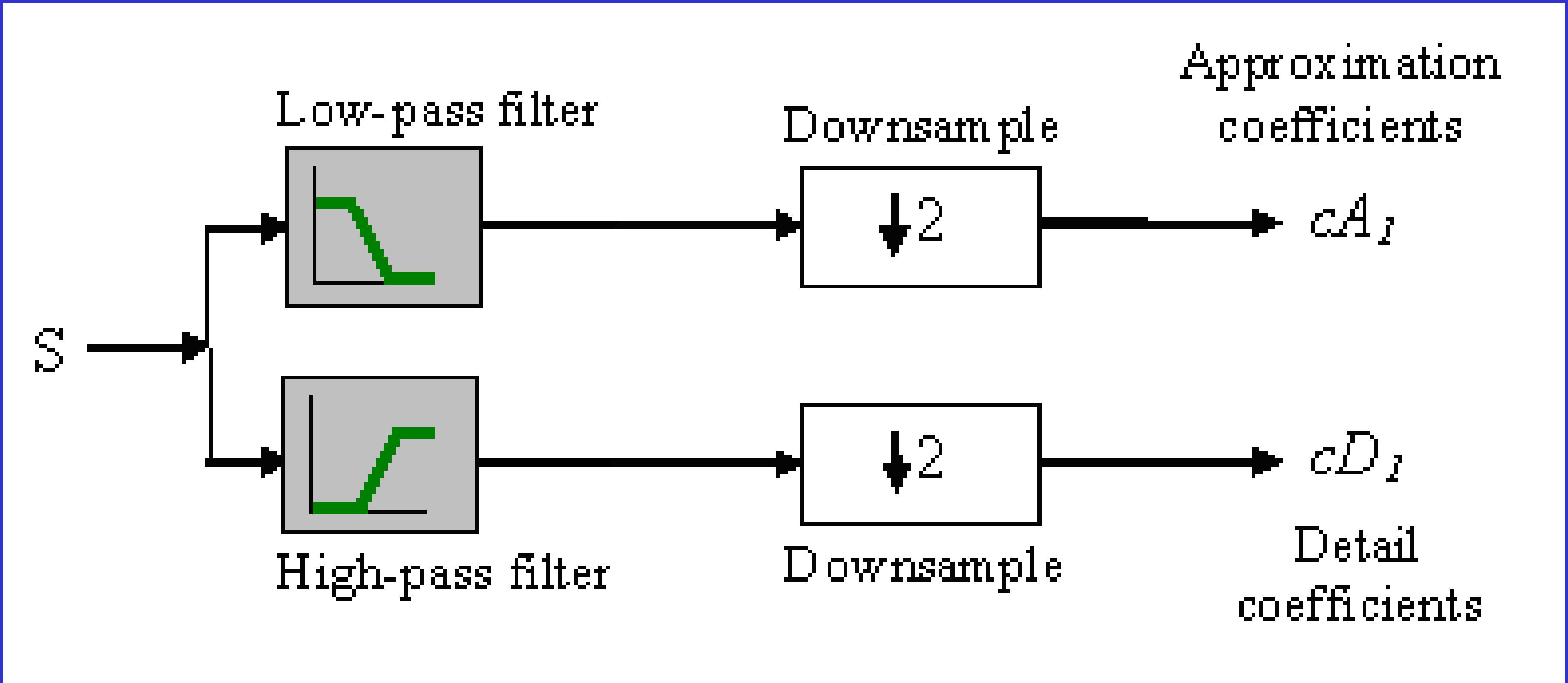
$$\psi_{j,k}(t) = \frac{1}{\sqrt{s_0^j}} \psi\left(\frac{t - k\tau_0 s_0^j}{s_0^j} \right).$$



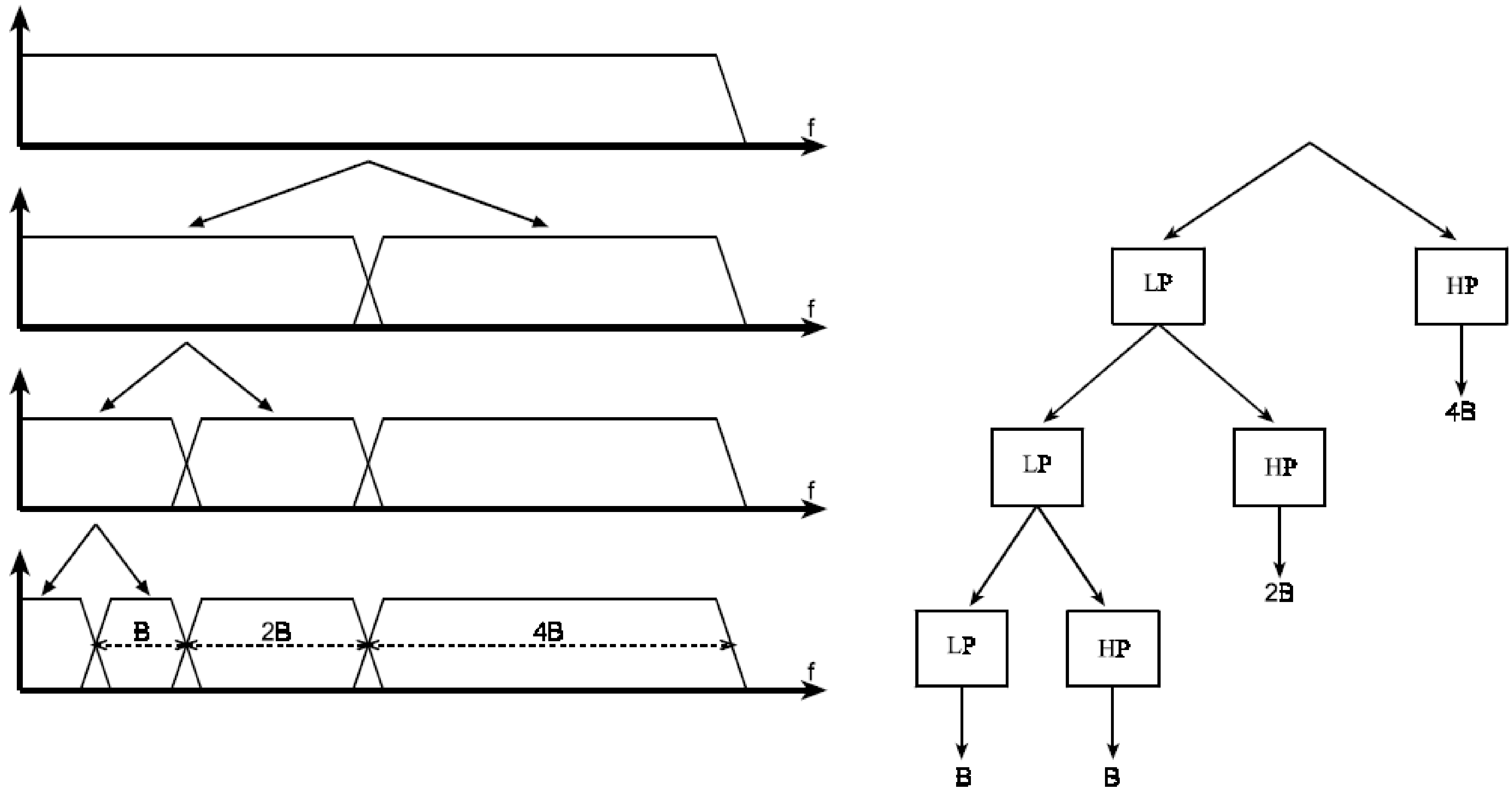
Discrete Wavelet Transform (DWT)



Discrete Wavelet Transform (DWT)



Discrete Wavelet Transform (DWT)



Wavelet

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\left(\frac{t-\tau}{s}\right)$$

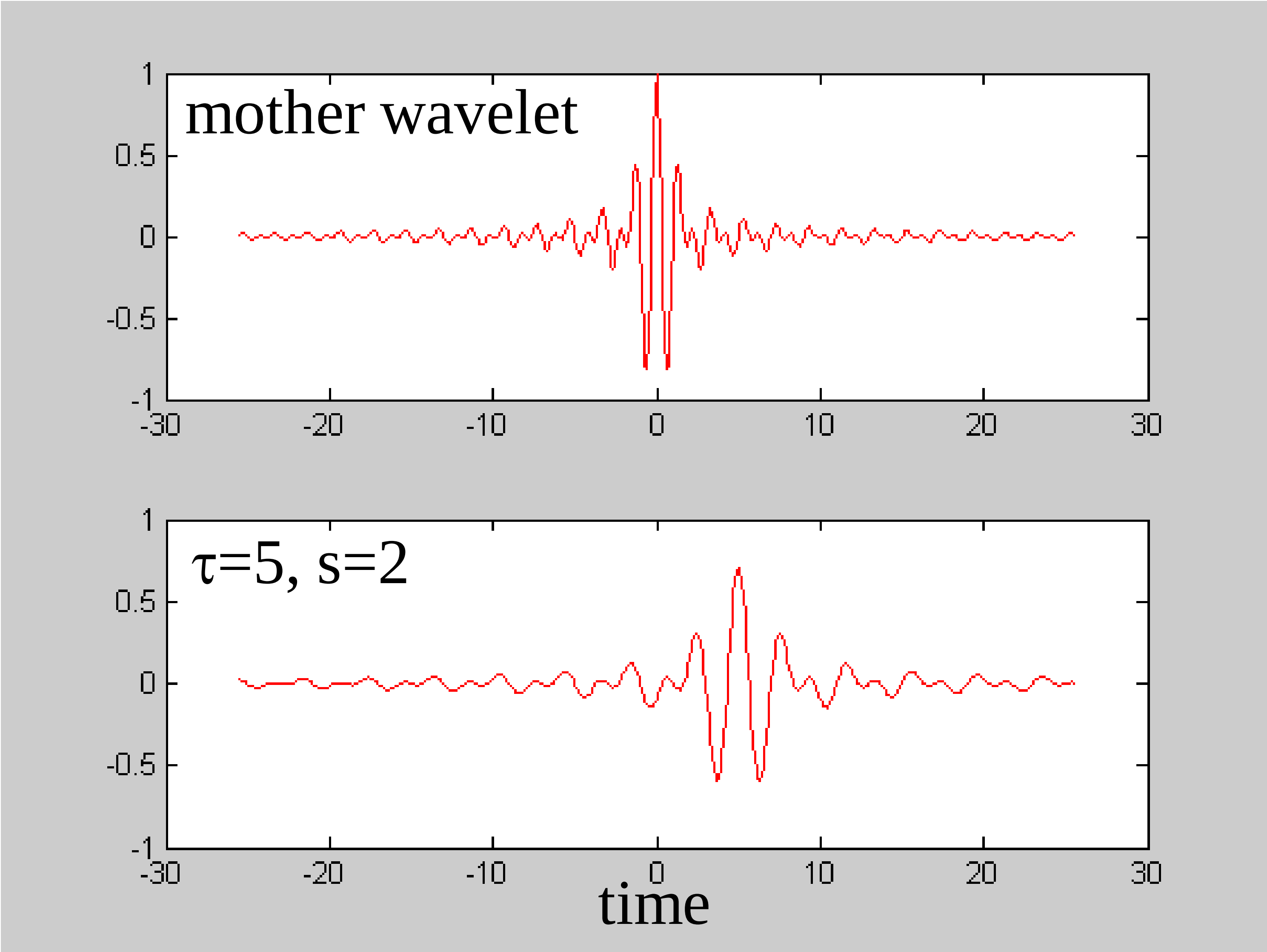
change in scale:
big s means long
wavelength

wavelet with
scale, s and time, τ

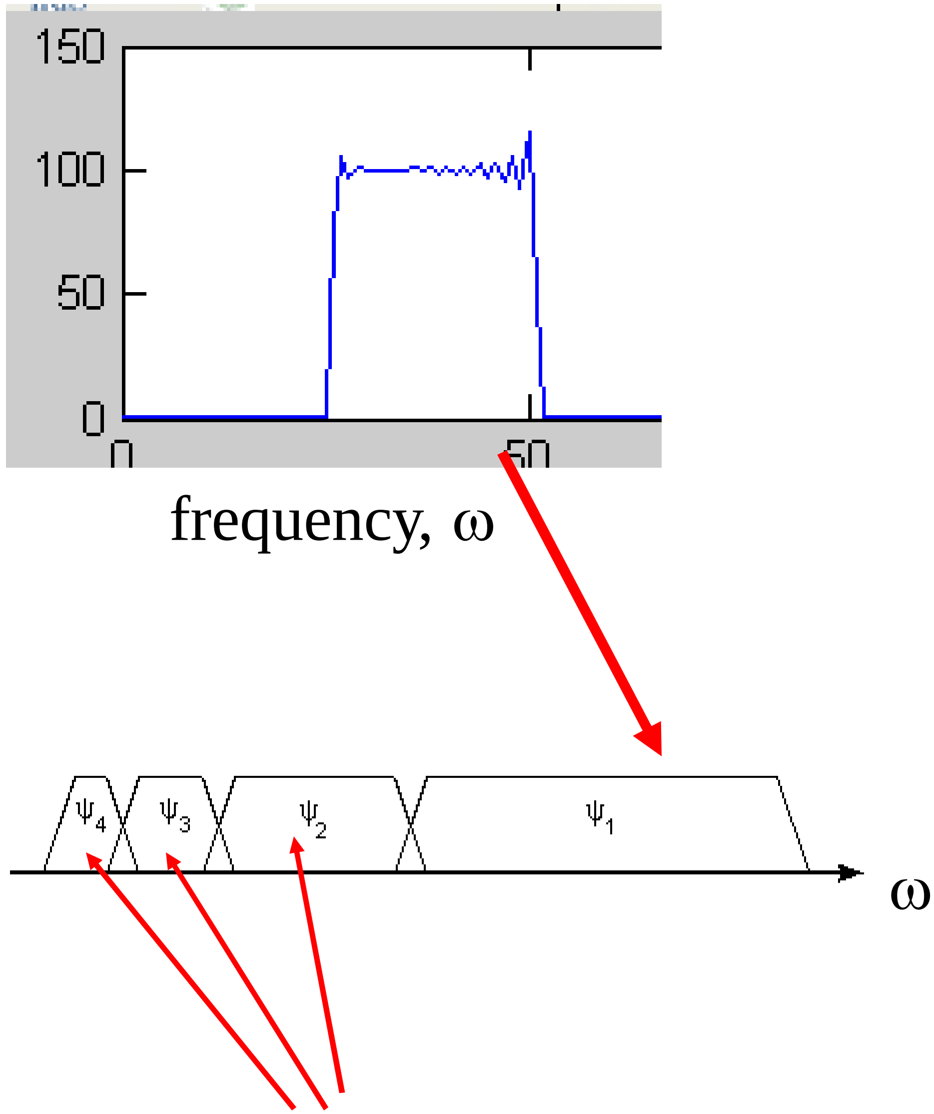
Mother wavelet

Shannon Wavelet

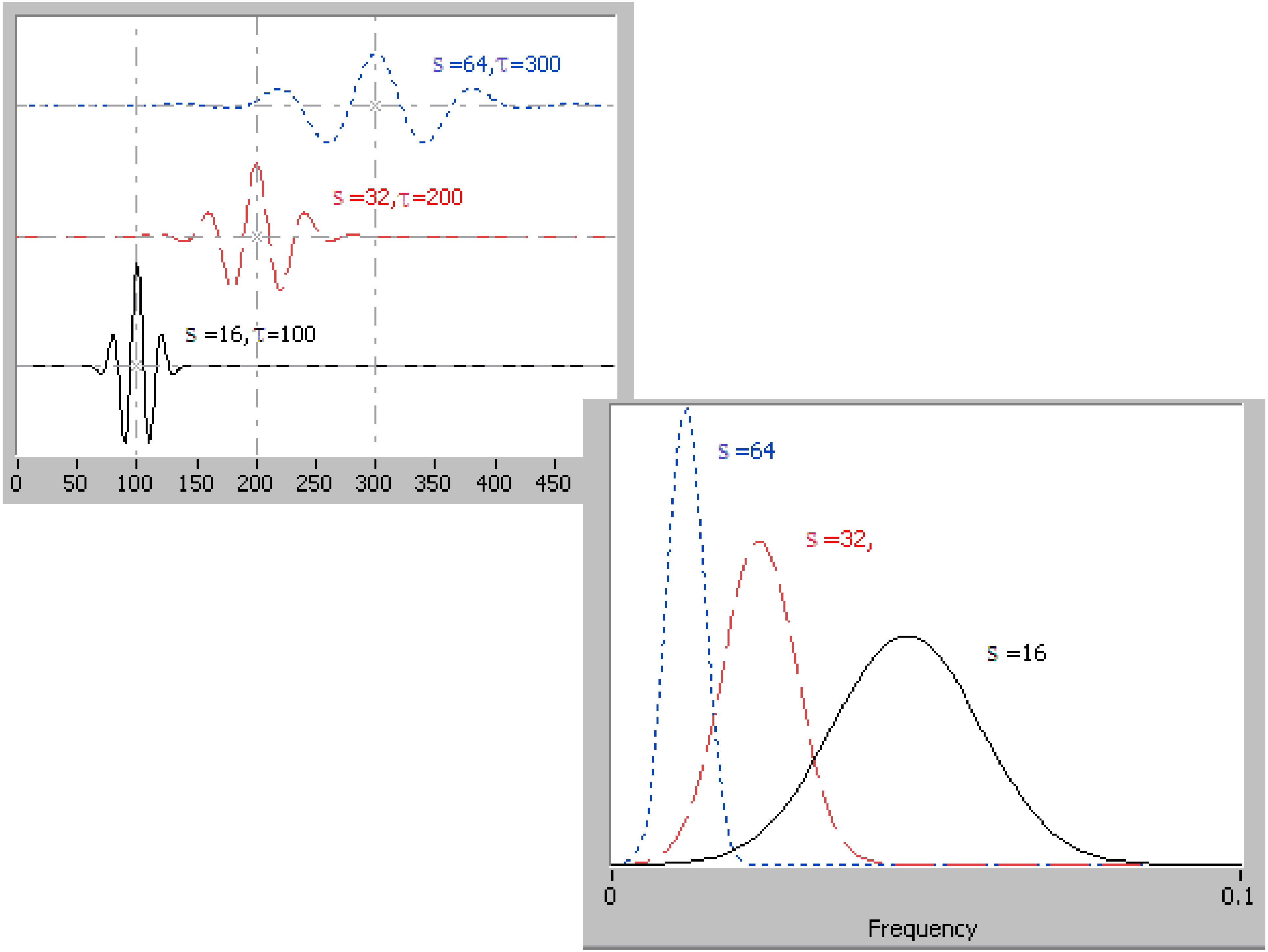
$$\Psi(t) = 2\ \mathrm{sinc}(2t) - \mathrm{sinc}(t)$$



Fourier spectrum of Shannon Wavelet



Spectrum of higher scale wavelets



s =64,τ=300
s =32,τ=200
s =16,τ=100
0
50
100
150
200
250
300
350
400
450
s =64
s =32,
s =16
0
0.1
Frequency