KALMAN CONTROLLER

Kalman's method of designing a digital controller is based on restrictions imposed on both the manipulated and controlled variables. This is unlike the Dahlin controller, where the output in response to a set point change is specified, without any constraint placed on manipulated variable. In Kalman's method, the specification for a unit step change in the set point is for the closed loop response to settle at the final value within a specific number of sampling periods with the actuation signal assuming only a specific number of values before reaching the final value. It can be shown that in the case of second-order system, a minimum of two values of manipulated variable are required before the set point can be reached. For a third-order system the minimum is three values, and so on. This type of controller design is known as a particular case of finite settling time controller.

The controlled and manipulated variables can be expressed as

$$C(z) = \sum_{n=0}^{\infty} c_n z^{-n} \qquad (1)$$

$$M(z) = \sum_{n=0}^{\infty} m_n Z^{-n} \qquad (2)$$

A second-order process has a z-transfer function of the form (including Z.O.H.)

$$\frac{\mathrm{C}(\mathrm{z})}{M(z)} = G(z) = \frac{Kz^{-N}\left(P_1 z^{-1} + P_2 z^{-2}\right)}{\left(1 - P_3 z^{-1}\right)\left(1 - P_4 z^{-1}\right)} \qquad (3)$$

Since the process has a dead-time of N sampling periods, we have

$$C(z) = c_{N+1} z^{-(N+1)} + c_{N+2} z^{-(N+2)} + ... \qquad (4)$$

The z-transform of the manipulated variable is however,

$$M(z) = m_o + m_1 z^{-1} + m_2 z^{-2} + ... \qquad (5)$$

Let us specify that the response should settle to the final value (unity) within two sampling periods. Then the manipulated variable can assume only two values at n = 0 and n = 1, before settling to the final value m_f at n = 2.

$$\therefore C(z) = c_{N+1} z^{-(N+1)} + z^{-(N+2)} + z^{-(N+3)} + ... \qquad (6)$$

$$M(z) = m_o + m_1 z^{-1} + m_f z^{-2} + m_f z^{-3} + \ldots\ldots \tag{7}$$

(m_f → final value of manipulated variable.)

Then for a unit step input,

$$R(z) = \frac{1}{1 - z^{-1}}$$

∴ Closed-loop transfer function is

$$\begin{aligned} H(z) = \frac{C(z)}{R(z)} &= \left(1 - z^{-1}\right)\left[c_{N+1} z^{-(N+1)} + z^{-(N+2)} + \ldots..\right] \\ &= c_{N+1} z^{-(N+1)} + \left(1 - c_{N+1}\right) z^{-(N+2)} \\ &= d_1 z^{-(N+1)} + d_2 z^{-(N+2)} \end{aligned} \tag{8}$$

and

$$\begin{aligned} Q(z) = \frac{M(z)}{R(z)} &= \left(1 - z^{-1}\right)\left(m_o + m_1 z^{-1} + m_f z^{-2} + \ldots..\right) \\ &= m_o + \left(m_1 - m_0\right) z^{-1} + \left(m_f - m_1\right) z^{-2} \\ &= q_0 + q_1 z^{-1} + q_2 z^{-2} \end{aligned} \tag{9}$$

Now, the z-transfer function of the controller is

$$G_c(z) = \frac{1}{G(z)} \frac{H(z)}{\left[1 - H(z)\right]} = \frac{Q(z)}{1 - H(z)} \tag{10}$$

$$\left[\because Q(z) = \frac{M(z)}{R(z)} = \frac{H(z)}{G(z)}\right]$$

[As

$$\frac{H(z)}{G(z)} = \frac{C(z)}{R(z)} \frac{M(z)}{C(z)} = \frac{M(z)}{R(z)}$$

Also, $Q(z) = \frac{M(z)}{R(z)} = \frac{C(z)}{R(z)} . \frac{M(z)}{C(z)} = \frac{H(z)}{G(z)}$]

$$G(z) = \frac{H(z)}{Q(z)} \tag{11}$$

Let us try to express G(z) as a ratio of 2 polynomials in z^{-1} such that the numerator polynomial gives H(z) and denominator polynomial gives Q(z). That is,

$$G(z)=\frac{H(z)}{Q(z)}=\frac{d_1 z^{-(N+1)}+d_2 z^{-(N+2)}}{q_0+q_1 z^{-1}+q_2 z^{-2}}$$

For zero steady state error, $d_1 + d_2 = 1$ $\left(\because \underset{z\to 1}{Lt}\, H(z)=1\right)$

$\therefore$ To fulfil this condition, G(z) can be expressed as (from eqn.3)

$$G(z)=\frac{\frac{P_1}{P_1+P_2}z^{-(N+1)}+\frac{P_2}{P_1+P_2}z^{-(N+2)}}{\frac{1}{K(P_1+P_2)}\left(1-P_3 z^{-1}\right)\left(1-P_4 z^{-1}\right)} \qquad (13)$$

$$\therefore H(z)=\frac{P_1}{P_1+P_2}z^{-(N+1)}+\frac{P_2}{P_1+P_2}z^{-(N+2)} \qquad (14)$$

and $$Q(z)=\frac{1}{K(P_1+P_2)}\left(1-P_3 z^{-1}\right)\left(1-P_4 z^{-1}\right) \qquad (15)$$

Hence $G_c(z)=\dfrac{Q(z)}{1-H(z)}$

$$=\frac{1}{K(P_1+P_2)}\frac{\left(1-P_3 z^{-1}\right)\left(1-P_4 z^{-1}\right)}{\left[1-\frac{P_1}{P_1+P_2}z^{-(N+1)}-\frac{P_2}{P_1+P_2}z^{-(N+2)}\right]} \qquad (16)$$

It is to be noted that the transfer function coefficients of the controller transfer function are directly related to the coefficients in the process transfer function.

A Kalman controller may exhibit ringing. In such case, ringing free controller can be obtained by removing the ringing poles and adjusting the static gain of the controller to be the same.

Problem:

A process is modeled by a second order lag plus dead time model. The time constants are 0.377 sec. and 0.132 sec. The dead time is 0.3 sec. and the steady state gain is 1.24. Design a ringing free digital controller for the above process using Kalman's method. Assume a sampling period of 0.3 sec.

$$G(z)=\frac{0.448\left(1+0.362z^{-1}\right)z^{-2}}{\left(1-0.451z^{-1}\right)\left(1-0.103z^{-1}\right)}$$

$$G_c(z)\{=D(z)\}=1.637\frac{\left(1-0.554z^{-1}+0.046z^{-2}\right)}{1-0.734z^{-2}-0.266z^{-3}}$$

$$=1.637\frac{\left(1-0.553z^{-1}+0.046z^{-2}\right)}{\left(1-z^{-1}\right)\left(1+z^{-1}+0.266z^{-2}\right)}$$

$$Z=1\rightarrow D(z)=0.72\frac{\left(1-0.554z^{-1}+0.046z^{-2}\right)}{\left(1-z^{-1}\right)}$$

Ref.: Industrial Digital Control : Warwick and Rees.

Ringing of Digital Controller

For a digital controller to be a stable system, the poles of its z-transfer function should lie on or within the unit circle centred at origin on the z-plane.

[“Ringing” means oscillation of output of digital controller]

Let us consider a digital controller with transfer function

$$G_c(z)=\frac{1}{1-bz^{-1}}=\frac{M(z)}{E(z)}$$

where b is a real pole of the transfer function.

$$\therefore E(z)=M(z)-bz^{-1}M(z)$$
$$or, M(z)=E(z)+bz^{-1}M(z)$$
$$\therefore m_n=e_n+bm_{n-1}$$

Let us consider that an error equal to unit impulse enters the controller at t = 0. i.e.,

$e_n = 1$ for n = 0.

$e_n = 0$ for n = 1,2,3,………..

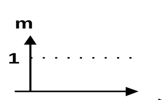
$\therefore\quad m_n = 1, \underset{\rightarrow}{\text{for}}\ n = 0$

$= b, \underset{\rightarrow}{\text{for}}\ n = 1$

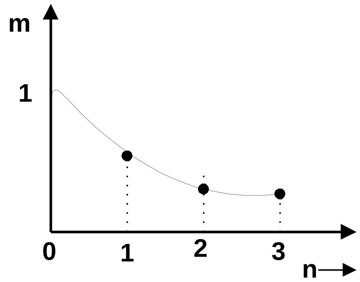
$= b^2, \underset{\rightarrow}{\text{for}}\ n = 2$

$\therefore\quad m_n = b^n$

If b = 1, m is a discrete-time step.



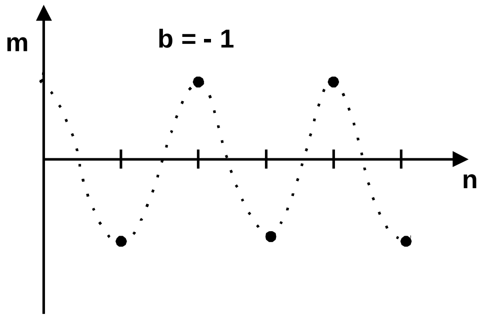
If b is a positive real number less than 1, m will gradually decay with time.

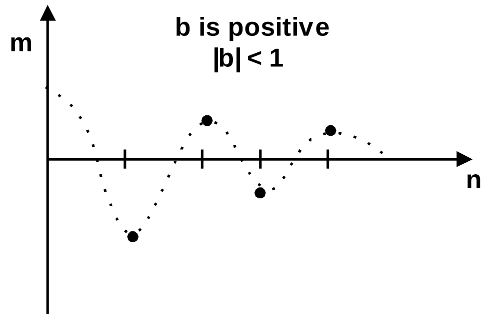


If b is a negative real number,

$$m_n = (-1)^n |b|^n$$

So there will be successive changes in the sign of the controller output.





(When b is negative, controller is ringing. When b is positive, controller is not ringing.)

This phenomenon is known as *controller ringing*. Smaller is the value of |b| more is the damping of the oscillations of the controller output.

n	$\mathbf{e_n}$	**b = - 0.9**	**b = - 0.3**
		$\mathbf{m_n}$	$\mathbf{m_n}$
0	1	1	1
1	0	- 0.9	- 0.3
2	0	0.81	0.09
3	0	- 0.729	- 0.027

Let us now consider a controller transfer function

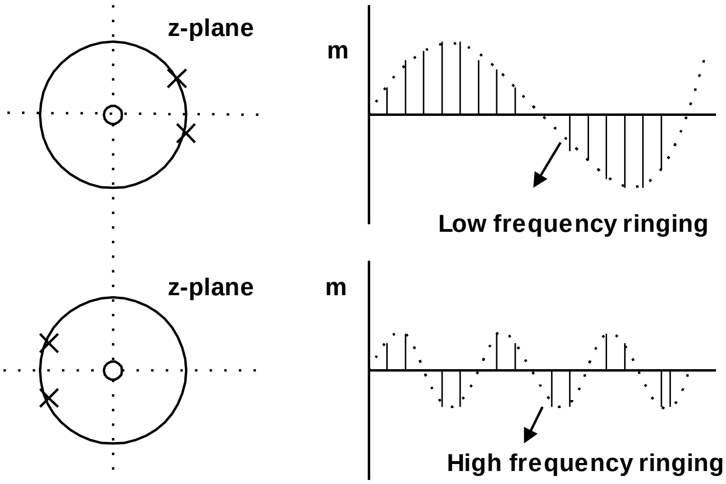
$$G_c(z) = \frac{M(z)}{E(z)} = \frac{ze^{-a\tau}\sin\omega\tau}{z^2 - ze^{-a\tau}z\cos\omega\tau + e^{-2a\tau}}$$

(For a discrete sequence of the nature $e^{-an\tau}\sin\omega n\tau$.)

If e is a discrete-time impulse, then E(z) = 1.

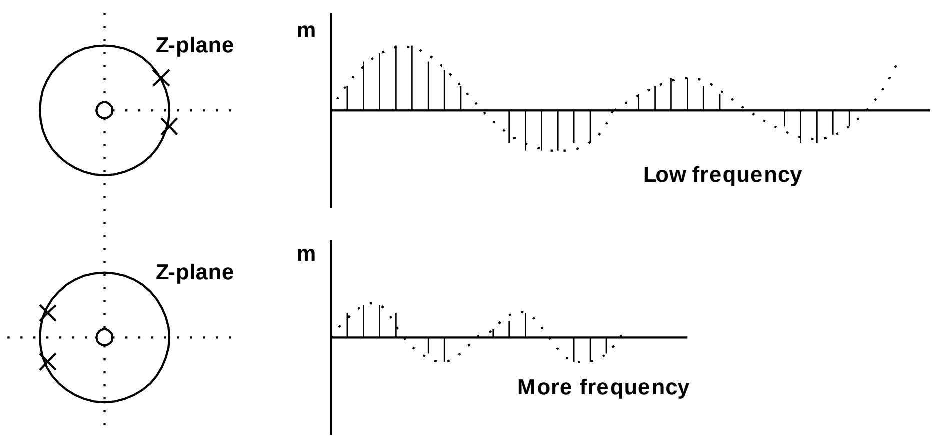
Then M(z) is the z-transform of the impulse response of the controller.

$G_c(z)$ has complex conjugate pole pair.



(With increasing ω, the poles shift from R.H.S. to the L.H.S.)

Location of the pole pair on the unit circle results in sustained oscillation of the controller output. However, if the pole pair is in the left half of z-plane, the frequency of oscillation is higher.



(As poles come nearer to the origin, damping increases.)

If the complex conjugate pole pair lies within the unit circle, there will be damped oscillation of the controller output.

So, complex conjugate pole pairs of the z-transfer function of a digital controller will also give rise to controller ringing.

It is clear from the discussions that,

1. Negative real poles and complex conjugate pole pairs of the controller's z-transfer function give rise to ringing. Ringing is more pronounced for complex conjugate pole pairs in left half of z-plane than that in right half of z-plane.
2. The closer a ringing pole is to the unit circle (i.e. the larger is its absolute value) the higher will be the ringing of the controller.

Ringing of the controller output will result in excessive movement of the control valve (or a similar piece of equipment) which is unacceptable in industrial practice. So some remedy to this problem must be found out.

Since Dahlin's design method does not take into consideration the controller output, it usually gives rise to controller transfer function poles that cause severe ringing.

Dahlin has suggested a method of overcoming the problem of controller ringing by removing the ringing poles and adjusting the static gain of the controller to be the same.

Example :- $G_c(z) = \dfrac{K\left(1-0.5z^{-1}\right)}{\left(1+0.6z^{-1}\right)\left(1-z^{-1}\right)\left(1-0.7z^{-1}\right)}$

Ringing pole $\Rightarrow$ z = - 0.6

$\therefore$ Ringing free controller is

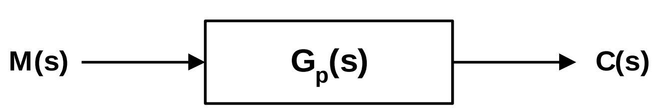
$$G_c(z) = \frac{K\left(1-0.5z^{-1}\right)\cancel{\left(1+0.6z^{-1}\right)}}{\lim\limits_{z\to 1}\left(1+0.6z^{-1}\right)\cancel{\left(1+0.6z^{-1}\right)}\left(1-z^{-1}\right)\left(1-0.7z^{-1}\right)}$$

$$= \frac{0.625K\left(1-0.5z^{-1}\right)}{\left(1-z^{-1}\right)\left(1-0.7z^{-1}\right)}$$

[From final value theorem, $\underset{n\to\infty}{Lt}\, f_n \Rightarrow \underset{z\to 1}{Lt}\, F(z)$

Only multiplying in numerator and denominator will change steady state gain and incorporate offset. So we keep $\left(\underset{z\to 1}{Lt}\right)$ to eliminate steady state error by using the final value theorem.]

s-domain



C(s) = G_P(s) M(s)

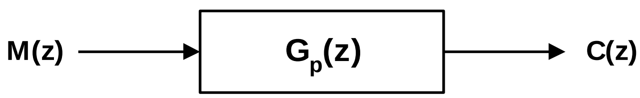
If m(t) is a unit step,

$$C(s) = \frac{1}{s} G_P(s)$$

Steady state gain = $c_{SS}(t)$

$$\underset{t\to\infty}{Lt}\, c(t) = \underset{s\to 0}{Lt}\, sC(s)$$
$$\Downarrow$$
$$\left[\text{Final value theorem}\right]$$

z-domain



C(z) = G_P(z) M(z)

If m_n is a unit step, response is

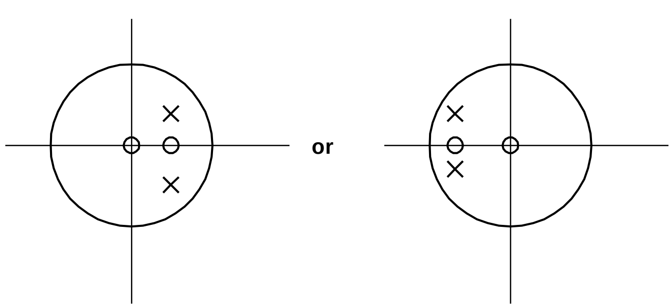
$$C(z) = \frac{1}{1 - z^{-1}} G_P(z)$$

Steady state gain c_{SSn}

$$= \underset{n\to\infty}{Lt}\, c_n = \underset{z\to 1}{Lt}\left(1 - z^{-1}\right) C(z)$$
$$\Downarrow$$
$$\left[\text{Final value theorem}\right]$$

$$= \underset{z\to 1}{Lt}\, G_P(z) \qquad G_c(z) = \frac{z\left(z - e^{-a\tau}\cos\omega\tau\right)}{z^2 - 2e^{-a\tau}\cos\omega\tau + e^{-2a\tau}}$$
$$= Z\left[e^{-an\tau}\cos\omega n\tau\right]$$

The graph will be a cosine graph and z-plane graph will have 2 zeroes.



The positions of the zeroes depend on both ω and a.