Discrete Fourier Transform

Prof. Anjan Rakshit and Prof. Amitava Chatterjee
Electrical Measurement and Instrumentation Laboratory,
Electrical Engineering Department,
Jadavpur University, Kolkata, India.

Discrete Fourier Transform

References

M. E. Van Valkenburg. Network Analysis. Prentice Hall.

J. G. Proakis and D. G. Manolakis. Digital Signal Processing: Principles, Algorithms and Applications. Prentice Hall. Fourth Edition.

Johnny R. Johnson. Introduction to Digital Signal Processing. Prentice Hall.

Li Tan. Digital Signal Processing: Fundamentals and Applications. Academic Press.

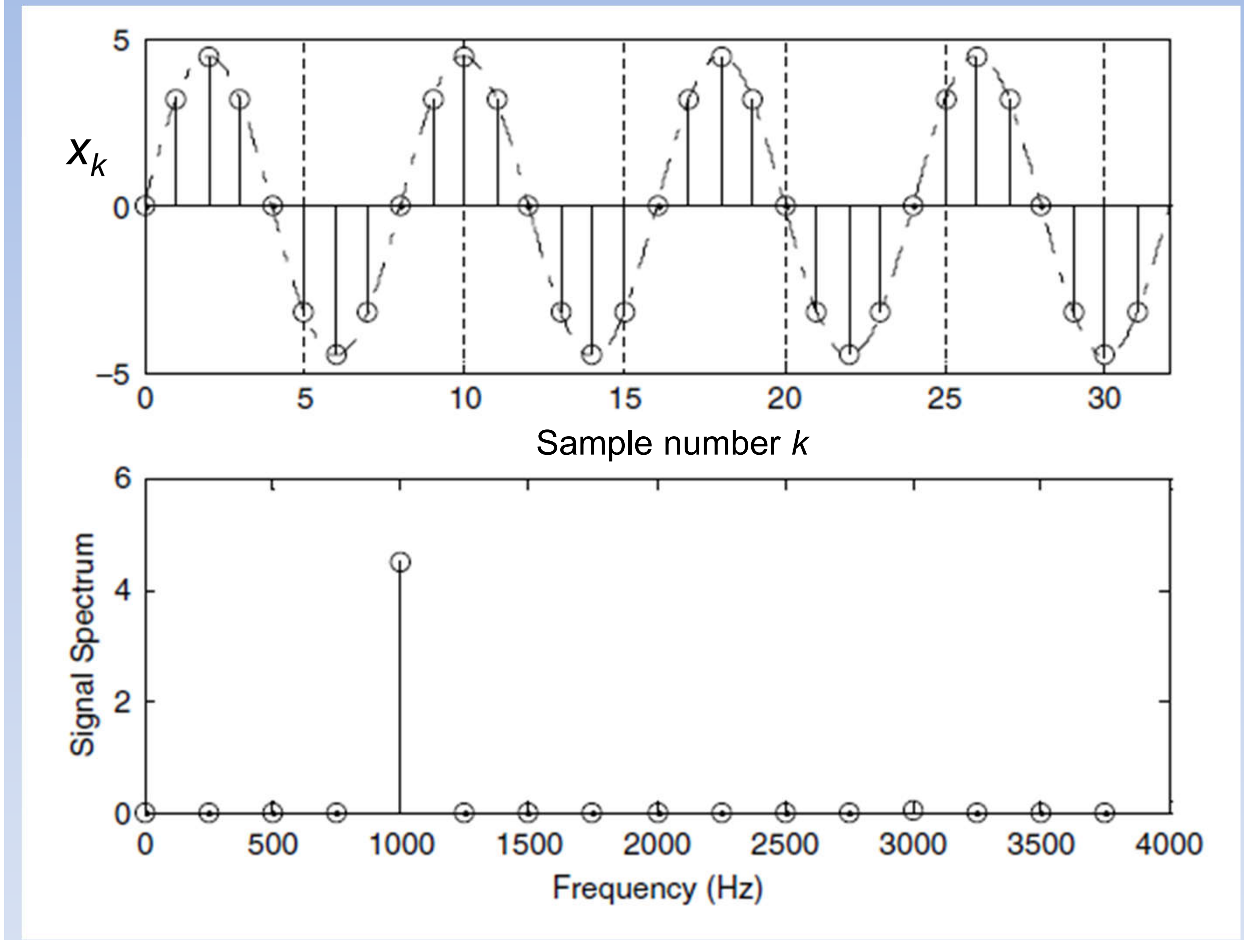
Discrete Fourier Transform

In time domain or sequence domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number.

However, in some applications, signal frequency content is more useful than the digital signal samples.

Hence representation of the digital signal in terms of its frequency components in frequency domain, i.e. the signal spectrum, needs to be developed.

Discrete Fourier Transform



Time domain representation of a 1,000-Hz sinusoid with 32 samples at a sampling rate of 8,000 Hz

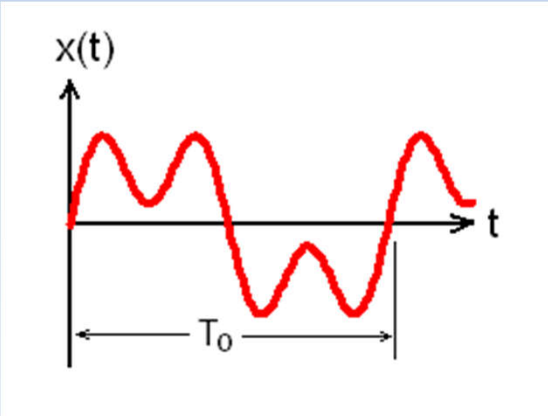
The corresponding signal spectrum i.e. the frequency domain representation

Conclusion:* *The spectral plot better displays frequency information of a digital signal.

Fourier series for a periodic signal

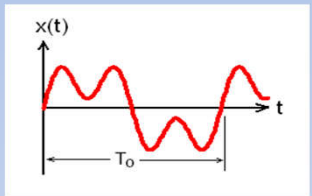
Let $x(t)$ be a periodic function of time having a time period T_0, then the fundamental frequency of $x(t)$ is

$$\omega_0 = \frac{2\pi}{T_0} \qquad \text{.......(1)}$$



Fourier series for a periodic signal

Let $x(t)$ be a periodic function of time having a time period T_0, then the fundamental frequency of $x(t)$ is



$$\omega_0 = \frac{2\pi}{T_0} \qquad \text{.......(1)}$$

The signal $x(t)$ may be expressed in terms of the Fourier series as

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \qquad \text{.......(2)}$$

Fourier series for a periodic signal

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \qquad \text{.......(2)}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,d(\omega_0 t) = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$, the average value

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos n\omega_0 t\,d(\omega_0 t) = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\,dt$$

for n = 1, 2, 3, …

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin n\omega_0 t\,d(\omega_0 t) = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\,dt$$

for n = 1, 2, 3, …

a_n*'*s are known as ***cosine coefficients*** and ***b_n*'*s*** are known as ***sine coefficients*.**

Fourier series for a periodic signal

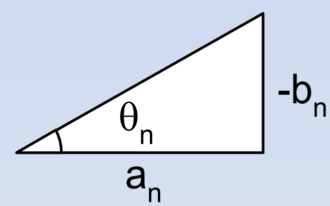
Relation (2) may be rewritten as

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t - \frac{(-b_n)}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t \right)$$

or $$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad \text{.......(3)}$$

$$\left[\cos(A+B) = \cos A \cos B - \sin A \sin B\right]$$

where $C_0 = a_0 \quad C_n = \sqrt{a_n^2 + b_n^2} \quad \theta_n = -\tan^{-1} \dfrac{b_n}{a_n}$



***C_n, n* = 1, 2, 3, … is the *amplitude* and *θ_n, n* = 1, 2, 3, … is the *phase* of the *n*th harmonic. *C_0* is the average value.**

Fourier series for a periodic signal

Expressing cosine and sine terms of relation (2) in terms of their complex exponential values as

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}) + b_n(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j})]$$

or $$x(t) = a_0 + \sum_{n=1}^{\infty}[e^{jn\omega_0 t}(\frac{a_n - jb_n}{2}) + e^{-jn\omega_0 t}(\frac{a_n + jb_n}{2})]$$

or $$x(t) = F_0 + \sum_{n=1}^{\infty}[F_n e^{jn\omega_0 t} + F_{-n} e^{-jn\omega_0 t}] \qquad \text{.......(4)}$$

where $F_0 = a_0$, $F_n = (\frac{a_n - jb_n}{2})$ and $F_{-n} = (\frac{a_n + jb_n}{2})$

Fourier series for a periodic signal

Now $$x(t) = F_0 + \sum_{n=1}^{\infty}[F_n e^{jn\omega_0 t} + F_{-n} e^{-jn\omega_0 t}]$$ ....(4)

where $F_0 = a_0$, $F_n = (\frac{a_n - jb_n}{2})$ and $F_{-n} = (\frac{a_n + jb_n}{2})$

Here $F_{-n} = \hat{F}_n$, conjugate of F_n.

Relation (4) may be expressed as

$$x(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} F_{-n} e^{-jn\omega_0 t}$$

Fourier series for a periodic signal

Now $$x(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} F_{-n} e^{-jn\omega_0 t}$$

$$x(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} F_n e^{jn\omega_0 t}$$

Hence, we can write, $$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad \text{.......(5)}$$

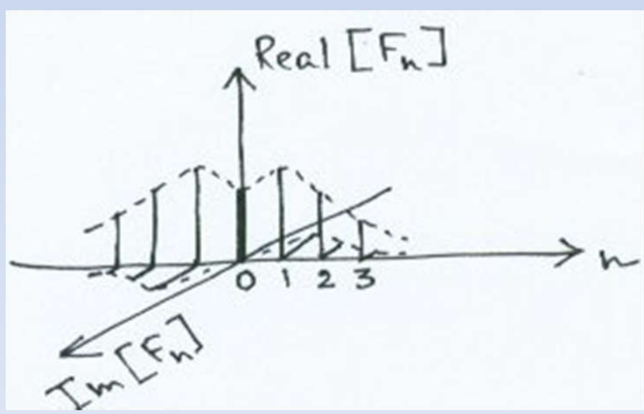
Thus $x(t)$ may be expressed in terms of Complex Fourier Series in relation (5). Here F_n is known as the *Complex Fourier coefficient*.

Fourier series for a periodic signal

Now $$x(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} F_{-n} e^{-jn\omega_0 t}$$

$$x(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} F_n e^{jn\omega_0 t}$$

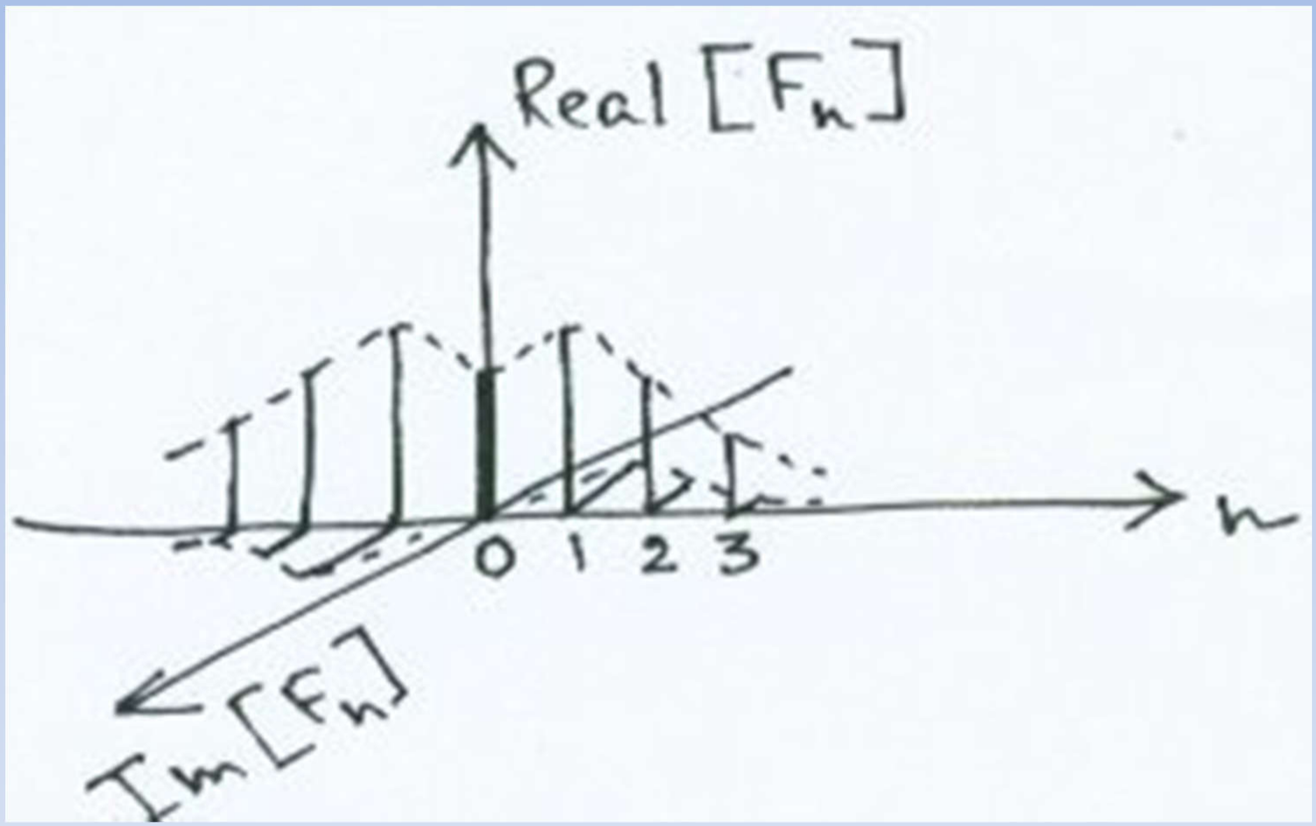
Hence, we can write, $$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad ....(5)$$



$$\leftarrow \quad F_{-n} = \hat{F}_n$$

Variation of F_n coefficients with n

Fourier series for a periodic signal



$$F_{-n} = \hat{F}_n$$

Variation of F_n coefficients with n

Fourier series for a periodic signal

The amplitudes C_n's of relation (3) may be related to F_n's as

$C_0 = F_0$, the average value

and $C_n = 2|F_n|$, for n = 1, 2, 3, … ....(6)

the amplitude of the nth harmonic.

and $$\theta_n = -\tan^{-1}\left(\frac{j(F_n - F_{-n})}{(F_n + F_{-n})}\right)$$

the phase of the nth harmonic.

Fourier series for a periodic signal

From relation (4), F_n may be expressed as

$$F_n = (\frac{a_n - jb_n}{2})$$

Substituting expressions of a_n and b_n from relation (2)

$$F_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}\,dt \qquad \text{.......(7)}$$

For *aperiodic signals*, the time period T_0 becomes infinite, and the *Fourier transform* of an aperiodic signal $x(t)$ is defined as

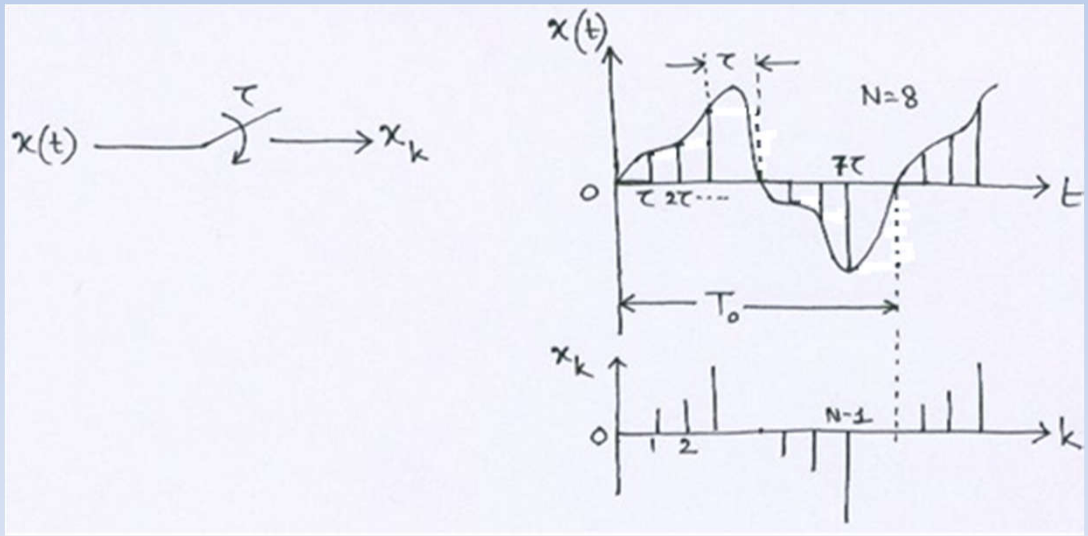
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad \text{.......(8)}$$

Fourier series for a periodic discrete sequence

Let x_k be a periodic discrete sequence obtained from a periodic signal $x(t)$ with a time period T_0.

Let N number of samples be available in the time period T_0 with a sampling interval τ. *The corresponding sampling frequency =* f_s *Hz.*

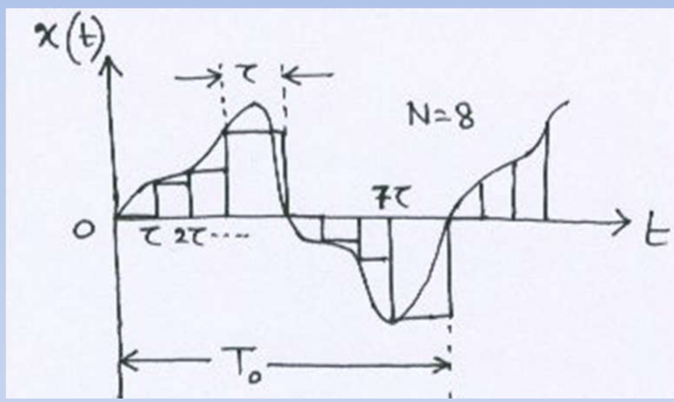
$$T_0 = N\tau \quad \text{and} \quad \tau = 1/f_s \quad \text{.......(9)}$$



***<u>Assumption</u>*: The periodic discrete sequence is band limited to have all harmonic frequencies less than the folding frequency (f_s/2) so that aliasing does not occur.**

Fourier series for a periodic discrete sequence

Using rectangular rule for integration, the Fourier coefficients may be obtained as



From relation (2)

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

$$a_0 = \frac{1}{T_0}\sum_{k=0}^{N-1} x_k \tau$$

or $$a_0 = \frac{1}{N\tau}\sum_{k=0}^{N-1} x_k \tau = \frac{1}{N}\sum_{k=0}^{N-1} x_k \quad \text{.......(10)}$$

Fourier series for a periodic discrete sequence

Using rectangular rule for integration, the Fourier coefficients may be obtained as

Similarly,

From relation (2)

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t dt$$

$$a_n = \frac{2}{N\tau}\sum_{k=0}^{N-1}\left[x_k \cos n\left(\frac{2\pi}{N\tau}\right)(k\tau)\right]\tau$$

or $$a_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \cos\left(\frac{2\pi kn}{N}\right)$$ …….(11)

(using the substitutions: $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N\tau}$ and $T_0 = N\tau$ and $t = k\tau$ in relation (2))

Fourier series for a periodic discrete sequence

Using rectangular rule for integration, the Fourier coefficients may be obtained as

and

From relation (2)

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\,dt$$

$$b_n = \frac{2}{N\tau}\sum_{k=0}^{N-1}\left[x_k \sin n\left(\frac{2\pi}{N\tau}\right)(k\tau)\right]\tau$$

or $$b_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \sin\left(\frac{2\pi kn}{N}\right) \qquad \text{.......(12)}$$

(using the substitutions: $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N\tau}$ and $T_0 = N\tau$ and $t = k\tau$ in relation (2))

Fourier series for a periodic discrete sequence

Now from relations (11) and (12),

$$\frac{a_n - jb_n}{2} = \frac{1}{N}\sum_{k=0}^{N-1} x_k \left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$

Hence, the **Fourier series coefficients** for the periodic discrete sequence are:

$F_0 = a_0$ and

$$F_n = \frac{a_n - jb_n}{2} = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}, \quad n = \pm 1, \pm 2, \pm 3, \cdots \qquad \text{.......(12a)}$$

Since the coefficients F_n are obtained from the Fourier series expansion in the complex form, the resultant spectrum F_n will have two sides.

Fourier series for a periodic discrete sequence

Now from relations (11) and (12),

$$\frac{a_n - jb_n}{2} = \frac{1}{N}\sum_{k=0}^{N-1} x_k \left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$

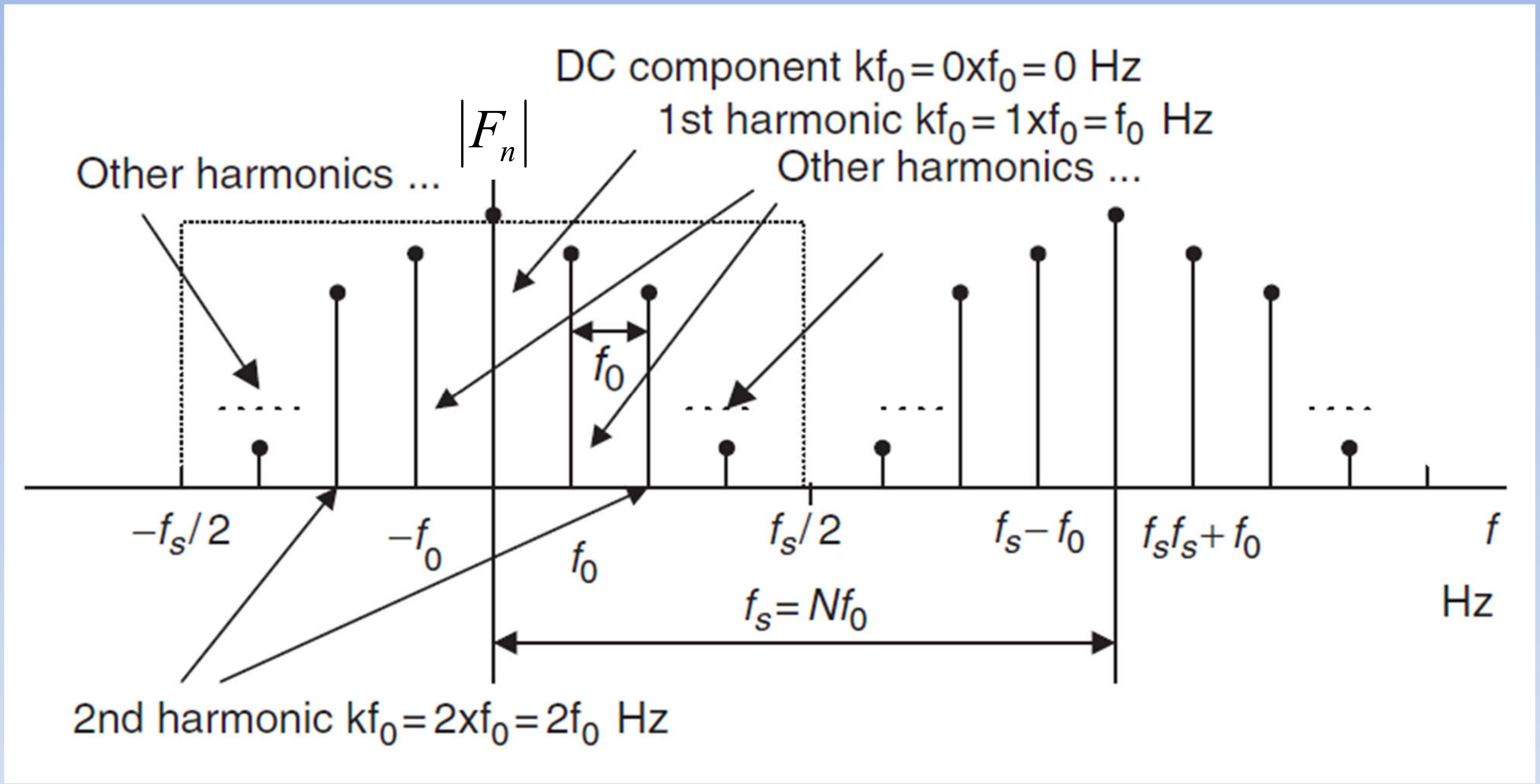
Hence, the **Fourier series coefficients** for the periodic discrete sequence are:

$F_0 = a_0$ and

$$F_n = \frac{a_n - jb_n}{2} = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}, \quad n = \pm 1, \pm 2, \pm 3, \cdots \qquad \text{.......(12a)}$$

It can be shown that $F_{n+N} = F_n$. Hence the Fourier series coefficients F_n are periodic having a periodicity of N.

Fourier series for a periodic discrete sequence

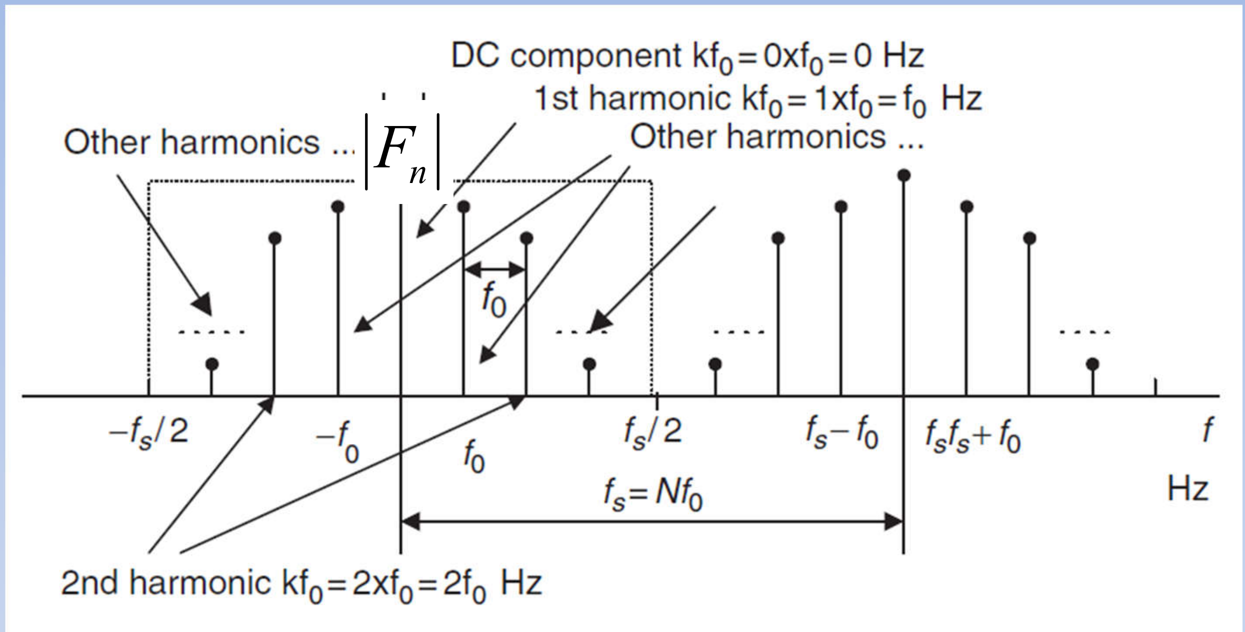


Amplitude spectrum of a representative periodic signal

For the *kth* harmonic, the frequency is $f=kf_0$. The frequency spacing between the consecutive spectral lines, called the frequency resolution, is f_0 Hz.

As $F_{n+N} = F_n$, the two-sided line amplitude spectrum $|F_n|$ is periodic.

Fourier series for a periodic discrete sequence

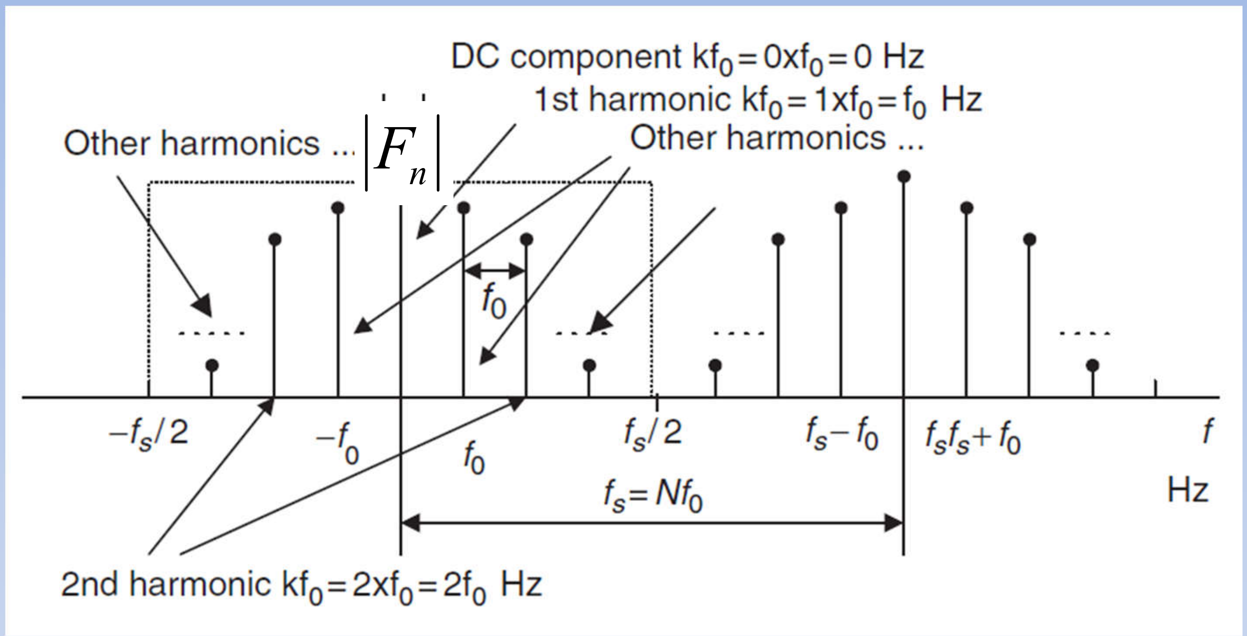


OBSERVATIONS:

- ***Only the line spectral portion between the frequency $-f_s/2$ and frequency $f_s/2$ (folding frequency) represents the frequency information of the periodic signal.***

- ***The spectral portion from $f_s/2$ to f_s is a copy of the spectrum in the negative frequency range from $-f_s/2$ to 0 Hz due to the spectrum being periodic for every Nf_0 Hz.***

Fourier series for a periodic discrete sequence



OBSERVATIONS:

- ***For convenience, we compute the spectrum over the range from 0 to f_s Hz with nonnegative indices, i.e.,***

$$F_n = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}, \quad n = 0,1,2,3,\cdots,N-1 \qquad \text{.......(12b)}$$

- ***If negative indexed spectral values are needed, those can be obtained using the relation: $F_{n+N} = F_n$.***

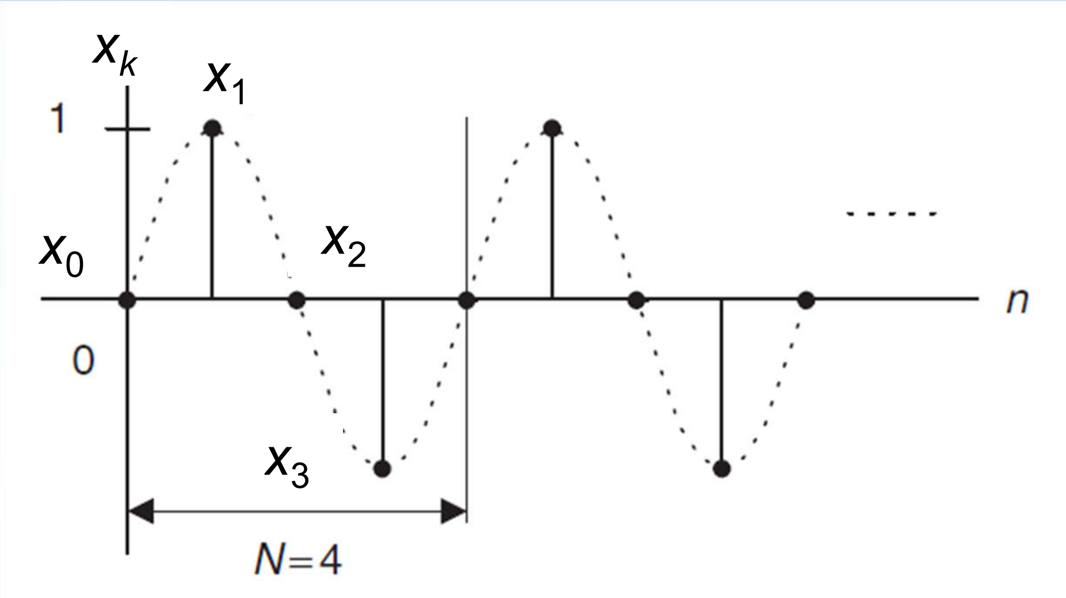
Fourier series for a periodic discrete sequence

Problem 1

Let us consider a periodic signal $x(t) = \sin(2\pi t)$, sampled using a sampling rate of $f_s = 4$ Hz.
(i) Compute the Fourier coefficients or spectrum F_n using the samples in one period.
(ii) Plot the two-sided amplitude spectrum $|F_n|$ over the range from -2 to 2 Hz.

Solution

From the analog signal, we get fundamental frequency $\omega_0 = 2\pi$ rad/s.
Hence $f_0 = (\omega_0/2\pi) = 1$ Hz and fundamental time period $T_0 = 1$ s.
Sampling interval $\tau = 1/f_s = 0.25$ s.
Hence sampled signal = $x_k = x(k\tau) = \sin(2\pi k\tau) = \sin(0.5\pi k)$



First eight samples of the periodic digital signal

Fourier series for a periodic discrete sequence

Problem 1

Let us consider a periodic signal $x(t) = \sin(2\pi t)$, sampled using a sampling rate of $f_s = 4$ Hz.
(i) Compute the Fourier coefficients or spectrum F_n using the samples in one period.
(ii) Plot the two-sided amplitude spectrum $|F_n|$ over the range from -2 to 2 Hz.

Solution (contd.)

For a duration of one period, $N = 4$. The sample values are: $x_0=0$, $x_1=1$, $x_2=0$, $x_3=-1$.
From the expression of F_n in relation (12a), we can compute:

$$F_0 = \frac{1}{4}\sum_{k=0}^{3} x_k = \frac{1}{4}\left(x_0 + x_1 + x_2 + x_3\right) = \frac{1}{4}\left(0+1+0-1\right) = 0$$

$$F_1 = \frac{1}{4}\sum_{k=0}^{3} x_k e^{-j2\pi\times\left(1k/4\right)} = \frac{1}{4}\left(x_0 + x_1 e^{-j\pi/2} + x_2 e^{-j\pi} + x_3 e^{-j3\pi/2}\right)$$

$$= \frac{1}{4}\left(x_0 - jx_1 - x_2 + jx_3\right) = \frac{1}{4}\left(0 - j1 - 0 + j(-1)\right) = -j0.5$$

Fourier series for a periodic discrete sequence

Problem 1

Let us consider a periodic signal $x(t) = \sin(2\pi t)$, sampled using a sampling rate of $f_s = 4$ Hz.
(i) Compute the Fourier coefficients or spectrum F_n using the samples in one period.
(ii) Plot the two-sided amplitude spectrum $|F_n|$ over the range from -2 to 2 Hz.

Solution (contd.)

Similarly we get:

$$F_2 = \frac{1}{4}\sum_{k=0}^{3} x_k e^{-j2\pi\times\left(2k/4\right)} = 0 \quad \text{and} \quad F_3 = \frac{1}{4}\sum_{k=0}^{3} x_k e^{-j2\pi\times\left(3k/4\right)} = j0.5$$

Using periodicity, it follows that:

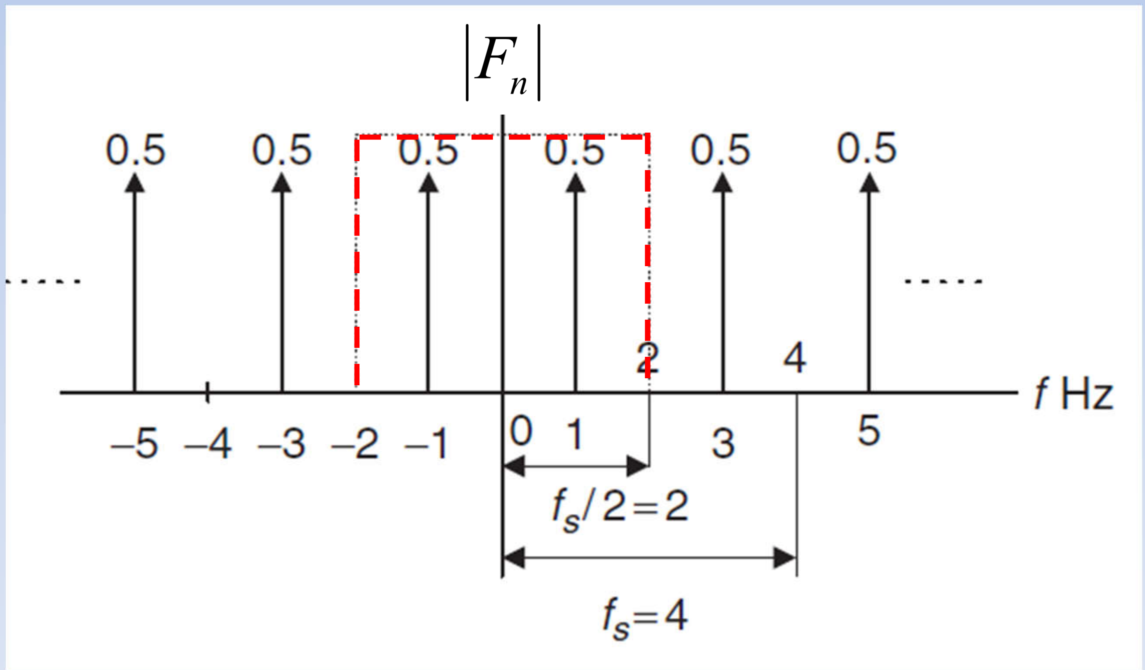
$$F_{-1} = F_3 = j0.5 \qquad \text{and} \qquad F_{-2} = F_2 = 0$$

Fourier series for a periodic discrete sequence

Problem 1

Let us consider a periodic signal $x(t) = \sin(2\pi t)$, sampled using a sampling rate of $f_s = 4$ Hz.
(*i*) Compute the Fourier coefficients or spectrum F_n using the samples in one period.
(*ii*) Plot the two-sided amplitude spectrum $|F_n|$ over the range from -2 to 2 Hz.

Solution (contd.)



Two sided amplitude spectrum $|F_n|$ for the periodic digital signal

Fourier series for a periodic discrete sequence

Now, from relation (12a), we can write,

$$\left(\frac{N}{2}\right)(a_n - jb_n) = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$

Substituting $Na_0 = X_0$ and $\left(\frac{N}{2}\right)(a_n - jb_n) = X_n$, for $n = \pm 1, \pm 2, \pm 3, \ldots$

$$X_n = NF_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$ for $n = 0, \pm 1, \pm 2, \ldots$ ....(13)

Fourier series for a periodic discrete sequence

From relation (13) $X_n = NF_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$ for $n = 0, \pm 1, \pm 2, \ldots$

Now, let us consider $n = N + m$, for $m = 0, \pm 1, \pm 2, \ldots$

$$X_n = \sum_{k=0}^{N-1} x_k e^{-j(N+m)\left(\frac{2\pi k}{N}\right)}$$

or $$X_{m+N} = \sum_{k=0}^{N-1} x_k e^{-j(2\pi k)} . e^{-jm\left(\frac{2\pi k}{N}\right)}$$

or $$X_{m+N} = \sum_{k=0}^{N-1} x_k e^{-jm\left(\frac{2\pi k}{N}\right)} = X_m \qquad \text{.......(14)}$$

***Conclusion*: X_n is periodic with a period *N*.**

Fourier series for a periodic discrete sequence

Then, within one period (i.e. for n = 0, 1, 2, …, N-1),

$$X_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$, for n = 0, 1, 2, …., N-1 …….(15)

***Conclusion*: Relation (15) is known as the Discrete Fourier Transform (DFT) of a finite sequence x_k, k = 0, 1, 2, …, N-1.**

The X_n constitutes the DFT coefficients.

Fourier series for a periodic discrete sequence

Relation (14) represents the **periodicity property of DFT**.

***X_n* repeats at the *N*th harmonic.**

The frequency corresponding to the *N*th harmonic is:

$$Nf_0 = \frac{N}{T_0} = \frac{N}{N\tau} = \frac{1}{\tau} = f_s$$, the sampling frequency.

***Conclusion*: *X_n* repeats at the sampling frequency *f_s*.**

Discrete Fourier Transform

The ***Discrete Fourier Transform*** (**DFT**) of a finite sequence x_k, k = 0, 1, 2, …, N-1 is defined as

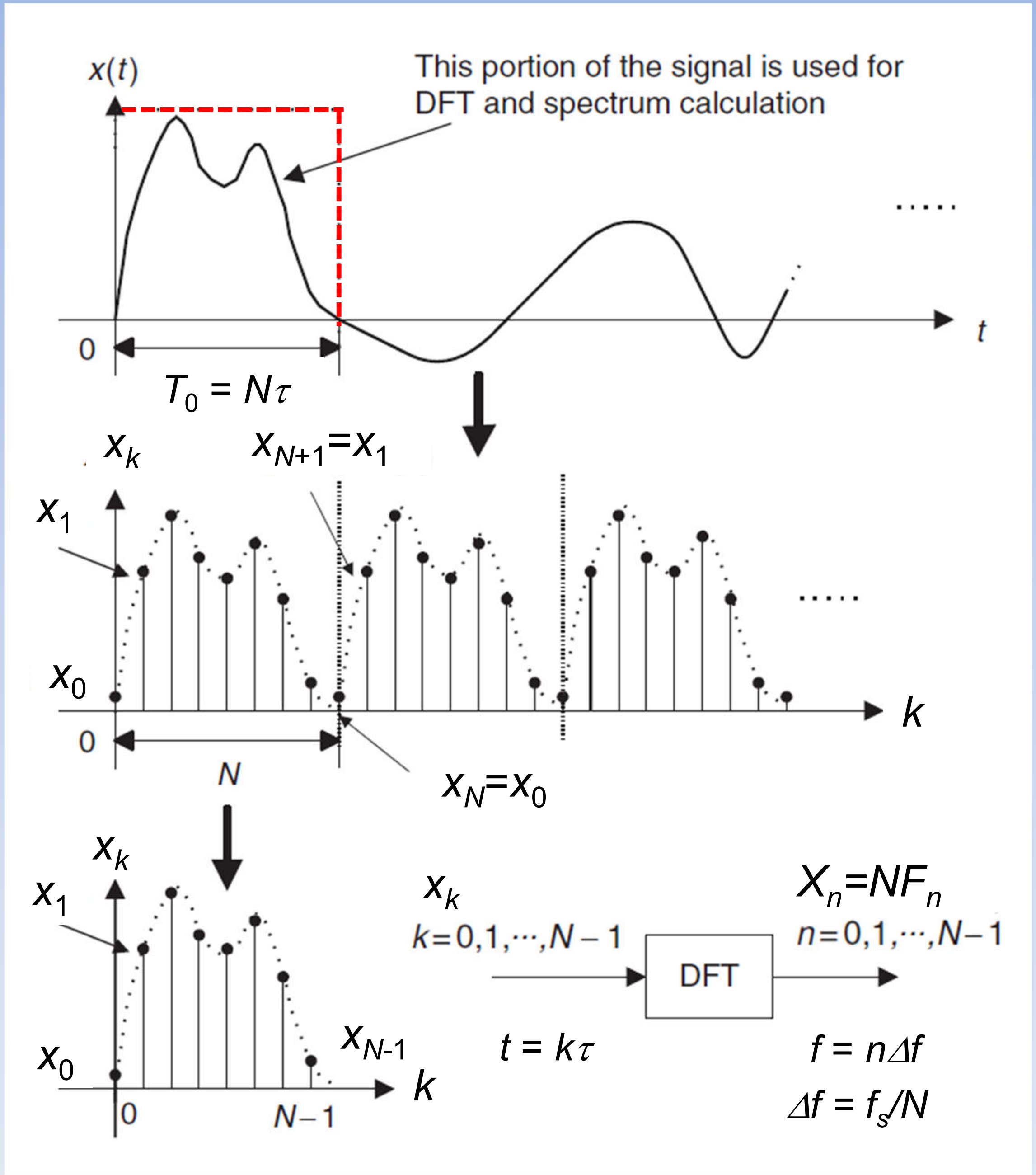
$$X_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$, for n = 0, 1, 2, …., N-1 …….(15)

Amplitude C_n (c.f. relation (3)) is related to X_n as

$$C_0 = \frac{1}{N}\left|X_0\right|$$, the average value

and $$C_n = \frac{2}{N}\left|X_n\right|$$, for n = 1, 2, 3, … …….(16)

Discrete Fourier Transform



The development of the DFT formula

Inverse Discrete Fourier Transform

Relation (15):

$$X_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}$$, for n = 0, 1, 2, …., N-1

By multiplying $\frac{1}{N} e^{jn\left(\frac{2\pi l}{N}\right)}$

on both sides of relation (15) and summing up from n = 0 to N-1 with $0 \le l < N$

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)} . e^{jn\left(\frac{2\pi l}{N}\right)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} x_k e^{jn(l-k)\frac{2\pi}{N}}$$

Now, changing the order of summation,

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = \sum_{k=0}^{N-1} x_k \left[\frac{1}{N}\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}\right]$$ …….(17)

Inverse Discrete Fourier Transform

Now, in $\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}$, when $(l-k) = pN$

where *p* is a positive integer, the expression becomes $\sum_{n=0}^{N-1} e^{jn2\pi p}$

As *np* is another integer, it becomes $\sum_{n=0}^{N-1} e^{j2\pi(np)} = \sum_{n=0}^{N-1} 1 = N$

In the present case, as *l* and *k* are limited within 0 and (*N* – 1), the possible value of *p* is zero, i.e. when **(*l* - *k*) = 0 or *l* = *k*, the summation becomes *N***.

Inverse Discrete Fourier Transform

Now, in $\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}$, let $\frac{2\pi(l-k)}{N} = \theta$

Then the summation becomes $\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}$ = $\sum_{n=0}^{N-1} e^{jn\theta}$

It may be expressed as

$$\sum_{n=0}^{N-1} e^{jn\theta} = \sum_{m=1}^{N} e^{j(m-1)\theta}$$, where $m = n + 1$

$$= \sum_{m=1}^{N} e^{jm\theta} . e^{-j\theta}$$

Inverse Discrete Fourier Transform

or $$\sum_{n=0}^{N-1} e^{jn\theta} = \sum_{m=1}^{N} e^{jm\theta} . e^{-j\theta}$$

$$= e^{-j\theta}\left[\sum_{m=1}^{N}\left(\cos m\theta + j\sin m\theta\right)\right]$$

$$= e^{-j\theta}\left[\frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}\cos\left(\frac{N+1}{2}\theta\right) + j\frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}\sin\left(\frac{N+1}{2}\theta\right)\right]$$

$$= e^{-j\theta}\left[\frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}} e^{j\frac{N+1}{2}\theta}\right]$$

Inverse Discrete Fourier Transform

or $$\sum_{n=0}^{N-1} e^{jn\theta} = e^{-j\theta}\left[\frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}} e^{j\frac{N+1}{2}\theta}\right]$$

$$= \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}} e^{j\frac{N-1}{2}\theta}$$

$$= \frac{\left(e^{\frac{jN\theta}{2}} - e^{\frac{-jN\theta}{2}}\right)\Big/ 2j}{\left(e^{\frac{j\theta}{2}} - e^{\frac{-j\theta}{2}}\right)\Big/ 2j} \cdot \frac{e^{\frac{jN\theta}{2}}}{e^{\frac{j\theta}{2}}} \qquad = \frac{e^{jN\theta} - 1}{e^{j\theta} - 1}$$

Inverse Discrete Fourier Transform

or $$\sum_{n=0}^{N-1} e^{jn\theta} = \frac{e^{jN\theta} - 1}{e^{j\theta} - 1}$$

Putting the value of θ,

$$\sum_{n=0}^{N-1} e^{jn2\pi\frac{(l-k)}{N}} = \frac{e^{j2\pi(l-k)} - 1}{e^{j2\pi\frac{(l-k)}{N}} - 1}$$

Now for ***l* ≠ *k*, the summation is zero**.

And for ***l* = *k*, it becomes indeterminate** $\left(\frac{0}{0}\right)$ **form.**

Inverse Discrete Fourier Transform

Thus, $$\sum_{n=0}^{N-1} e^{jn2\pi\frac{(l-k)}{N}} = N, \text{for } l = k$$

$$= 0, \text{for } l \neq k$$

considering $0 \leq l, k < N$

Thus all terms on the right hand side of relation (17) vanishes except when *l* = *k*.

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = \sum_{k=0}^{N-1} x_k \left[\frac{1}{N}\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}\right] \quad \text{.......(17)}$$

Inverse Discrete Fourier Transform

Thus, $$\sum_{n=0}^{N-1} e^{jn2\pi\frac{(l-k)}{N}} = N, \text{for}\, l = k$$

$$= 0, \text{for}\, l \neq k$$

considering $0 \leq l, k < N$

Thus all terms on the right hand side of relation (17) vanishes except when *l* = *k*.

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = \sum_{k=0}^{N-1} x_k \left[\frac{1}{N}\sum_{n=0}^{N-1} e^{jn\frac{2\pi(l-k)}{N}}\right] \quad \text{........(17)}$$

Therefore,

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = x_l\left(\frac{N}{N}\right) = x_l, \text{ for } l = 0,1,2,...,N\text{-}1$$

Inverse Discrete Fourier Transform

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi l}{N}\right)} = x_l\left(\frac{N}{N}\right) = x_l, \text{ for } l = 0,1,2,...,N\text{-}1$$

Now, changing the suffix *l* to *k*,

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi k}{N}\right)}, \text{ for } k = 0,1,2,...,N\text{-}1 \qquad ....(18)$$

Relation (18) is known as the **Inverse Discrete Fourier Transform (IDFT)**.

Relations (15) and (18) are called ***N*-point DFT pair**.

N-point DFT pair

N-point DFT:

$$X_n = \sum_{k=0}^{N-1} x_k e^{-jn\left(\frac{2\pi k}{N}\right)}, \text{ for } n = 0,1,2,\ldots, N\text{-}1 \qquad \text{.......(15)}$$

N-point IDFT:

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{jn\left(\frac{2\pi k}{N}\right)}, \text{ for } k = 0,1,2,\ldots,N\text{-}1 \qquad \text{.......(18)}$$

Replacing the expression $e^{-j\left(\frac{2\pi}{N}\right)}$

by the term W_N, the DFT pair takes the form

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1 \qquad \text{.......(19)}$$

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n W_N^{-nk}, \text{ for } k = 0,1,2,\ldots,N\text{-}1 \qquad \text{.......(20)}$$

N-point DFT pair

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1$$ ....(19)

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n W_N^{-nk}, \text{ for } k = 0,1,2,\ldots,N\text{-}1$$ ....(20)

where $W_N = e^{-j\left(\frac{2\pi}{N}\right)}$

a complex operator (**twiddle factor**), which rotates any vector through $\left(-2\pi/N\right)$ Radians.

$$W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

***Here, n* = harmonic number and *k* = sample number.**

DFT and IDFT

X = fft(x)	% Calculate DFT coefficients
x = ifft(X)	% Inverse DFT
x = input vector	
X = DFT coefficient vector	

MATLAB FFT functions

DFT and IDFT

Problem 2

A sequence x_k, for k = 0,1,2,3, is given as: $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$. Evaluate its DFT X_n.

Solution

Here $N = 4$. Hence $W_N = W_4 = e^{-j\left(\frac{2\pi}{4}\right)} = e^{-j\left(\frac{\pi}{2}\right)}$

Therefore, $X_n = \sum_{k=0}^{3} x_k W_4^{nk} = \sum_{k=0}^{3} x_k e^{-j\frac{\pi nk}{2}}$

For $n = 0$, $X_0 = \sum_{k=0}^{3} x_k e^{-j0} = x_0 e^{-j0} + x_1 e^{-j0} + x_2 e^{-j0} + x_3 e^{-j0}$

$$= x_0 + x_1 + x_2 + x_3 = 1 + 2 + 3 + 4 = 10$$

For $n = 1$, $X_1 = \sum_{k=0}^{3} x_k e^{-j\frac{\pi k}{2}} = x_0 e^{-j0} + x_1 e^{-j\frac{\pi}{2}} + x_2 e^{-j\pi} + x_3 e^{-j\frac{3\pi}{2}}$

$$= x_0 - jx_1 - x_2 + jx_3 = 1 - j2 - 3 + j4 = -2 + j2$$

DFT and IDFT

Problem 2

A sequence x_k, for k = 0,1,2,3, is given as: $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.
Evaluate its DFT X_n.

Solution (contd.)

Here $N = 4$. Hence $W_N = W_4 = e^{-j\left(\frac{2\pi}{4}\right)} = e^{-j\left(\frac{\pi}{2}\right)}$

Therefore, $X_n = \sum_{k=0}^{3} x_k W_4^{nk} = \sum_{k=0}^{3} x_k e^{-j\frac{\pi nk}{2}}$

For $n = 2$, $X_2 = \sum_{k=0}^{3} x_k e^{-j\frac{2\pi k}{2}} = x_0 e^{-j0} + x_1 e^{-j\pi} + x_2 e^{-j2\pi} + x_3 e^{-j3\pi}$

$$= x_0 - x_1 + x_2 - x_3 = 1 - 2 + 3 - 4 = -2$$

For $n = 3$, $X_3 = \sum_{k=0}^{3} x_k e^{-j\frac{3\pi k}{2}} = x_0 e^{-j0} + x_1 e^{-j\frac{3\pi}{2}} + x_2 e^{-j3\pi} + x_3 e^{-j\frac{9\pi}{2}}$

$$= x_0 + jx_1 - x_2 - jx_3 = 1 + j2 - 3 - j4 = -2 - j2$$

DFT and IDFT

Problem 2

A sequence x_k, for k = 0,1,2,3, is given as: $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.
Evaluate its DFT X_n.

Solution (contd.)

This result can be verified in **MATLAB®** as:

$$\gg X = \text{fft}([1\ 2\ 3\ 4])$$

$$X = 10.0000 \quad -2.0000 + 2.0000i \quad -2.0000 \quad -2.0000 - 2.0000i$$

DFT and IDFT

Problem 3

Using the DFT coefficients X_n, for n = 0,1,2,3, computed in the previous problem, evaluate its inverse DFT to determine the time domain sequence x_k.

Solution

Here N = 4. Hence $W_N^{-1} = W_4^{-1} = e^{j\left(\frac{2\pi}{4}\right)} = e^{j\left(\frac{\pi}{2}\right)}$

Therefore, $x_k = \frac{1}{4}\sum_{n=0}^{3} X_n W_4^{-nk} = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{\pi nk}{2}}$

For k = 0,

$$x_0 = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j0} = \frac{1}{4}\left(X_0 e^{j0} + X_1 e^{j0} + X_2 e^{j0} + X_3 e^{j0}\right)$$

$$= \frac{1}{4}\left(X_0 + X_1 + X_2 + X_3\right)$$

$$= \frac{1}{4}\left(10 + (-2 + j2) - 2 + (-2 - j2)\right) = 1$$

DFT and IDFT

Problem 3

Using the DFT coefficients X_n, for n = 0,1,2,3, computed in the previous problem, evaluate its inverse DFT to determine the time domain sequence x_k.

Solution (contd.)

Here N = 4. Hence $W_N^{-1} = W_4^{-1} = e^{j\left(\frac{2\pi}{4}\right)} = e^{j\left(\frac{\pi}{2}\right)}$

Therefore, $x_k = \frac{1}{4}\sum_{n=0}^{3} X_n W_4^{-nk} = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{\pi nk}{2}}$

For k = 1,

$$x_1 = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{n\pi}{2}} = \frac{1}{4}\left(X_0 e^{j0} + X_1 e^{j\frac{\pi}{2}} + X_2 e^{j\pi} + X_3 e^{j\frac{3\pi}{2}} \right)$$

$$= \frac{1}{4}\left(X_0 + jX_1 - X_2 - jX_3 \right)$$

$$= \frac{1}{4}\left(10 + j(-2 + j2) + 2 - j(-2 - j2)\right) = 2$$

DFT and IDFT

Problem 3

Using the DFT coefficients X_n, for n = 0,1,2,3, computed in the previous problem, evaluate its inverse DFT to determine the time domain sequence x_k.

Solution (contd.)

Here N = 4. Hence $W_N^{-1} = W_4^{-1} = e^{j\left(\frac{2\pi}{4}\right)} = e^{j\left(\frac{\pi}{2}\right)}$

Therefore, $x_k = \frac{1}{4}\sum_{n=0}^{3} X_n W_4^{-nk} = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{\pi nk}{2}}$

For k = 2,

$$x_2 = \frac{1}{4}\sum_{n=0}^{3} X_n e^{jn\pi} = \frac{1}{4}\left(X_0 e^{j0} + X_1 e^{j\pi} + X_2 e^{j2\pi} + X_3 e^{j3\pi}\right)$$

$$= \frac{1}{4}\left(X_0 - X_1 + X_2 - X_3\right)$$

$$= \frac{1}{4}\left(10 - \left(-2 + j2\right) + \left(-2\right) - \left(-2 - j2\right)\right) = 3$$

DFT and IDFT

Problem 3

Using the DFT coefficients X_n, for n = 0,1,2,3, computed in the previous problem, evaluate its inverse DFT to determine the time domain sequence x_k.

Solution (contd.)

Here N = 4. Hence $W_N^{-1} = W_4^{-1} = e^{j\left(\frac{2\pi}{4}\right)} = e^{j\left(\frac{\pi}{2}\right)}$

Therefore, $x_k = \frac{1}{4}\sum_{n=0}^{3} X_n W_4^{-nk} = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{\pi nk}{2}}$

For k = 3,

$$x_3 = \frac{1}{4}\sum_{n=0}^{3} X_n e^{j\frac{3n\pi}{2}} = \frac{1}{4}\left(X_0 e^{j0} + X_1 e^{j\frac{3\pi}{2}} + X_2 e^{j3\pi} + X_3 e^{j\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}\left(X_0 - jX_1 - X_2 + jX_3\right)$$

$$= \frac{1}{4}\left(10 - j(-2 + j2) - (-2) + j(-2 - j2)\right) = 4$$

DFT and IDFT

Problem 3

Using the DFT coefficients X_n, for n = 0,1,2,3, computed in the previous problem, evaluate its inverse DFT to determine the time domain sequence x_k.

Solution (contd.)

This result can be verified in **MATLAB®** as:

$$\gg \mathrm{x} = \mathrm{ifft}([10 \quad -2+2j \quad -2 \quad -2-2j])$$

$$\mathrm{x} = 1 \qquad 2 \qquad 3 \qquad 4.$$

Important Properties of DFT

Periodicity

From relation (19),

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1, \text{ where } W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

Then,

$$X_{n+pN} = \sum_{k=0}^{N-1} x_k W_N^{(n+pN)k} \quad \text{for} \quad p = 0, \pm 1, \pm 2, \ldots.$$

$$= \sum_{k=0}^{N-1} x_k W_N^{nk}, \quad \text{as } W_N^{pNk} = W_N^{N(pk)} = 1$$

$$= X_n$$

i.e. $X_{n+pN} = X_n \quad \text{for} \quad p = 0, \pm 1, \pm 2, \ldots.$ ....(21)

Thus X_n is **periodic** with a period ***N***, i.e. the ***pN***th harmonic or at the ***p*** times sampling frequency, the DFT repeats.

Important Properties of DFT

Linearity

If $x_{1k} \xleftrightarrow[N]{\text{DFT}} X_{1n}$ and $x_{2k} \xleftrightarrow[N]{\text{DFT}} X_{2n}$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1 x_{1k} + a_2 x_{2k} \xleftrightarrow[N]{\text{DFT}} a_1 X_{1n} + a_2 X_{2n} \quad \text{....}$$

This property follows immediately from the definition of DFT given in (19).

Important Properties of DFT

Circular symmetries of a sequence

The *N*-point DFT of a finite duration sequence x_k of length $L \leq N$, is equivalent to the *N*-point DFT of a periodic sequence x_{pk} of period *N*, which is obtained by periodically extending x_k i.e.

$$x_{pk} = \sum_{l=-\infty}^{\infty} x_{k-lN} \qquad \text{.......(21a)}$$

Let us assume that the periodic sequence x_{pk} is shifted by m units to the right. Thus we obtain another periodic sequence, given as:

$$x'_{pk} = x_{p(k-m)} = \sum_{l=-\infty}^{\infty} x_{k-m-lN} \qquad \text{.......(21b)}$$

The finite duration sequence

$$x'_k = \begin{cases} x'_{pk}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{.......(21c)}$$

Is related to the original sequence x_k by a **circular shift**.

Important Properties of DFT

Circular symmetries of a sequence

In general, the circular shift of the sequence can be represented as the index modulo *N*. Thus we can write,

$$x'_k = x_{(k-m,\ \text{modulo}\ N)} \equiv x_{(k-m)_N} \qquad \text{.......(21d)}$$

For example, let us assume $m = 2$ and $N = 4$. Then we have,

$$x'_k = x_{(k-2)_4}$$

This implies that

$$x'_0 = x_{(-2)_4} = x_2$$

$$x'_1 = x_{(-1)_4} = x_3$$

$$x'_2 = x_{(0)_4} = x_0$$

$$x'_3 = x_{(1)_4} = x_1$$

Hence x'_k is simply x_k shifted circularly by two units in time, where counterclockwise direction has been arbitrarily selected as the positive direction.

Important Properties of DFT

Circular symmetries of a sequence

Hence we can conclude that **a circular shift of an *N*-point sequence is equivalent to a linear shift of its periodic extension, and vice versa**.

The inherent periodicity resulting from the arrangement of the *N*-point sequence on the circumference of a circle dictates a different definition of even and odd symmetry, and time reversal of a sequence.

An ***N*-point sequence is called circularly even** if it is symmetric about the point zero on the circle i.e.

$$x_{N-k} = x_k \qquad 1 \leq k \leq N-1 \qquad \text{.......(21e)}$$

An ***N*-point sequence is called circularly odd** if it is antisymmetric about the point zero on the circle i.e.

$$x_{N-k} = -x_k \qquad 1 \leq k \leq N-1 \qquad \text{.......(21f)}$$

The **time reversal of an *N*-point sequence** is attained by reversing its samples about the point zero on the circle i.e.

$$x_{(-k)_N} = x_{(N-k)} \qquad 1 \leq k \leq N-1 \qquad \text{.......(21g)}$$

Important Properties of DFT

Circular symmetries of a sequence

This **time reversal** is equivalent to plotting x_k in a clockwise direction on a circle.

An equivalent definition of even and odd sequences for the associated periodic sequence x_{pk} is given as:

$$\text{even}: \quad x_{pk} = x_{p(-k)} = x_{p(N-k)}$$

$$\text{odd}: \quad x_{pk} = -x_{p(-k)} = -x_{p(N-k)} \qquad \text{.......(21h)}$$

If the periodic sequence is complex valued, then:

$$\text{conjugate even}: \quad x_{pk} = x^*_{p(N-k)}$$

$$\text{conjugate odd}: \quad x_{pk} = -x^*_{p(N-k)} \qquad \text{.......(21i)}$$

Important Properties of DFT

Circular symmetries of a sequence

Hence we can decompose the sequence x_{pk} as:

$$x_{pk} = x_{pe(k)} + x_{po(k)} \quad \text{.......(21j)}$$

where

$$x_{pe(k)} = \frac{1}{2}\left(x_{pk} + x^{*}_{p(N-k)}\right)$$

$$x_{po(k)} = \frac{1}{2}\left(x_{pk} - x^{*}_{p(N-k)}\right) \quad \text{.......(21k)}$$

Important Properties of DFT

Symmetry

From relation (19),

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1, \text{ where } W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

Then,

$$X_{pN-n} = \sum_{k=0}^{N-1} x_k W_N^{(pN-n)k} \quad \text{for } p = 0,\pm 1,\pm 2, \ \ldots$$

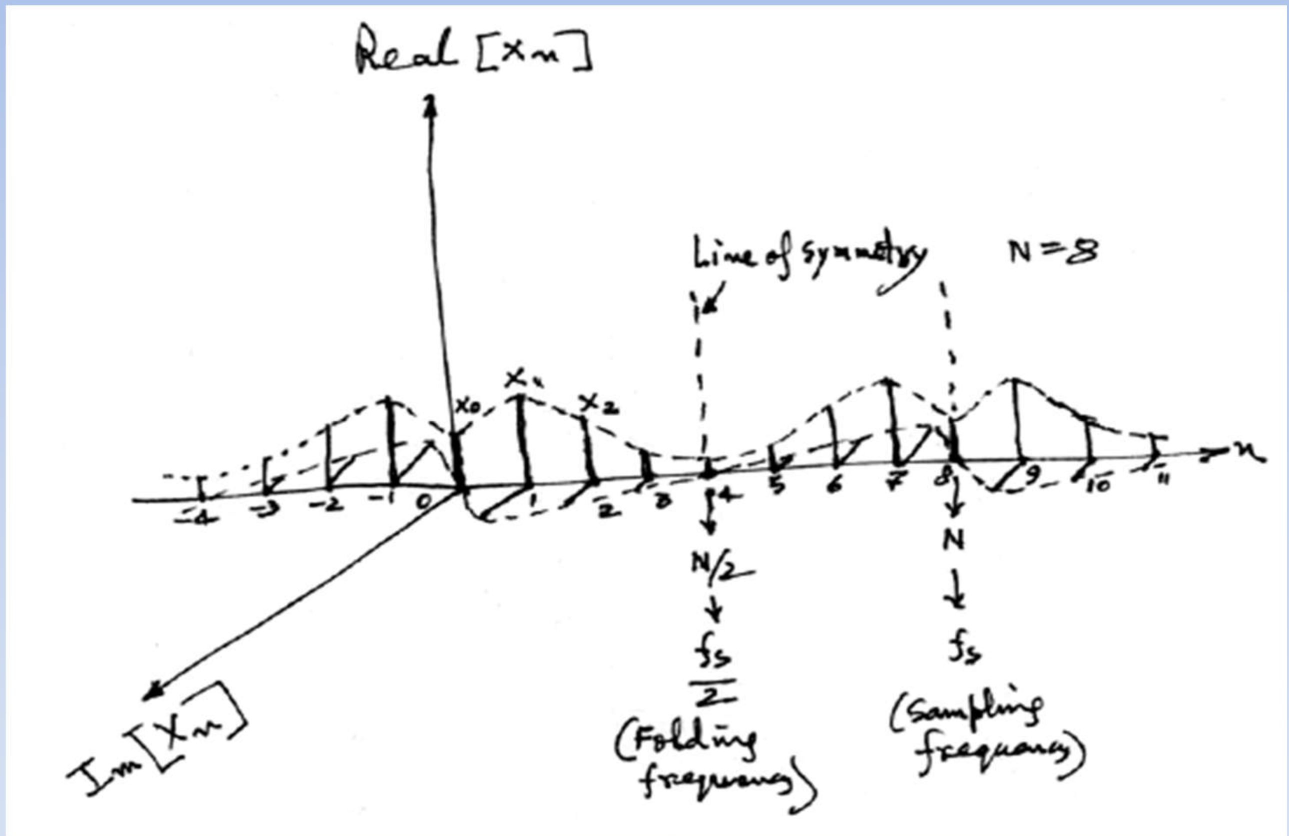
$$= \sum_{k=0}^{N-1} x_k W_N^{-nk}, \quad \text{as } W_N^{pNk} = W_N^{N(pk)} = 1$$

$= \hat{X}_n$, conjugate of X_n, if x_k is a real sequence.

Thus, $X_{pN-n} = \hat{X}_n$, for $p = 0,\pm 1,\pm 2, \ldots$ ....(22)

For $p = 0$, $X_{-n} = \hat{X}_n$ and for $p = 1$, $X_{N-n} = \hat{X}_n$

Real and imaginary parts of X_n



Multiplication of two DFTs and Circular Convolution

Let us assume that we have two finite duration sequences of length *N*, x_{1k} and x_{2k}. Their respective *N*-point DFTs are:

$$X_{1n} = \sum_{k=0}^{N-1} x_{1k} e^{\frac{-j2\pi kn}{N}}, \quad n = 0,1,\cdots,N-1 \qquad \text{.......(22a)}$$

$$X_{2n} = \sum_{k=0}^{N-1} x_{2k} e^{\frac{-j2\pi kn}{N}}, \quad n = 0,1,\cdots,N-1 \qquad \text{.......(22b)}$$

If these two DFTs are multiplied together, the resultant will be a DFT X_{3n} of a sequence x_{3k} of length *N*.

Now our objective is to determine the relationship between x_{3k} and sequences x_{1k} and x_{2k}

Now, we have:

$$X_{3n} = X_{1n} X_{2n} \quad n = 0,1,\cdots,N-1 \qquad \text{.......(22c)}$$

The IDFT of $\{X_{3n}\}$ is:

$$x_{3m} = \frac{1}{N} \sum_{n=0}^{N-1} X_{3n} e^{\frac{j2\pi nm}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} X_{1n} X_{2n} e^{\frac{j2\pi nm}{N}} \qquad \text{.......(22d)}$$

Multiplication of two DFTs and Circular Convolution

Substituting X_{1n} and X_{2n} in (22d) using the DFTs in (22a) and (22b), we get:

$$x_{3m} = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} x_{1k} e^{\frac{-j2\pi nk}{N}}\right]\left[\sum_{l=0}^{N-1} x_{2l} e^{\frac{-j2\pi nl}{N}}\right] e^{\frac{j2\pi nm}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} x_{1k} \sum_{l=0}^{N-1} x_{2l}\left[\sum_{n=0}^{N-1} e^{\frac{j2\pi n(m-k-l)}{N}}\right] \quad \text{.......(22e)}$$

The inner sum in the brackets in (22e) has the form:

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \quad \text{.......(22f)}$$

where a is defined as:

$$a = e^{\frac{j2\pi(m-k-l)}{N}} \quad \text{.......(22g)}$$

Multiplication of two DFTs and Circular Convolution

We observe that $a = 1$, when m-k-l is a multiple of N.
On the other hand, $a^N = 1$, for any value of $a \neq 0$. Hence (22f) gets reduced to:

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & l = m - k + pN = (m-k)_N \\ 0, & \text{otherwise} \end{cases} \qquad \text{.......(22h)}$$

If we substitute this result in (22e), we obtain the desired expression of x_{3m} as:

$$x_{3m} = \sum_{k=0}^{N-1} x_{1k} x_{2(m-k)_N}, \qquad m = 0, 1, \cdots, N-1 \qquad \text{.......(22i)}$$

The expression in (22i) has the form of a convolution sum.

However it is not the ordinary linear convolution. Instead, the convolution sum in (22i) Involves the index $(m\text{-}k)_N$ and is called **circular convolution**.

Conclusion: **The *multiplication* of the DFTs of two sequences is equivalent to the *circular convolution* of the two sequences in the time domain.**

Important Properties of DFT

Circular Convolution

If $x_{1k} \xleftrightarrow[N]{\text{DFT}} X_{1n}$ and $x_{2k} \xleftrightarrow[N]{\text{DFT}} X_{2n}$

then

$$x_{1k} (\mathrm{N}) x_{2k} \xleftrightarrow[N]{\text{DFT}} X_{1n} X_{2n}$$

where $x_{1k} (\mathrm{N}) x_{2k}$ denotes the circular convolution of the sequences x_{1k} and x_{2k}.

Computation of DFT

From relation (19),

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1, \text{ where } \quad W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

It may be represented in matrix form as

$$\left[\boldsymbol{X}_n\right] = \left[\boldsymbol{W}_N^{nk}\right]\left[\boldsymbol{X}_k\right] \qquad \text{.......(23)}$$

where $\left[\boldsymbol{X}_n\right]$ and $\left[\boldsymbol{X}_k\right]$ are $N \times 1$ column matrices and $\left[\boldsymbol{W}_N^{nk}\right]$ is an $N \times N$ square matrix.

Computation of DFT

$$[\boldsymbol{X}_n] = [\boldsymbol{W}_N^{nk}][\boldsymbol{X}_k] \qquad \text{.......(23)}$$

Here, $[\boldsymbol{X}_n] = \begin{bmatrix} X_0 \\ X_1 \\ \\ \\ X_{N-1} \end{bmatrix}$, $[\boldsymbol{X}_k] = \begin{bmatrix} x_0 \\ x_1 \\ \\ \\ x_{N-1} \end{bmatrix}$

and $[\boldsymbol{W}_N^{nk}] = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$

Computation of DFT

$$[X_n] = [W_N^{nk}][X_k] \qquad \text{.......(23)}$$

For N = 4, relation (23) becomes

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Frequency) (Time)

Computation of DFT

$$[X_n] = [W_N^{nk}][X_k] \qquad \text{.......(23)}$$

For N = 4, relation (23) becomes

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Frequency) (Time)

Hence, computation of X_0 requires 4 complex multiplications and 4 complex additions.

Computation of DFT

$$[X_n] = [W_N^{nk}][X_k] \qquad \text{.......(23)}$$

For N = 4, relation (23) becomes

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Frequency) (Time)

Hence, computation of X_0 requires 4 complex multiplications and 4 complex additions.

In general, execution of relation (23) requires ***N^2* complex multiplications** and ***N^2* complex additions**. Thus computational load increases rapidly with increasing ***N***. **Fast Fourier Transform (FFT)** algorithms allow computation of DFT with reduced computational burden.

Fast Fourier Transform (FFT)

From relation (19),

$$X_n = \sum_{k=0}^{N-1} x_k W_N^{nk}, \text{ for } n = 0,1,2,\ldots, N\text{-}1$$

Assuming *N* to be a power of 2, *N*-point data sequence x_k in relation (19) may be split into two N/2 point data sequences as follows:

$$X_n = \sum_{k=0}^{\frac{N}{2}-1} x_k W_N^{nk} + \sum_{k=\frac{N}{2}}^{N-1} x_k W_N^{nk}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} x_k W_N^{nk} + \sum_{k=0}^{\frac{N}{2}-1} x_{k+\frac{N}{2}} W_N^{n\left(k+\frac{N}{2}\right)}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} x_k W_N^{nk} + W_N^{\frac{nN}{2}} \sum_{k=0}^{\frac{N}{2}-1} x_{k+\frac{N}{2}} W_N^{nk}$$

Fast Fourier Transform (FFT)

or $$X_n = \sum_{k=0}^{\frac{N}{2}-1} x_k W_N^{nk} + W_N^{\frac{nN}{2}} \sum_{k=0}^{\frac{N}{2}-1} x_{k+\frac{N}{2}} W_N^{nk}$$

Now, $$W_N^{\frac{nN}{2}} = e^{-jn\left(\frac{2\pi}{N}\right)\frac{N}{2}} = e^{-jn\pi} = (-1)^n$$

Then, $$X_n = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k W_N^{nk} + (-1)^n x_{k+\frac{N}{2}} W_N^{nk} \right]$$

or $$X_n = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k + (-1)^n x_{k+\frac{N}{2}} \right] W_N^{nk} \qquad \text{.......(24)}$$

Fast Fourier Transform (FFT)

or $$X_n = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k + (-1)^n x_{k+\frac{N}{2}} \right] W_N^{nk}$$ ....(24)

Now, splitting (or decimating) X_n into even and odd harmonics,

for even harmonics, $n = 2p$, for p = 0,1,2,…,(N/2-1) and

for odd harmonics, $n = 2p+1$, for p = 0,1,2,…,(N/2-1).

For even harmonics,

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k + x_{k+\frac{N}{2}} \right] W_N^{2pk} = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k + x_{k+\frac{N}{2}} \right] W_{\frac{N}{2}}^{pk}$$

as $W_N^{2pk} = W_{\frac{N}{2}}^{pk}$

Fast Fourier Transform (FFT)

Now, $$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} \left[x_k + x_{k+\frac{N}{2}} \right] W_{\frac{N}{2}}^{pk}$$

Let, $g_k = x_k + x_{k+\frac{N}{2}}$, for k = 0,1,2,…(N/2-1)

Then, $$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk} \qquad \text{.......(25)}$$

This is an N/2 point DFT sequence g_k, k = 0,1,2,….(N/2 - 1)

Fast Fourier Transform (FFT)

Now, for odd harmonics [c.f. relation (24)],

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1}\left[x_k - x_{k+\frac{N}{2}}\right] W_N^{(2p+1)k}$$

$$= \sum_{k=0}^{\frac{N}{2}-1}\left[x_k - x_{k+\frac{N}{2}}\right] W_N^{k} W_N^{2pk}$$

$$= \sum_{k=0}^{\frac{N}{2}-1}\left[x_k - x_{k+\frac{N}{2}}\right] W_N^{k} W_{\frac{N}{2}}^{pk} \quad \text{as} \quad W_N^{2pk} = W_{\frac{N}{2}}^{pk}$$

Let, $g_k^{/} = \left(x_k - x_{k+\frac{N}{2}}\right) W_N^{k}$ for k = 0,1,2,…,(N/2-1)

Fast Fourier Transform (FFT)

Then, $$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk}$$ …….(26)

This is an N/2 point DFT sequence g'_k, $k = 0,1,2,....(N/2 - 1)$

Thus an *N*-point DFT may be split into two N/2-point DFTs.

This process of splitting may be continued up to 2-point transforms as *N* is a power of 2.

4-point FFT

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk} \qquad \text{.......(25)}$$

Let N = 4. Then from relation (25),

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk}$$, for *p* = 0,1,2,…,(N/2-1)

Where, $g_k = x_k + x_{k+\frac{N}{2}}$, for *k* = 0,1,2,…,(N/2-1)

or, $X_{2p} = \sum_{k=0}^{1} g_k W_2^{pk}$, for $p = 0,1$

and, $g_k = x_k + x_{k+2}$, for $k = 0,1$ ....(27)

4-point FFT

Now, $X_{2p} = \sum_{k=0}^{1} g_k W_2^{pk}$, for $p = 0,1$

and, $g_k = x_k + x_{k+2}$, for $k = 0,1$ ....(27)

Then for $p = 0,$

$$X_0 = \sum_{k=0}^{1} g_k W_2^0 = \sum_{k=0}^{1} g_k = g_0 + g_1 \quad \text{.......(28)}$$

and for $p = 1,$

$$X_2 = \sum_{k=0}^{1} g_k W_2^k = g_0 W_2^0 + g_1 W_2^1$$

$$= g_0 + (-1) g_1 = g_0 - g_1 \quad \text{.......(29)}$$

4-point FFT

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk} \quad \text{.......(26)}$$

Now from relation (26),

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk}$$, for p = 0,1,2,…,(N/2-1)

where $g_k^{/} = \left(x_k - x_{k+\frac{N}{2}} \right) W_N^k$, for k = 0,1,2,…,(N/2-1)

or, $X_{2p+1} = \sum_{k=0}^{1} g_k^{/} W_2^{pk}$, for $p = 0,1$

and, $g_k^{/} = (x_k - x_{k+2}) W_4^k$, for $k = 0,1$ ....(30)

4-point FFT

Now, $X_{2p+1} = \sum_{k=0}^{1} g_k^{/} W_2^{pk}$, for $p = 0,1$

and, $g_k^{/} = (x_k - x_{k+2}) W_4^k$, for $k = 0,1$ ....(30)

Then for $p = 0,$

$$X_1 = \sum_{k=0}^{1} g_k^{/} W_2^{0} = g_0^{/} + g_1^{/} \quad ....(31)$$

and for $p = 1,$

$$X_3 = \sum_{k=0}^{1} g_k^{/} W_2^{k} = g_0^{/} + g_1^{/} W_2^{1}$$

$$= g_0^{/} - g_1^{/} \quad ....(32)$$

4-point FFT

Now from relations (27) and (30), for $k = 0,1$

$$g_0 = x_0 + x_2$$

$$g_1 = x_1 + x_3$$

$$g_0^{/} = (x_0 - x_2)W_4^0 = x_0 - x_2 = x_0 + x_2 W_4^2$$

$$g_1^{/} = (x_1 - x_3)W_4^1 = x_1 W_4^1 - x_3 W_4^1 = x_1 W_4^1 + x_3 W_4^3$$

Then in matrix form,

$$\begin{bmatrix} g_0 \\ g_1 \\ g_0^{/} \\ g_1^{/} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & W_4^2 & 0 \\ 0 & W_4^1 & 0 & W_4^3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{.......(33)}$$

4-point FFT

From relations (28), (29), (31) and (32),

$$X_0 = g_0 + g_1$$

$$X_2 = g_0 - g_1$$

$$X_1 = g_0^/ + g_1^/$$

$$X_3 = g_0^/ - g_1^/$$

Then in matrix form,

$$\begin{bmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_0^/ \\ g_1^/ \end{bmatrix} \qquad \text{.......(34)}$$

4-point FFT

From relations (33) and (34),

$$\begin{bmatrix} g_0 \\ g_1 \\ g_0^{/} \\ g_1^{/} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & W_4^2 & 0 \\ 0 & W_4^1 & 0 & W_4^3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Time history ↓

Frequency

$$\begin{bmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_0^{/} \\ g_1^{/} \end{bmatrix}$$

Bit reversed order

4-point FFT

Frequency

$$\begin{bmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_0^{/} \\ g_1^{/} \end{bmatrix}$$

Bit reversed order

since,

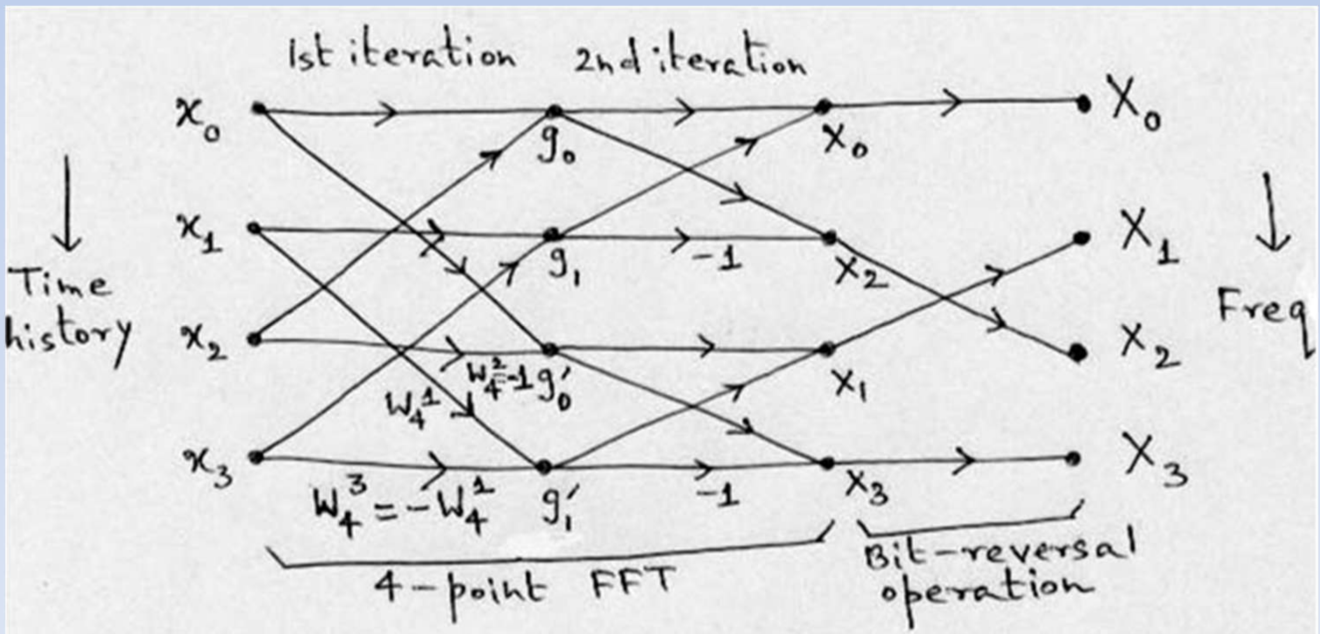
$$\left.\begin{aligned} X_0 &= X_{00} \\ X_2 &= X_{10} \\ X_1 &= X_{01} \\ X_3 &= X_{11} \end{aligned}\right\}$$

in terms of binary bits

Signal Flow Graph for N = 4

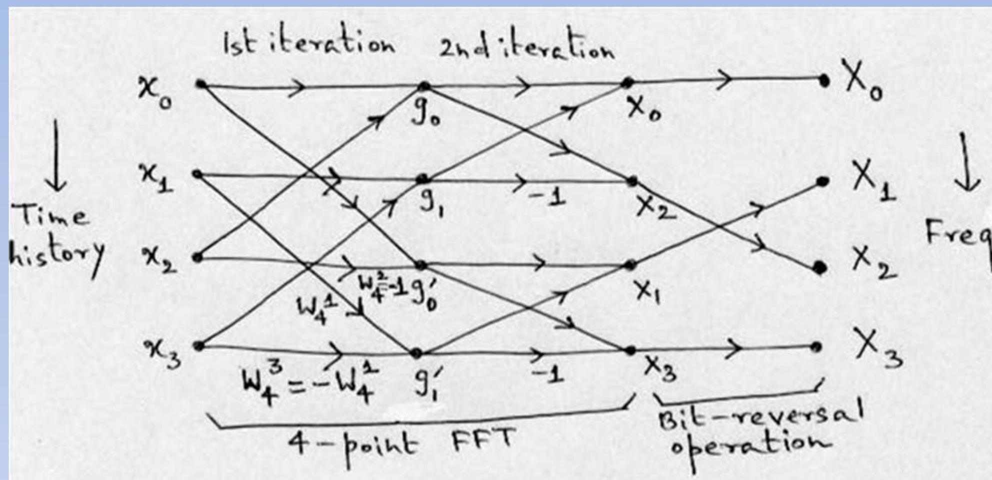
From relations (33) and (34),

$$\begin{bmatrix} g_0 \\ g_1 \\ g_0' \\ g_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & W_4^2 & 0 \\ 0 & W_4^1 & 0 & W_4^3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_0' \\ g_1' \end{bmatrix}$$



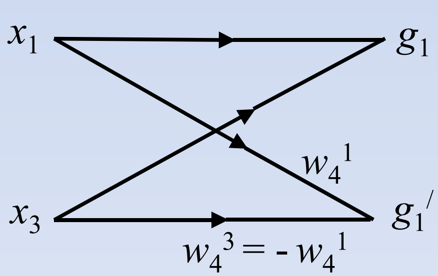
Number of iterations = M, where $M = \log_2 N$ [as $N = 2^M$], here N = 4 and M = 2

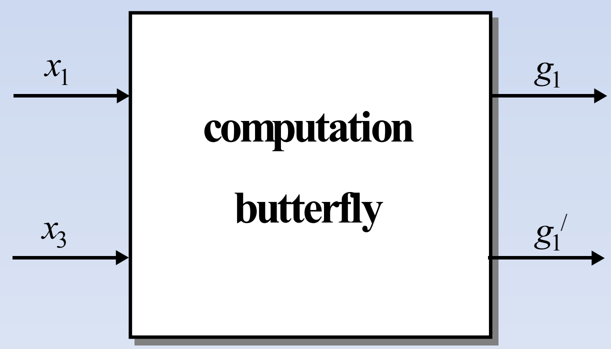
Signal Flow Graph for N = 4

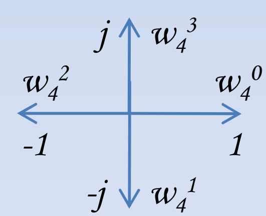


Each iteration involves N/2 number of butterfly computations.

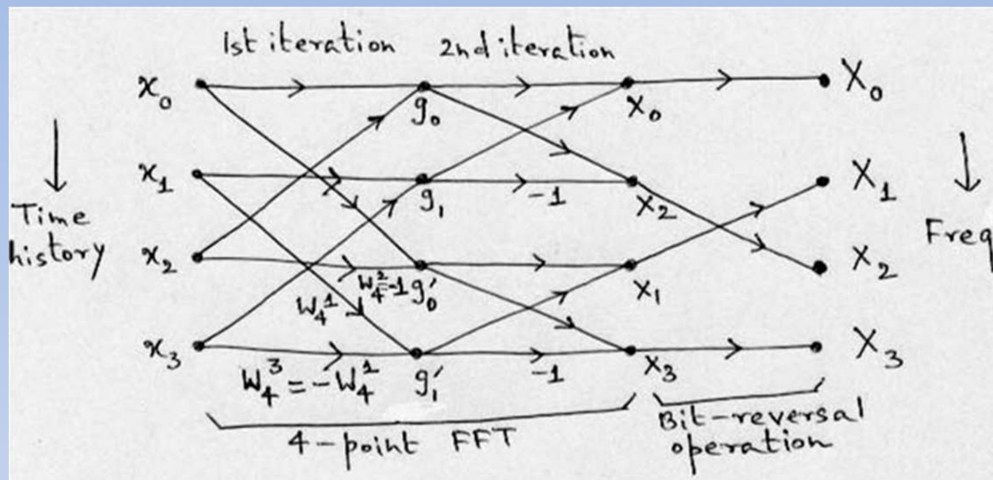
Computation of g_1 and $g_1^{/}$ may be represented as:





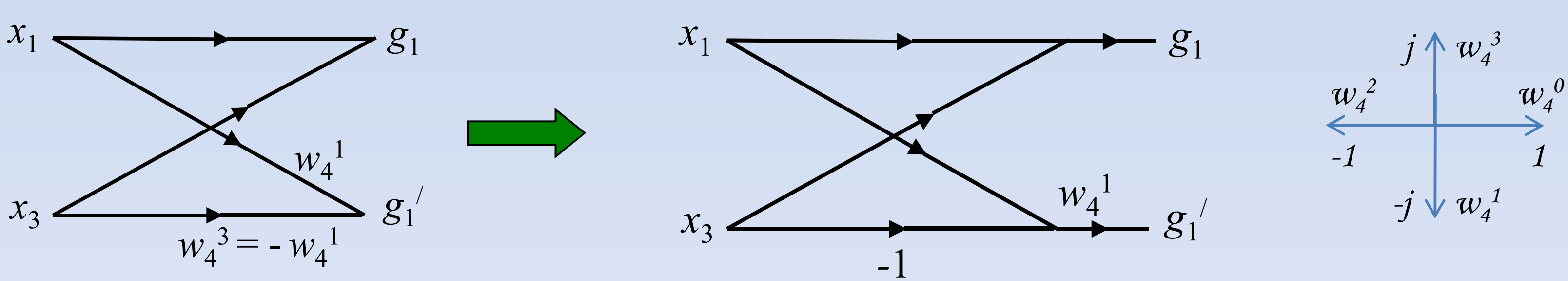


Signal Flow Graph for N = 4



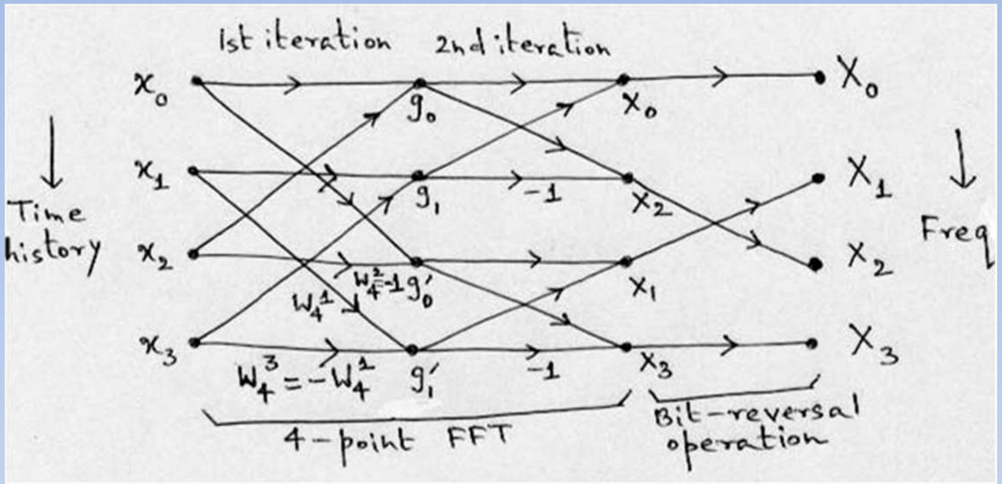
Each iteration involves N/2 number of butterfly computations.

Computation of g_1 and g_1' may be represented as:



This involves **two** complex additions and **one** complex multiplication.
This is true for all butterflies.

Signal Flow Graph for N = 4



The procedure can be summarized as,

No. of iterations = $M = \log_2 N$

Total no. of butterflies = $\frac{NM}{2} = \frac{N}{2}\log_2 N$

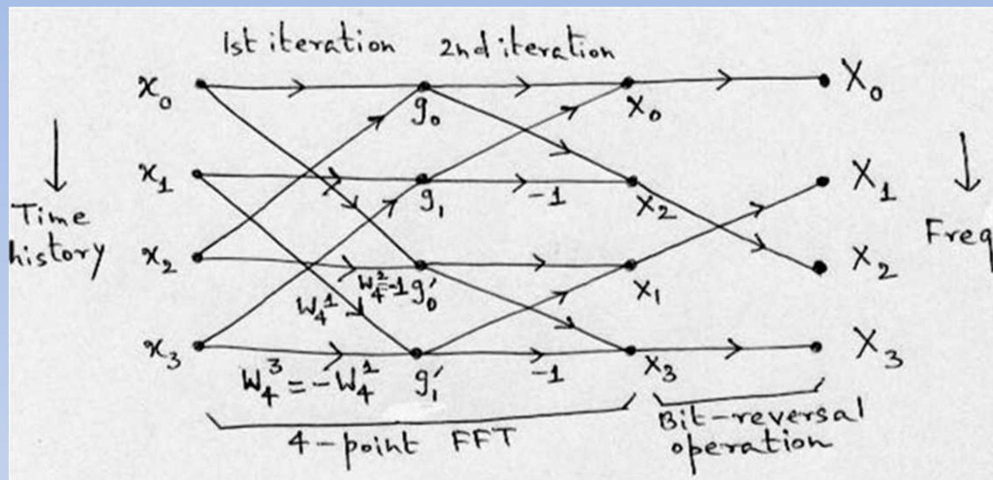
No. of complex multiplications per butterfly = 1

No. of complex additions per butterfly = 2

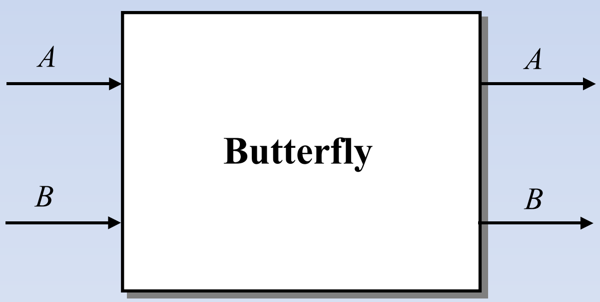
Total no. of complex multiplications = $\frac{NM}{2} = \frac{N}{2}\log_2 N$

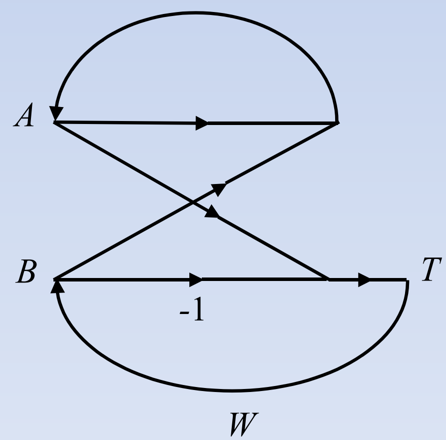
Total no. of complex additions = $NM = N\log_2 N$

Signal Flow Graph for N = 4



Computation of each butterfly may be carried out *in-place* to reduce memory requirement as follows:

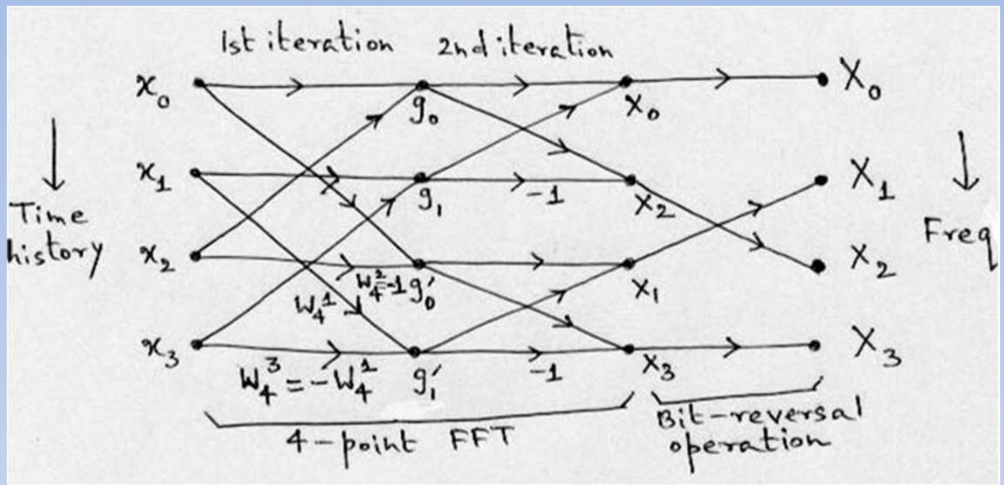




$$T = A - B$$
$$A = A + B$$
$$B = T * W$$

Here ***T*** is a scratch-pad variable and ***W*** is the twiddle factor.

Signal Flow Graph for N = 4

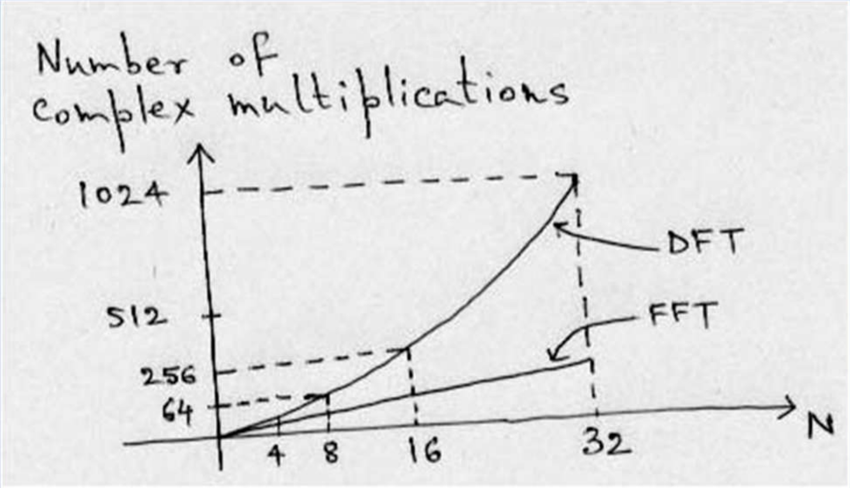


The above algorithm for the computation of FFT of sequence x_k, k = 0,1,2,….,(N-1) may be called ***radix-2 decimation-in-frequency in-place*** **FFT algorithm**.
Here, **N should be a power of 2**.

Similarly, **radix-2 decimation-in-time in-place FFT algorithm** may be derived with same computation load.

Comparison of computational loads of DFT and FFT

N	DFT		FFT	
	complex additions	complex multiplications	complex additions	complex multiplications
4	16	16	8	4
8	64	64	24	12
16	256	256	64	32
32	1024	1024	160	80



8-point FFT

Relations (25) and (26) may be split further (i.e. decimated) into N/2-point DFTs as follows:

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk} \quad ....(25)$$

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk} \quad ....(26)$$

8-point FFT

Relations (25) and (26) may be split further (i.e. decimated) into N/2-point DFTs as follows:

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk} \qquad \text{.......(25)}$$

In relation (25), splitting N/2-point sequence g_k into two N/4-point sequences,

$$X_{2p} = \sum_{k=0}^{\frac{N}{4}-1} g_k W_{\frac{N}{2}}^{pk} + \sum_{k=\frac{N}{4}}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk}$$

$$= \sum_{k=0}^{\frac{N}{4}-1} g_k W_{\frac{N}{2}}^{pk} + \sum_{k=0}^{\frac{N}{4}-1} g_{k+\frac{N}{4}} W_{\frac{N}{2}}^{p\left(k+\frac{N}{4}\right)}$$

$$= \sum_{k=0}^{\frac{N}{4}-1} g_k W_{\frac{N}{2}}^{pk} + \sum_{k=0}^{\frac{N}{4}-1} g_{k+\frac{N}{4}} W_{\frac{N}{2}}^{pk} W_{\frac{N}{2}}^{\frac{pN}{4}}$$

8-point FFT

Now, $$X_{2p} = \sum_{k=0}^{\frac{N}{4}-1} g_k W_{\frac{N}{2}}^{pk} + \sum_{k=0}^{\frac{N}{4}-1} g_{k+\frac{N}{4}} W_{\frac{N}{2}}^{pk} W_{\frac{N}{2}}^{\frac{pN}{4}}$$

Here, $$W_{\frac{N}{2}}^{\frac{pN}{4}} = (-1)^p$$

Therefore, $$X_{2p} = \sum_{k=0}^{\frac{N}{4}-1} \left[g_k + (-1)^p g_{k+\frac{N}{4}} \right] W_{\frac{N}{2}}^{pk} \qquad \text{.......(35)}$$

Now splitting X_{2p} into even and odd harmonics,

for even harmonics, $p = 2r$, for r = 0,1,2,…,(N/4-1)

and for odd harmonics, $p = (2r+1)$, for r = 0,1,2,…,(N/4-1)

8-point FFT

Now, for even harmonics,

$$X_{4r} = \sum_{k=0}^{\frac{N}{4}-1} \left[g_k + g_{k+\frac{N}{4}} \right] W_{\frac{N}{2}}^{2rk}$$

$$= \sum_{k=0}^{\frac{N}{4}-1} \left[g_k + g_{k+\frac{N}{4}} \right] W_{\frac{N}{4}}^{rk}$$

Let $h_k = g_k + g_{k+\frac{N}{4}}$, for k = 0,1,2,…,(N/4-1)

Then, $X_{4r} = \sum_{k=0}^{\frac{N}{4}-1} h_k W_{\frac{N}{4}}^{rk}$, for r = 0,1,2,…,(N/4-1) …….(36)

This is an N/4-point DFT of sequence h_k , k = 0,1,2,…,(N/4-1)

8-point FFT

Now, for odd harmonics,

$$X_{4r+2} = \sum_{k=0}^{\frac{N}{4}-1} \left[g_k - g_{k+\frac{N}{4}} \right] W_{\frac{N}{2}}^{(2r+1)k}$$

$$= \sum_{k=0}^{\frac{N}{4}-1} \left[g_k - g_{k+\frac{N}{4}} \right] W_N^{2k} W_{\frac{N}{4}}^{rk}$$

Let $h_k^{/} = \left(g_k - g_{k+\frac{N}{4}} \right) W_N^{2k}$, for k = 0,1,2,…,(N/4-1)

Then, $X_{4r+2} = \sum_{k=0}^{\frac{N}{4}-1} h_k^{/} W_{\frac{N}{4}}^{rk}$, for r = 0,1,2,…,(N/4-1) …….(37)

This is an N/4-point DFT of sequence $h_k^{/}$, k = 0,1,2,…,(N/4-1)

Thus the *N/2*-point DFT as represented in relation (25), may be split into two N/4-point DFTs, as represented in relations (36) and (37).

8-point FFT

Similarly, the N/2-point DFT in relation (26) may be split into two even and odd harmonic N/4-point DFTs as follows:

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk} \quad \text{.......(26)}$$

For even harmonics,

$$X_{4r+1} = \sum_{k=0}^{\frac{N}{4}-1} l_k W_{\frac{N}{4}}^{rk}$$, for r = 0,1,2,…,(N/4-1) ....(38)

This is an N/4-point DFT where $l_k = g_k^{/} + g_{k+\frac{N}{4}}^{/}$, for k = 0,1,2,…,(N/4-1)

8-point FFT

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk} \quad ....(26)$$

Similarly for odd harmonics,

$$X_{4r+3} = \sum_{k=0}^{\frac{N}{4}-1} l_k^{/} W_{\frac{N}{4}}^{rk}$$, for r = 0,1,2,…,(N/4-1) ....(39)

This is another N/4-point DFT where

$$l_k^{/} = \left(g_k^{/} - g_{k+\frac{N}{4}}^{/} \right) W_N^{2k}$$, for k = 0,1,2,…,(N/4-1)

8-point FFT

$$X_{2p} = \sum_{k=0}^{\frac{N}{2}-1} g_k W_{\frac{N}{2}}^{pk} \qquad \text{.......(25)}$$

Let N = 8 (= 2^3) for 8-point FFT.

In relation (25),

$$g_k = x_k + x_{k+\frac{N}{2}}$$, for k = 0,1,2,…,(N/2-1)

Possible values of k are k = 0,1,2,3.

Then,

$$\left.\begin{aligned} g_0 &= x_0 + x_4 \\ g_1 &= x_1 + x_5 \\ g_2 &= x_2 + x_6 \\ g_3 &= x_3 + x_7 \end{aligned}\right\} \qquad \text{.......(40)}$$

8-point FFT

$$X_{2p+1} = \sum_{k=0}^{\frac{N}{2}-1} g_k^{/} W_{\frac{N}{2}}^{pk} \quad \text{.......(26)}$$

Now in relation (26),

$$g_k^{/} = \left(x_k - x_{k+\frac{N}{2}}\right) W_N^k$$, for k = 0,1,2,…,(N/2-1)

Possible values of k are k = 0,1,2,3.

Then,

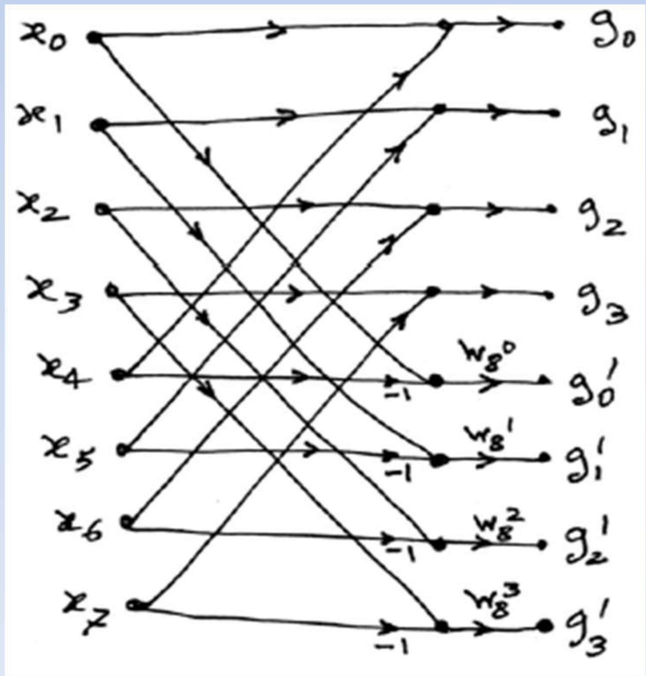
$$\left.\begin{aligned} g_0^{/} &= \left(x_0 - x_4\right) W_8^0 \\ g_1^{/} &= \left(x_1 - x_5\right) W_8^1 \\ g_2^{/} &= \left(x_2 - x_6\right) W_8^2 \\ g_3^{/} &= \left(x_3 - x_7\right) W_8^3 \end{aligned}\right\} \quad \text{.......(41)}$$

8-point FFT

$$\left.\begin{aligned} g_0 &= x_0 + x_4 \\ g_1 &= x_1 + x_5 \\ g_2 &= x_2 + x_6 \\ g_3 &= x_3 + x_7 \end{aligned}\right\} \quad \text{.......(40)}$$

$$\left.\begin{aligned} g_0^/ &= (x_0 - x_4)W_8^0 \\ g_1^/ &= (x_1 - x_5)W_8^1 \\ g_2^/ &= (x_2 - x_6)W_8^2 \\ g_3^/ &= (x_3 - x_7)W_8^3 \end{aligned}\right\} \quad \text{.......(41)}$$

From relations (40) and (41), signal flow graph for computations of g_{0-3} and $g^/_{0-3}$ may be represented as:



8-point FFT

Now from relation (36),

$$h_k = g_k + g_{k+\frac{N}{4}}$$, for k = 0,1,2,…,(N/4-1)

Possible values of K are K = 0,1.

Then,

$$\left.\begin{aligned} h_0 &= g_0 + g_2 \\ h_1 &= g_1 + g_3 \end{aligned}\right\} \qquad \text{.......(42)}$$

8-point FFT

And from relation (37),

$$h_k^{/} = \left(g_k - g_{k+\frac{N}{4}} \right) W_N^{2k}$$, for k = 0,1,2,…,(N/4-1)

Possible values of K are K = 0,1.

Then,

$$\left. \begin{aligned} h_0^{/} &= (g_0 - g_2) W_8^{0} \\ h_1^{/} &= (g_1 - g_3) W_8^{2} \end{aligned} \right\} \qquad \text{…….(43)}$$

8-point FFT

Now from relation (38),

$$l_k = g_k^{/} + g_{k+\frac{N}{4}}^{/}$$, for k = 0,1,2,…,(N/4-1)

Possible values of K are K = 0,1.

Then,

$$\left.\begin{aligned} l_0 &= g_0^{/} + g_2^{/} \\ l_1 &= g_1^{/} + g_3^{/} \end{aligned}\right\} \quad \text{.......(44)}$$

8-point FFT

And from relation (39),

$$l_k^{/} = \left(g_k^{/} - g_{k+\frac{N}{4}}^{/} \right) W_N^{2k}$$, for k = 0,1,2,…,(N/4-1)

Possible values of K are K = 0,1.

Then,

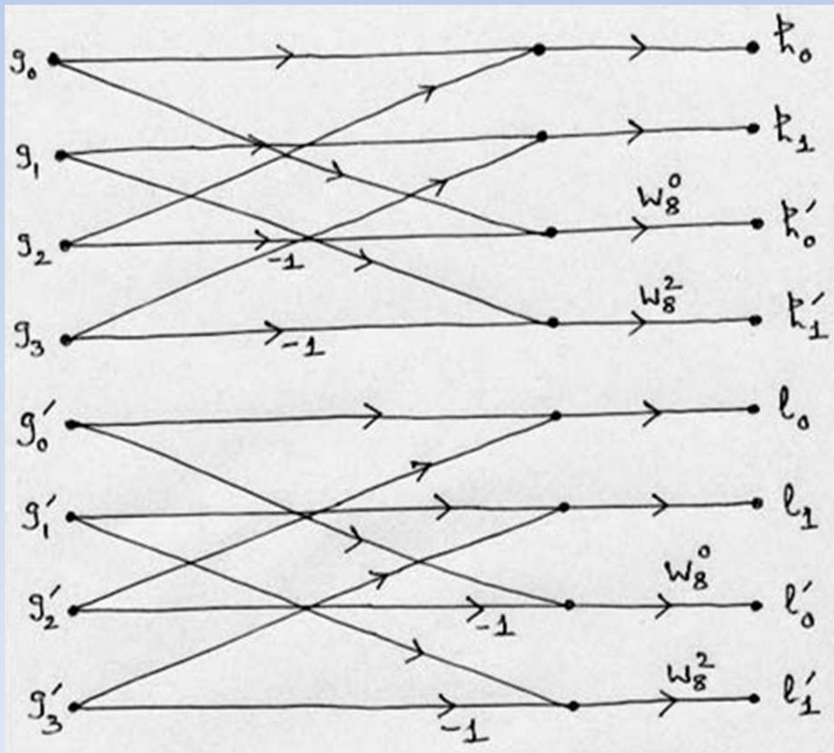
$$\left. \begin{aligned} l_0^{/} &= \left(g_0^{/} - g_2^{/} \right) W_8^{0} \\ l_1^{/} &= \left(g_1^{/} - g_3^{/} \right) W_8^{2} \end{aligned} \right\} \qquad \text{.......(45)}$$

8-point FFT

$$\left.\begin{aligned} h_0 &= g_0 + g_2 \\ h_1 &= g_1 + g_3 \end{aligned}\right\} \text{.......(42)} \qquad \left.\begin{aligned} h_0^{/} &= (g_0 - g_2)W_8^0 \\ h_1^{/} &= (g_1 - g_3)W_8^2 \end{aligned}\right\} \text{.......(43)}$$

$$\left.\begin{aligned} l_0 &= g_0^{/} + g_2^{/} \\ l_1 &= g_1^{/} + g_3^{/} \end{aligned}\right\} \text{.......(44)} \qquad \left.\begin{aligned} l_0^{/} &= (g_0^{/} - g_2^{/})W_8^0 \\ l_1^{/} &= (g_1^{/} - g_3^{/})W_8^2 \end{aligned}\right\} \text{.......(45)}$$

From relations (42), (43), (44) and (45), signal flow graph for computation of h_{0-1}, $h_{0-1}^{/}$, l_{0-1} and $l_{0-1}^{/}$ may be represented as:



8-point FFT

Now, from relation (36),

possible values of *r* are *r* = 0,1.

$$X_{4r} = \sum_{k=0}^{\frac{N}{4}-1} h_k W_{\frac{N}{4}}^{rk} \text{, for } r = 0,1,2,\dots,(N/4-1) \quad \text{.......(36)}$$

Then,

$$\left.\begin{aligned} X_0 &= h_0 W_2^0 + h_1 W_2^0 = h_0 + h_1 \\ X_4 &= h_0 W_2^0 + h_1 W_2^1 = h_0 - h_1 \end{aligned}\right\} \quad \text{.......(46)}$$

And, from relation (37),

possible values of *r* are *r* = 0,1.

$$X_{4r+2} = \sum_{k=0}^{\frac{N}{4}-1} h_k' W_{\frac{N}{4}}^{rk} \text{, for } r = 0,1,2,\dots,(N/4-1) \quad \text{.......(37)}$$

Then,

$$\left.\begin{aligned} X_2 &= h_0' W_2^0 + h_1' W_2^0 = h_0' + h_1' \\ X_6 &= h_0' W_2^0 + h_1' W_2^1 = h_0' - h_1' \end{aligned}\right\} \quad \text{.......(47)}$$

8-point FFT

From relation (38),

possible values of *r* are *r* = 0,1.

Then,

$$\left.\begin{aligned} X_1 &= l_0 W_2^0 + l_1 W_2^0 = l_0 + l_1 \\ X_5 &= l_0 W_2^0 + l_1 W_2^1 = l_0 - l_1 \end{aligned}\right\} \quad \text{.......(48)}$$

$$X_{4r+1} = \sum_{k=0}^{\frac{N}{4}-1} l_k W_{\frac{N}{4}}^{rk} \text{ , for } r = 0,1,2,\ldots,(N/4-1) \quad \text{.......(38)}$$

And, from relation (39),

possible values of *r* are *r* = 0,1.

Then,

$$\left.\begin{aligned} X_3 &= l_0^{/} W_2^0 + l_1^{/} W_2^0 = l_0^{/} + l_1^{/} \\ X_7 &= l_0^{/} W_2^0 + l_1^{/} W_2^1 = l_0^{/} - l_1^{/} \end{aligned}\right\} \quad \text{.......(49)}$$

$$X_{4r+3} = \sum_{k=0}^{\frac{N}{4}-1} l_k^{/} W_{\frac{N}{4}}^{rk} \text{ , for } r = 0,1,2,\ldots,(N/4-1) \quad \text{.......(39)}$$

8-point FFT

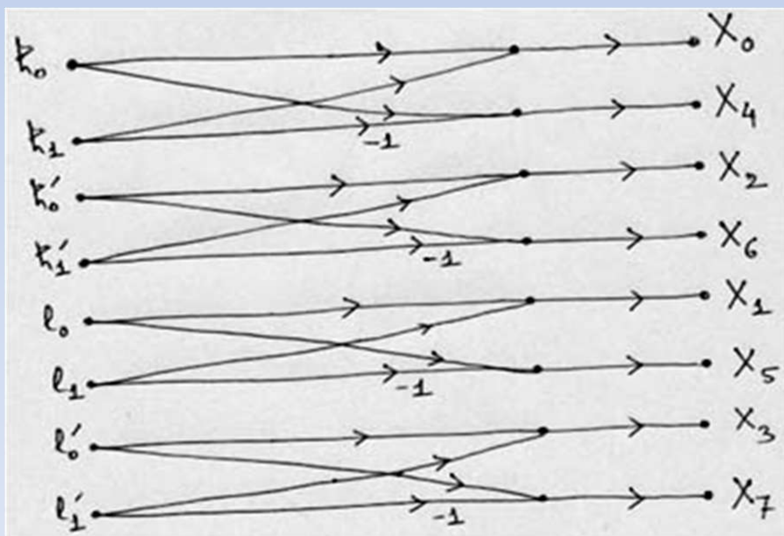
$$\left.\begin{aligned} X_0 &= h_0 W_2^0 + h_1 W_2^0 = h_0 + h_1 \\ X_4 &= h_0 W_2^0 + h_1 W_2^1 = h_0 - h_1 \end{aligned}\right\} \text{.......(46)}$$

$$\left.\begin{aligned} X_2 &= h_0^{/} W_2^0 + h_1^{/} W_2^0 = h_0^{/} + h_1^{/} \\ X_6 &= h_0^{/} W_2^0 + h_1^{/} W_2^1 = h_0^{/} - h_1^{/} \end{aligned}\right\} \text{.......(47)}$$

$$\left.\begin{aligned} X_1 &= l_0 W_2^0 + l_1 W_2^0 = l_0 + l_1 \\ X_5 &= l_0 W_2^0 + l_1 W_2^1 = l_0 - l_1 \end{aligned}\right\} \text{.......(48)}$$

$$\left.\begin{aligned} X_3 &= l_0^{/} W_2^0 + l_1^{/} W_2^0 = l_0^{/} + l_1^{/} \\ X_7 &= l_0^{/} W_2^0 + l_1^{/} W_2^1 = l_0^{/} - l_1^{/} \end{aligned}\right\} \text{.......(49)}$$

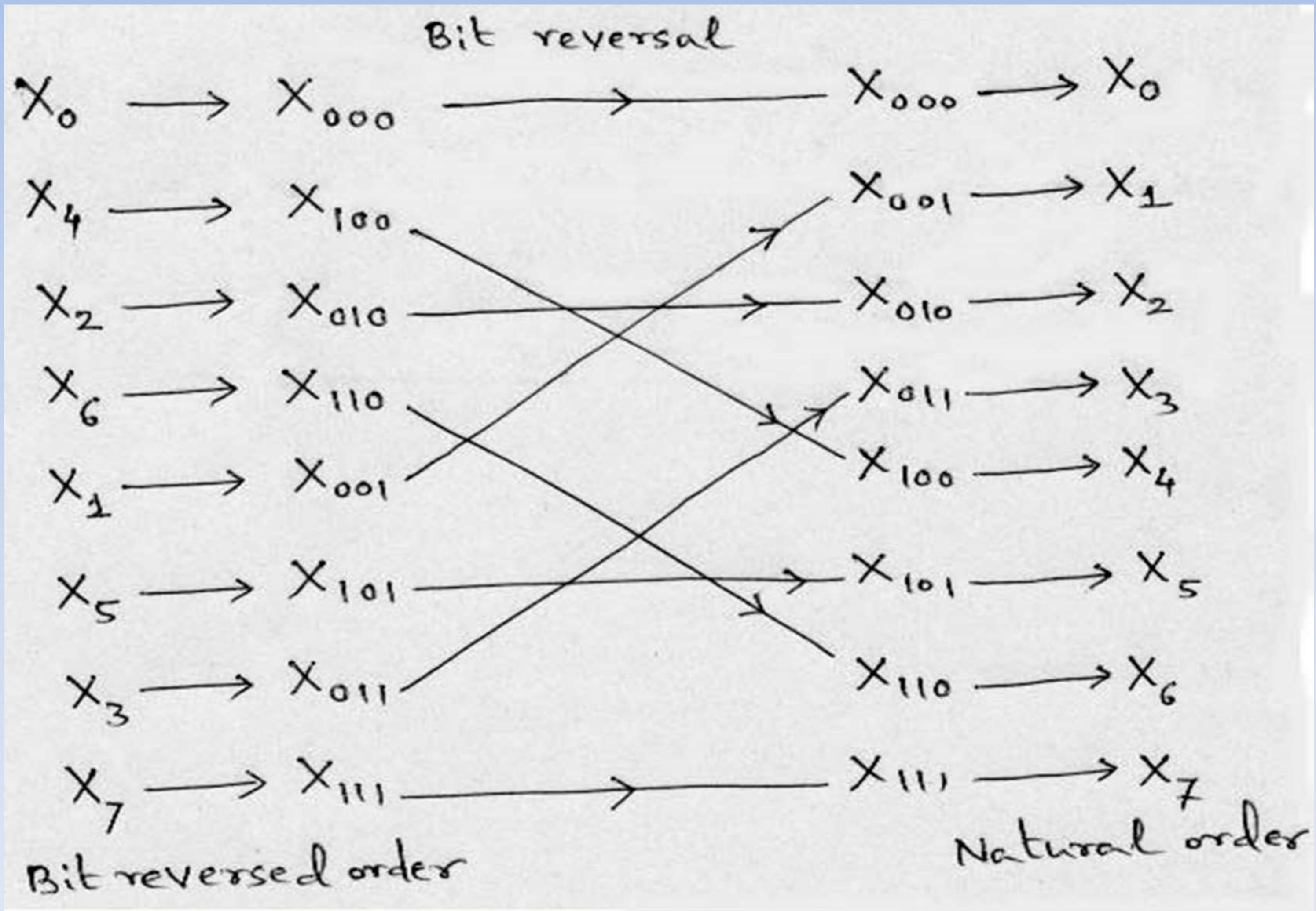
From relations (46), (47), (48) and (49), signal flow graph for computation of X_{0-7} may be represented as:



not in natural order, hence bit-reversal should be carried out to bring it in natural order

8-point FFT

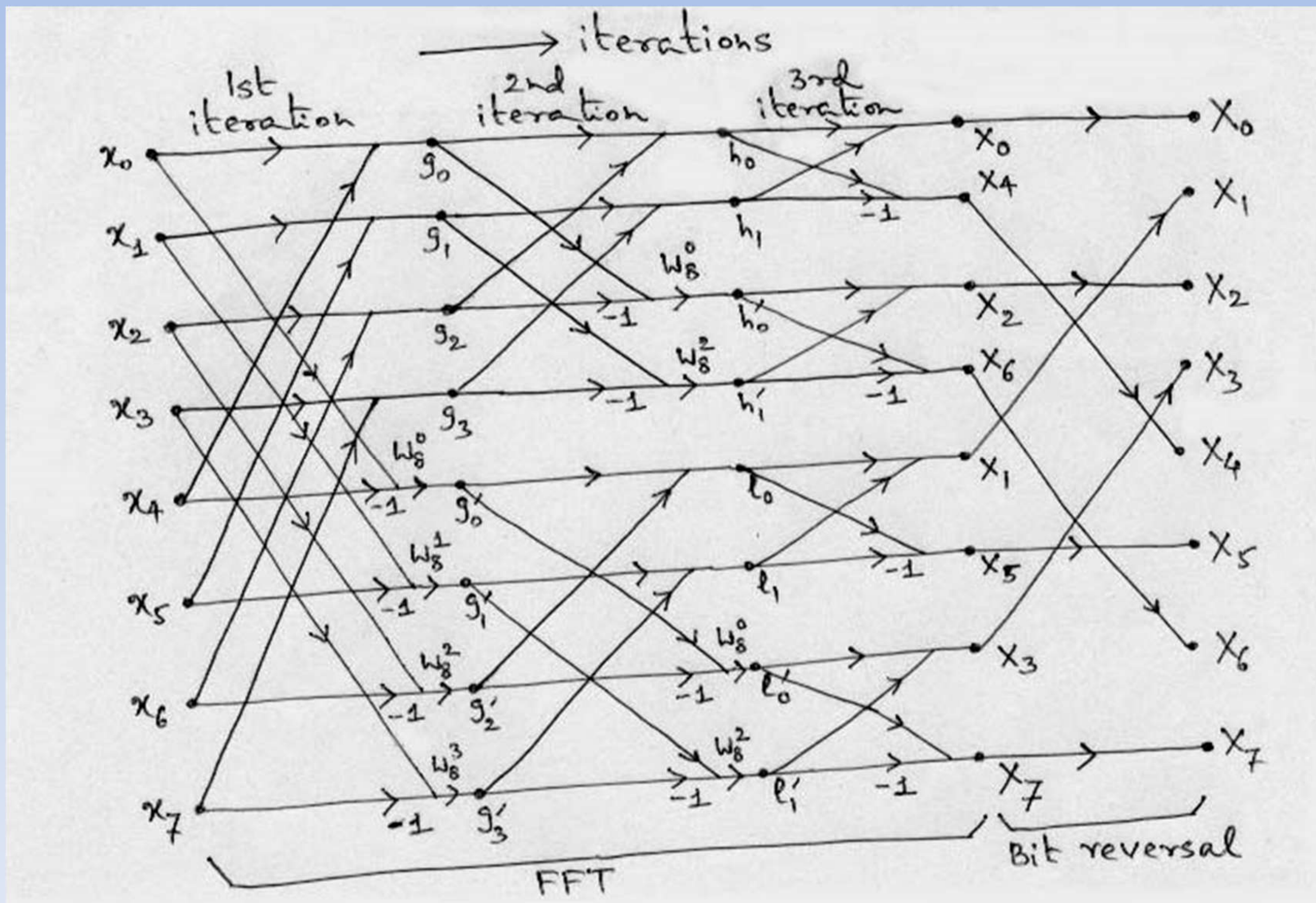
Bit Reversal procedure



8-point FFT

Complete Signal Flow Graph

Time history



Frequency

FORTRAN subroutine to compute radix-2 FFT

```
C ***Subroutine to compute radix-2 FFT***
C Decimation-in-frequency in-place algorithm
            SUBROUTINE FFT(A,N,INV)
C N: Dimension of Array (must be a power of 2)
C A: Complex array containing data sequence
C DFT coefficients are returned in the array
C INV = 0 for forward FFT
C INV = 1 for inverse FFT
            DIMENSION A(N)
            COMPLEX T,W,A
            IF (INV.EQ.0) GO TO 8
C Divide sequence by N for inverse FFT
            DO 7 I=1,N
7           A(I)=A(I)/CMPLX(FLOAT(N),0.0)
8           S=-1.0
            IF (INV.EQ.1) S=1.0
```

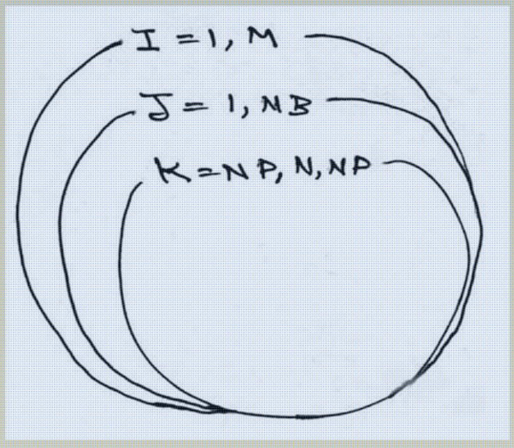
FORTRAN subroutine to compute radix-2 FFT

```
C Calculate number of iterations
C M: Number of iterations (log(N) to the base 2)
        M=1
        K=N
2       K=K/2
        IF (K.EQ.1) GO TO 1
        M=M+1
        GO TO 2
C Compute for each iteration
C NP: Number of points in each partition
1       NB=N
        DO 3 I=1,M
        NP=NB
        NB=NP/2
        PHI=3.14159265/FLOAT(NB)
```

FORTRAN subroutine to compute radix-2 FFT

```
C Compute for each iteration
C NP: Number of points in each partition
1        NB=N
         DO 3 I=1,M
         NP=NB
         NB=NP/2
         PHI=3.14159265/FLOAT(NB)
C Calculate the twiddle factor W for each butterfly
C NB: Number of butterflies for each partition
         DO 3 J=1,NB
         ARG=FLOAT(J-1)*PHI
         W=CMPLX(COS(ARG),S*SIN(ARG))
C Compute butterfly for each partition
         DO 3 K=NP,N,NP
         J1=K-NP+J
         J2=J1+NB
         T=A(J1)-A(J2)
         A(J1)=A(J1)+A(J2)
         A(J2)=T*W
3        CONTINUE
```

Nested DO loop



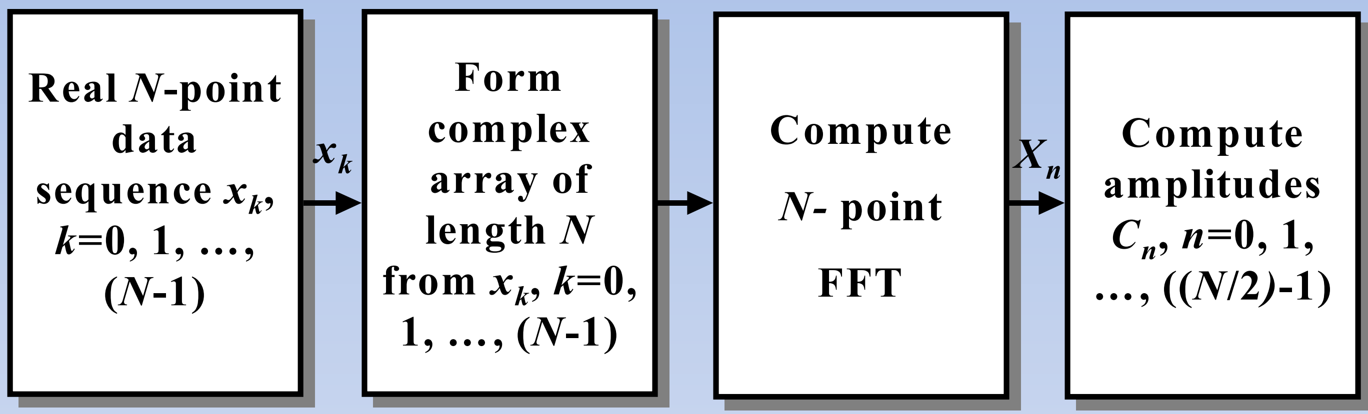
FORTRAN subroutine to compute radix-2 FFT

```
C Bit reversal operation
        N2=N/2
        N1=N-1
        J=1
        DO 4 I=1,N1
        IF (I.GE.J) GO TO 5
        T=A(J)
        A(J)=A(I)
        A(I)=T
5       K=N2
6       IF (K.GE.J) GO TO 4
        J=J-K
        K=K/2
        GO TO 6
4       J=J+K
        RETURN
        END
```

During the bit-reversal operation, N/2 DFT coefficients remain unchanged and the remaining N/2 coefficients are exchanged in place as required.

Applications of FFT

Computation of amplitude spectrum of a finite real data sequence



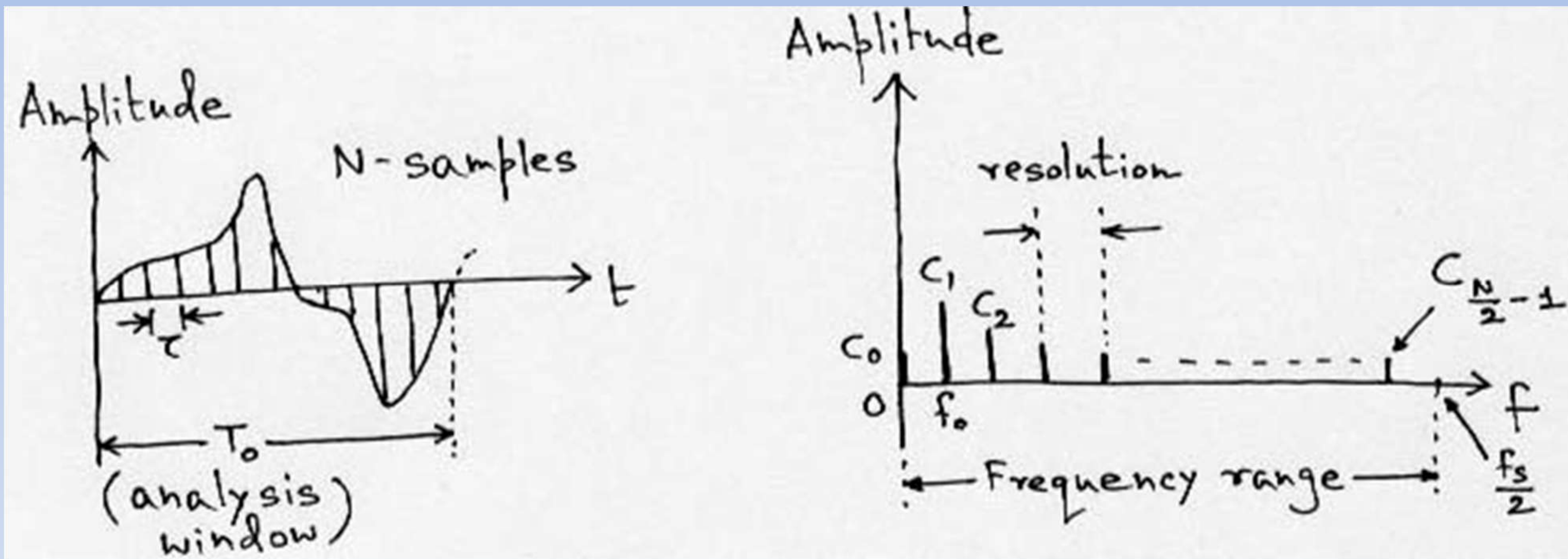
(N must be a power of 2)

$$C_0 = \frac{1}{N}\left|X_0\right|$$, the average value

and $$C_n = \frac{2}{N}\left|X_n\right|$$, for n = 1,2,…,(N/2-1), the nth harmonic amplitude.

Applications of FFT

Computation of amplitude spectrum of a finite real data sequence



The range of frequency may be expressed as $f_s/2$ where f_s is the sampling frequency $\left(=\frac{1}{\tau}\right)$.

The frequency resolution may be estimated as f_0 where f_0 is the fundamental frequency ($=1/T_0$), where T_0 is the time period of fundamental frequency and also the width of the analysis window.

Applications of FFT

FORTRAN program for computation of amplitude spectrum

```
C ***Amplitude spectrum analysis program using FFT***
        DIMENSION A(1024),B(1024),C(512),PHASE(512)
        COMPLEX A
        CHARACTER*64 FNAME
        WRITE(*,10)
10      FORMAT(1X,'Enter file name - '\)
        READ(*,20)FNAME
20      FORMAT(A)
        OPEN(2,FILE=FNAME)
        READ(2,*,END=100)(B(I),I=1,1024)
100      N=I-1
        CLOSE(2)
        WRITE(*,200)N
200     FORMAT(1X,'Data points = ',I4)
```

Applications of FFT

FORTRAN program for computation of amplitude spectrum

```
        DO 15 I=1,10
        IF(N-2**I)24,25,15
15      CONTINUE
24      WRITE(*,5)
5       FORMAT(1X,'Incorrect size - it must be a power of 2')
        STOP
25      DO 30 I=1,N
30      A(I)=CMPLX(B(I),0.0)/CMPLX(FLOAT(N),0.0)
        WRITE(*,300)
300     FORMAT(1X,'FFT analysis in progress')
        CALL FFT(A,N,0)
```

Applications of FFT

FORTRAN program for computation of amplitude spectrum

```
        NA=N/2
        C(1)=CABS(A(1))
        DO 40 I=2,NA
40      C(I)=CABS(A(I))*2.0
        D=180.0/3.141592654
        DO 80 I=2,NA
        R=REAL(A(I))
        X=AIMAG(A(I))
        ALPHA=ATAN2(X,R)
80      PHASE(I)=D*ALPHA
```

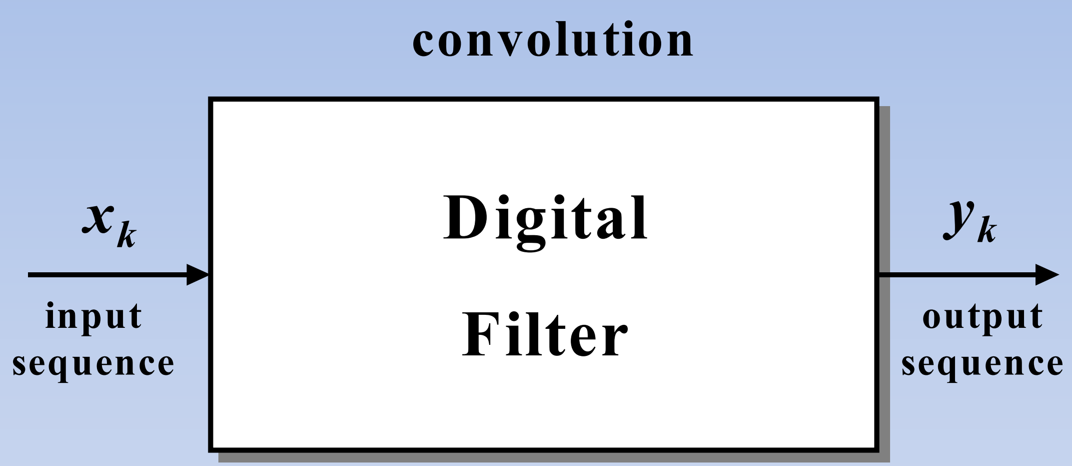
Applications of FFT

FORTRAN program for computation of amplitude spectrum

```
      WRITE(*,60)
60    FORMAT('0','Harmonic no.',7X,'Amplitude',12X,'Phase (deg)')
      WRITE(*,70)
70    FORMAT(1X,'------------',7X,'---------',12X,'-----------'//)
      NB=0
      WRITE(*,75)NB,C(1)
75    FORMAT(5X,I3,9X,1P,E13.6)
      DO 85 I=2,NA
      NB=I-1
85    WRITE(*,90)NB,C(I),PHASE(I)
90    FORMAT(5X,I3,9X,1P,E13.6,9X,E13.6)
      END
```

Applications of FFT

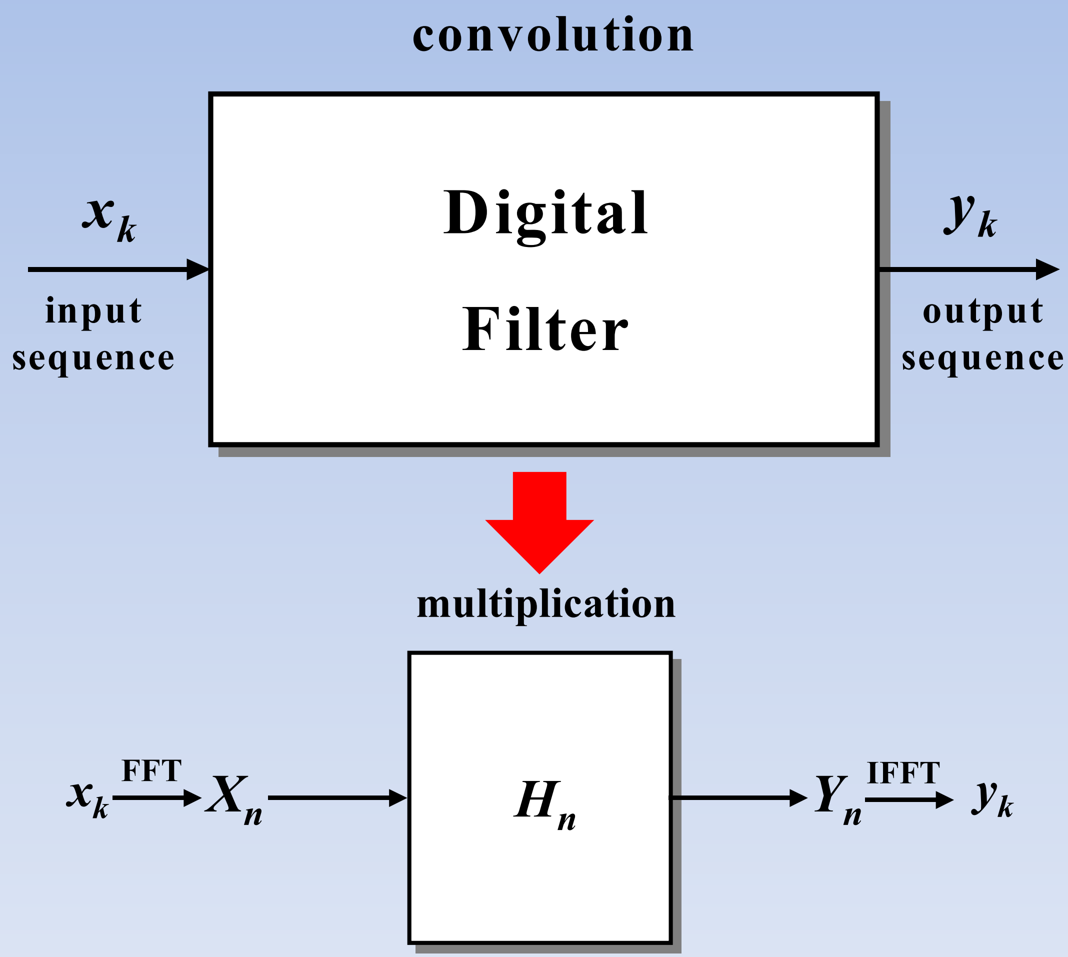
FFT-based digital filtering of a finite real data sequence



$y_k = h_k * x_k$, h_k is the impulse sequence of the digital filter

Applications of FFT

FFT-based digital filtering of a finite real data sequence

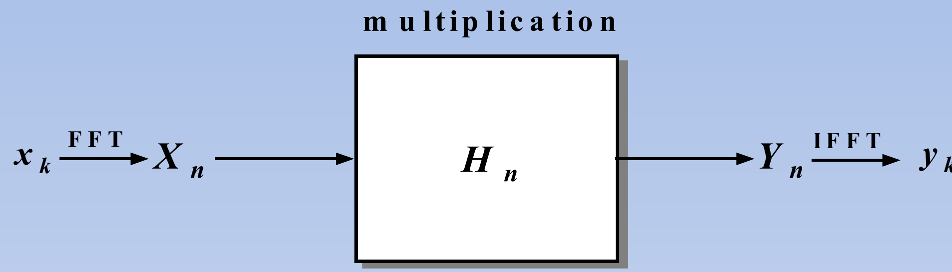


$y_k = h_k * x_k$

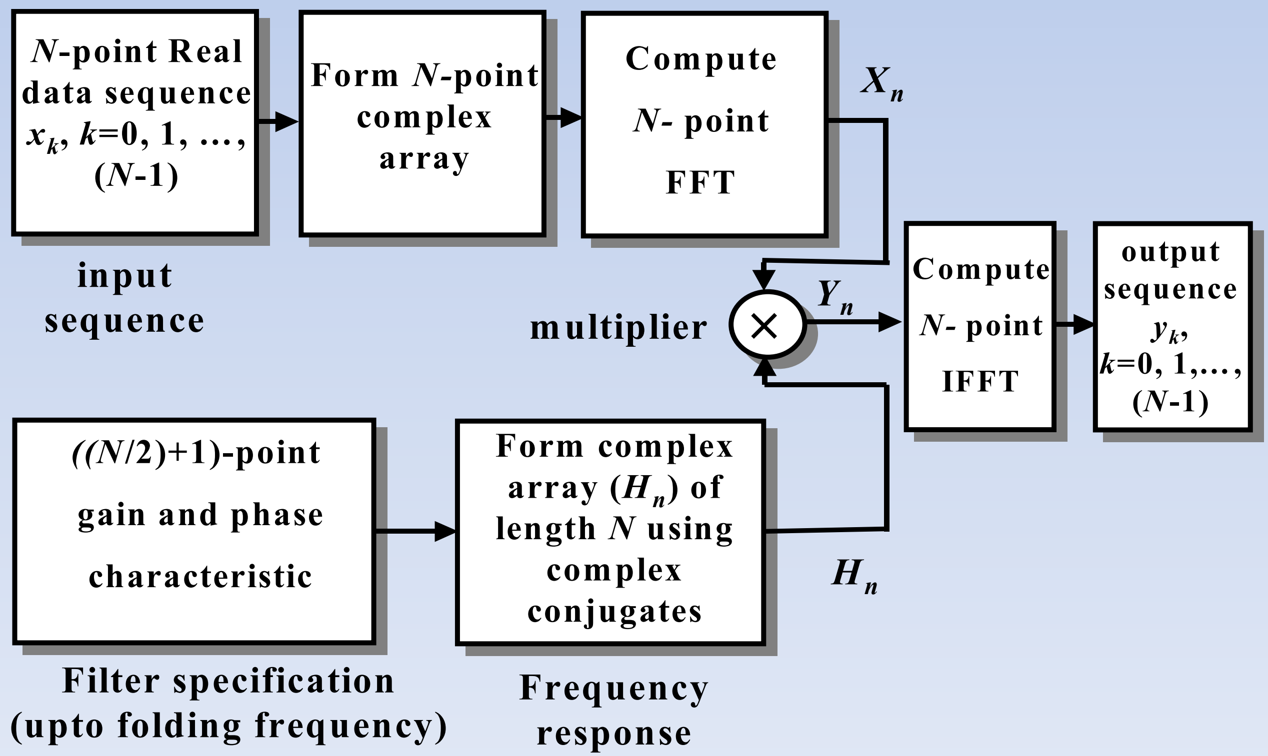
$Y_n = H_n \cdot X_n$, H_n is the complex gain of the digital filter

Applications of FFT

FFT-based digital filtering of a finite real data sequence

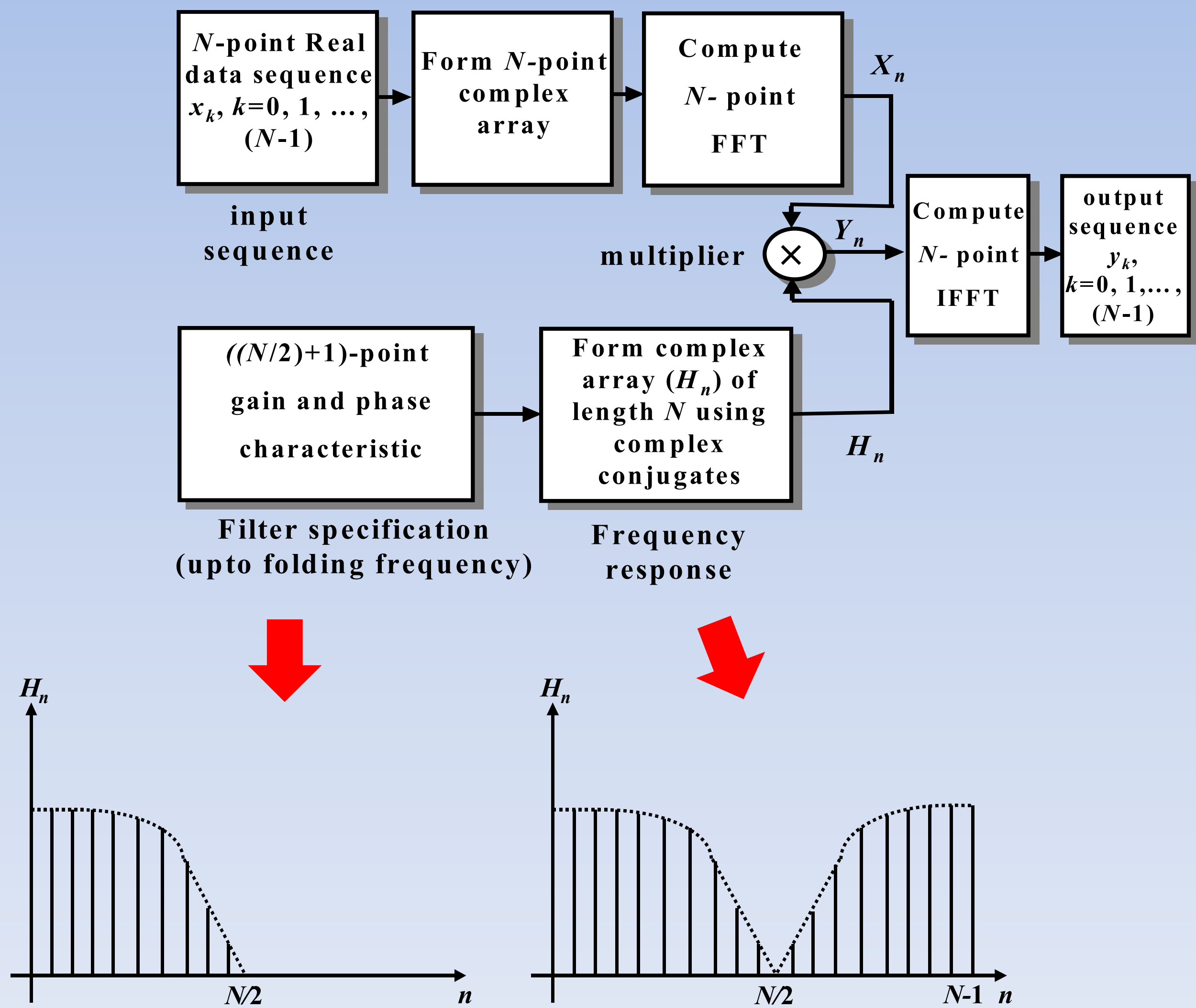


$$Y_n = H_n \,.\, X_n$$



Applications of FFT

FFT-based digital filtering of a finite real data sequence



Applications of FFT

FORTRAN program for FFT-based digital filtering

```
C ***FFT based digital filter program***
        DIMENSION X(1024),A(513),PH(513)
        COMPLEX H(1024),CX(1024)
        CHARACTER*64 FNAME1,FNAME2
        WRITE(*,20)
20      FORMAT(1X,'Enter input file name - '\)
        READ(*,30)FNAME1
30      FORMAT(A)
        WRITE(*,40)
40      FORMAT(1X,'Enter output file name - '\)
        READ(*,30)FNAME2
```

Applications of FFT

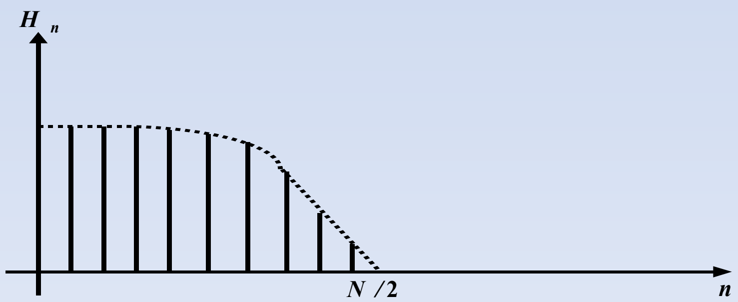
FORTRAN program for FFT-based digital filtering

```
          OPEN(1,FILE=FNAME1)
          READ(1,*,END=50)(X(I),I=1,1024)
          GOTO 60
50        I=I-1
60        N=I
          WRITE(*,70)N
70        FORMAT(1X,'Data points = ',I4)
          DO 375 I=1,10
          IF(N-2**I)380,390,375
375       CONTINUE
380       WRITE(*,400)
400       FORMAT(1X,'Incorrect size of data')
          STOP
```

Applications of FFT

FORTRAN program for FFT-based digital filtering

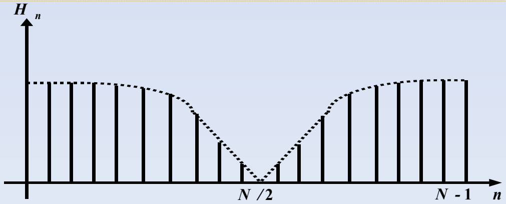
```
390     WRITE(*,100)N/2+1
100     FORMAT(1X,'Enter filter gain (',I3,' points)')
        READ(*,*)(A(I),I=1,N/2+1)
        PH(1)=0.0
        WRITE(*,110)N/2
110     FORMAT(1X,'Enter filter phase (',I3,' points)')
        READ(*,*)(PH(I),I=2,N/2+1)
C Form complex filter gain array
        H(1)=CMPLX(A(1)*N,0.0)
        DO 200 I=2,N/2+1
200     H(I)=CMPLX(A(I)*N*COS(PH(I))/2.0,A(I)*N*SIN(PH(I))/2.0)
```



Applications of FFT

FORTRAN program for FFT-based digital filtering

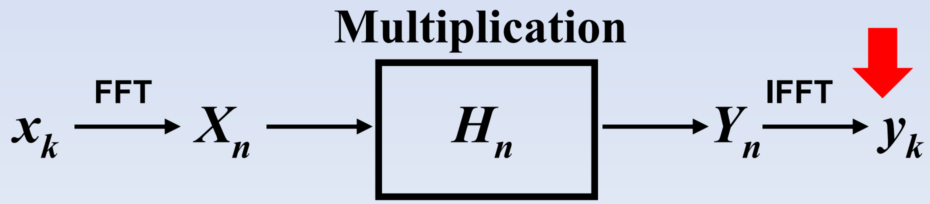
```
390     WRITE(*,100)N/2+1
100     FORMAT(1X,'Enter filter gain (',I3,' points)')
        READ(*,*)(A(I),I=1,N/2+1)
        PH(1)=0.0
        WRITE(*,110)N/2
110     FORMAT(1X,'Enter filter phase (',I3,' points)')
        READ(*,*)(PH(I),I=2,N/2+1)
C Form complex filter gain array
        H(1)=CMPLX(A(1)*N,0.0)
        DO 200 I=2,N/2+1
200     H(I)=CMPLX(A(I)*N*COS(PH(I))/2.0,A(I)*N*SIN(PH(I))/2.0)
C Form rest of the gain array by complex conjugate
        DO 340 I=2,N/2
        J=N+2-I
340     H(J)=CONJG(H(I))
```



Applications of FFT

FORTRAN program for FFT-based digital filtering

```
C Form complex input data array
          DO 350 I=1,N
350       CX(I)=CMPLX(X(I),0.0)
C Compute FFT
          CALL FFT(CX,N,0)
C Perform filtering in frequency domain
          DO 360 I=1,N
360       CX(I)=CX(I)*H(I)
C Back to time
          CALL FFT(CX,N,1)
          DO 370 I=1,N
370       X(I)=REAL(CX(I))
```



Applications of FFT

FORTRAN program for FFT-based digital filtering

```
C Save output
        IF(FNAME1.EQ.FNAME2)CLOSE(1)
        OPEN(2,FILE=FNAME2,STATUS='NEW')
        WRITE(2,*)(X(I),I=1,N)
        END
```

Thank You