WIENER FILTERING THEORY

In conventional filtering technique, a signal can be separated from unwanted noise when the signal and the noise spectra do not overlap (shown in Fig.1).

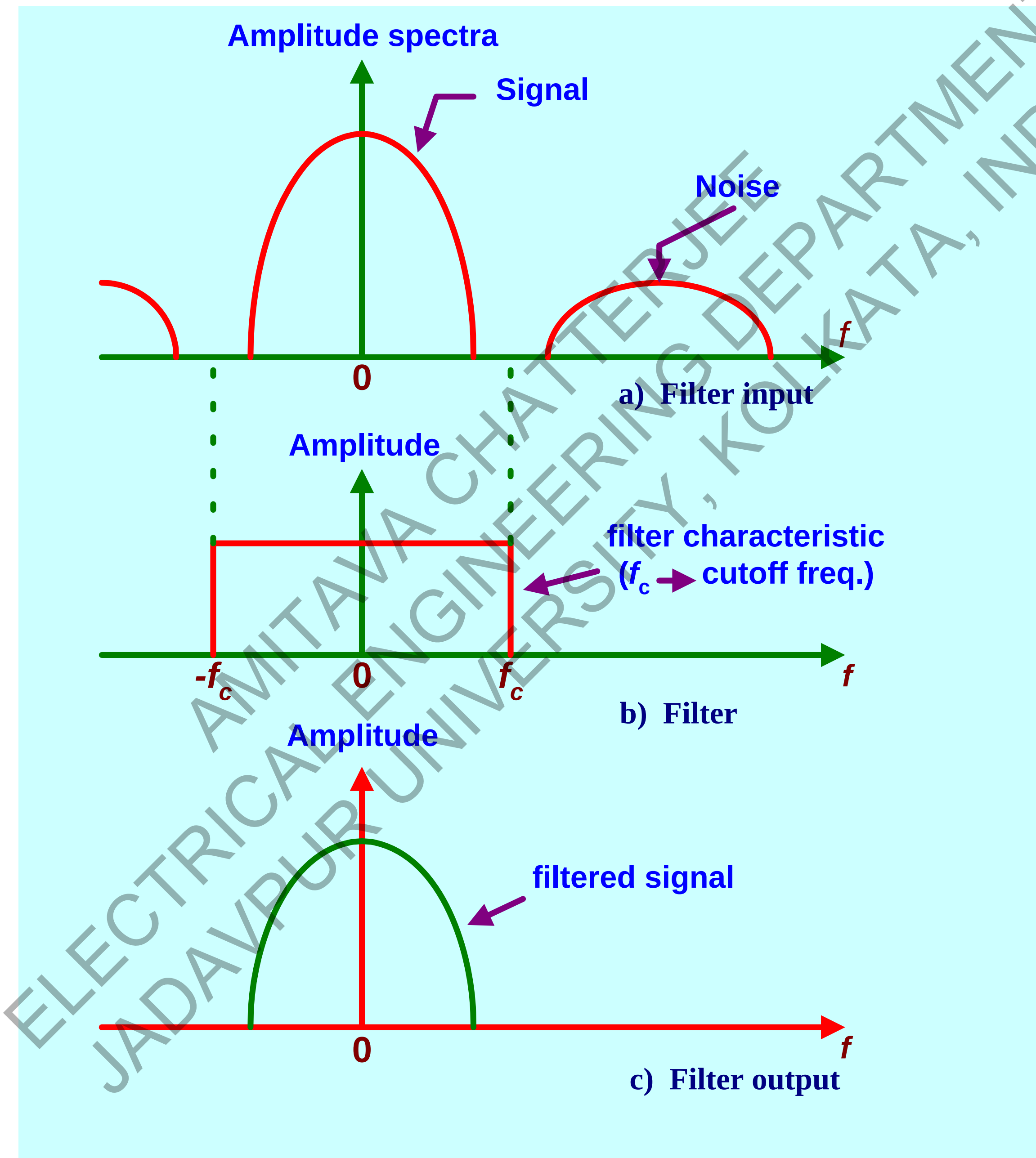

Fig. 1. Conventional filtering.

But when the signal and the noise spectra overlap, conventional filtering technique fails to separate the signal and the additive noise (shown in Fig.2).

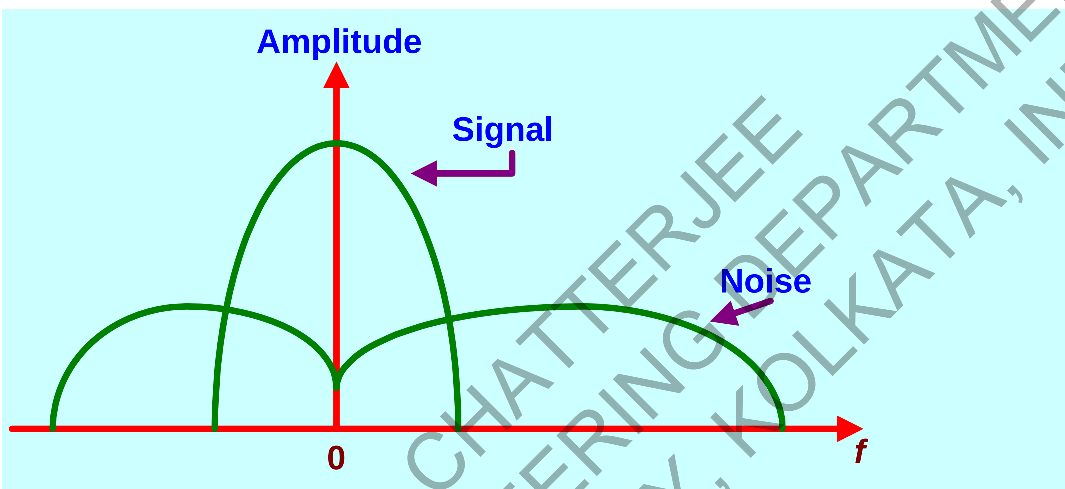

Fig. 2. Signal and noise having overlapping amplitude spectra.

When both the signal s(t) and the additive noise n(t) are stochastic and stationary processes, the signal s(t) can be estimated optimally with **Wiener filtering technique**, so that the noise part is reduced as much as possible when the spectral densities of the signal and the noise are known quantities. Filtering is possible with overlapping or non-overlapping signal and noise spectra.

Fig. 3 illustrates the principle of operation of **Wiener filter**. An optimal filter estimates the signal $\hat{s}(t)$ so that the mean square error $E\left[e(t)^2\right]$ is a minimum.

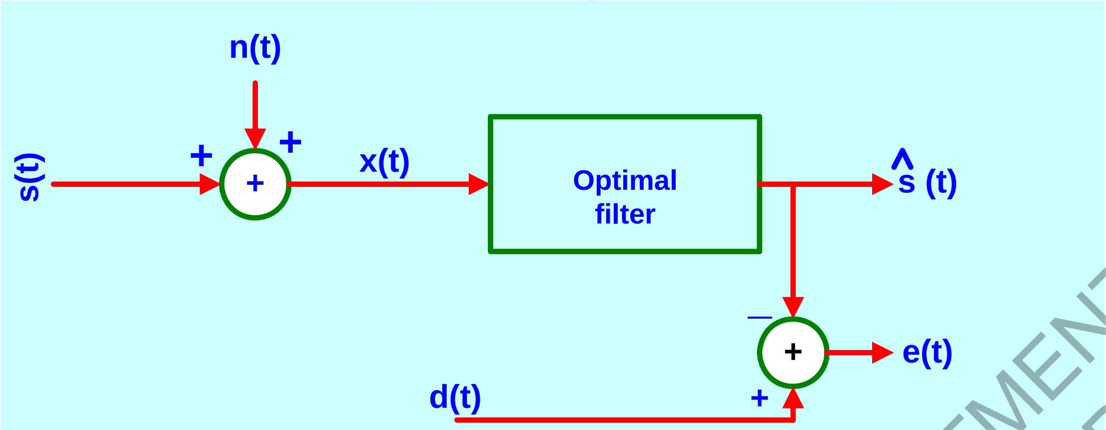

Fig. 3. Wiener filter model

$s(t)$: Signal

$n(t)$: Noise

$x(t)$: Filter input

$\hat{s}(t)$: Filter output

$d(t)$: Desired output

$e(t)$: Error

$$e(t) = d(t) - \hat{s}(t)$$

$$x(t) = s(t) + n(t)$$

If a filter can be designed that has **minimum mean square error**, the filter is then called the **Optimal filter** in the mean square error sense. It is also known as the **Wiener filter**.

When the filter is causal with a finite duration impulse response it is called a constrained Wiener FIR filter.

Let a Wiener FIR filter H(z) contain N number of taps, i.e. (N-1) number of delay stages. Then, at the nth instant, the input-output relation can be expressed as,

$$\hat{s}_n = \sum_{m=0}^{N-1} h_m \, x_{n-m} \qquad (1)$$

for all n

where h_m, $m = 0, 1, \ldots\ldots N-1$, is the finite impulse response of the causal FIR filter. In matrix form, relation (1) can be rewritten as

$$\hat{S}_n = H^T X_n \qquad (2)$$

where

$$H^T = [h_o, h_1 \ldots\ldots\ldots\ldots\ldots, h_{N-1}]$$

and $$X_n^T = \lfloor x_n, x_{n-1} \ldots\ldots\ldots\ldots, x_{n-(N-1)} \rfloor$$

The problem is, for a given s_n and x_n, design h_m, $m=0, 1, \ldots\ldots N-1$, such that the mean square error $E[e_n^2] = E\lfloor (d_n - \hat{s}_n)^2 \rfloor$ is a minimum.

Fig. 4 shows the **Discrete Wiener Filter** in its schematic form.

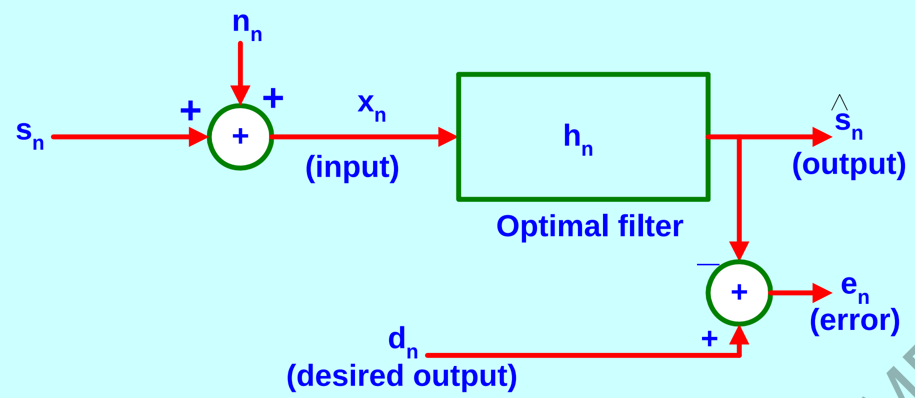

Fig.4: Discrete Wiener Filter

The mean square error can be expressed as

$$E\left[e_n^2\right] = E\left[\left\{d_n - \sum_{m=0}^{N-1} h_m x_{n-m}\right\}\left\{d_n - \sum_{k=0}^{N-1} h_k x_{n-k}\right\}\right]$$
$$= \sum_{m=0}^{N-1}\sum_{k=0}^{N-1} h_m h_k E\left[x_{n-m}x_{n-k}\right] - 2\sum_{k=0}^{N-1} h_k E\left[x_{n-k}d_n\right] + E\left[d_n^2\right] \qquad (3)$$
$$= \sum_{m=0}^{N-1}\sum_{k=0}^{N-1} h_m h_k R_{xx}(m-k) - 2\sum_{k=0}^{N-1} h_k R_{xd}(k) + R_{dd}(0)$$

Note:

$$E\left[x_{n-m}x_{n-k}\right] = E\left[x_{n'}x_{\{n'+(m-k)\}}\right]$$
$$= R_{xx}(m-k) \qquad (\text{where } n' = n-m)$$

$R_{xx}(k)$: Auto-correlation of the input sequence x_n

$R_{xd}(k)$: Cross-correlation between the input sequence x_n and the desired output sequence d_n

Now, for a minimum mean square error,

$$\frac{\partial E\left[e_n^2\right]}{\partial h_j}=0, \quad \text{for } j = 0, 1, \ldots\ldots\ldots\ldots, N-1$$

or $$2\sum_{m=0}^{N-1} h_m R_{xx}(m-j) - 2R_{xd}(j) = 0$$

or $$\sum_{m=0}^{N-1} h_m R_{xx}(m-j) = R_{xd}(j) \qquad (4)$$

j = 0, 1, ………………, N – 1

Note:

Let us consider the quantity,

$$\frac{\partial \sum_{m=0}^{N-1}\sum_{k=0}^{N-1} h_m h_k R_{xx}(m-k)}{\partial h_j}$$

Now for each j, say j = 2, there is one term of m = 2 in the first summation, and one term of k = 2 in the second summation. Thus, the above quantity becomes,

$$\begin{aligned}
&= \sum_{m=0}^{N-1} h_m R_{xx}(m-j) + \sum_{k=0}^{N-1} h_k R_{xx}(j-k) \\
&= \sum_{m=0}^{N-1} h_m R_{xx}(m-j) + \sum_{m=0}^{N-1} h_m R_{xx}(j-m) \\
&= \sum_{m=0}^{N-1} h_m R_{xx}(m-j) + \sum_{m=0}^{N-1} h_m R_{xx}(m-j) \\
&= 2\sum_{m=0}^{N-1} h_m R_{xx}(m-j)
\end{aligned}$$

as $R_{xx}(l-k)=R_{xx}(k-l)$, an even function.

When x_n is stationary, $R_{xx}(n) = R_{xx}(-n)$, and relation (4) can be expressed in matrix form as

$$\begin{bmatrix} R_{xx}(0) & R_{xx}(1) & \cdots & R_{xx}(N-1) \\ R_{xx}(1) & R_{xx}(0) & \cdots & R_{xx}(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(N-1) & R_{xx}(N-2) & \cdots & R_{xx}(0) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{N-1} \end{bmatrix} = \begin{bmatrix} R_{xd}(0) \\ R_{xd}(1) \\ \vdots \\ R_{xd}(N-1) \end{bmatrix} \quad (5)$$

or, $\quad R\,H = P \quad (6)$

where,

$$R = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & \cdots & R_{xx}(N-1) \\ R_{xx}(1) & R_{xx}(0) & \cdots & R_{xx}(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(N-1) & R_{xx}(N-2) & \cdots & R_{xx}(0) \end{bmatrix}$$

, the data correlation matrix,

$$H^T = [h_o \; h_1 \cdots h_{N-1}]$$

and $P^T = [R_{xd}(0) \; \cdots \; R_{xd}(N-1)]$

These relations (4), (5) or (6) are the finite causal form of **Discrete Wiener–Hopf Equation**. This set of linear equations specifies the optimal filter.

From relations (4), (5) or (6), the optimal impulse response h_m, $m=0, 1, \ldots, N-1$ of the filter $H(z)$ can be obtained when correlation functions $R_{xx}(m-j)$ and $R_{xd}(j)$ are known quantities {c.f. relation (4)}.

The mean square error with optimal h_m can be obtained as (from relation (3)),

$$E\left[e_n^2\right] = \sum_{k=0}^{N-1} h_k \left[\sum_{m=0}^{N-1} h_m R_{xx}(m-k) - 2R_{xd}(k) \right] + R_{dd}(0)$$

Substituting the value of $\sum_{m=0}^{N-1} h_m R_{xx}(m-k)$ from relation (4),

$$\begin{aligned} E\left[e_n^2\right]_{optimal} &= R_{dd}(0) + \sum_{k=0}^{N-1} h_k \left[R_{xd}(k) - 2R_{xd}(k)\right] \\ &= R_{dd}(0) - \sum_{k=0}^{N-1} h_k R_{xd}(k) \end{aligned} \quad (7)$$

When s_n and n_n are statistically independent and n_n has a zero mean, then

$$\begin{aligned} R_{xd}(k) &= E\left[x_n d_{n+k}\right] = E\left[\{s_n + n_n\} d_{n+k}\right] \\ &= E\left[s_n d_{n+k}\right] + E\left[n_n d_{n+k}\right] \\ &= R_{sd}(k) + R_{nd}(k) \end{aligned} \quad (8)$$

and
$$\begin{aligned} R_{xx}(k) &= E\left[x_n x_{n+k}\right] = E\left[(s_n + n_n)(s_{n+k} + n_{n+k})\right] \\ &= E\left[s_n s_{n+k}\right] + E\left[n_n n_{n+k}\right] \\ &= R_{ss}(k) + R_{nn}(k) \end{aligned} \quad (9)$$

as $E[s_n n_{n+k}] = E[n_n s_{n+k}] = 0$

Now, relation (4) can be expressed in terms of convolution operation as

$$h_j * R_{xx}(j) = R_{xd}(j) \qquad (10)$$

$$j = 0, 1, \ldots, N-1$$

The **unconstrained Wiener filtering operation** can be expressed as

$$h_j * R_{xx}(j) = R_{xd}(j) \qquad (11)$$

$$j = -\infty \ldots \text{ to } +\infty$$

where * represents convolution operation.

Taking z-transform of relation (11) and by **Wiener – Khintchine theorem**,

$$H(z)\, S_{xx}(z) = S_{xd}(z) \qquad (12)$$

where H(z) is the optimal system function of the unconstrained Wiener filter, $S_{xx}(z)$ is the discrete power spectral density of the filter input, and $S_{xd}(z)$ is the discrete cross spectral density between the filter input and the desired output.

Thus the optimal system function of the filter is (from relation (12)),

$$H(z) = \frac{S_{xd}(z)}{S_{xx}(z)} \qquad (13)$$

Now,

$$S_{xx}(z) = S_{ss}(z) + S_{nn}(z) \tag{14}$$

when s_n and n_n are statistically independent.

Normally, for filtering problems, the desired output is the signal itself, $d_n = s_n$. Furthermore s_n and n_n are uncorrelated random sequences as is usually the case in practice.

$\therefore$ From relation (8),

$$R_{xd}(k) = R_{ss}(k) + R_{ns}(k) = R_{ss}(k) \tag{15}$$

Now, taking z-transform,

$$S_{xd}(z) = S_{ss}(z) \tag{16}$$

Therefore the optimal system function becomes (when $d_n = s_n$),

$$H(z) = \frac{S_{ss}(z)}{S_{ss}(z) + S_{nn}(z)} \tag{17}$$

To obtain the frequency response of the filter, in the frequency domain, we substitute $z = e^{j\omega\tau} = e^{j2\pi f\tau}$, where τ is the sampling interval. Then the **optimal filter gain** is

$$H(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{nn}(\omega)} \tag{18}$$

It can be seen from relation (18), that H(ω) is a frequency dependent scalar quantity. Therefore the **optimal estimate of signal output** is

$$\hat{s}(t) = F^{-1}[X(\omega)H(\omega)] \tag{19}$$

where $F^{-1}[\bullet]$ represents the inverse Fourier transform operation and X(ω) is the Fourier transform of the filter input x(t). Therefore,

$$X(\omega) = F[x(t)] \tag{20}$$

where F[•] is the Fourier transform operation.

The Wiener filter weights the spectral components in X(f) in accordance with the relation (18), as shown in Fig. 5. In region where there is no signal power, the spectral components are entirely suppressed, and if there is no noise power, the components are entirely passed. In the overlapping region, the filter not only affects the noise components, but the signal components as well.

Therefore, the smaller the spectral overlap between the signal and the noise, the more effective the Wiener filter is.

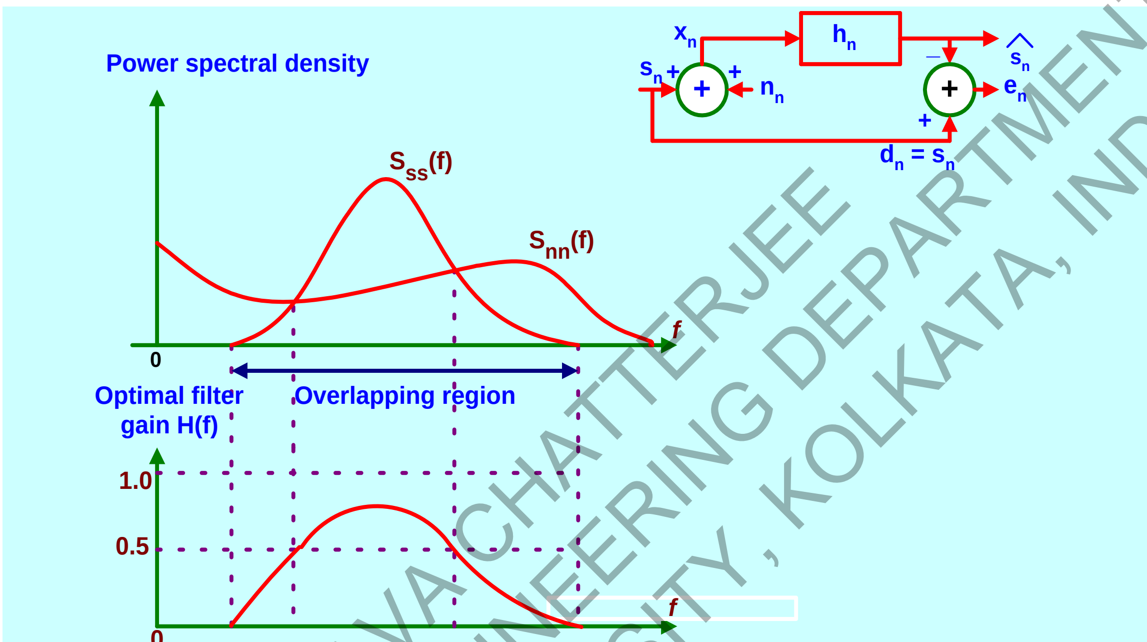

Fig.5: Relation between optimal filter gain and power spectral densities