# Variable Structure Control: An Introduction

by

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# What is Variable Structure Control?

- ✓ Variable Structure Control (VSC) is a viable **high-speed switching feedback control** where the gains in each feedback path switch between two values according to some rule.

- ✓ This variable structure control law provides an **effective and robust means of controlling nonlinear plants**.

- ✓ VSC utilizes a high-speed switching control law to drive the nonlinear plant's state trajectory onto a specified and user-chosen surface in the state space (called the **sliding or switching surface**), and to maintain the plant's state trajectory on this surface for all subsequent time.

# What is Variable Structure Control? (contd…)

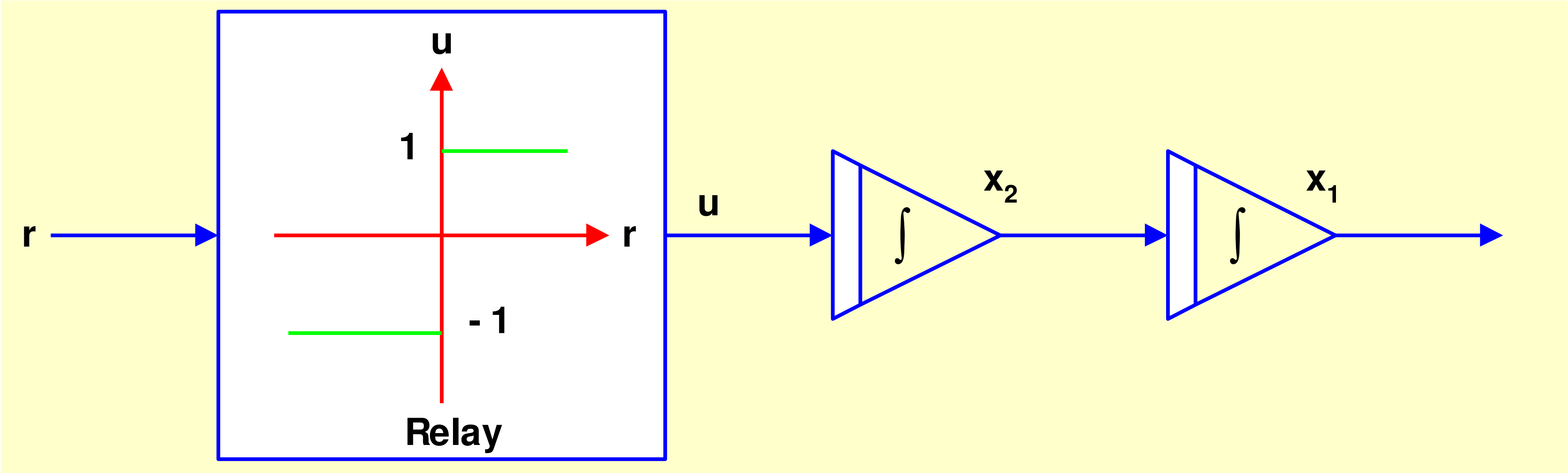
- ✓ This surface is called the **switching surface** because if the state trajectory of the plant is “above” the surface a control path has one gain and a different gain if the trajectory drops ”below” the surface.

- ✓ The plant dynamics restricted to this surface represent the controlled system’s behavior.

- ✓ **VSC** has a great ability to result in very robust control systems, which means it **can provide** such **a control system which is insensitive to parametric uncertainty and external disturbances**.

# VSC - Example 1

*A plant with two accessible states and one control input, controlled by a relay:*



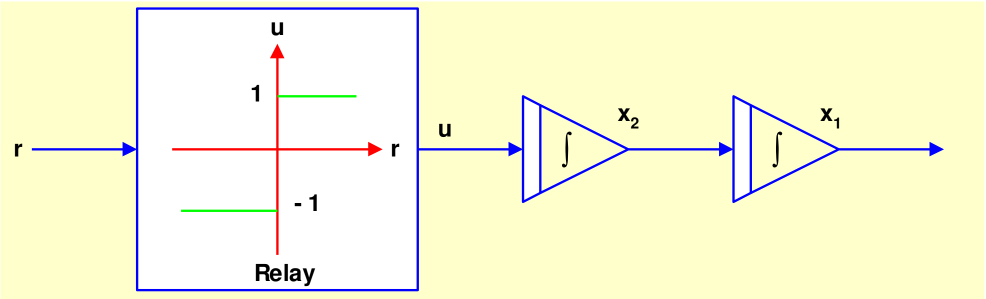

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad |u| \le 1$$

The second order system under consideration

# VSC - Example 1

*A plant with two accessible states and one control input, controlled by a relay:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad |u| \le 1$$




The second order system under consideration

**Let the switching surface be:**

$$\sigma(x_1, x_2) = s_1 x_1 + x_2 = 0$$

**and the control law be:**

$$u = \mathbf{sgn}[\sigma(x_1, x_2)]$$

$$\mathbf{sgn}(\sigma) = \begin{cases} 1 & \sigma > 0 \\ -1 & \sigma < 0 \end{cases}$$

# VSC - Example 1 (contd…)

*A plant with two accessible states and one control input, controlled by a relay:*

$$u = \mathbf{sgn}[\sigma(x_1, x_2)]$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad |u| \le 1$$

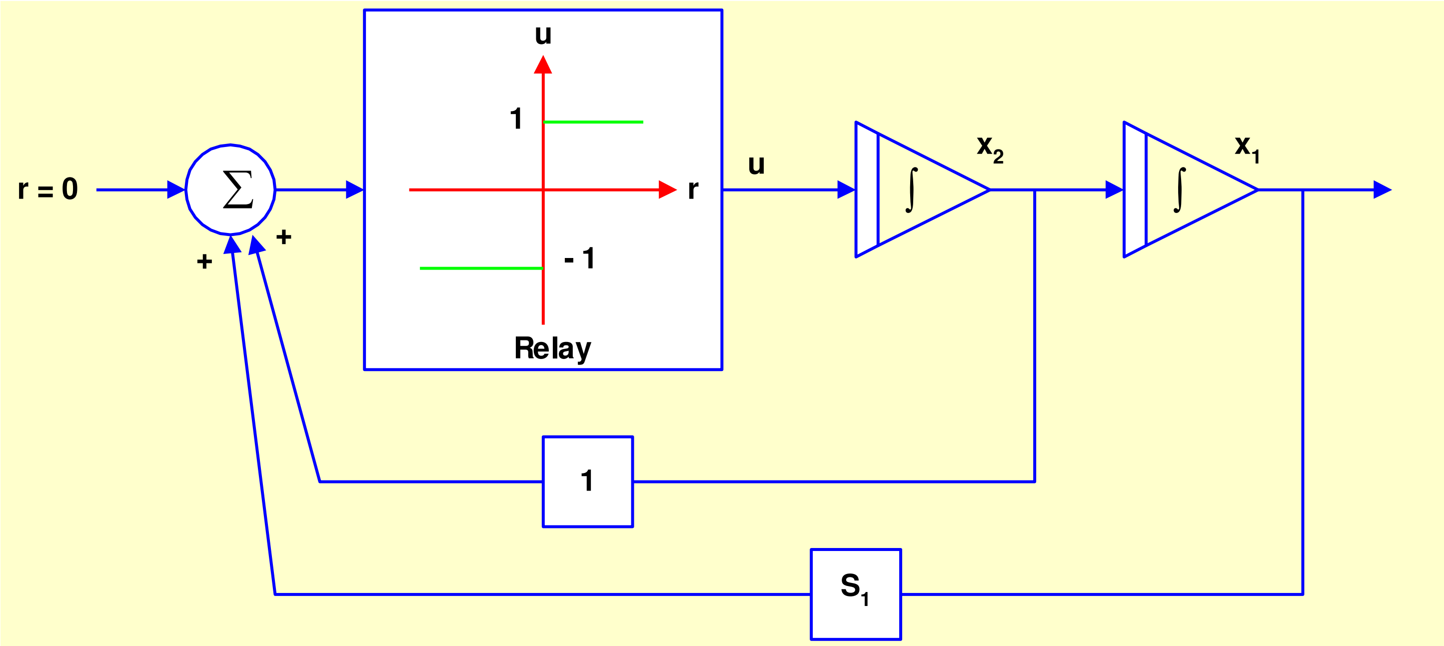
The Plant

$$\sigma(x_1, x_2) = s_1 x_1 + x_2 = 0$$

The Switching Surface

$$\mathbf{sgn}(\sigma) = \begin{cases} 1 & \sigma > 0 \\ -1 & \sigma < 0 \end{cases}$$

The Control Law




Block diagram of the closed loop system with the control strategy

# VSC – Example 1 (contd…)

*A plant with two accessible states and one control input, controlled by a relay:*

$$u = \mathbf{sgn}\left[\sigma(x_1, x_2)\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad |u| \le 1$$

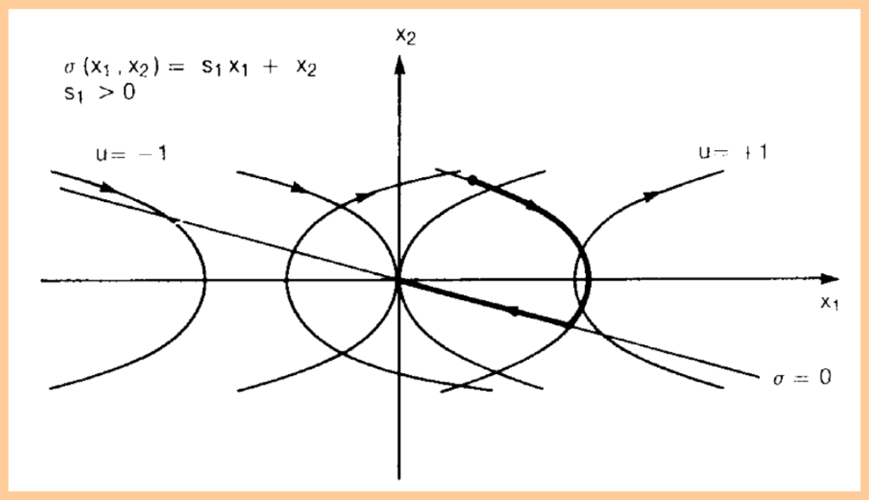
The Plant

$$\sigma(x_1, x_2) = s_1 x_1 + x_2 = 0$$

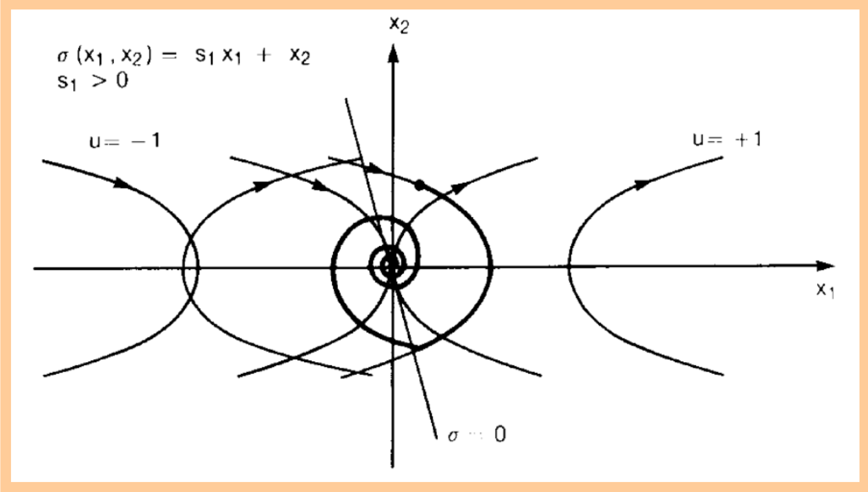
The Switching Surface

$$\mathbf{sgn}(\sigma) = \begin{cases} 1 & \sigma > 0 \\ -1 & \sigma < 0 \end{cases}$$

The Control Law




$s_1$ is small




$s_1$ is large

Phase-plane diagrams of the closed loop system for different values of $s_1$

# VSC – Example 2

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

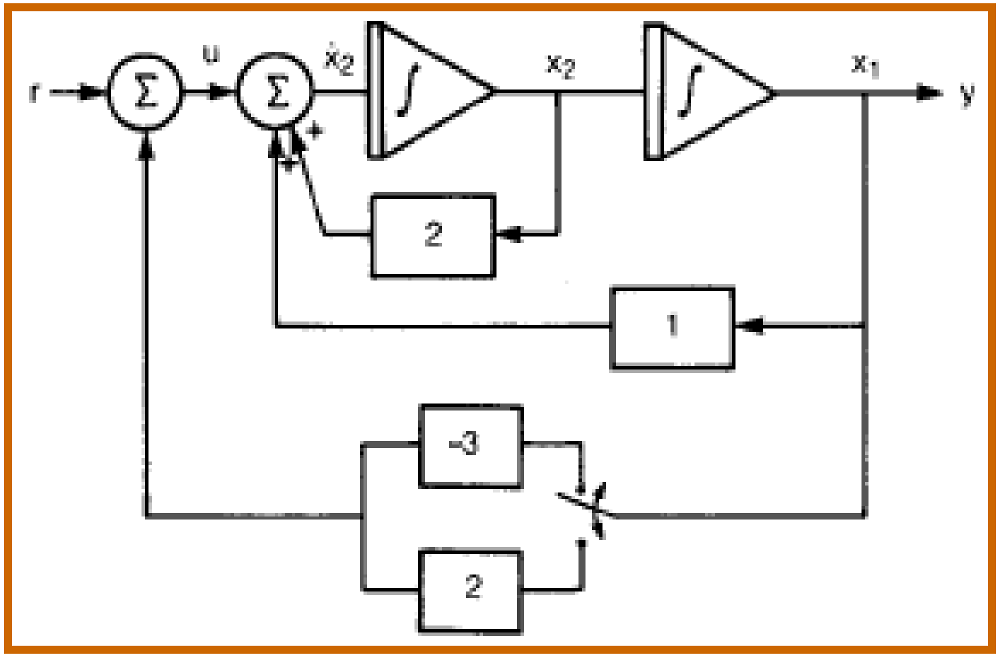
$$u(t) = k(x_1)x_1(t)$$

$$k(x_1) = +2$$

or

$$k(x_1) = -3$$

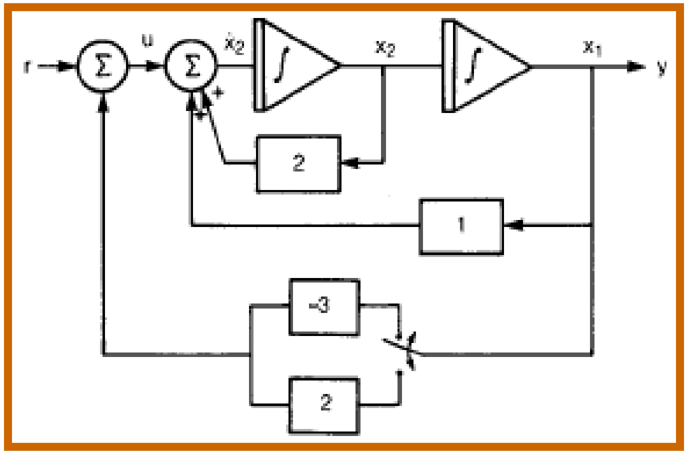
The variable structure control law




Block diagram of the second-order system with variable structure control

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*

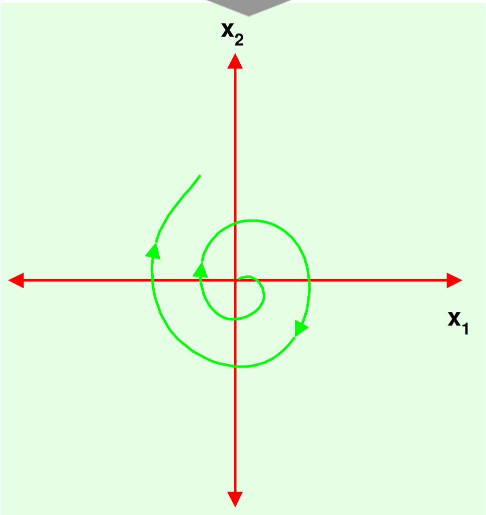



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

When switch is in upper position

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

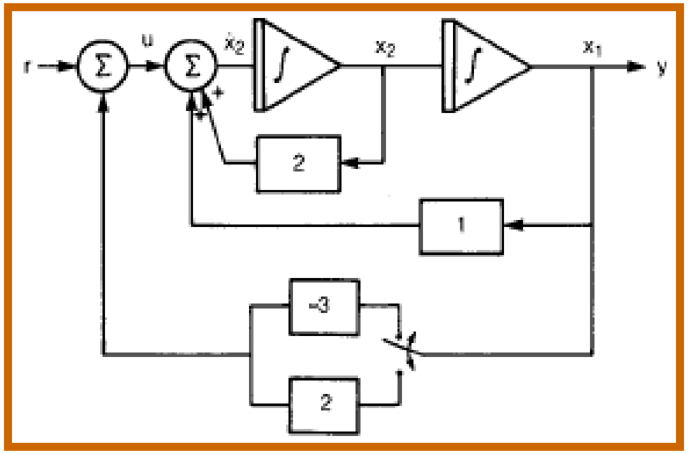



Phase plane plot

**Feedback produces an unstable free motion.**

# VSC - Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*

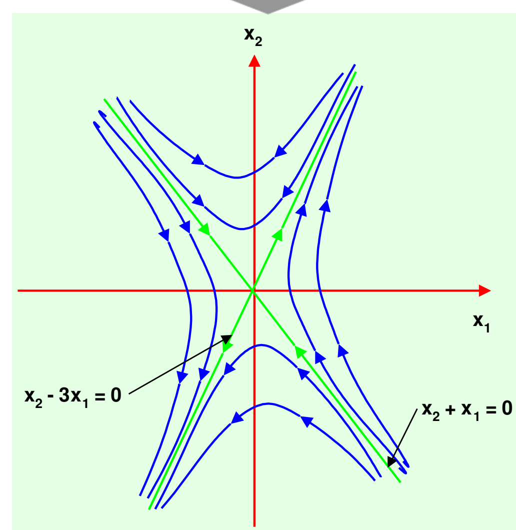



$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

When switch is in lower position

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

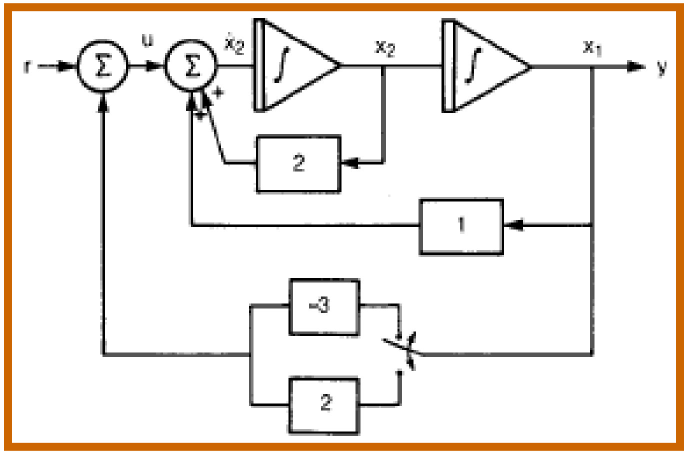



Phase plane plot

**The origin (0,0) is a saddle point with asymptotes: $x_2 = 3x_1$ and $x_2 = -x_1$.**

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*




$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

$$k(x_1) = \begin{cases} -3, & \text{if } \sigma(x_1, x_2)x_1 > 0 \\ +2, & \text{if } \sigma(x_1, x_2)x_1 < 0 \end{cases}$$

- ✓ However, **switching of the controller is not a random phenomenon.** It occurs with respect to a sliding or **switching surface.**

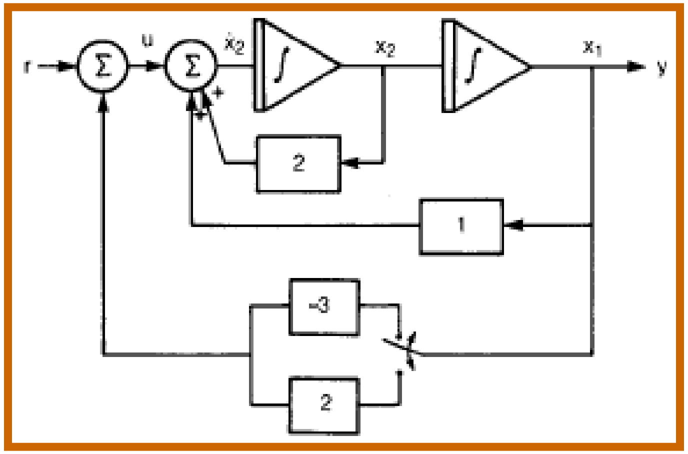
The switching surface considered

$$\sigma = \sigma(x_1, x_2) = s_1 x_1 + x_2 = 0$$

Feedback switching takes place according to the relation

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*



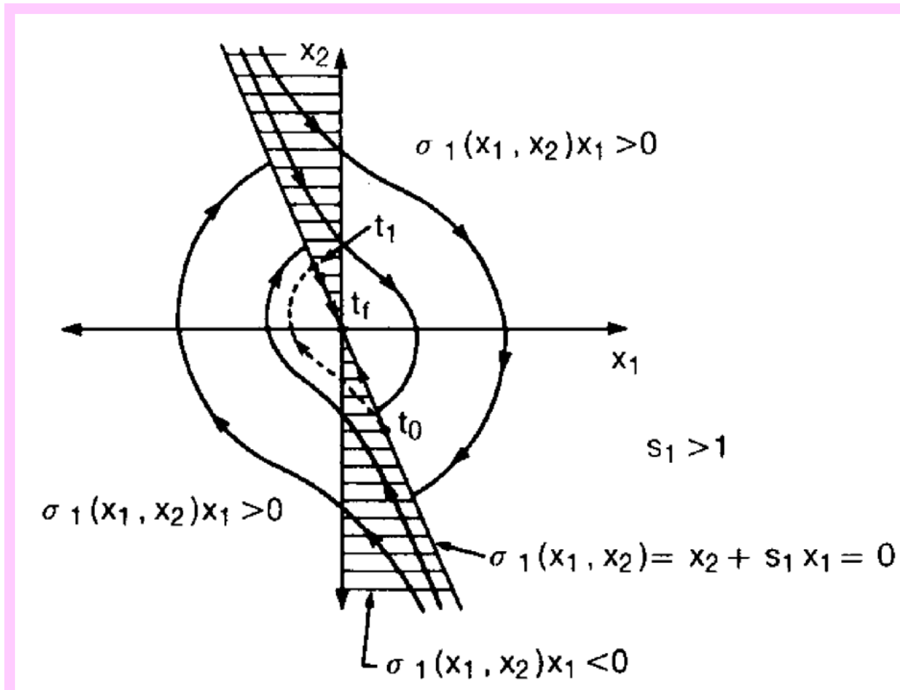

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

$$k(x_1) = \begin{cases} -3, & \text{if } \sigma_1(x_1, x_2)x_1 > 0 \\ +2, & \text{if } \sigma_1(x_1, x_2)x_1 < 0 \end{cases}$$

*Case I*: Switching Surface with $s_1 > 1$

$$\sigma = \sigma_1(x_1, x_2) = s_1 x_1 + x_2 = 0$$

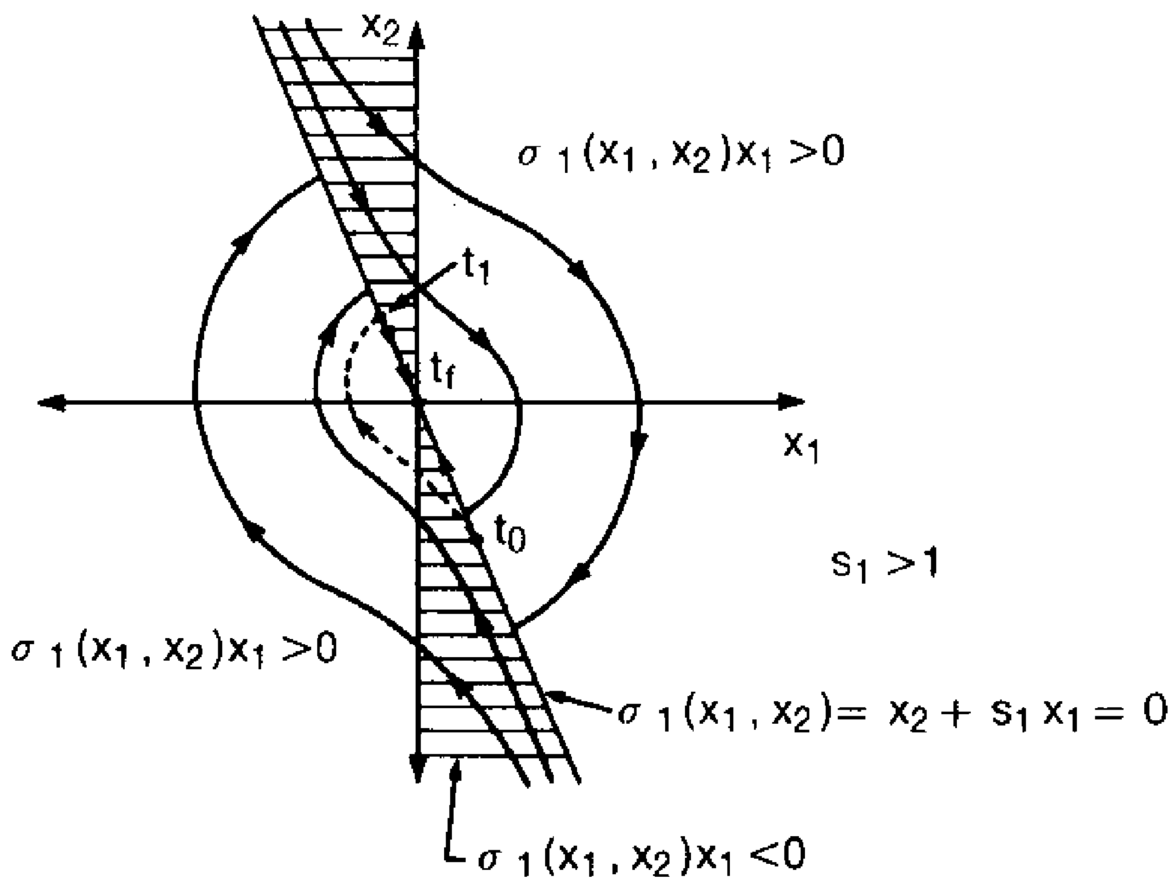



State trajectory of the controlled system when perturbed slightly below the asymptote

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*

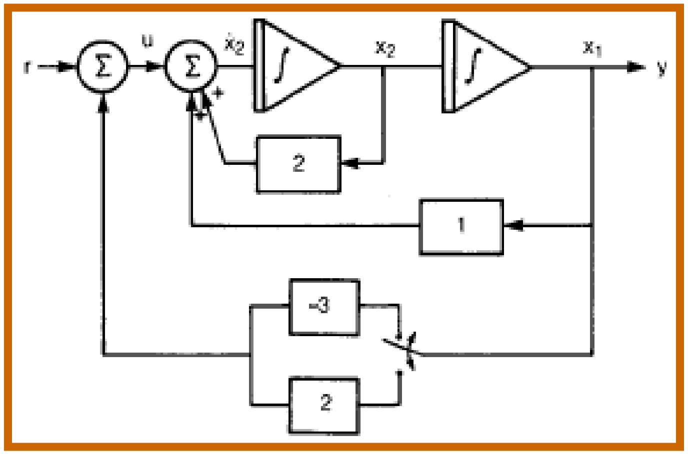
*Case I:* Switching Surface with $s_1 > 1$




State trajectory of the controlled system when perturbed slightly below the asymptote

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*



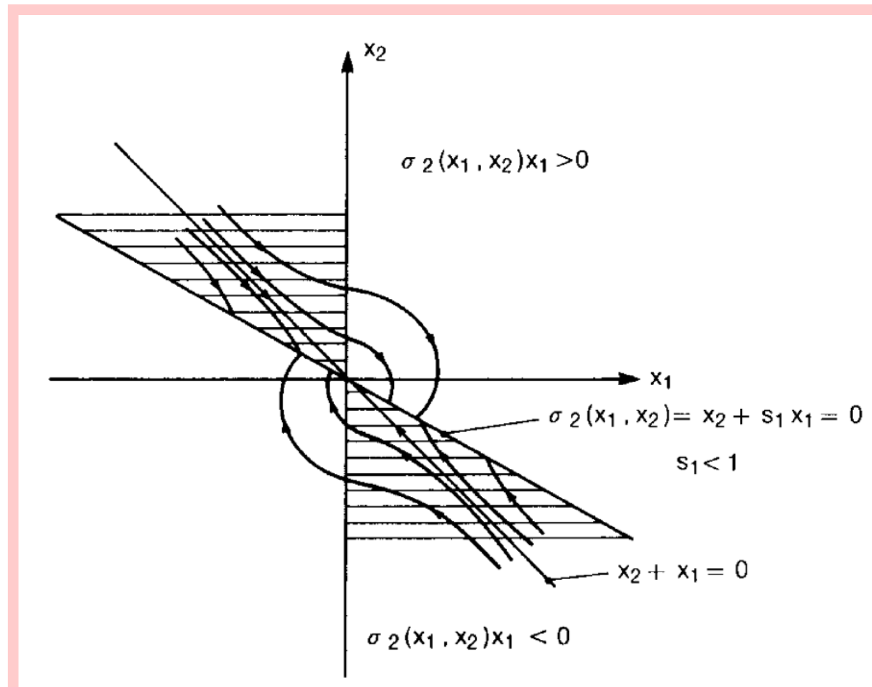

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The second order plant

$$k(x_1) = \begin{cases} -3, & \text{if } \sigma_2(x_1, x_2)x_1 > 0 \\ +2, & \text{if } \sigma_2(x_1, x_2)x_1 < 0 \end{cases}$$

*Case II:* Switching Surface with $s_1 < 1$

$$\sigma = \sigma_2(x_1, x_2) = s_1 x_1 + x_2 = 0$$

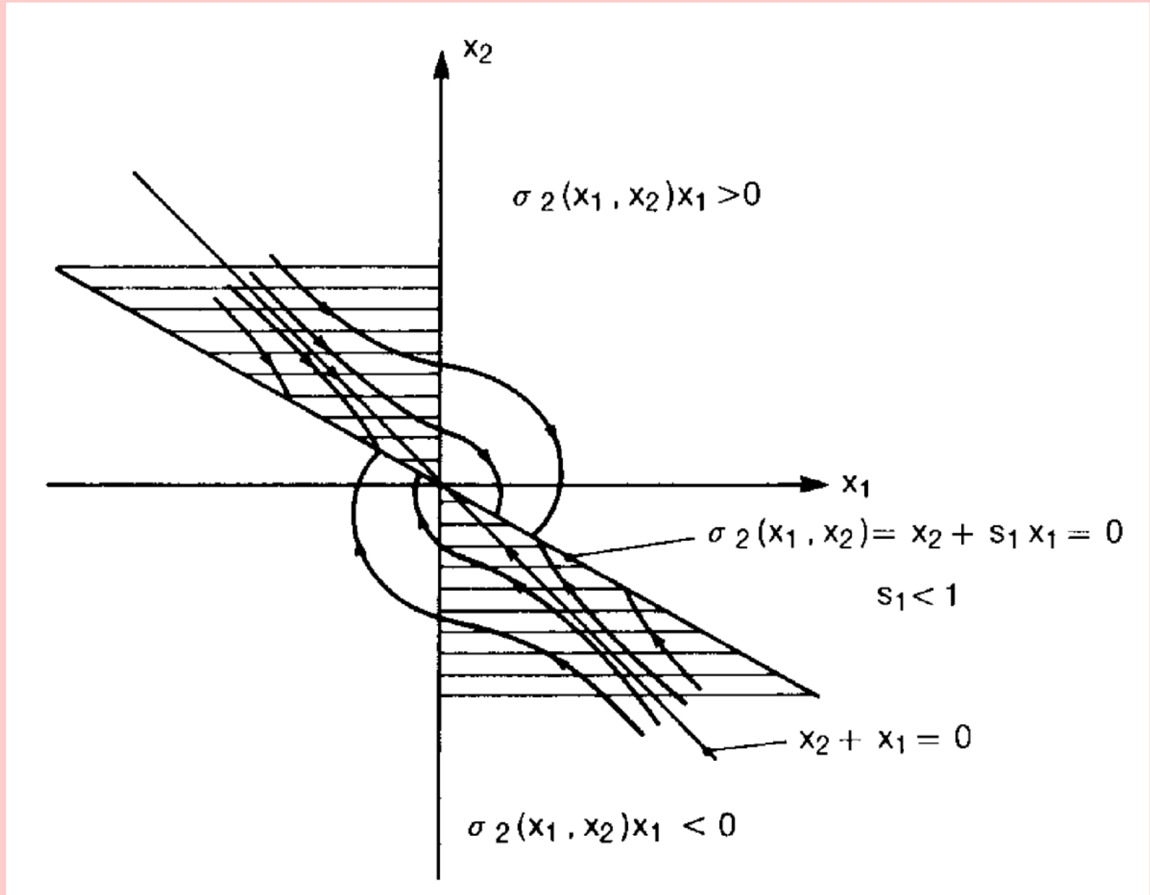



Phase plane plot of the controlled system when a sliding mode exists on the switching surface

# VSC – Example 2 (contd…)

*A plant with two accessible states and one control input, controlled by a partial state feedback controller:*

*Case II:* Switching Surface with $s_1 < 1$




Phase plane plot of the controlled system when a sliding mode exists on the switching surface

# Variable Structure Control

## *Conclusions from Example 1 and Example 2*

- ✓ **Different choices of switching surface produce radically different system responses.**

- ✓ **Case II of Example 2 illustrates an important notion that once a state trajectory intercepts the switching surface, it remains on the surface for all subsequent time.**

- ✓ **This property of remaining on the switching surface once intercepted is called a sliding mode.**

# Variable Structure Control

## *When will a sliding mode exist?*

- ✓ A sliding mode will exist for a system, **if, in the vicinity of the switching surface**, the **state velocity vector is directed towards the surface.**

## *Why was there lack of a sliding mode in case I of example 2?*

- ✓ This is because the system utilized a partial state feedback control law.

## *Any Solution ???*

- ✓ Use a **full-state feedback control law.** → $u(x) = k_1(x_1, x_2)x_1 + k_2(x_1, x_2)x_2$

- ✓ With **appropriate choice of gains**, the **original system can always be forced to have a sliding mode** on an arbitrary surface $\sigma = s_1 x_1 + x_2 = 0$.

# Key Aspects of VSC Design

*Phase 1 …*

- ✓ The **construction of a suitable switching surface,** so that the original plant restricted to the surface responds in a desired manner.

*Phase 2 …*

- ✓ The **development of a suitable switching control law** that satisfies a set of **sufficient conditions** for the existence and reachability of a sliding mode.

# The System Model

✓ For subsequent discussions, we consider a class of systems where the **state model is nonlinear in the state vector $x(\cdot)$ and linear in the control vector $u(\cdot)$.**

$$x(t) = \hat{f}(t,x,u) = f(t,x) + B(t,x)u(t)$$

… (1)

*where* …

state vector $x(t) \in R^n$,
control vector $u(t) \in R^m$,
$f(t,x) \in R^n$, and $B(t,x) \in R^{n\times m}$

✓ Each entry $u_i(t)$ of the switched control $u(t) \in R^m$ has the form:

$$u_i(t,x) = \begin{cases} u_i^+(t,x) & \text{with} \quad \sigma_i(x) > 0 \\ u_i^-(t,x) & \text{with} \quad \sigma_i(x) < 0 \end{cases} \qquad i = 1, \ldots, m$$

✓ $\sigma_i(x) = 0$ is the $i$th switching surface associated with the switching surface:

$$\sigma(x) = [\sigma_1(x), \ldots, \sigma_m(x)]^T = 0$$

# Design of Sliding Surface

## The Method of Equivalent Control

### *Initial Assumption*

- ✓ Let us assume that at $t_0$, the state trajectory of the plant intercepts the switching surface and a sliding mode exists for $t \geq t_0$.

### *The Implication*

- ✓ The existence of a sliding mode implies *i*) $\sigma(x(t)) = 0$ and *ii*) $\sigma(x(t)) = 0$, for all $t \geq t_0$.

*Chain Rule …*

$$\left[\frac{\partial \sigma}{\partial x}\right] x = 0 \quad \Rightarrow \quad \left[\frac{\partial \sigma}{\partial x}\right] x = \left[\frac{\partial \sigma}{\partial x}\right] \left[ f(t,x) + B(t,x) u_{eq} \right] = 0$$

$u_{eq}$: **the equivalent control**

- ✓ Substitution of this $u_{eq}$ into **eq. (1)** describes the behavior of the system restricted to the switching surface (**provided the initial condition $x(t_0)$ satisfies $\sigma(x(t_0)) = 0$**).

# Design of Sliding Surface (contd…)

## The Method of Equivalent Control (contd…)

### *How to compute $u_{eq}$??*

$$u_{eq} = -\left[\left[\frac{\partial\sigma}{\partial x}\right]B(t,x)\right]^{-1}\frac{\partial\sigma}{\partial x}f(t,x)$$

***Assumption:*** $\left[\frac{\partial\sigma}{\partial x}\right]B(t,x)$ **is nonsingular for all $t$ and $x$.**

### *The Implication*

- ✓ **Given $\sigma(x(t_0)) = 0$, the dynamics of the system on the switching surface for all $t \geq t_0$ :**

$$x = \left[I - B(t,x)\left[\frac{\partial\sigma}{\partial x}B(t,x)\right]^{-1}\frac{\partial\sigma}{\partial x}\right]f(t,x) \qquad \ldots (2)$$

### *Special Case: Linear Switching Surface*

$$\sigma(x) = Sx = 0 \quad \Rightarrow \quad \frac{\partial\sigma}{\partial x} = S \quad \Rightarrow \quad x = \left[I - B(t,x)[SB(t,x)]^{-1}S\right]f(t,x)$$

- ✓ ***Note:*** **Eq. (2) in conjunction with the constraint $\sigma(x) = 0$ determines the system motion on the switching surface.**

# The Method of Equivalent Control (contd…)

## *An Example…*

**Let us consider a system:** $x(t) = A(t,x)x(t) + Bu(t)$

$$A(t,x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_{11}(t,x) & a_{12}(t,x) & a_{13}(t,x) & a_{14}(t,x) & a_{15}(t,x) \\ 0 & 0 & 0 & 0 & 1 \\ a_{21}(t,x) & a_{22}(t,x) & a_{23}(t,x) & a_{24}(t,x) & a_{25}(t,x) \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a_{ij}^{\min} \le a_{ij}(t,x) \le a_{ij}^{\max} \text{ for all } x \in R^n \text{ and } t \in [t_0, \infty)$$

**The method of equivalent control produces the following equivalent system:**

$$x(t) = \left[I - B[SB]^{-1}S\right]A(t,x)x(t)$$

**provided $\sigma(x(t_0)) = 0$ for some $t_0$**

# The Method of Equivalent Control (contd…)

## *Example contd…*

**The linear switching surface parameters are chosen as:**

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \end{bmatrix}$$

$$SB = \begin{bmatrix} s_{13} & s_{15} \\ s_{23} & s_{25} \end{bmatrix}$$

## *How to Simplify the Example ??*

**Let us choose:** $s_{13}s_{25} - s_{15}s_{23} = 1$

**Specifically:** $s_{13} = 2, s_{15} = s_{23} = s_{25} = 1.$

$$(SB)^{-1} = \frac{\begin{bmatrix} s_{25} & -s_{15} \\ -s_{23} & s_{13} \end{bmatrix}}{s_{13}s_{25} - s_{15}s_{23}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

# The Method of Equivalent Control (contd…)

*Example contd…*

**Hence,**

$$\dot{x}(t) = \left[I - B(SB)^{-1}S\right]A(t,x)x(t)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & s_{21}-s_{11} & s_{22}-s_{12} & 0 & s_{24}-s_{14} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & s_{11}-2s_{21} & s_{12}-2s_{22} & 0 & s_{14}-2s_{24} \end{bmatrix} x(t)$$

**subject to $\sigma(x) = 0$**

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = -\begin{bmatrix} s_{11} & s_{12} & s_{14} \\ s_{21} & s_{22} & s_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = -\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} s_{11} & s_{12} & s_{14} \\ s_{21} & s_{22} & s_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$$

# The Method of Equivalent Control (contd…)

## *Example contd…*

**The reduced order equivalent LTI system**

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ s_{21} - s_{11} & s_{22} - s_{12} & s_{24} - s_{14} \\ s_{21} - 2s_{11} & s_{12} - 2s_{22} & s_{14} - 2s_{24} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \quad \ldots (3)$$

*where* $\tilde{x}_1 = x_1, \tilde{x}_2 = x_2, \tilde{x}_3 = x_4$

## *How to Accomplish the Control Design ??*

**Let us assume that a design constraint requires the spectrum of the equivalent system be {1, 2, 3}.**

**The Desired Characteristic Polynomial**

$$\pi_A(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

**The Characteristic Polynomial of the equivalent system in Eq. (3)**

$$\pi_A(\lambda) = \lambda^3 + (s_{12} - s_{22} + 2s_{24} - s_{14})\lambda^2 + (s_{12}s_{24} - s_{14}s_{22} + s_{11} - s_{21})\lambda + (s_{11}s_{24} - s_{14}s_{21})$$

# The Method of Equivalent Control (contd…)

*Example contd…*

**Equating co-efficients of like powers of λ**

$$\begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 2 \\ 1 & s_{24} & -s_{22} & -1 & 0 & 0 \\ s_{24} & 0 & 0 & -s_{14} & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{12} \\ s_{14} \\ s_{21} \\ s_{22} \\ s_{24} \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix}$$

**A solution achieving the control design objective is:**

$$S = \begin{bmatrix} 1 & 1.8333 & 2 & -6 & 1 \\ 1 & 1.8333 & 1 & 0 & 1 \end{bmatrix}$$

**The Reduced order equivalent system with the desired Eigenvalues**

$$\dot{\tilde{x}} = A\tilde{x}$$ *where* $$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 6 \\ -1 & -1.83333 & -6 \end{bmatrix}$$

# Design of the Controller

## What is our Goal ??

- ✓ To determine **switched feedback gains** which will drive the plant state trajectory to the switching surface and maintain a sliding mode condition.

## Any Presumption ??

- ✓ *Yes*, the presumption is that **the sliding surface has already been designed**.

## What will be the control structure ??

**The control is an *m*-vector *u*(*t*):**

$$u_i = \begin{cases} u_i^+(t,x) & \text{for} \quad \sigma_i(x) > 0 \\ u_i^-(t,x) & \text{for} \quad \sigma_i(x) < 0 \end{cases} \qquad i = 1, \quad , m$$

*where*

$$\sigma(x) = [\sigma_1(x), \quad , \sigma_m(x)]^T = 0$$

# Design of the Controller (contd…)

## Diagonalization Method

*The first Step …*

- ✓ We construct a new control vector $u^*$, from the original control $u$, employing a nonsingular transformation:

$$Q^{-1}(t,x)\left[\frac{\partial\sigma}{\partial x}\right]B(t,x)$$

*result of the transformation*

$$u^*(t) = Q^{-1}(t,x)\left[\frac{\partial\sigma}{\partial x}(x)\right]B(t,x)u(t)$$

*where* $Q(t, x)$ is an arbitrary $m \times m$ diagonal matrix with elements $q_i(t, x)$ $(i = 1, \ldots, m)$ such that $\inf|q_i(t, x)| > 0$ for all $t \geq 0$ and all $x$.

*What is the practical implication of this transformation ??*

- ✓ The diagonal entries of $Q^{-1}(t, x)$ allow flexibility in the design. This makes it possible to weight various channels of $u^*$.

# Design of the Controller (contd…)

## Diagonalization Method (contd…)

*The New State Dynamics in terms of u* …*

$$x(t) = f(t,x) + B(t,x)\left[\frac{\partial\sigma}{\partial x}(x)B(t,x)\right]^{-1} Q(t,x)u^*(t)$$

*The New Control Structure looks more complicated …*

- ✓ ***Agreed,*** **but the structure of** $\sigma(x) = 0$ **permits us to independently choose the** ***m*****-entries of** ***u**** **to satisfy the sufficient conditions for the existence and the reachability of a sliding mode.**

- ✓ ***Note:*** **Once** ***u**** **is known, we can obtain the desired** ***u*** **by inverting the transformation.**

# Design of the Controller (contd…)

## Diagonalization Method (contd…)

*What is the Condition to be Satisfied for Existence and Reachability of a Sliding Mode ??*

$$\sigma^T(x)\sigma(x) < 0$$

*How will this Relation Translate in terms of u* ??*

$$\sigma(x) = \frac{\partial \sigma}{\partial x}(x)f(t,x) + Q(t,x)u^*(t)$$

# Design of the Controller (contd…)

## Diagonalization Method (contd…)

*How can the Sufficient Conditions for Existence and Reachability be Satisfied ??*

- ✓ **By suitable choices of the entries $u_i^{*+}$ and $u_i^{*-}$ satisfying the relations:**

$$
\begin{cases}
q_i(t,x)u_i^{*+} < -\nabla\sigma_i(x)f(t,x) \\
\qquad = -\sum_{j=1}^{n} s_{ij} f_j(t,x) \quad \text{when} \quad \sigma_i(x) > 0 \\
q_i(t,x)u_i^{*-} > -\nabla\sigma_i(x)f(t,x) \\
\qquad = -\sum_{j=1}^{n} s_{ij} f_j(t,x) \quad \text{when} \quad \sigma_i(x) < 0
\end{cases}
\quad \ldots (4)
$$

*where* **$s_{ij}$: the $j$th entry of $\nabla\sigma_i(x)$ i.e. the $j$th row of $(\partial\sigma/\partial x)$**

- ✓ ***Note:*** **The above conditions force each term in the summation of $\sigma^T\sigma$ to be negative definite.**

# Design of the Controller (contd…)

## Diagonalization Method (contd…)

*What Next ??*

- ✓ **The last step will be to calculate the actual control signal *u*.**

$$u(t) = \left[\frac{\partial \sigma}{\partial x} B(t, x)\right]^{-1} Q(t, x) u^*(t)$$

# Diagonalization Method (contd…)

## *An Example…*

**Let us consider the system, utilized in the previous example:**

$$x(t) = A(t,x)x(t) + Bu(t)$$

$$A(t,x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_{11}(t,x) & a_{12}(t,x) & a_{13}(t,x) & a_{14}(t,x) & a_{15}(t,x) \\ 0 & 0 & 0 & 0 & 1 \\ a_{21}(t,x) & a_{22}(t,x) & a_{23}(t,x) & a_{24}(t,x) & a_{25}(t,x) \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

**The switching surface $\sigma(x) = Sx = 0$ was previously designed as:**

$$S = \begin{bmatrix} 1 & 1.8333 & 2 & -6 & 1 \\ 1 & 1.8333 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

# Diagonalization Method (contd…)

*Example contd…*

## *What is the Objective ??*

- ✓ To demonstrate **Phase 2** of the VSC controller design utilizing **Diagonalization Method**.

## *How to Proceed ??*

- ✓ The control $u$ is transformed according to the relation:

$$u^*(t) = Q^{-1}(t,x)\left[\frac{\partial \sigma}{\partial x}(x)\right]B(t,x)u(t)$$

*where* $Q(t,x)$ is a nonsingular diagonal matrix such that $\inf|q_i(t,x)| > 0$.

## *How to Simplify the Design ??*

**Let us choose:** $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ → $Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$

# Diagonalization Method (contd…)

*Example contd…*

*The State Dynamics driven by u* …*

$$x(t) = A(t,x)x(t) + B(t,x)[SB(t,x)]^{-1} Q(t,x)u^{*}(t)$$

*Next, Compute the Feedback Gains, meeting the existence conditions …*

$$\sigma(t) = Sx(t) = SA(t,x)x(t)Q(t,x)u^{*}(t)$$

**Since $Q(t,x)$ is diagonal, from Eq. (4), sufficient conditions for sliding mode are:**

$$\begin{cases} q_i(t,x)u_i^{*+} < -[s_{i1}, \quad , s_{i5}]A(t,x)x(t) \\ \qquad \text{if} \quad \sigma_i(x) > 0, \ i = 1,2 \\ q_i(t,x)u_i^{*-} > -[s_{i1}, \quad , s_{i5}]A(t,x)x(t) \\ \qquad \text{if} \quad \sigma_i(x) < 0, \ i = 1,2 \end{cases}$$

… (5)

# Diagonalization Method (contd…)

*Example contd…*

*For the First Switching Surface …*

$$\sigma_1(x) = [s_{11}, \quad , s_{15}]x(t) = S_1 x(t)$$

$$S_1 A(t,x)x(t) = (2a_{11} + a_{21})x_1 + (1 + 2a_{12} + a_{22}x_2)x_2 + (1.8333 + 2a_{13} + a_{23})x_3 + (2a_{14} + a_{24})x_4 + (2a_{15} + a_{25} - 6)x_5$$

**Recall the assumption:** $a_{ij}^{\mathbf{min}} \le a_{ij}(t,x) \le a_{ij}^{\mathbf{max}}, \ i = 1,2, \ j = 1, \quad ,5$

**and the control law:** $u^* = Kx, \quad \text{where} \quad K = \lfloor k_{ij}^{\pm} \rfloor$

# Diagonalization Method (contd…)

*Example contd…*

**In accordance with Eq. (5), $k_{1j}$ must satisfy:**

$$k_{11} = \begin{cases} < -\left(2a_{11}^{\max} + a_{21}^{\max}\right), & \text{if } \sigma_1 x_1 > 0 \\ > -\left(2a_{11}^{\min} + a_{21}^{\min}\right), & \text{if } \sigma_1 x_1 < 0 \end{cases}$$

$$k_{12} = \begin{cases} < -\left(1 + 2a_{12}^{\max} + a_{22}^{\max}\right), & \text{if } \sigma_1 x_2 > 0 \\ > -\left(1 + 2a_{12}^{\min} + a_{22}^{\min}\right), & \text{if } \sigma_1 x_2 < 0 \end{cases}$$

$$k_{13} = \begin{cases} < -\left(1.8333 + 2a_{13}^{\max} + a_{23}^{\max}\right), & \text{if } \sigma_1 x_3 > 0 \\ > -\left(1.8333 + 2a_{13}^{\min} + a_{23}^{\min}\right), & \text{if } \sigma_1 x_3 < 0 \end{cases}$$

$$k_{14} = \begin{cases} < -\left(2a_{14}^{\max} + a_{24}^{\max}\right), & \text{if } \sigma_1 x_4 > 0 \\ > -\left(2a_{14}^{\min} + a_{24}^{\min}\right), & \text{if } \sigma_1 x_4 < 0 \end{cases}$$

$$k_{15} = \begin{cases} < -\left(2a_{15}^{\max} + a_{25}^{\max} - 6\right), & \text{if } \sigma_1 x_5 > 0 \\ > -\left(2a_{15}^{\min} + a_{25}^{\min} - 6\right), & \text{if } \sigma_1 x_5 < 0 \end{cases}$$

… (6)

# Diagonalization Method (contd…)

*Example contd…*

*For the Second Switching Surface …*

$$\sigma_2(x) = [s_{21}, \quad , s_{25}]x(t) = S_2 x(t)$$

$$S_2 A(t, x)x(t) = (a_{11} + a_{21})x_1 + (1 + a_{12} + a_{22}x_2)x_2 + (1.8333 + a_{13} + a_{23})x_3 + (a_{14} + a_{24})x_4 + (a_{15} + a_{25})x_5$$

**Let us determine the control law:** $u_2^* = K_2 x$

# Diagonalization Method (contd…)

*Example contd…*

**In accordance with Eq. (5), $k_{2j}$ must satisfy:**

$$k_{21} = \begin{cases} < -\frac{1}{2}\left(a_{11}^{\max} + a_{21}^{\max}\right), & \text{if } \sigma_2 x_1 > 0 \\ > -\frac{1}{2}\left(a_{11}^{\min} + a_{21}^{\min}\right), & \text{if } \sigma_2 x_1 < 0 \end{cases}$$

$$k_{22} = \begin{cases} < -\frac{1}{2}\left(1 + a_{12}^{\max} + a_{22}^{\max}\right), & \text{if } \sigma_2 x_2 > 0 \\ > -\frac{1}{2}\left(1 + a_{12}^{\min} + a_{22}^{\min}\right), & \text{if } \sigma_2 x_2 < 0 \end{cases}$$

$$k_{23} = \begin{cases} < -\frac{1}{2}\left(1.8333 + a_{13}^{\max} + a_{23}^{\max}\right), & \text{if } \sigma_2 x_3 > 0 \\ > -\frac{1}{2}\left(1.8333 + a_{13}^{\min} + a_{23}^{\min}\right), & \text{if } \sigma_2 x_3 < 0 \end{cases}$$

$$k_{24} = \begin{cases} < -\frac{1}{2}\left(a_{14}^{\max} + a_{24}^{\max}\right), & \text{if } \sigma_2 x_4 > 0 \\ > -\frac{1}{2}\left(a_{14}^{\min} + a_{24}^{\min}\right), & \text{if } \sigma_2 x_4 < 0 \end{cases}$$

$$k_{25} = \begin{cases} < -\frac{1}{2}\left(a_{15}^{\max} + a_{25}^{\max}\right), & \text{if } \sigma_2 x_5 > 0 \\ > -\frac{1}{2}\left(a_{15}^{\min} + a_{25}^{\min}\right), & \text{if } \sigma_2 x_5 < 0 \end{cases}$$

… (7)

# Diagonalization Method (contd…)

*Example contd…*

*Achievement, so far …*

✓ We obtained the control law, $u^*(t) = Kx(t)$ where the co-efficients of the $K$ matrix i.e. $k_{1j}$s and $k_{2j}$s are obtained from Eq. (6) and Eq. (7).

*Finally we obtain the Original Control u from u*,*

*Inverting the Transformation …*

**We know:** $u(t) = [SB]^{-1} Q u^*(t, x)$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} \Rightarrow = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} k_{11}^{\pm} & k_{12}^{\pm} & k_{13}^{\pm} & k_{14}^{\pm} & k_{15}^{\pm} \\ k_{21}^{\pm} & k_{22}^{\pm} & k_{23}^{\pm} & k_{24}^{\pm} & k_{25}^{\pm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

# The Phenomenon of Chattering

## *What is Chattering??*

- ✓ The design of VSC control systems is based on the assumption that the control can be switched from one value to another at will, infinitely fast, that will ensure an ideal sliding mode i.e. the state trajectory $x(t)$ of the controlled plant satisfies $\sigma(x(t)) = 0$, at every $t \geq t_0$ for some $t_0$.

## *In Practice, there is a Problem …*

- ✓ In reality, it is impossible to achieve this very high frequency switching control.

## *Possible Reasons …*

- ➢ The presence of finite time delays for control computation.
- ➢ The limitation of the physical actuators.

## *Result ??*

- ✓ Hence, in the sliding and steady state-modes, the representative point oscillates within the neighborhood of the switching surface. This oscillation is called **chattering**.

# The Phenomenon of Chattering

## *How to Overcome Chattering??*

- **One popular mechanism employs *The Continuation Approach*.**

## *What is its Essence ??*

- ✓ **Here a *Boundary Layer* around the switching surface is introduced.**

## *What is its Effect ??*

- ✓ **Within the boundary layer, the control is chosen to be a continuous approximation of the switching function. The robustness of the system becomes a function of the width of the boundary layer.**

# The Phenomenon of Chattering

## *Demonstration of the effect of Continuation Approach …*

- ✓ **We demonstrate utilizing four common types of switching characteristics.**

## *The System and the Switching Surface considered …*

**We consider a simple second order system:**

$$x+\alpha x+\beta x=-ku(s)$$

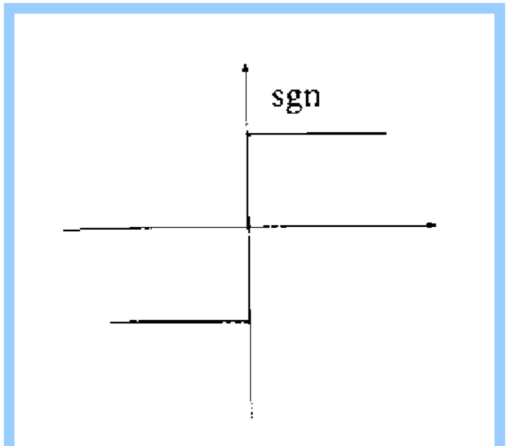
**and an associated switching surface:**

$$s=x+cx$$

# The Phenomenon of Chattering

## *Switching Characteristic – Type I:*

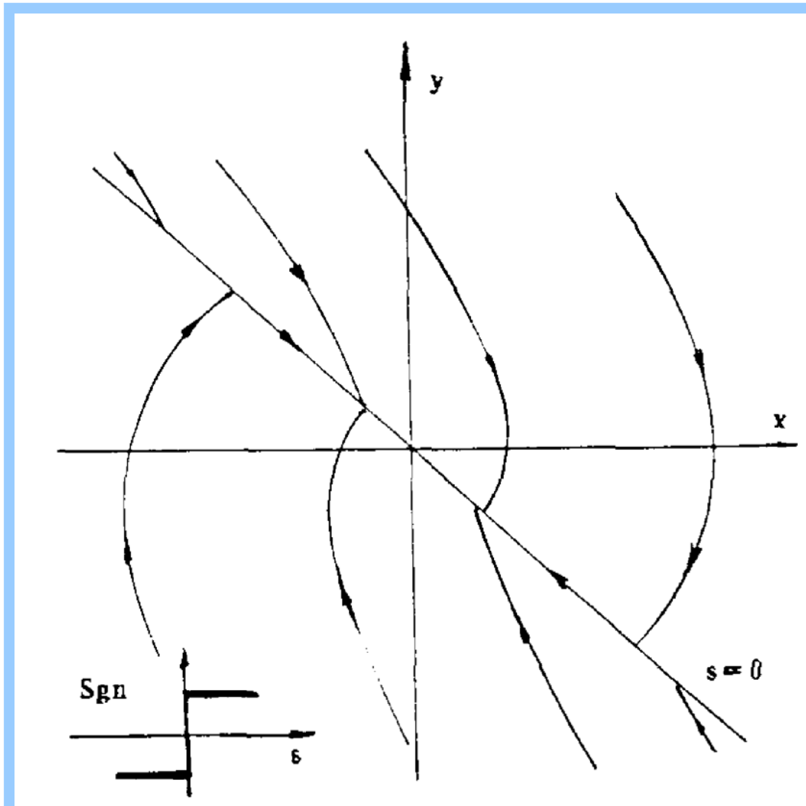
### The Ideal Relay Control




Switching Function

The Control Law:

$$u(s) = \mathbf{sgn}(s) = +1 \quad \text{when} \quad s > 0$$
$$= -1 \quad \text{when} \quad s < 0$$

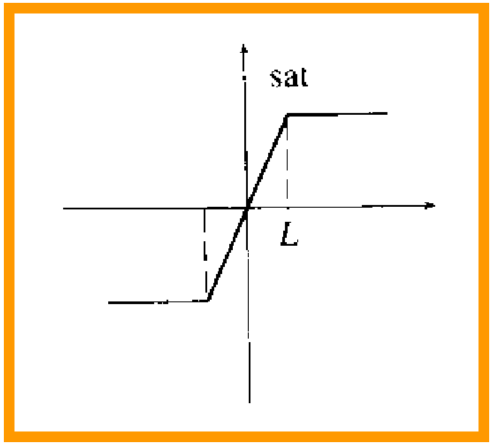



Phase Portrait in Sliding and Quasi-Sliding Modes

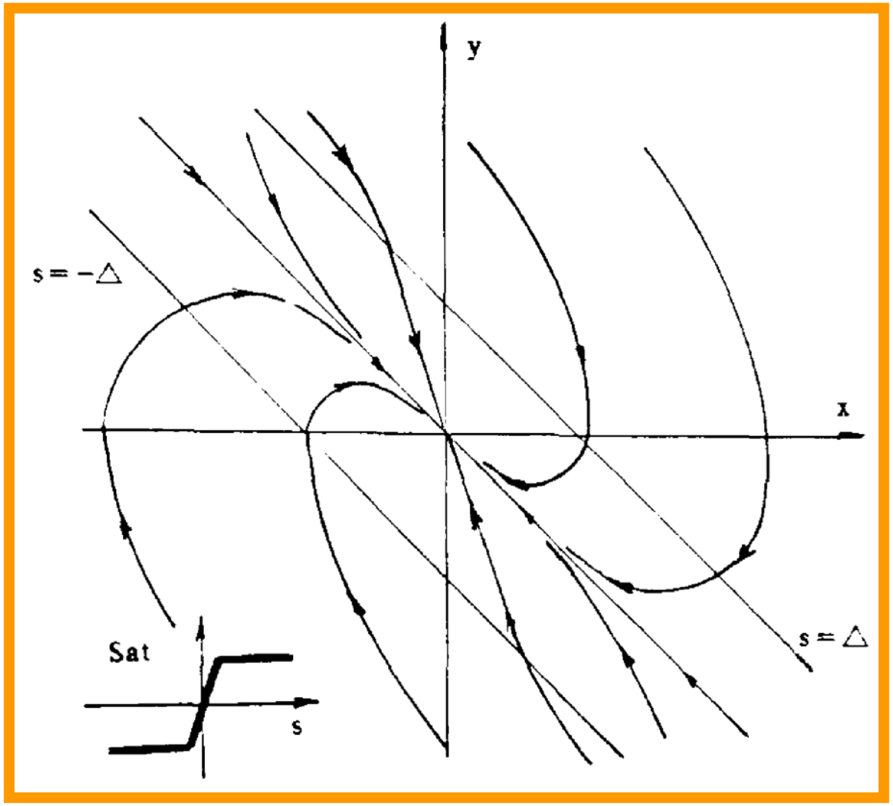
# The Phenomenon of Chattering

## *Switching Characteristic – Type II:*

### The Ideal Saturation Control




Switching Function




Phase Portrait in Sliding and Quasi-Sliding Modes

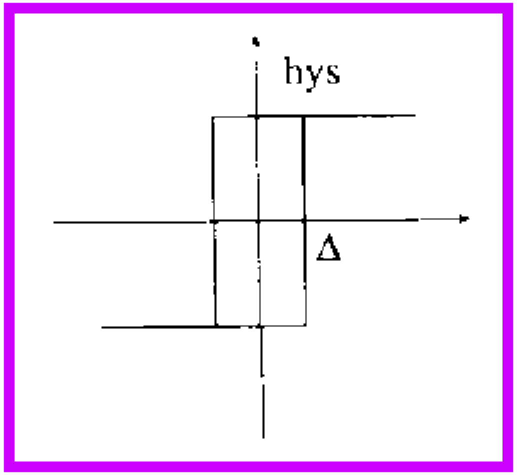
The Control Law:

$$
\begin{aligned}
u(s) = sat(s) &= +1 \quad \text{when} \quad s > L \\
&= \frac{s}{L} \quad \text{when} \quad |s| \le L \\
&= -1 \quad \text{when} \quad s < -L
\end{aligned}
$$

# The Phenomenon of Chattering

## *Switching Characteristic – Type III:*

### The Practical Relay Control




Switching Function

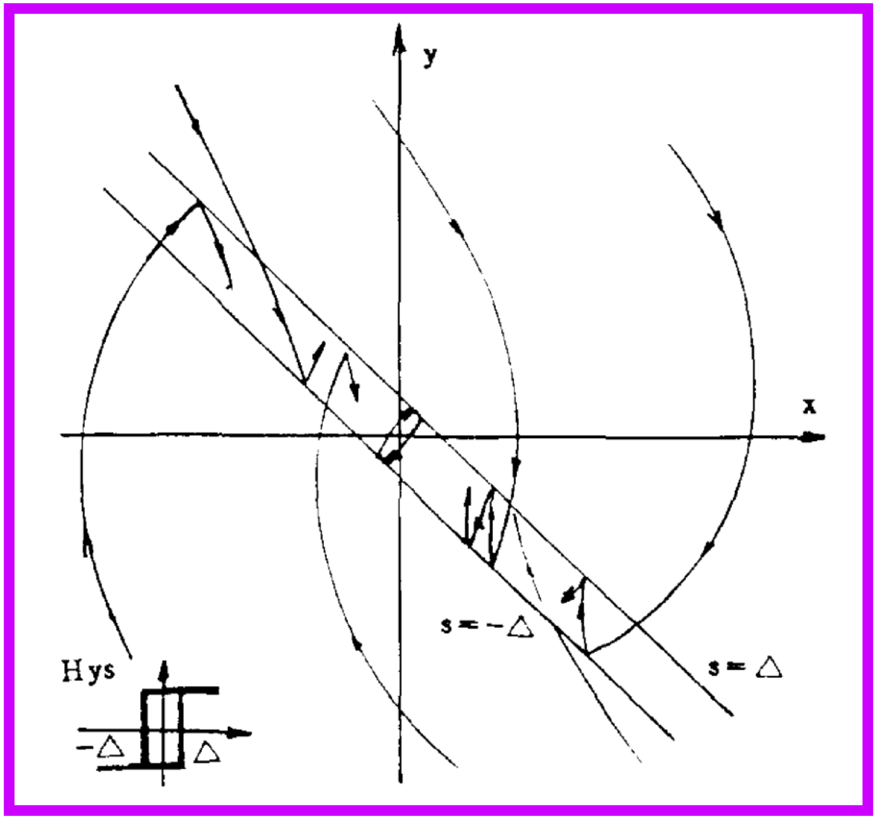
The Control Law:

$$u(s) = hys(s) = +1 \quad \text{when} \quad s > \Delta, \text{ or}$$

$$\text{when} \quad s < 0 \quad \text{and} \quad |s| < \Delta$$

$$= -1 \quad \text{when} \quad s < \Delta, \text{ or}$$

$$\text{when} \quad s > 0 \quad \text{and} \quad |s| < \Delta$$

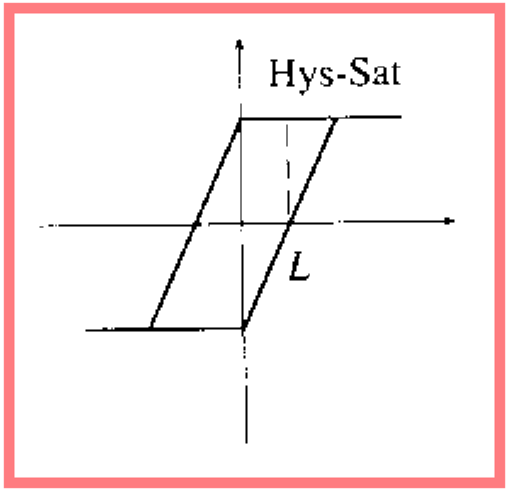



Phase Portrait in Sliding and Quasi-Sliding Modes

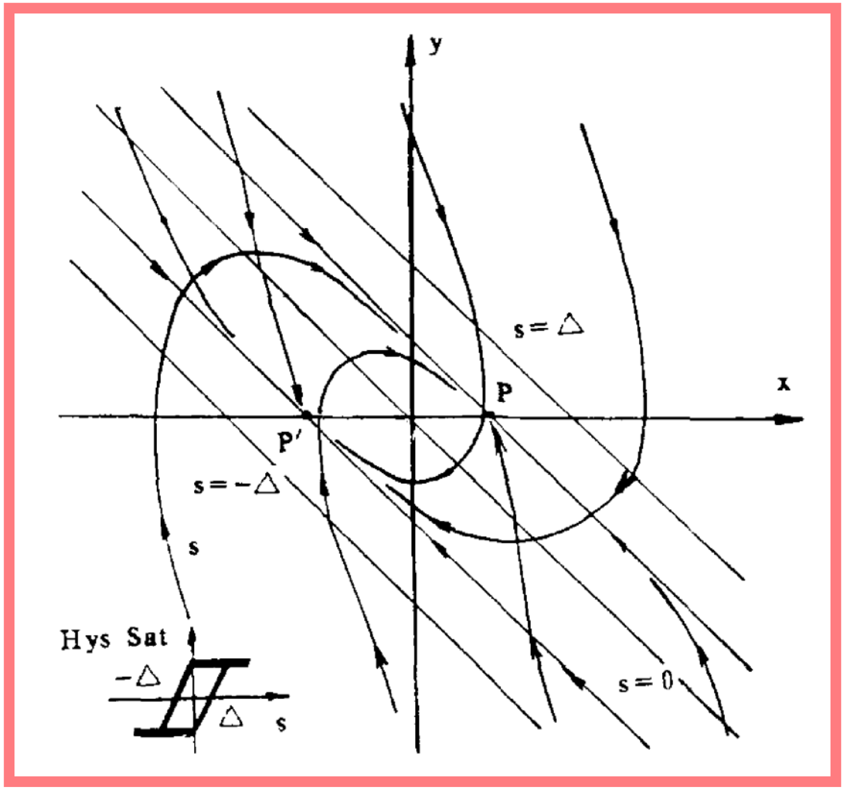
# The Phenomenon of Chattering

## *Switching Characteristic – Type IV:*

### The Practical Saturation Control




Switching Function




Phase Portrait in Sliding and Quasi-Sliding Modes

# Variable Structure Control

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# Thank You…