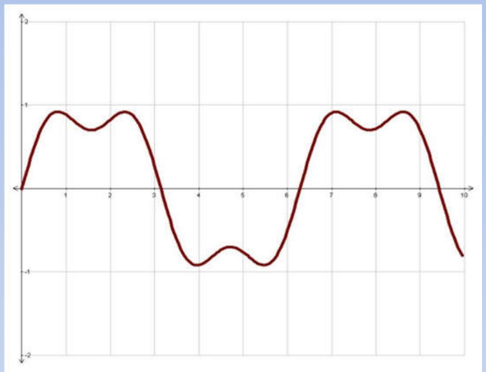
FIR Digital Filters

Prof. Anjan Rakshit and Prof. Amitava Chatterjee
Electrical Measurement and Instrumentation Laboratory,
Electrical Engineering Department,
Jadavpur University, Kolkata, India.

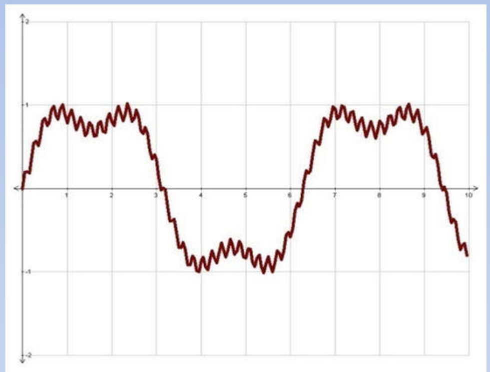
Distortion-less Transmission of Signal through a Filter

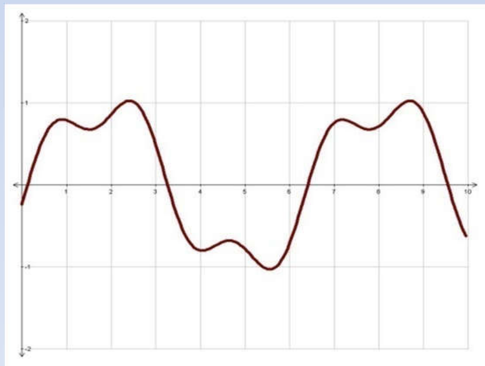
In distortion-less transmission, the inter-harmonic phase relations must remain same before and after transmission.



Periodic signal

Interference

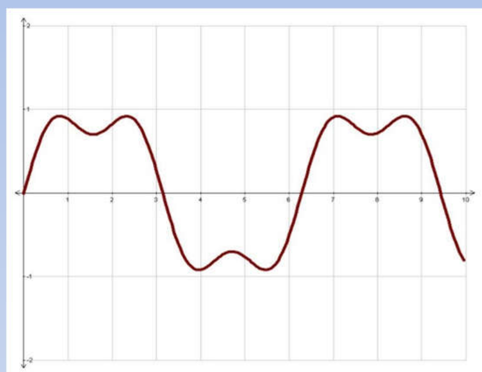




Conventional low-pass filter output

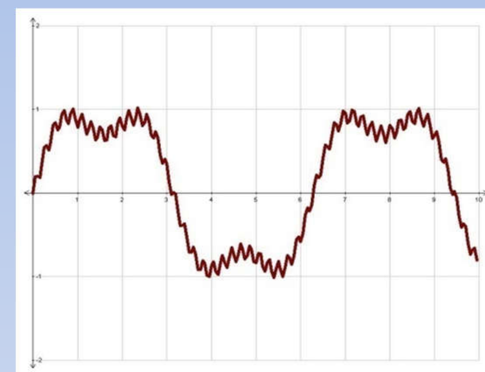
Distortion-less Transmission of Signal through a Filter

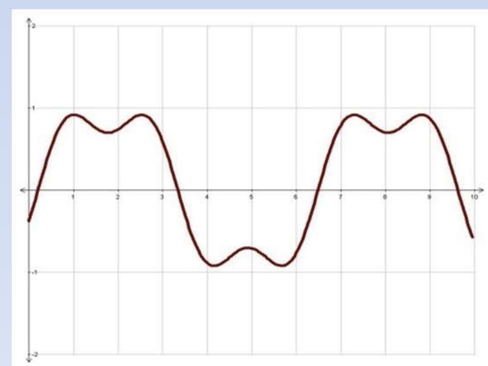
In distortion-less transmission, the inter-harmonic phase relations must remain same before and after transmission.



Periodic signal

Interference





Distortion-less low-pass filter output

Distortion-less Transmission of Signal through a Filter

For distortion-less transmission of signal through a filter within its pass-band region let the magnitude of the steady state gain of the filter (assuming it be an ideal low-pass) be:

$$|H(\omega)| = K, \text{ for } 0 \le \omega \le \omega_c \quad \text{....(1)}$$

and the output of the filter be:

$$y(t) = Kx(t-t_d) \quad \text{....(2)}$$

where $x(t)$ is the input signal, band limited to ω_c, and t_d is a constant delay.

Taking Fourier transform of relation (2),

$$Y(\omega) = KX(\omega)exp(-j\omega t_d) \quad \text{....(3)}$$

using delay property of Fourier transform.

Distortion-less Transmission of Signal through a Filter

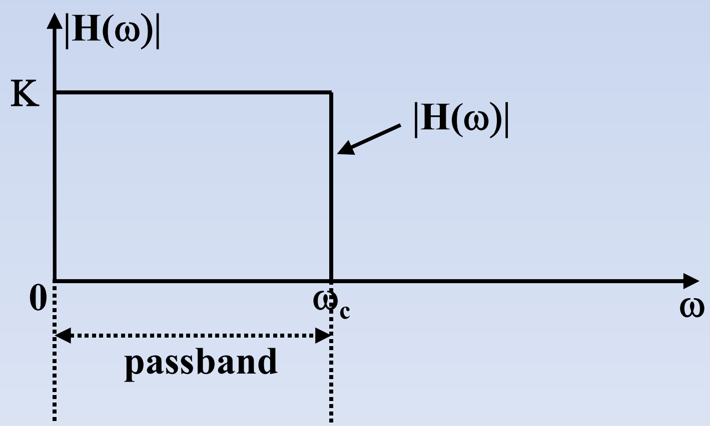
From relation (3), the steady-state transfer function is:

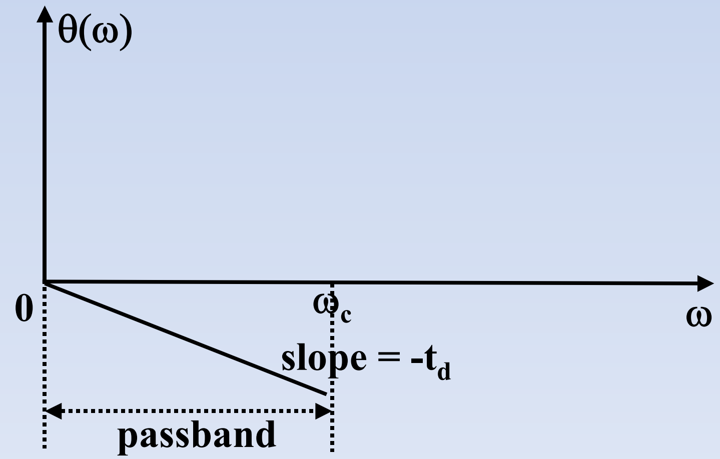
$$H(\omega) = Y(\omega) / X(\omega) = K\exp(-j\omega t_d) \quad \text{....(4)}$$

Thus, $|H(\omega)| = K$ ....(5)

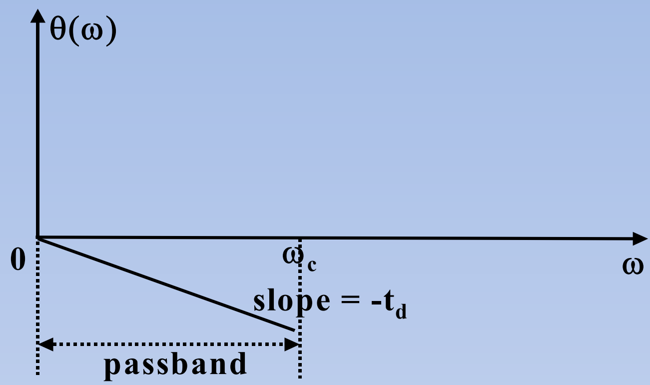
and $\angle H(\omega) = -\omega t_d = \theta(\omega)$, *say*

Then, $\theta(\omega) = -\omega t_d$ ....(6)





Distortion-less Transmission of Signal through a Filter

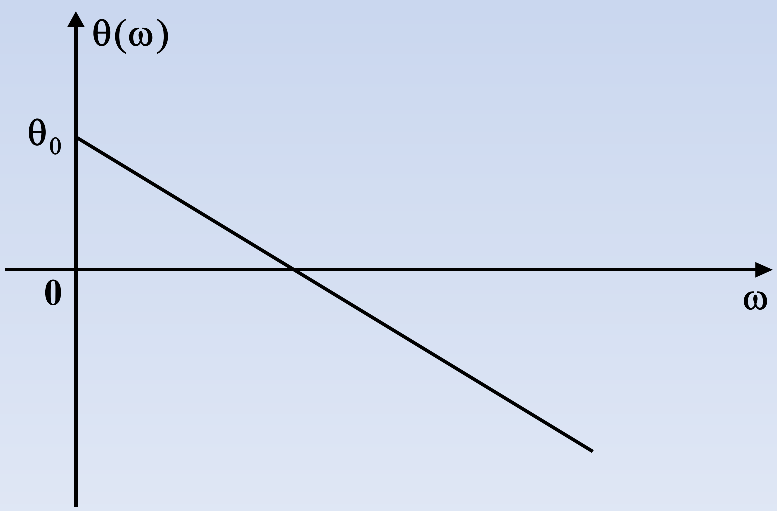


Thus, for distortion-less transmission, phase shift $\theta(\omega)$ is **lagging and proportional to frequency** within the pass band region.

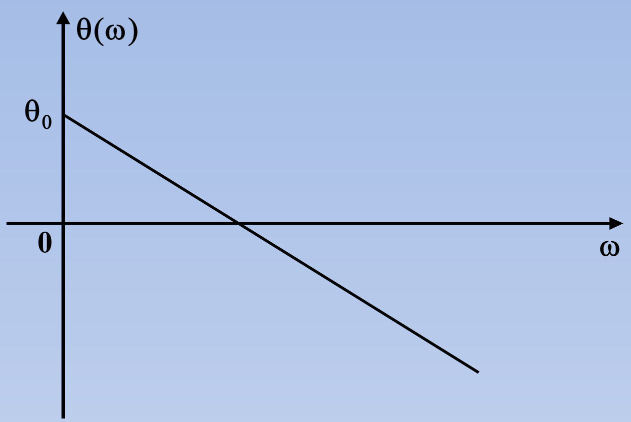
This characteristic is called **linear phase** characteristic.

If the phase shift $\theta(\omega)$ includes any constant offset, say θ_0, then let

$$\theta(\omega) = \theta_0 - \omega t_d \quad \text{....(7)}$$



Distortion-less Transmission of Signal through a Filter



Thus the steady state filter gain may be expressed as,

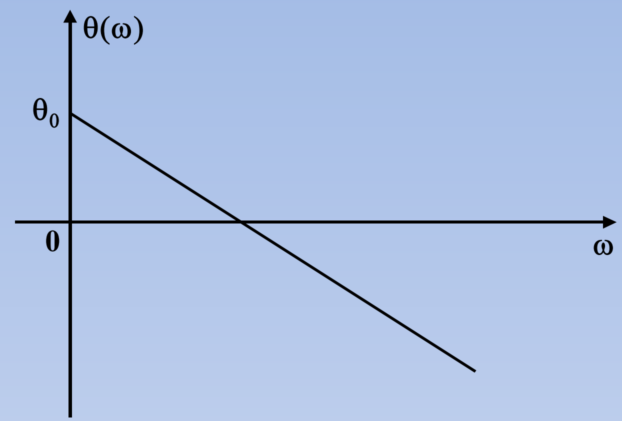
$$H(\omega) = K\exp(j(\theta_0 - \omega t_d))$$

$$= K\exp(j\theta_0).\exp(-j\omega t_d)$$

or $$H(\omega) = K'.\exp(-j\omega t_d) \qquad \text{....(8)}$$

where K' $(= K\exp(j\theta_0))$ is the complex gain of the filter.

Distortion-less Transmission of Signal through a Filter



Now, $H(\omega) = K'.exp(-j\omega t_d)$

Therefore,

$$Y(\omega) / X(\omega) = K'.exp(-j\omega t_d)$$

or $Y(\omega) = K' X(\omega).exp(-j\omega t_d)$

Taking inverse Fourier transform,

$$y(t) = K' x(t - t_d) \quad \text{....(9)}$$

Thus **distortion-less transmission** is possible with a **linear phase characteristic with offset**.

Phase Delay of a distortion-less filter

The phase delay of a filter is defined as:

$$\tau_p(\omega) = -\theta(\omega) / \omega \qquad \text{....(10)}$$

Now, for a linear phase filter,

$$\theta(\omega) = -\omega t_d$$

Thus, $\tau_p(\omega) = -\omega t_d /\omega = t_d$, a constant.

Now, for a linear phase filter with offset,

$$\theta(\omega) = \theta_0 - \omega t_d$$

Thus, $\tau_p(\omega) = -(\theta_0 - \omega t_d)/\omega = -\theta_0 /\omega + t_d$, not a constant quantity.

Group Delay of a distortion-less filter

The group delay of a filter is defined as:

$$\tau_g(\omega) = -d\theta(\omega) / d\omega \quad \text{….(11)}$$

Now, for a linear phase filter,

$$\theta(\omega) = -\omega t_d$$

Thus, $\tau_g(\omega) = -d(\omega t_d)/d\omega = t_d$, a constant.

Now, for a linear phase filter with offset,

$$\theta(\omega) = \theta_0 - \omega t_d$$

Thus, $\tau_g(\omega) = -d(\theta_0 - \omega t_d)/d\omega = t_d$, a constant.

Thus it may be concluded that a filter with constant group delay in the pass band region is a distortion-less filter.

Linear Phase Digital Filter

The *z*-transfer function of a digital filter may be expressed as:

$$H(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n} \qquad \text{....(12)}$$

where h_n is the discrete impulse response (impulse sequence) of the digital filter.

Putting $z = e^{j\omega\tau}$, the steady state frequency response may be obtained as:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h_n e^{-jn\omega\tau} \qquad \text{....(13)}$$

Linear Phase Digital Filter

Properties of $H(\omega)$

Periodicity:

Let $\omega = p\omega_s + \omega'$, where $0 \le \omega' < \omega_s/2$ and $p = 0, \pm 1, \pm 2, \ldots$ with ω_s as the sampling frequency.

Then, $z = exp(j\omega\tau)$

$$= exp(j(p\omega_s + \omega')\tau)$$

$$= exp(jp\omega_s\tau + j\omega'\tau)$$

$$= exp(jp2\pi + j\omega'\tau) \text{ as } \omega_s = 2\pi/\tau$$

$$= exp(jp2\pi).exp(j\omega'\tau) = exp(j\omega'\tau)$$

Thus,

$$H(\omega) = H(\omega') \quad \text{….(14)}$$

Linear Phase Digital Filter

Properties of $H(\omega)$

Symmetry:

Let $\omega = p\omega_s - \omega'$, where $0 \leq \omega' < \omega_s/2$ and $p = 0, \pm 1, \pm 2, \dots$ with ω_s as the sampling frequency.

Then, $z = exp(j\omega\tau)$

$= exp(j(p\omega_s - \omega')\tau)$

$= exp(jp\omega_s\tau - j\omega'\tau)$

$= exp(jp2\pi - j\omega'\tau)$ as $\omega_s = 2\pi/\tau$

$= exp(jp2\pi).exp(-j\omega'\tau) = exp(-j\omega'\tau)$,

complex conjugate of $exp(j\omega'\tau)$

Thus,

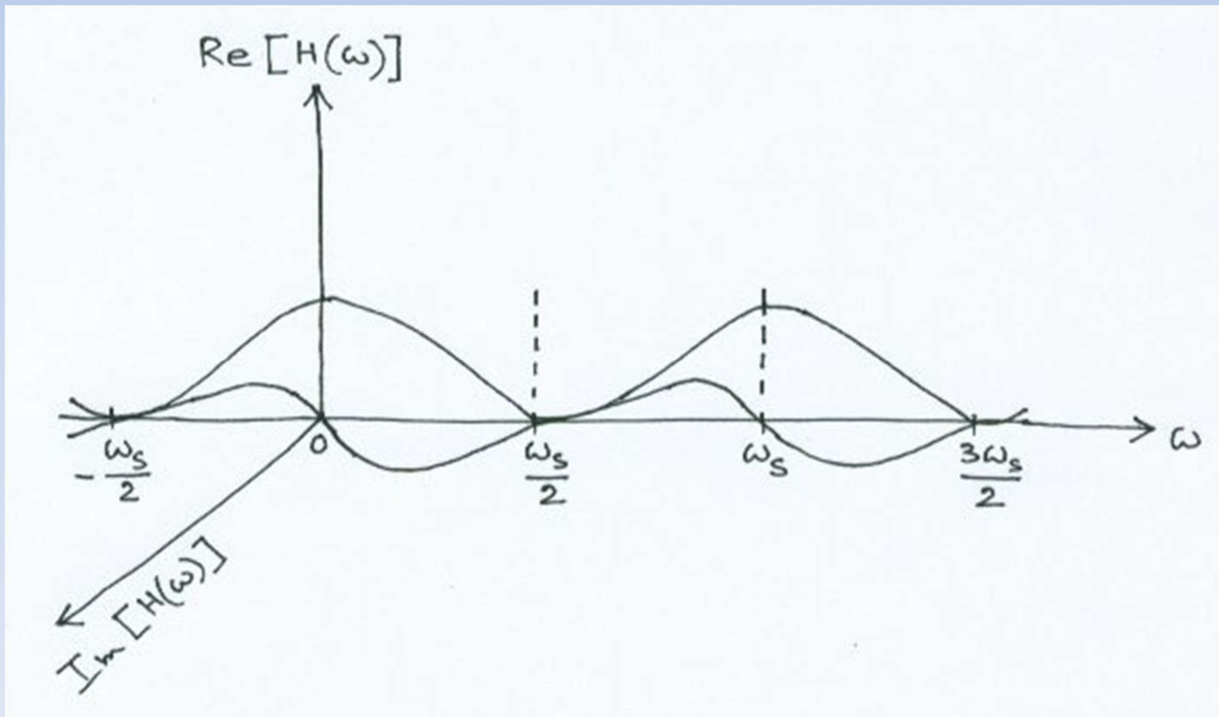
$H(\omega) = \hat{H}(\omega')$, complex conjugate of $H(\omega')$ ….(15)

Linear Phase Digital Filter

Properties of *H(ω)*

Periodicity: $H(\omega) = H(\omega')$, $\omega = p\omega_s + \omega'$ ….(14)

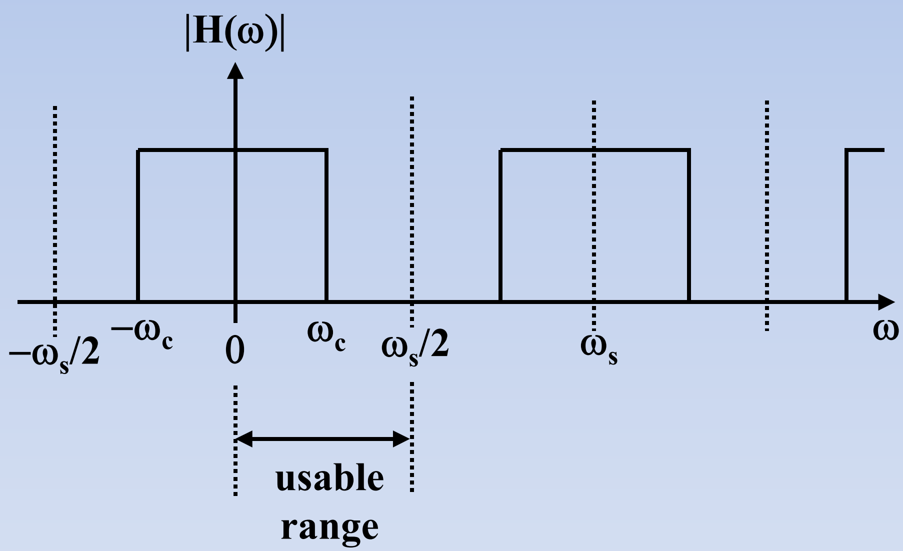
Symmetry: $H(\omega) = \hat{H}(\omega')$, $\omega = p\omega_s - \omega'$ ….(15)



Linear Phase Digital Filter

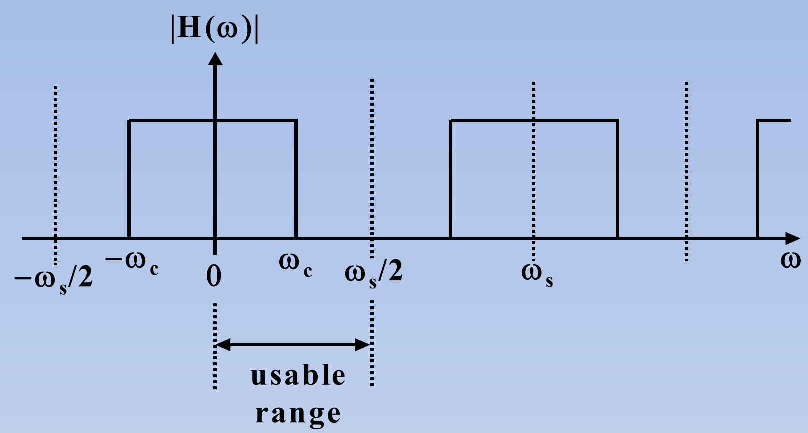
Design of digital filter by Fourier series method

Let the digital filter be an ideal low-pass filter with a cutoff frequency of ω_c, then $|H(\omega)|$ may be represented as:



Thus, $H(\omega)$ is found to be periodic with period ω_s (in frequency domain).

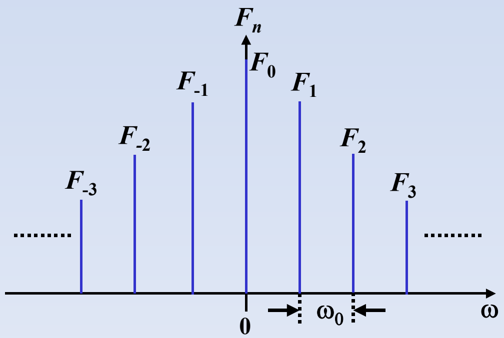
Design of digital filter by Fourier series method



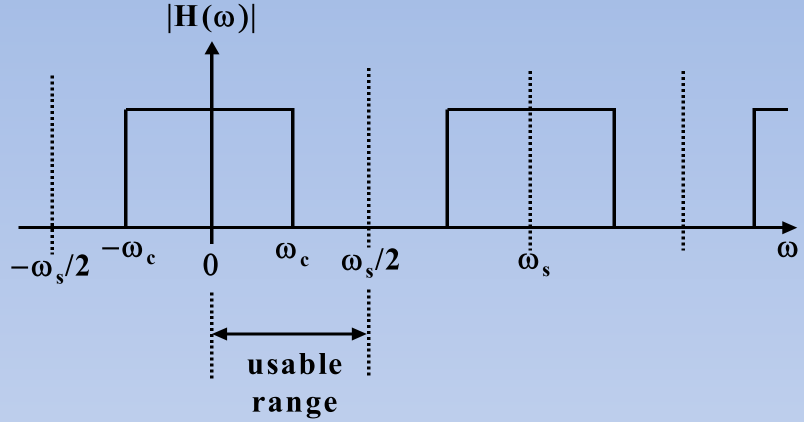
Comparing this periodic property with that of a periodic signal, complex Fourier coefficients of a periodic signal may be evaluated as follows:

For a periodic time signal $x(t)$, with T_0 as the time period,

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad \text{with} \quad F_n = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jn\omega_0 t}\, dt$$



Design of digital filter by Fourier series method



For a periodic time signal

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt$$

Comparing this periodic property with that of a periodic signal, complex Fourier coefficients of a periodic signal may be evaluated as follows:

From relation (13),

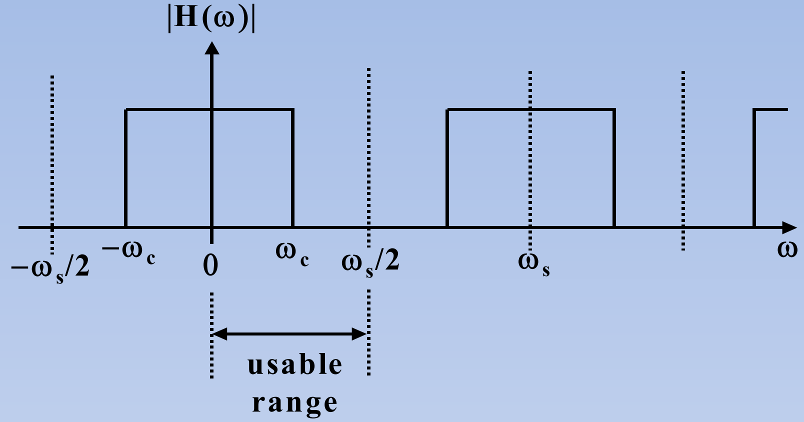
$$H(\omega) = \sum_{n=-\infty}^{\infty} h_n e^{-jn\omega\tau}$$

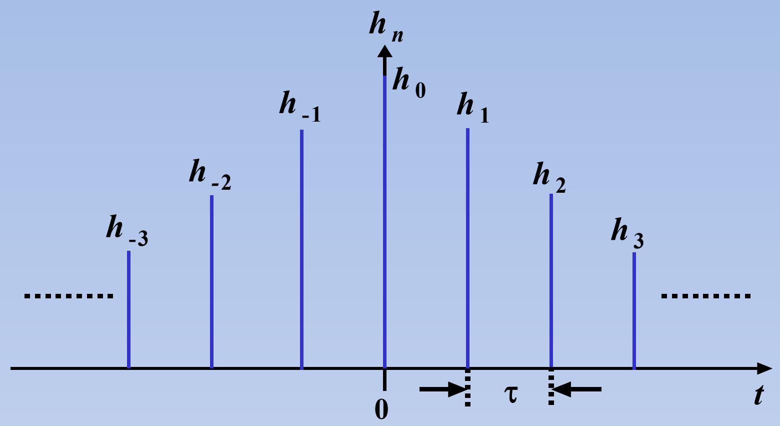
$x(t) \leftrightarrow H(\omega)$ $F_n \leftrightarrow h_n$ $T_0 \leftrightarrow \omega_s$ $\omega_0 \leftrightarrow -\tau$

h_n may be evaluated as:

$$h_n = \frac{1}{\omega_s} \int_0^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega \quad \text{....(16)}$$

Design of digital filter by Fourier series method





Comparing this periodic property with that of a periodic signal, complex Fourier coefficients of a periodic signal may be evaluated as follows:

From relation (13),

$$H(\omega) = \sum_{n=-\infty}^{\infty} h_n e^{-jn\omega\tau}$$

h_n may be evaluated as: $$h_n = \frac{1}{\omega_s} \int_0^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega \quad \text{....(16)}$$

Design of digital filter by Fourier series method

Digital filter impulse sequence h_n :

$$h_n = \frac{1}{\omega_s} \int_0^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega \qquad \text{....(16)}$$

Now, relation (16) may be re-written as:

$$h_n = \frac{1}{\omega_s} \int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} H(\omega) e^{jn\tau\omega} d\omega \qquad \text{....(17)}$$

by shifting the limits of integration and using the periodicity property of H(ω).

Design of digital filter by Fourier series method

Proof of relation (17):

$$\frac{1}{\omega_s}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} H(\omega)e^{jn\tau\omega}\,d\omega$$

$$=\frac{1}{\omega_s}\left[\int_{-\frac{\omega_s}{2}}^{0} H(\omega)e^{jn\tau\omega}\,d\omega+\int_{0}^{\frac{\omega_s}{2}} H(\omega)e^{jn\tau\omega}\,d\omega\right]$$

Now, from periodicity property,

$$\int_{-\frac{\omega_s}{2}}^{0} H(\omega)e^{jn\tau\omega}\,d\omega=\int_{\frac{\omega_s}{2}}^{\omega_s} H(\omega)e^{jn\tau\omega}\,d\omega$$

Design of digital filter by Fourier series method

Proof of relation (17):

$$\int_{-\frac{\omega_s}{2}}^{0} H(\omega) e^{jn\tau\omega} d\omega = \int_{\frac{\omega_s}{2}}^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega$$

Thus,

$$\frac{1}{\omega_s}\left[\int_{0}^{\frac{\omega_s}{2}} H(\omega) e^{jn\tau\omega} d\omega + \int_{\frac{\omega_s}{2}}^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega\right]$$

$$= \frac{1}{\omega_s}\int_{0}^{\omega_s} H(\omega) e^{jn\tau\omega} d\omega$$

$$= h_n$$

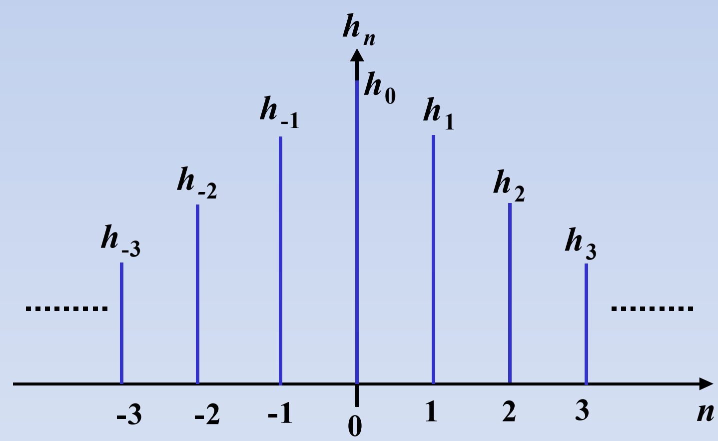
Frequency response of digital filter

From relation (13), the frequency response H(ω) may be expressed as:

$$H(\omega) = \sum_{n=-\propto}^{\propto} h_n e^{-jn\tau\omega}$$

$$= \sum_{n=-1}^{-\propto} h_n e^{-jn\tau\omega} + h_0 + \sum_{n=1}^{\propto} h_n e^{-jn\tau\omega}$$

$$= h_0 + \sum_{n=1}^{\propto} h_{-n} e^{jn\tau\omega} + \sum_{n=1}^{\propto} h_n e^{-jn\tau\omega}$$



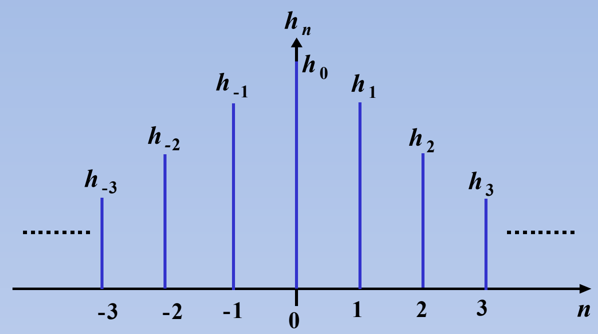
Let h_n be real and symmetric, i.e. $h_n = h_{-n}$.

Then,

$$H(\omega) = h_0 + \sum_{n=1}^{\propto} h_n \left(e^{jn\tau\omega} + e^{-jn\tau\omega} \right)$$

Frequency response of digital filter

For real and symmetric h_n,



$$H(\omega) = h_0 + \sum_{n=1}^{\infty} h_n \left(e^{jn\tau\omega} + e^{-jn\tau\omega} \right)$$

$$= h_0 + 2\sum_{n=1}^{\infty} h_n \cos(n\tau\omega) \qquad ....(18)$$

This is a real quantity.

Thus, with a ***real*** and ***symmetric*** h_n, a ***real frequency response*** may be obtained. This results in a ***distortion-less*** filter with zero phase shift.

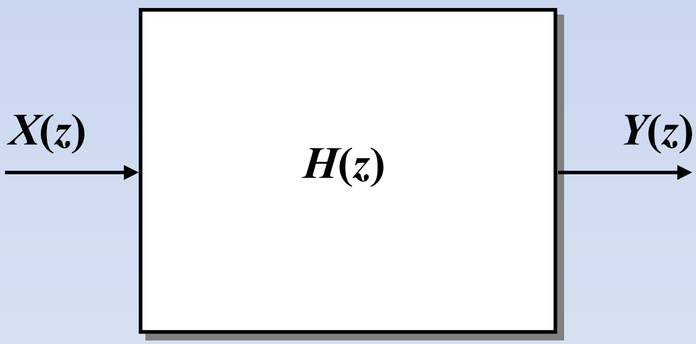
Realization problems

From relation (12),

$$H(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n}$$

In terms of input and output,

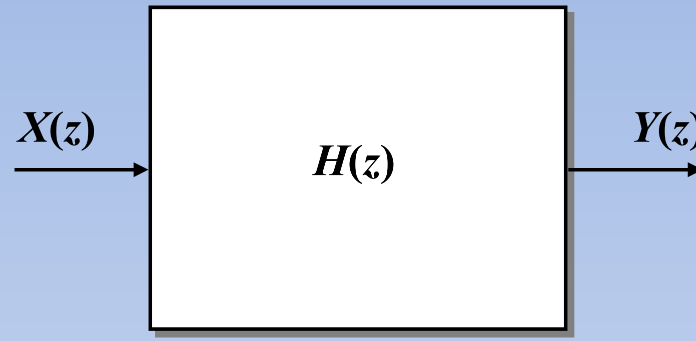
$$\frac{Y(z)}{X(z)} = \sum_{n=-\infty}^{\infty} h_n z^{-n}$$



Digital Filter

Realization problems

$$\frac{Y(z)}{X(z)} = \sum_{n=-\infty}^{\infty} h_n z^{-n}$$



Digital Filter

Then,

$$Y(z) = X(z) \sum_{n=-\infty}^{\infty} h_n z^{-n}$$

or, $$Y(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n} X(z)$$

Now, using the shifting property,

$$z^{-n} X(z) = \sum_{k=-\infty}^{\infty} x_{k-n} z^{-k}$$, the z-transform of sequence x_k delayed by n.

Realization problems

Substituting,

$$Y(z) = \sum_{n=-\infty}^{\infty} h_n \sum_{k=-\infty}^{\infty} x_{k-n} z^{-k}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n} X(z)$$

$$z^{-n} X(z) = \sum_{k=-\infty}^{\infty} x_{k-n} z^{-k}$$

Realization problems

Substituting,

$$Y(z) = \sum_{n=-\infty}^{\infty} h_n \sum_{k=-\infty}^{\infty} x_{k-n} z^{-k}$$

Changing the order of summation,

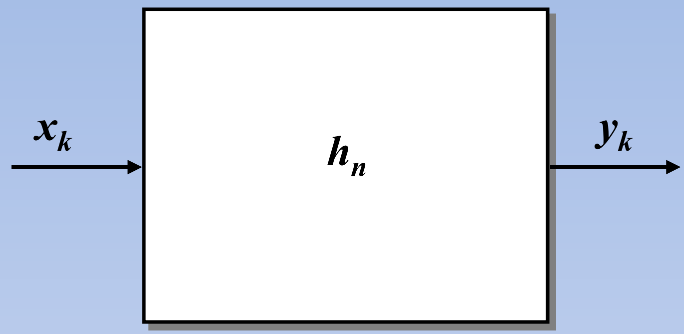
$$Y(z) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} h_n x_{k-n} \right] z^{-k} \qquad \text{....(19)}$$

Now $Y(z)$ may be expressed as:

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} \qquad \text{....(20)}$$

where y_k is the filter output sequence.

Realization problems



Digital Filter

$$Y(z) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} h_n x_{k-n} \right] z^{-k} \quad \text{....(19)}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} \quad \text{....(20)}$$

Comparing relations (19) and (20),

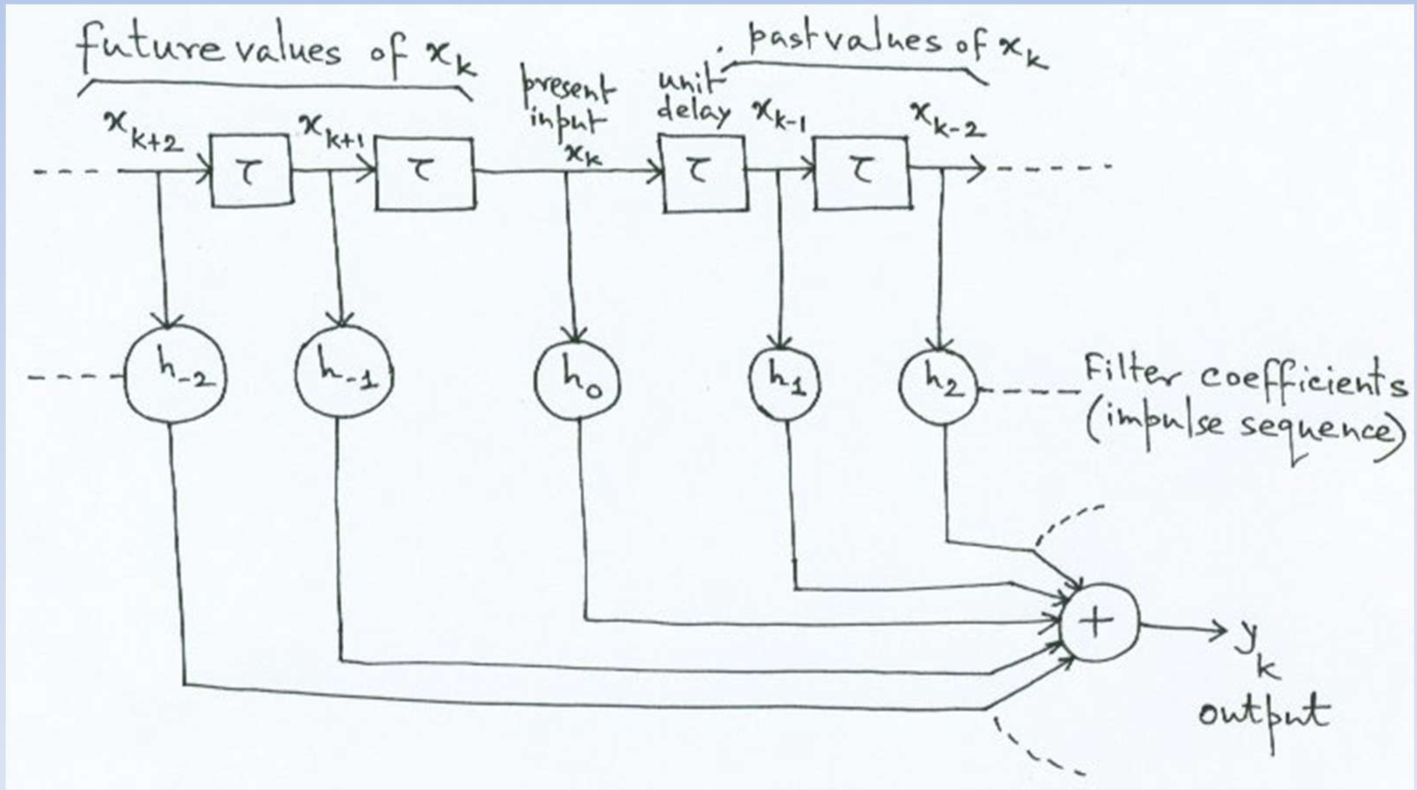
$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n} \quad \text{, for } k = 0,1,2,\ldots \qquad \text{....(21)}$$

Relation (21) is the ***discrete convolution summation***.

Realization problems

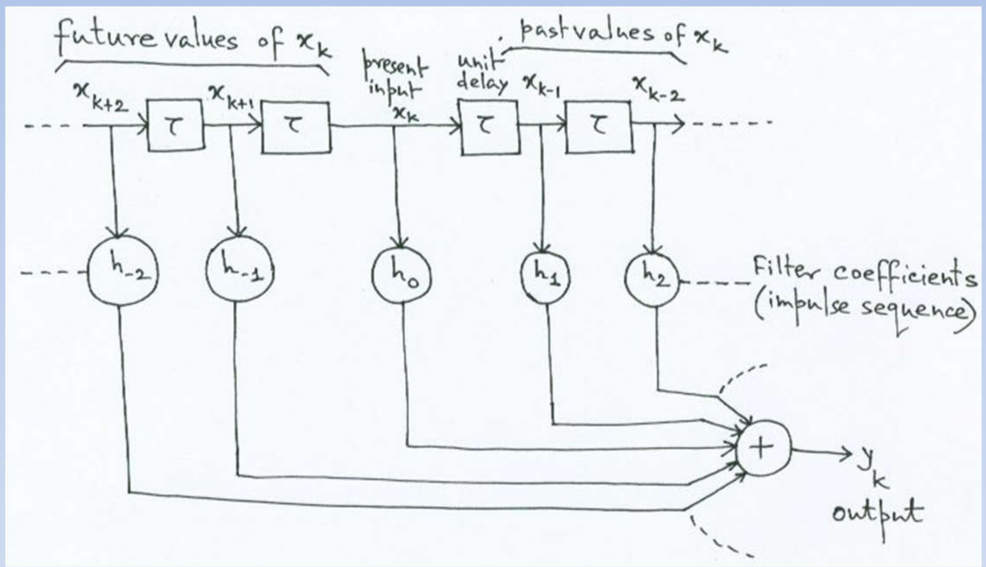
$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n}$$, for k = 0,1,2,… ….(21)

Realization of relation (21) is shown below:



Realization problems

$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n}$$, for k = 0,1,2,… ….(21)



Practical realization of relation (21) requires that,

- **summation to finite number of terms, which means applying some approximation to relation (21)**
- **h_n to be causal (i.e. h_n = 0, for n < 0)**

Realization problems

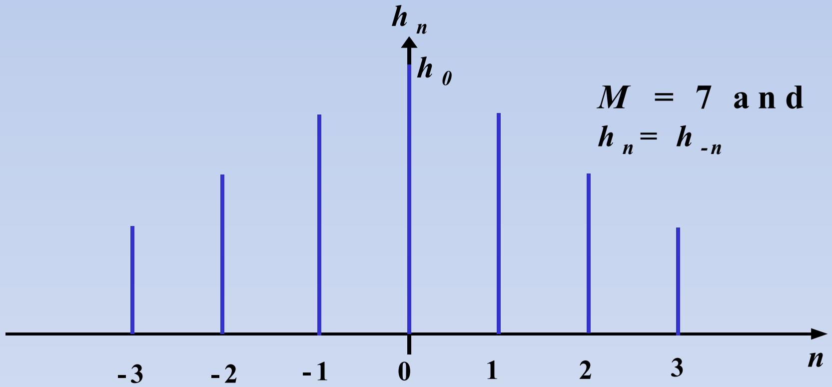
First requirement: summation to finite number of terms

Considering *M* number of finite terms (assuming *M* to be an odd number),

the truncated h_n extends from $n = -(M-1)/2, \ldots, 0, \ldots, (M-1)/2$.

Assuming h_n to be real and symmetric, from relation (21),

$$\hat{y}_k = \sum_{n=-(M-1)/2}^{n=(M-1)/2} h_n x_{k-n}$$



Truncated real and symmetric impulse sequence

where $\hat{y}_k$ is an estimate of y_k

The corresponding frequency response may be expressed as from relation (18),

$$\hat{H}(\omega) = h_0 + 2\sum_{n=1}^{(M-1)/2} h_n \cos(n\tau\omega)$$

, a real quantity, where $\hat{H}(\omega)$ is an estimate of $H(\omega)$.

Realization problems

Second requirement: h_n to be causal

To make h_n to be causal, let the impulse sequence be delayed by *(M-1)/2* delay units (one unit is τ, the sampling interval).

Let the delayed impulse sequence be h_l, where $l = n + (M-1)/2$.

Truncated real & symmetric impulse sequence

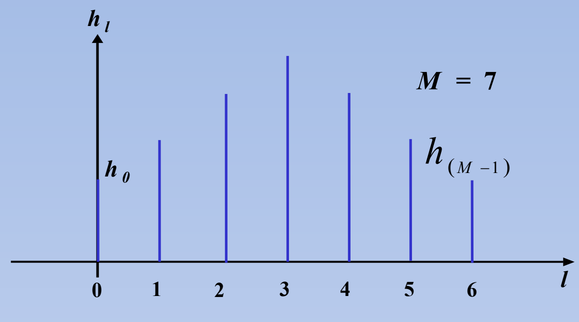
Delayed & truncated real & symmetric impulse sequence

Symmetry property: $\boldsymbol{h_n = h_{-n}}$

Symmetry property: **?**

Realization problems

Symmetry property of h_l



The symmetry property of h_l may be expressed as follows:

Before shifting, $h_n = h_{-n}$

Now shifting by $(M-1)/2$ units,

$$h_{n+(M-1)/2} = h_{-n+(M-1)/2}$$

Putting $l = n + (M-1)/2$,

$$h_l = h_{(-l+(M-1)/2)+(M-1)/2}$$

or $$h_l = h_{M-1-l} \qquad \text{....(22)}$$

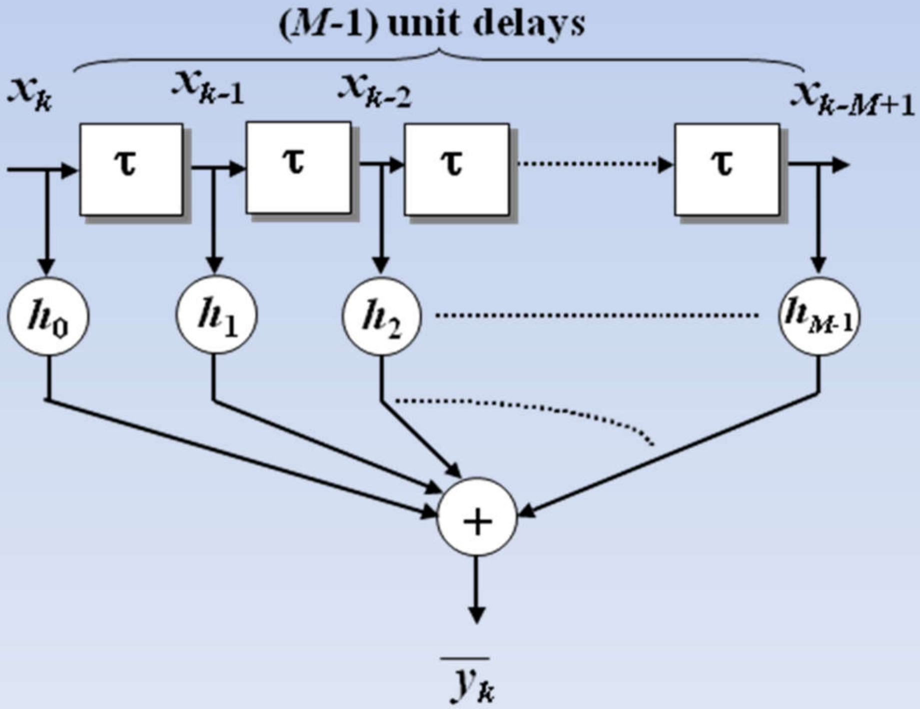
Relation (22) signifies a symmetric property of a causal filter whose impulse sequence is h_l.

Realization of a causal digital filter

The filter output sequence may be expressed as,

$$\bar{y}_k = \sum_{l=0}^{M-1} h_l x_{k-l} \quad \text{, for k = 0,1,2…} \qquad \text{….(23)}$$

with the causal impulse sequence h_l.



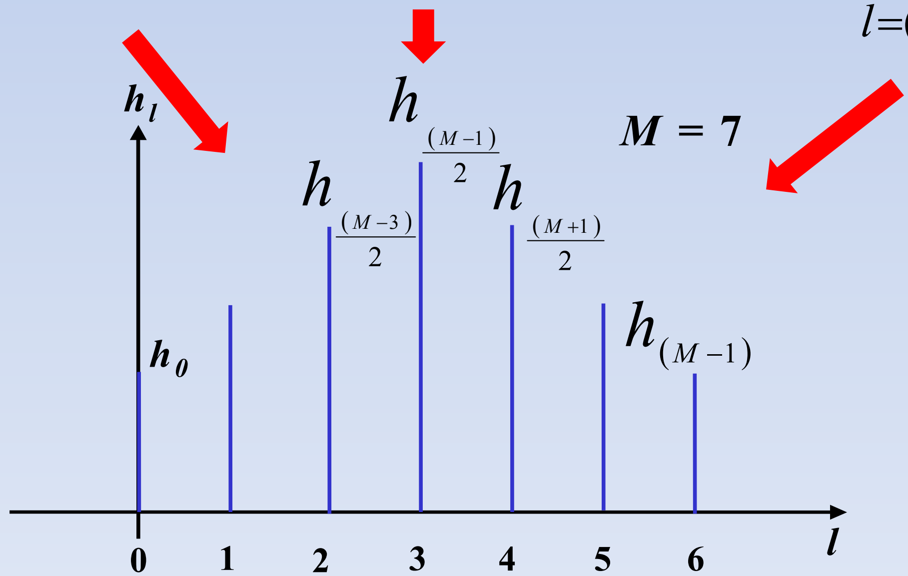
As the duration of the impulse response is finite, it is called **Finite duration Impulse Response (FIR)** digital filter.

Frequency response

The frequency response of the causal digital filter may be expressed as:

$$\overline{H}(\omega) = \sum_{l=0}^{M-1} h_l e^{-jl\tau\omega}$$

or
$$\overline{H}(\omega) = \sum_{l=0}^{(M-3)/2} h_l e^{-jl\tau\omega} + h_{(M-1)/2} e^{-j\tau\omega(M-1)/2} + \sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega} \quad \text{....(24)}$$



Frequency response

Using the symmetry property of h_l, the last term of relation (24) becomes,

$$\bar{H}(\omega) = \sum_{l=0}^{(M-3)/2} h_l e^{-jl\tau\omega} + h_{(M-1)/2} e^{-j\tau\omega(M-1)/2} + \sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega} \quad \text{....(24)}$$

$$\sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega} = \sum_{l=(M+1)/2}^{(M-1)} h_{M-1-l} e^{-jl\tau\omega}$$

By substituting $p = M - 1 - l$,

$$\sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega} = \sum_{l=(M+1)/2}^{(M-1)} h_p e^{-j(M-1-p)\tau\omega}$$

When $l = (M+1)/2$, $p = M-1-l = M-1-(M+1)/2 = (M-3)/2$

Changing index l to p,

$$\sum_{l=(M+1)/2}^{(M-1)} h_p e^{-j(M-1-p)\tau\omega} = \sum_{p=(M-3)/2}^{0} h_p e^{-j(M-1-p)\tau\omega} = \sum_{p=0}^{(M-3)/2} h_p e^{-j(M-1-p)\tau\omega}$$

Frequency response

$$\overline{H}(\omega)=\sum_{l=0}^{(M-3)/2} h_l e^{-jl\tau\omega}+h_{(M-1)/2}e^{-j\tau\omega(M-1)/2}+\sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega} \quad \text{....(24)}$$

Now, changing index *p* to *l*,

$$\sum_{l=(M+1)/2}^{(M-1)} h_l e^{-jl\tau\omega}=\sum_{l=0}^{(M-3)/2} h_l e^{-j(M-1-l)\tau\omega}$$

Substituting this in relation (24),

$$\overline{H}(\omega)=h_{(M-1)/2}e^{-j\tau\omega(M-1)/2}+\sum_{l=0}^{(M-3)/2} h_l\left(e^{-jl\tau\omega}+e^{-j(M-1-l)\tau\omega}\right)$$

Frequency response

Now,

$$\overline{H}(\omega) = h_{(M-1)/2} e^{-j\tau\omega(M-1)/2} + \sum_{l=0}^{(M-3)/2} h_l \left(e^{-jl\tau\omega} + e^{-j(M-1-l)\tau\omega} \right)$$

or,

$$\overline{H}(\omega) = e^{-j\tau\omega(M-1)/2} \left[h_{(M-1)/2} + \sum_{l=0}^{(M-3)/2} h_l \left(e^{-j\tau\omega(l-(M-1)/2)} + e^{j\tau\omega(l-(M-1)/2)} \right) \right]$$

or,

$$\overline{H}(\omega) = e^{-j\tau\omega(M-1)/2} \left[h_{(M-1)/2} + 2 \sum_{l=0}^{(M-3)/2} h_l \cos\{\omega\tau(l-(M-1)/2)\} \right]$$

….(25)

Frequency response

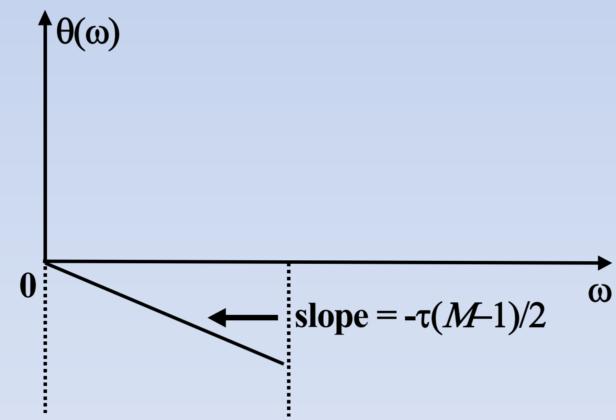
$$\overline{H}(\omega) = e^{-j\tau\omega(M-1)/2}\left[h_{(M-1)/2} + 2\sum_{l=0}^{(M-3)/2} h_l \cos\{\omega\tau(l-(M-1)/2)\}\right]$$

….(25)

Relation (25) may be expressed as

$$\overline{H}(\omega) = \left|\overline{H}(\omega)\right| \angle\theta(\omega)$$

where $\theta(\omega) = -\omega\tau(M-1)/2$, a linear phase characteristic.

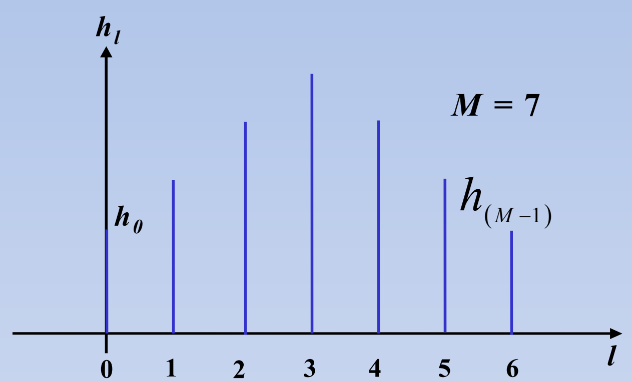


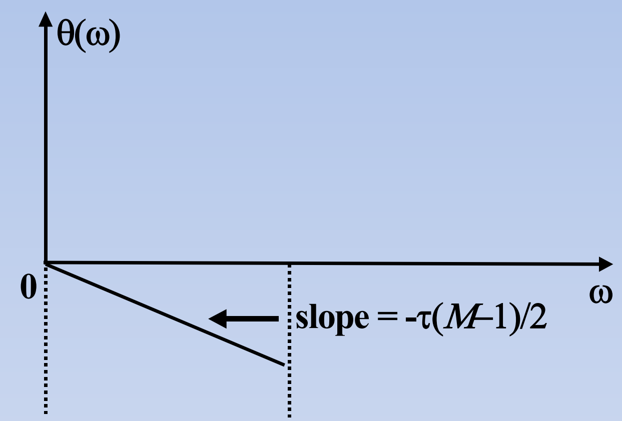
Thus the frequency response of the FIR filter has a **linear phase** characteristic (which implies a distortion-less filter).

The group delay of the FIR filter is: $\tau_g(\omega) = -d\theta(\omega) / d\omega = \tau(M-1)/2$, a constant

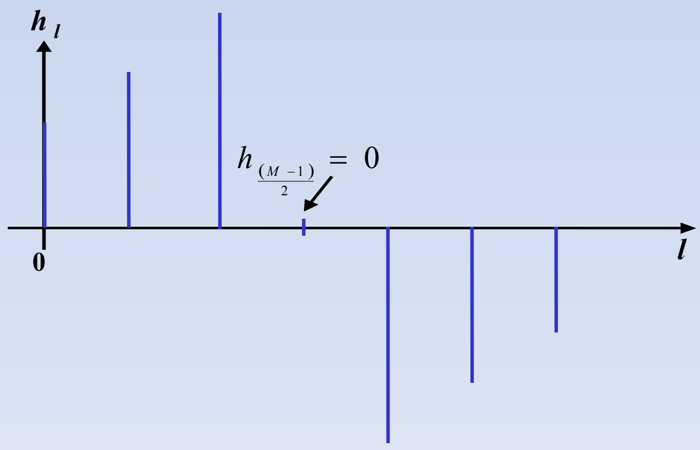
Frequency response

Thus, any FIR digital filter, with a real and symmetric impulse response ($h_l = h_{M-1-l}$) has a linear phase characteristic with a constant group delay.





It may be noted that, an FIR digital filter with anti-symmetric impulse response (i.e., $h_l = -h_{M-1-l}$) also results in a linear phase characteristic with offset (piece-wise linear) and constant group delay.



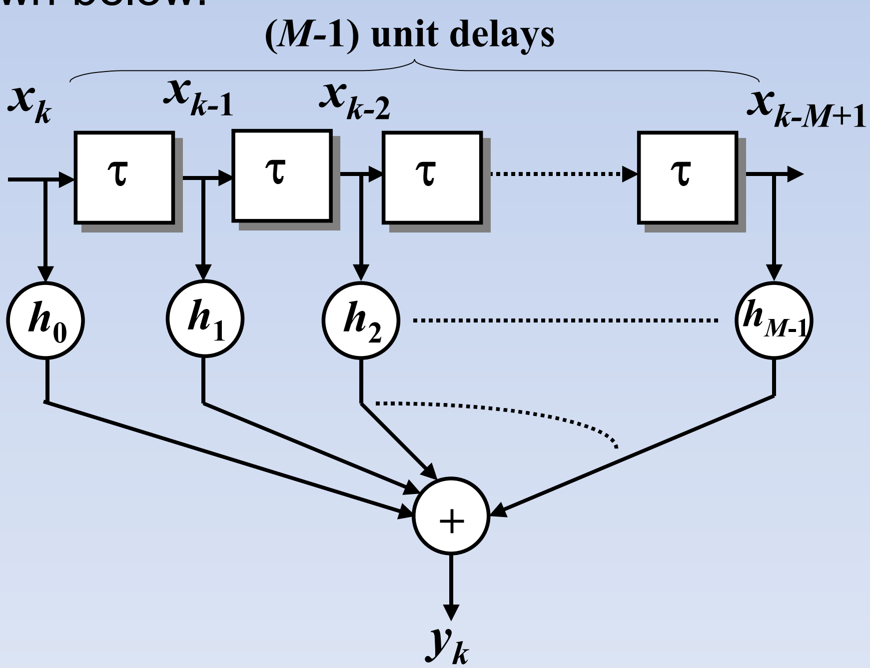
Reference: ***J. R. Johnson,*** **Introduction to Digital Signal Processing**

Direct realization of linear-phase FIR digital filters

From relation (23), the output sequence from the causal FIR digital filter may be expressed as:

$$y_k = \sum_{l=0}^{M-1} h_l x_{k-l} \quad \text{, for } K = 0,1,2,\ldots \qquad \text{....(26)}$$

Direct realization of relation (26) may be carried out with a tapped delay line having (M-1) unit delays as shown below:



Here number of multiplications required is *M*.

Direct realization of linear-phase FIR digital filters

The output sequence from the causal FIR digital filter may be expressed as:

$$y_k = \sum_{l=0}^{M-1} h_l x_{k-l} \quad \text{, for } K = 0,1,2,\ldots \qquad \text{….(26)}$$

Considering the symmetry property, $h_l = h_{M-1-l}$, assuming an odd M, the input output sequence relation may be expressed as, from relation (26),

$$y_k = \sum_{l=0}^{(M-3)/2} h_l x_{k-l} + h_{(M-1)/2} x_{k-(M-1)/2} + \sum_{l=(M+1)/2}^{M-1} h_l x_{k-l}$$

Substituting $h_l = h_{M-1-l}$ in the last term,

$$\sum_{l=(M+1)/2}^{M-1} h_l x_{k-l} = \sum_{l=(M+1)/2}^{M-1} h_{M-1-l} x_{k-l}$$

Direct realization of linear-phase FIR digital filters

The last term: $$\sum_{l=(M+1)/2}^{M-1} h_l x_{k-l} = \sum_{l=(M+1)/2}^{M-1} h_{M-1-l} x_{k-l}$$

Let *p = l - (M+1)/2*. Then the last term becomes,

$$\sum_{p=0}^{(M-3)/2} h_{M-1-\left(p+(M+1)/2\right)} x_{k-\left(p+(M+1)/2\right)}$$

$$= \sum_{p=0}^{(M-3)/2} h_{((M-3)/2)-p} x_{k-\left(p+(M+1)/2\right)}$$

Direct realization of linear-phase FIR digital filters

The last term: $\sum_{p=0}^{(M-3)/2} h_{((M-3)/2)-p} x_{k-(p+(M+1)/2)}$

Let *q = (M-3)/2 - p*. Then after rearrangement, the last term becomes

$$\sum_{q=0}^{(M-3)/2} h_q x_{k-((M+1)/2+((M-3)/2)-q)}$$

$$= \sum_{q=0}^{(M-3)/2} h_q x_{k-(M-1-q)}$$

Direct realization of linear-phase FIR digital filters

The last term: $\sum_{q=0}^{(M-3)/2} h_q x_{k-(M-1-q)}$

Now, replacing *q* by *l*, and putting it in the main relation, we get:

$$y_k = \sum_{l=0}^{(M-3)/2} h_l x_{k-l} + \sum_{l=0}^{(M-3)/2} h_l x_{k-(M-1-l)} + h_{(M-1)/2} x_{k-(M-1)/2}$$

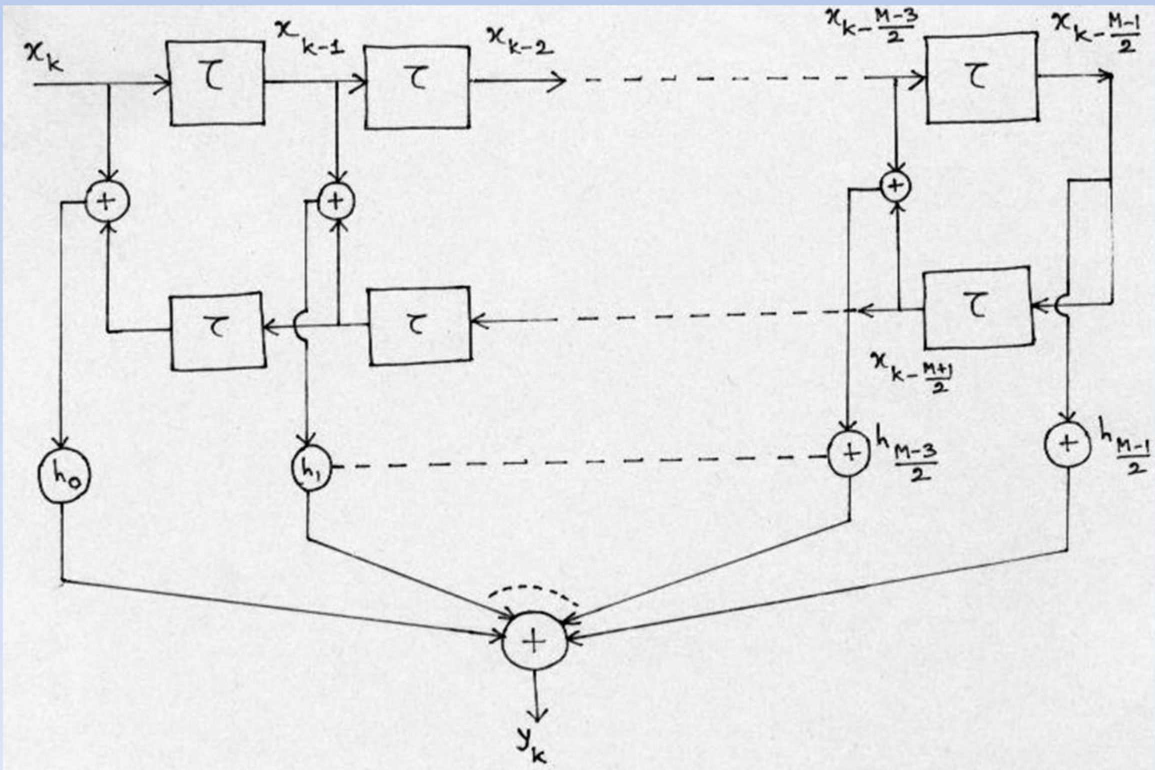
or

$$y_k = \sum_{l=0}^{(M-3)/2} h_l \left(x_{k-l} + x_{k-(M-1-l)} \right) + h_{(M-1)/2} x_{k-(M-1)/2} \qquad \text{....(27)}$$

Direct realization of linear-phase FIR digital filters

$$y_k = \sum_{l=0}^{(M-3)/2} h_l \left(x_{k-l} + x_{k-(M-1-l)} \right) + h_{(M-1)/2} x_{k-(M-1)/2} \qquad \text{....(27)}$$

Realization of relation (27) is shown below:



Here number of multiplications required is (M+1)/2. Similarly, for even M, the number of multiplications required is M/2.

Effect of truncation of impulse response

Relation between the desired frequency response of the FIR digital filter, considering infinite impulse response, and the frequency response obtained by truncating the impulse response may be expressed as follows:

Let *H(ω)* be the desired frequency response and be expressed in terms of the infinite impulse sequence h_n, $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$ as

$$H(\omega) = \sum_{n=-\infty}^{\infty} h_n e^{-jn\omega\tau} \qquad \text{….(28)}$$

where $$h_n = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\omega) e^{jn\omega\tau} \, d\omega$$

$$= \frac{1}{\omega_s} \int_{0}^{\omega_s} H(\omega) e^{jn\omega\tau} \, d\omega$$ [from relations (16) & (17)]

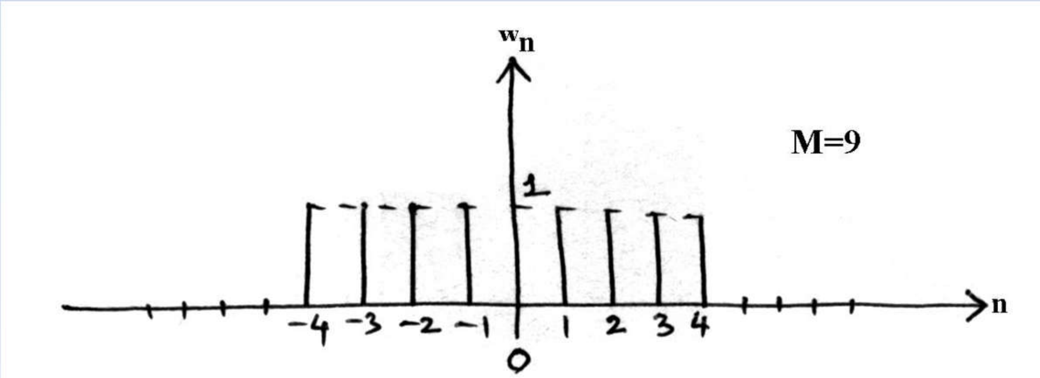
Effect of truncation of impulse response

Let $\acute{H}(\omega)$ be the frequency response of the filter with truncated impulse sequence (without considering the delay for causality) h_n, $n = 0, \pm 1, \pm 2, \ldots, \pm (M-1)/2$ and be expressed as

$$\acute{H}(\omega) = \sum_{n=-(M-1)/2}^{(M-1)/2} h_n e^{-jn\omega\tau} \qquad \ldots.(29)$$

Now, $\acute{H}(\omega)$ may be expressed in terms of infinite impulse sequence, considering a rectangular window sequence w_n, $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$ defined as

$$w_n = 1 \text{ for } |n| \leq (M-1)/2$$
$$= 0, \text{ otherwise} \qquad \ldots.(30)$$



and

$$\acute{H}(\omega) = \sum_{n=-\infty}^{\infty} h_n w_n e^{-jn\omega\tau} \qquad \ldots.(31)$$

Effect of truncation of impulse response

Now, by replacing h_n with its value,

$$\tilde{H}(\omega) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) e^{jn\Omega\tau} d\Omega \right] w_n e^{-jn\omega\tau}$$

with Ω as a dummy variable for integration.

Now, changing the order of summation and integration,

$$\tilde{H}(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) \left[\sum_{n=-\infty}^{\infty} w_n e^{-jn\tau(\omega-\Omega)} \right] d\Omega$$

Now $\sum_{n=-\infty}^{\infty} w_n e^{-jn\tau(\omega-\Omega)}$ is the Fourier series representation of $W(\omega-\Omega)$.

Therefore, $$\tilde{H}(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) W(\omega - \Omega) d\Omega \qquad \text{....(32)}$$

Effect of truncation of impulse response

$$\acute{H}(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) W(\omega - \Omega) d\Omega \qquad \text{….(32)}$$

Effect of truncation of impulse response

$$\tilde{H}(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) W(\omega - \Omega) d\Omega \qquad \text{....(32)}$$

Relation (32) is known as the ***Circular Complex Convolution Integral***.

Now,

$$W(\omega) = \sum_{n=-\infty}^{\infty} w_n e^{-jn\omega\tau} = \sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\omega\tau} \quad \text{(considering the sequence } w_n\text{)}$$

Effect of truncation of impulse response

$$\acute{H}(\omega) = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} H(\Omega)W(\omega-\Omega)\,d\Omega \qquad \text{....(32)}$$

Relation (32) is known as the ***Circular Complex Convolution Integral***.
Now,

$$W(\omega)=\sum_{n=-\infty}^{\infty} w_n e^{-jn\omega\tau} = \sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\omega\tau}$$ (considering the sequence w_n)

Let $k = n + (M-1)/2$. Then,

$$W(\omega)=\sum_{k=0}^{M-1} e^{-j(k-(M-1)/2)\omega\tau} = e^{j\omega\tau(M-1)/2}\sum_{k=0}^{M-1} e^{-jk\omega\tau}$$

Changing index k to n,

$$W(\omega)=e^{j\omega\tau(M-1)/2}\sum_{n=0}^{M-1} e^{-jn\omega\tau} \qquad \text{....(33)}$$

Effect of truncation of impulse response

$$W(\omega) = e^{j\omega\tau(M-1)/2} \sum_{n=0}^{M-1} e^{-jn\omega\tau} \quad \text{....(33)}$$

Now $\sum_{n=0}^{M-1} e^{-jn\omega\tau}$ may be expressed as

$$\sum_{n=0}^{M-1} e^{-jn\omega\tau} = \frac{1-e^{-jM\omega\tau}}{1-e^{-j\omega\tau}}$$

Thus,

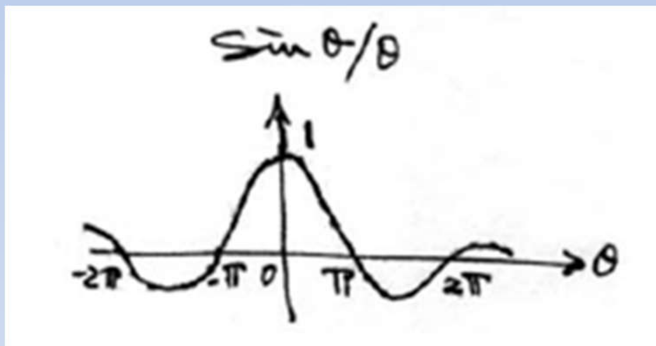
$$W(\omega) = e^{j\omega\tau(M-1)/2}\left(\frac{1-e^{-j\omega M\tau}}{1-e^{-j\omega\tau}}\right) = \frac{e^{j\omega\tau M/2} - e^{-j\omega\tau M/2}}{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}$$

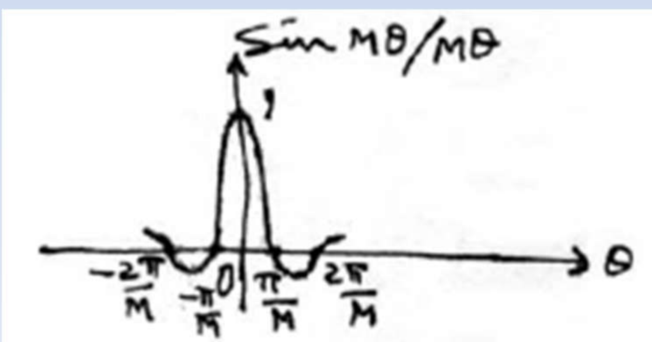
$$= \frac{(e^{j\omega\tau M/2} - e^{-j\omega\tau M/2})/2j}{(e^{j\omega\tau/2} - e^{-j\omega\tau/2})/2j} = \sin\left(\frac{\omega\tau M}{2}\right) / \sin\left(\frac{\omega\tau}{2}\right)$$

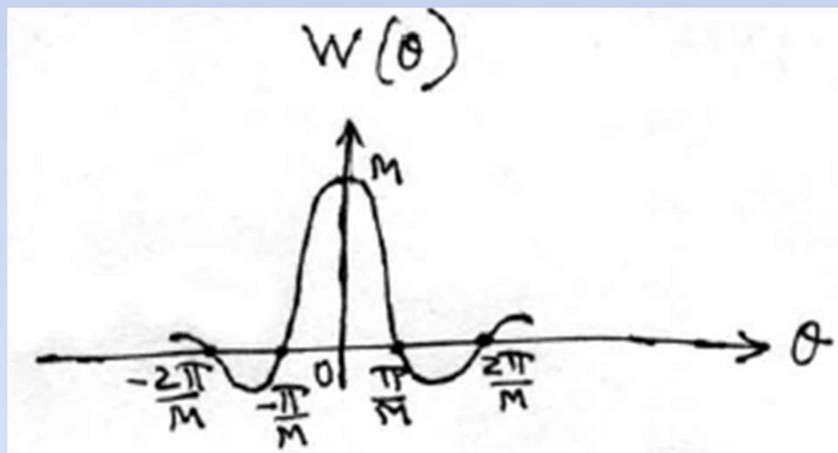
Effect of truncation of impulse response

Plot of *W(ω)*:

$$W(\omega) = \frac{\sin M\theta}{\sin\theta} = M\left(\frac{\sin M\theta}{M\theta} \Big/ \frac{\sin\theta}{\theta} \right)$$



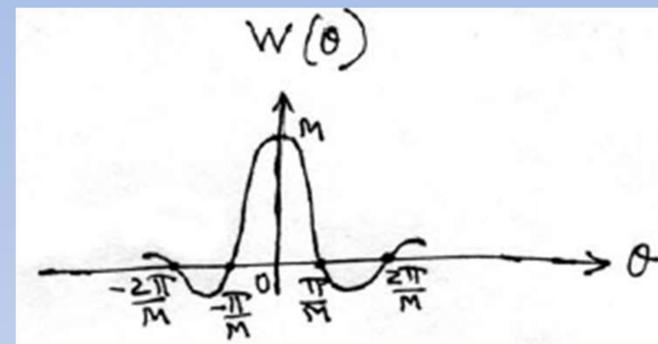




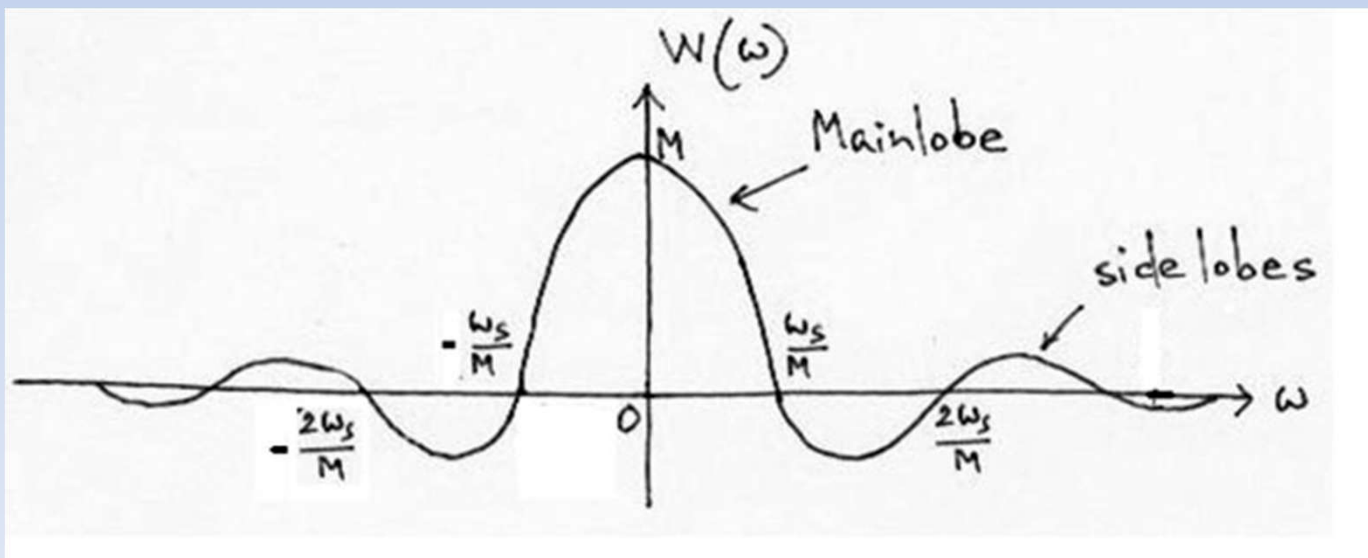
Effect of truncation of impulse response

Plot of *W(ω)*:

$$W(\omega)=\frac{\sin M\theta}{\sin\theta}=M\left(\frac{\sin M\theta}{M\theta}\Big/\frac{\sin\theta}{\theta}\right)$$



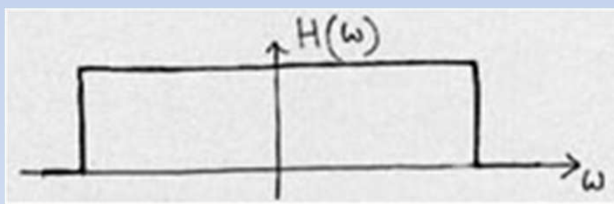
Substituting the value of *θ (= ωτ/2)*,

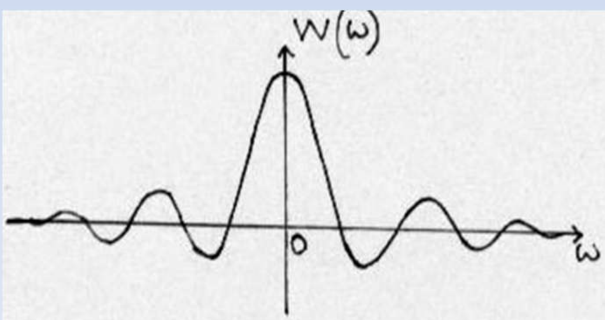


Effect of truncation of impulse response

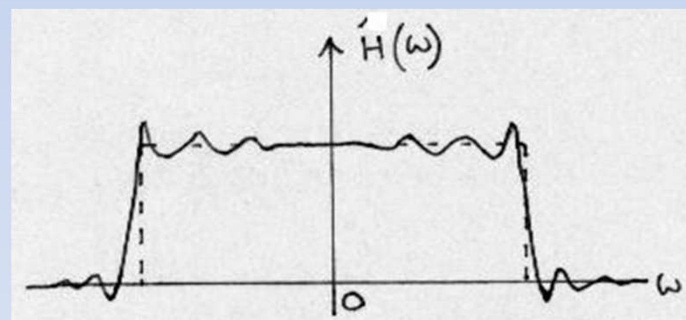
When $W(\omega)$ is convolved with $H(\Omega)$, in relation (32), overshoots and undershoots occur in $\acute{H}(\omega)$ as shown below:

$$\acute{H}(\omega) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\Omega) W(\omega - \Omega) d\Omega \qquad \text{....(32)}$$

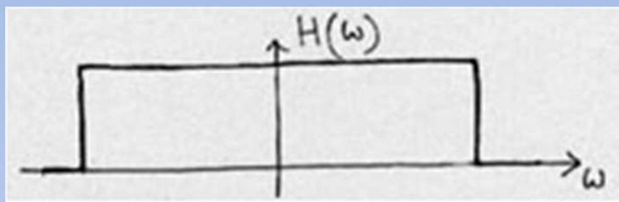


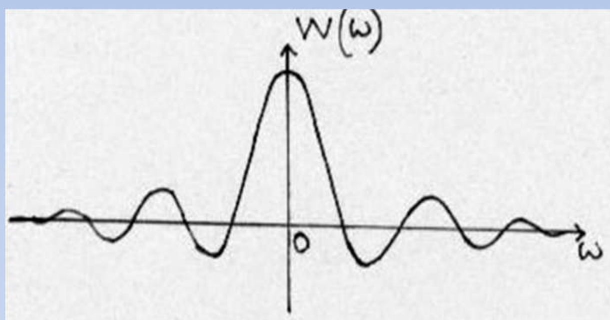


Convolution

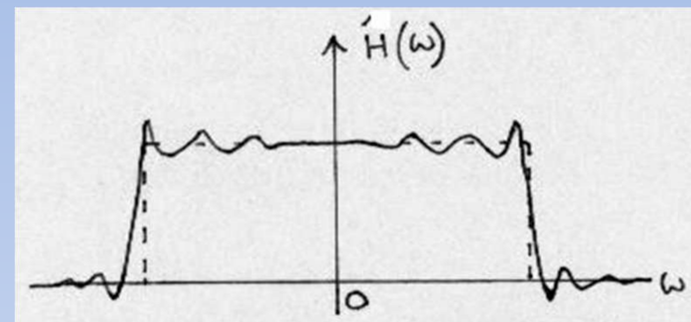


Effect of truncation of impulse response





Convolution

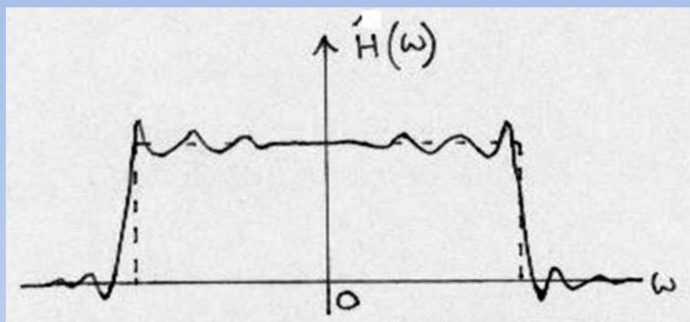


This is known as **Gibbs phenomenon**.

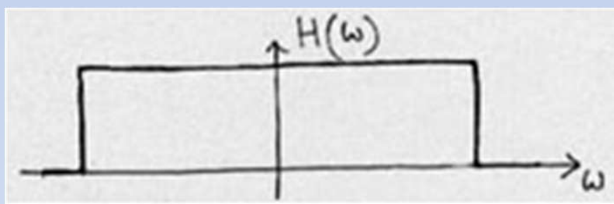
In this case, the peak overshoot is about 9%.

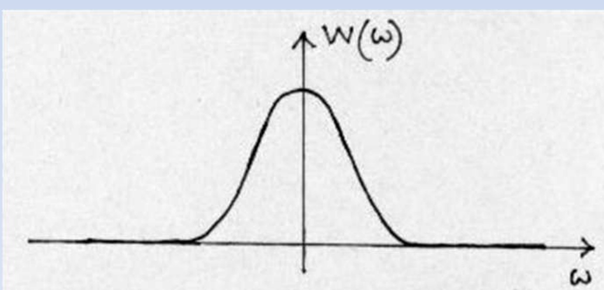
The major part of the ripple in $\acute{H}(\omega)$ is mainly due to the last component of the truncated Fourier series, as obtained in relation (33).

Effect of truncation of impulse response

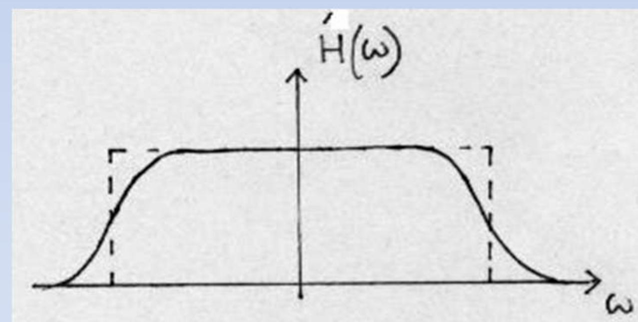


These overshoots and undershoots (ripples) may be minimized by replacing the rectangular window function by an appropriate smooth window function whose amount of side lobes are minimum.





Convolution



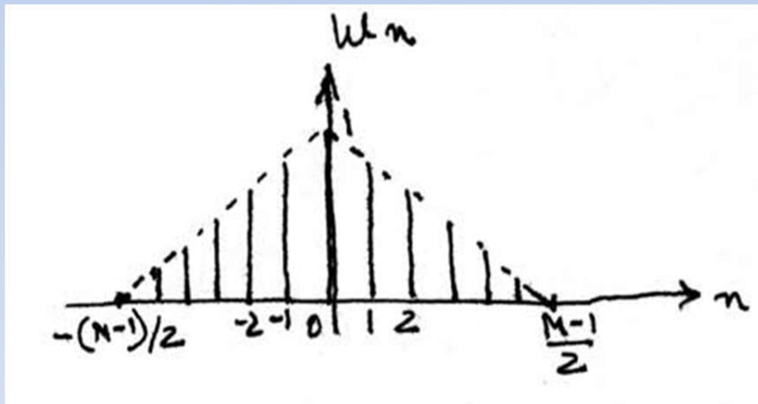
Common window functions for FIR filter design

Bartlett or Triangular Window

Non causal

$w_n = 1 - |n|/((M-1)/2)$ for $|n| \leq (M-1)/2$

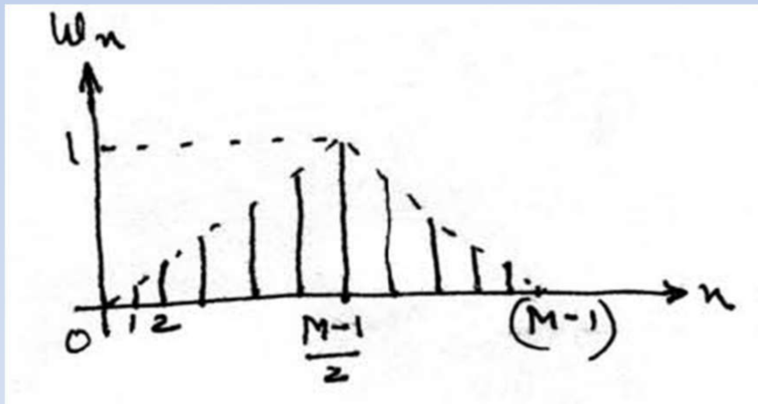
$= 0$, otherwise



Causal

$w_n = 2n/(M-1)$ for $0 \leq n \leq (M-1)/2$

$= 2 - 2n/(M-1)$ for $(M-1)/2 \leq n \leq (M-1)$



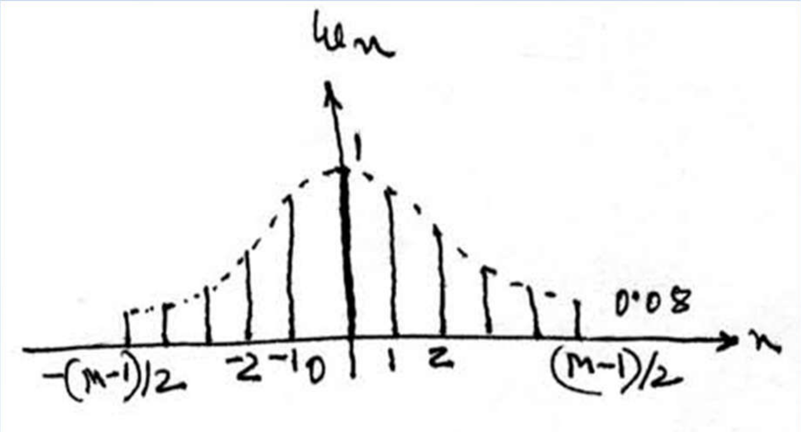
Common window functions for FIR filter design

Hamming or raised-cosine window

Non causal

$w_n = 0.54+0.46\cos(2\pi n/(M-1))$ for $|n|\leq(M-1)/2$

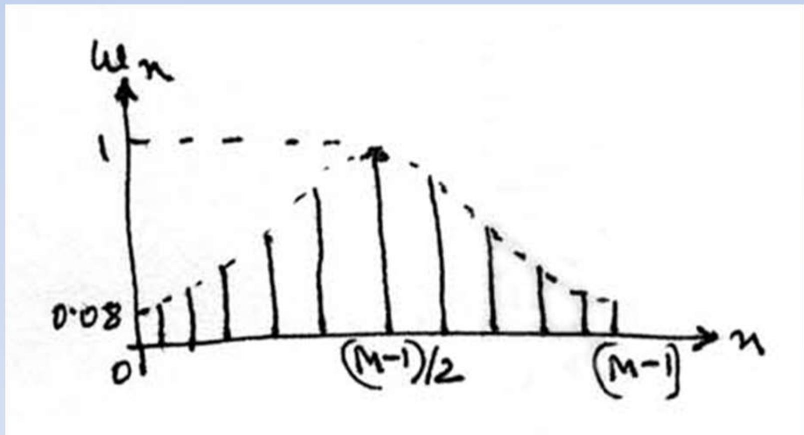
$= 0$, otherwise



Causal

$w_n = 0.54-0.46\cos(2\pi n/(M-1))$ for $|n|\leq(M-1)$

$= 0$, otherwise



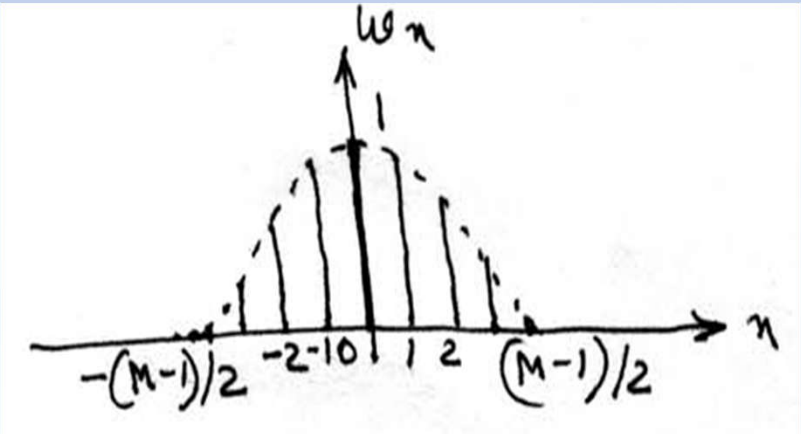
Common window functions for FIR filter design

Hann window

Non causal

$w_n = 0.5+0.5\cos(2\pi n/(M-1))$ for $|n| \leq (M-1)/2$

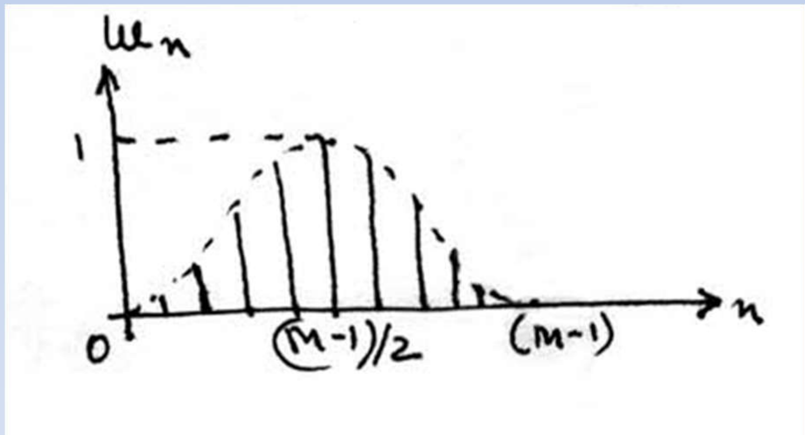
$= 0$, otherwise



Causal

$w_n = 0.5-0.5\cos(2\pi n/(M-1))$ for $|n| \leq (M-1)$

$= 0$, otherwise



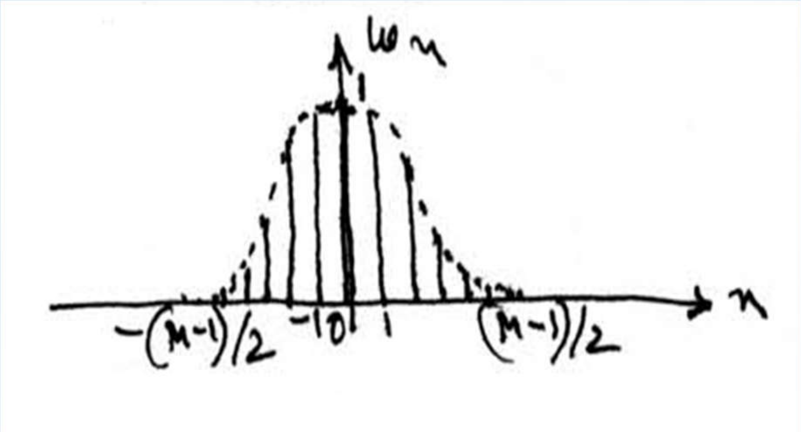
Common window functions for FIR filter design

Blackman window

Non causal

$w_n = 0.42+0.5\cos(2\pi n/(M-1))$
$\quad +0.08\cos(4\pi n/(M-1))$ for $|n|\leq(M-1)/2$

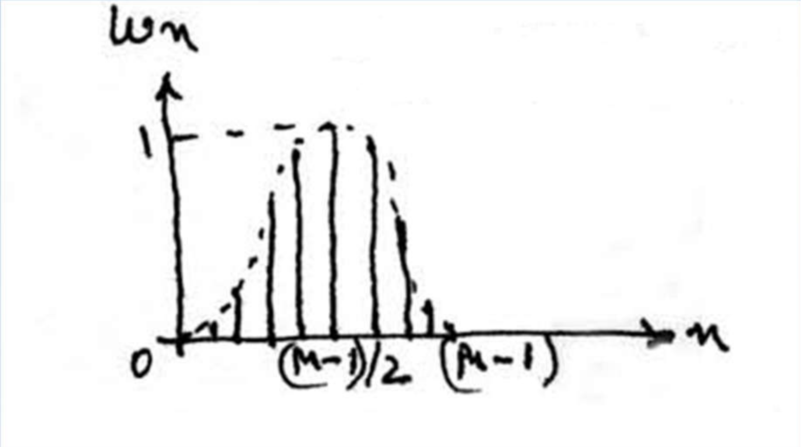
$\quad = 0$, otherwise



Causal

$w_n = 0.42-0.5\cos(2\pi n/(M-1))$
$\quad +0.08\cos(4\pi n/(M-1))$ for $|n|\leq(M-1)$

$\quad = 0$, otherwise



Frequency response of Hann window

For a non causal Hann window,

w_n = 0.5+0.5cos(2πn/(M-1)) for |n|≤(M-1)/2

Then,

$$W(\omega)=\sum_{n=-(M-1)/2}^{(M-1)/2} w_n e^{-jn\omega\tau}$$

$$=\sum_{n=-(M-1)/2}^{(M-1)/2}\left\{0.5+0.5\cos\left(\frac{2\pi n}{M-1}\right)\right\}e^{-jn\omega\tau}$$

$$=0.5\sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\omega\tau}+0.5\sum_{n=-(M-1)/2}^{(M-1)/2}\left\{\frac{e^{j2\pi n/(M-1)}+e^{-j2\pi n/(M-1)}}{2}\right\}e^{-jn\omega\tau}$$

Frequency response of Hann window

$$W(\omega)=0.5\sum_{n=-(M-1)/2}^{(M-1)/2}e^{-jn\omega\tau}+0.5\sum_{n=-(M-1)/2}^{(M-1)/2}\left\{\frac{e^{j2\pi n/(M-1)}+e^{-j2\pi n/(M-1)}}{2}\right\}e^{-jn\omega\tau}$$

$$=0.5\sum_{n=-(M-1)/2}^{(M-1)/2}e^{-jn\omega\tau}+0.25\sum_{n=-(M-1)/2}^{(M-1)/2}e^{-jn\left(\omega\tau-\frac{2\pi}{M-1}\right)}+0.25\sum_{n=-(M-1)/2}^{(M-1)/2}e^{-jn\left(\omega\tau+\frac{2\pi}{M-1}\right)}$$

Frequency response of Hann window

$$W(\omega) = 0.5 \sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\omega\tau} + 0.25 \sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\left(\omega\tau - \frac{2\pi}{M-1}\right)} + 0.25 \sum_{n=-(M-1)/2}^{(M-1)/2} e^{-jn\left(\omega\tau + \frac{2\pi}{M-1}\right)}$$

or $W(\omega) = 0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]$
$+ 0.25[\sin(\omega\tau - (2\pi/(M-1))M/2)/\sin(\omega\tau - (2\pi/(M-1))/2)]$
$+ 0.25[\sin(\omega\tau + (2\pi/(M-1))M/2)/\sin(\omega\tau + (2\pi/(M-1))/2)]$

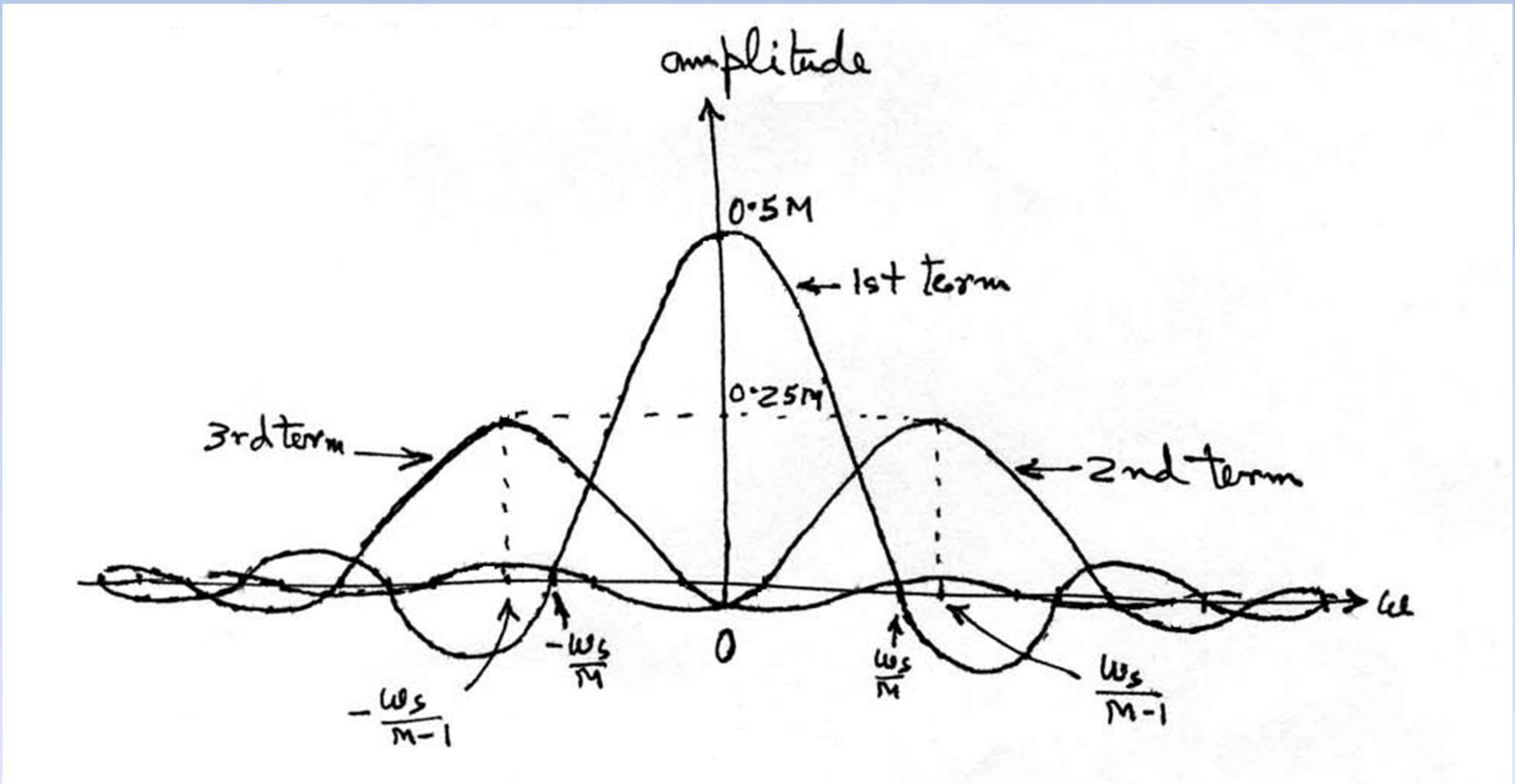
Now, $2\pi/(M-1) = 2\pi f_s/(M-1)f_s = \omega_s\tau/(M-1)$

Therefore,

$W(\omega) = 0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]$
$+ 0.25[\sin(\omega - (\omega_s/(M-1))\tau M/2)/\sin(\omega - (\omega_s/(M-1))\tau/2)]$
$+ 0.25[\sin(\omega + (\omega_s/(M-1))\tau M/2)/\sin(\omega\tau + (\omega_s/(M-1))\tau/2)]$

Frequency response of Hann window

$$W(\omega) = 0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]$$
$$+ 0.25[\sin(\omega - (\omega_s/(M-1))\tau M/2)/\sin(\omega - (\omega_s/(M-1))\tau/2)]$$
$$+ 0.25[\sin(\omega + (\omega_s/(M-1))\tau M/2)/\sin(\omega\tau + (\omega_s/(M-1))\tau/2)]$$



Frequency response of Hann window

$$W(\omega) = \mathbf{0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]}$$
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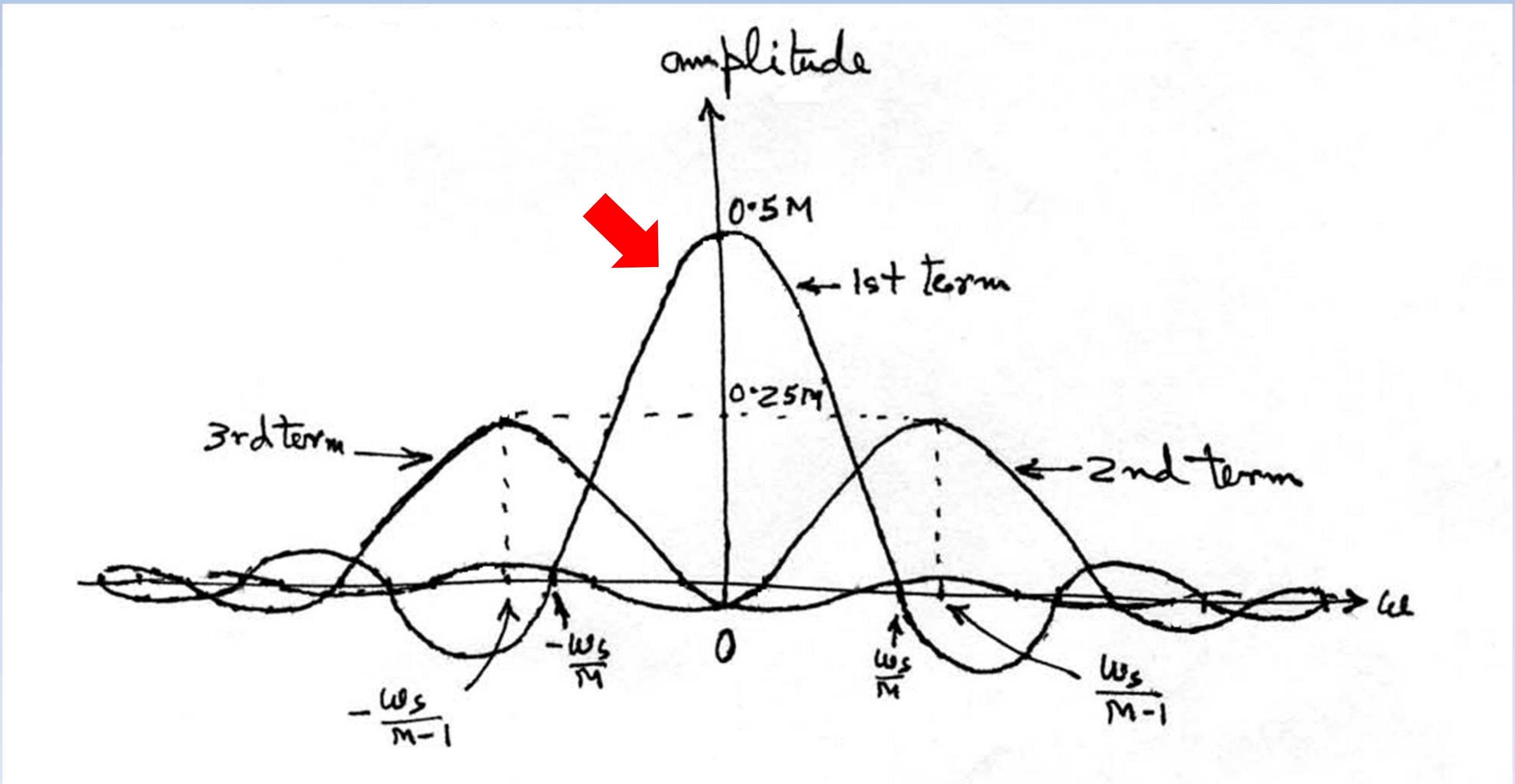
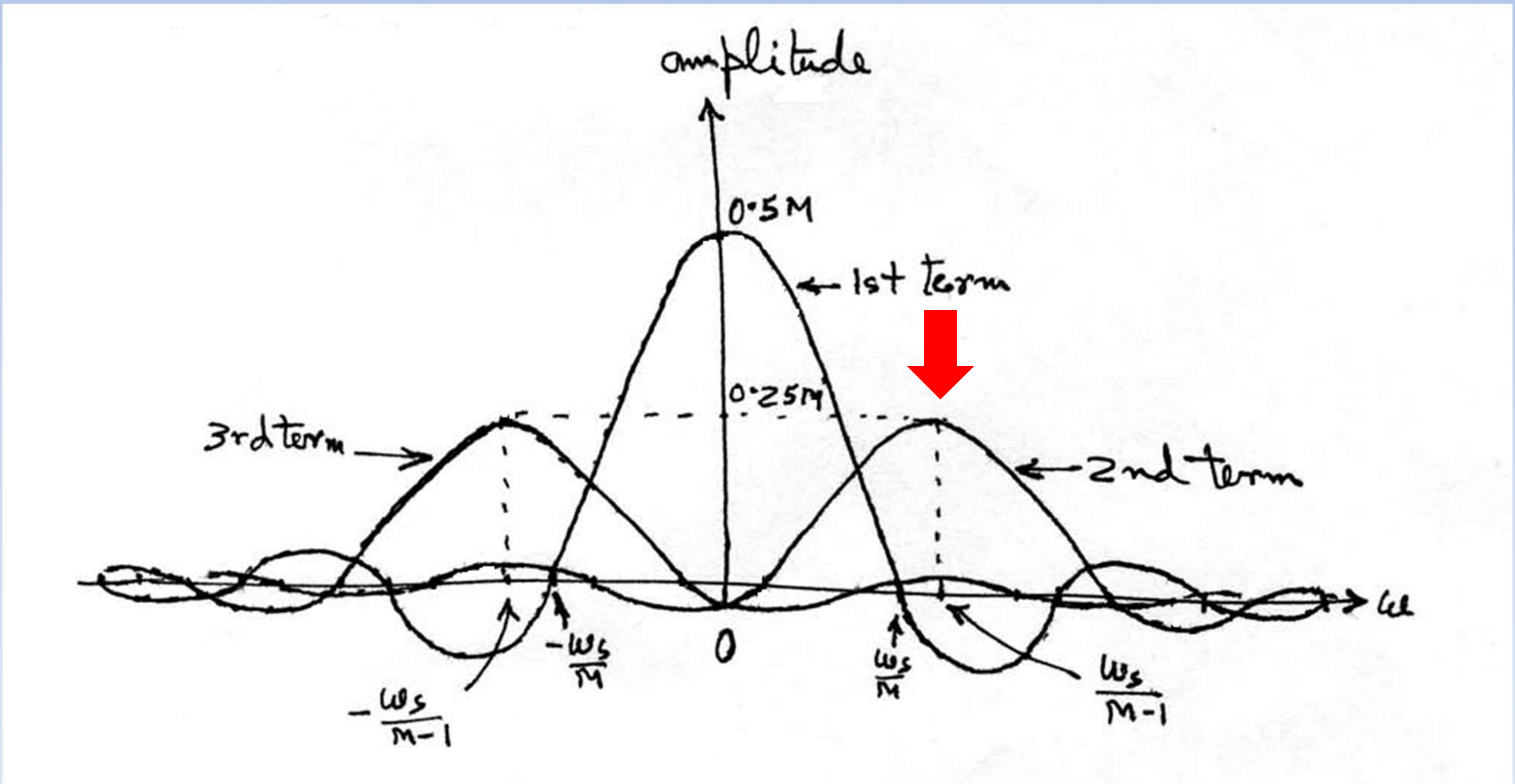
$$+ 0.25[\sin(\omega + (\omega_s/(M-1))\tau M/2)/\sin(\omega\tau + (\omega_s/(M-1))\tau/2)]$$



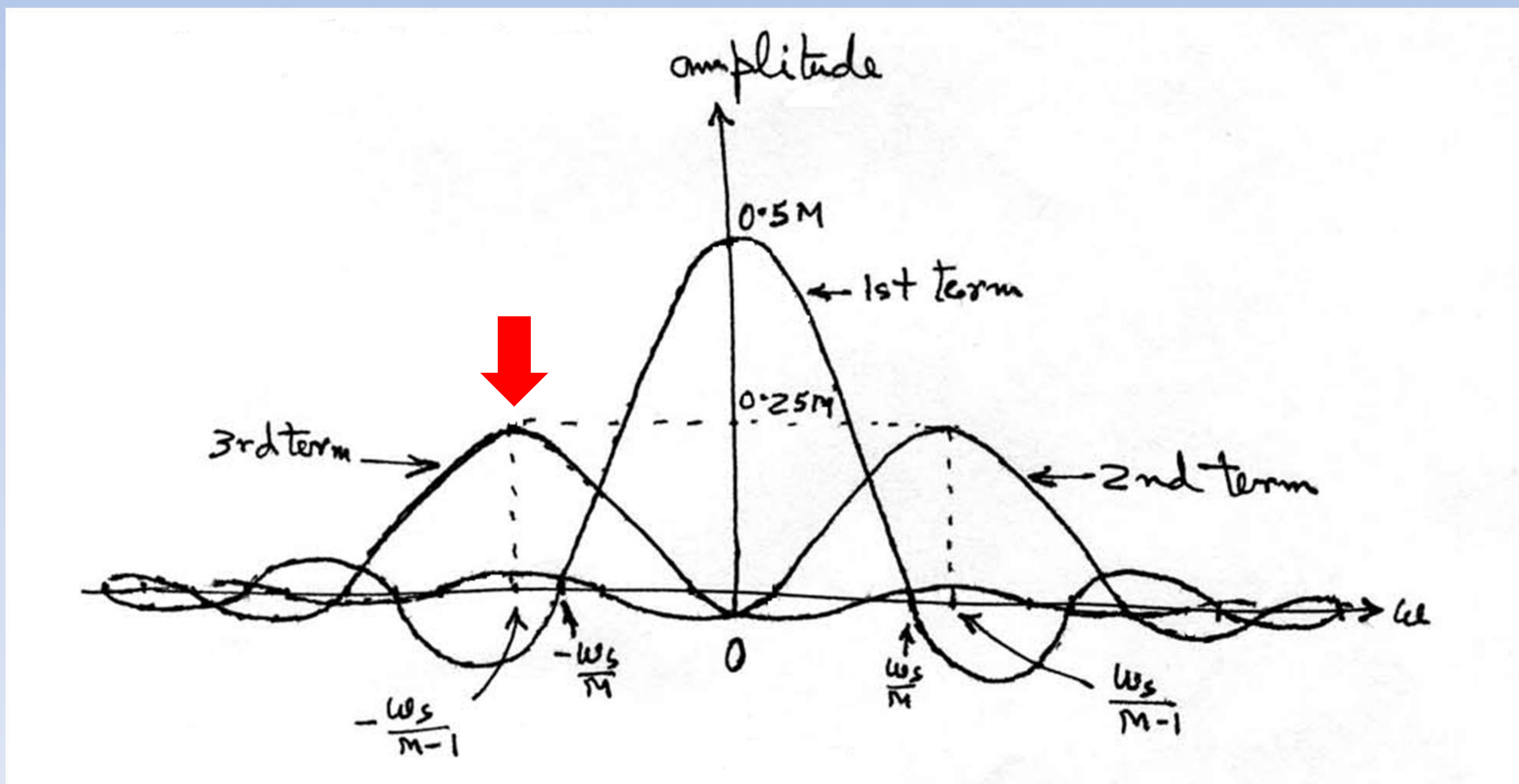
Frequency response of Hann window

$$W(\omega) = 0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]$$
$$+ \mathbf{0.25[\sin(\omega - (\omega_s/(M\text{-}1))\tau M/2)/\sin(\omega - (\omega_s/(M\text{-}1))\tau/2)]}$$
$$+ 0.25[\sin(\omega + (\omega_s/(M\text{-}1))\tau M/2)/\sin(\omega\tau + (\omega_s/(M\text{-}1))\tau/2)]$$

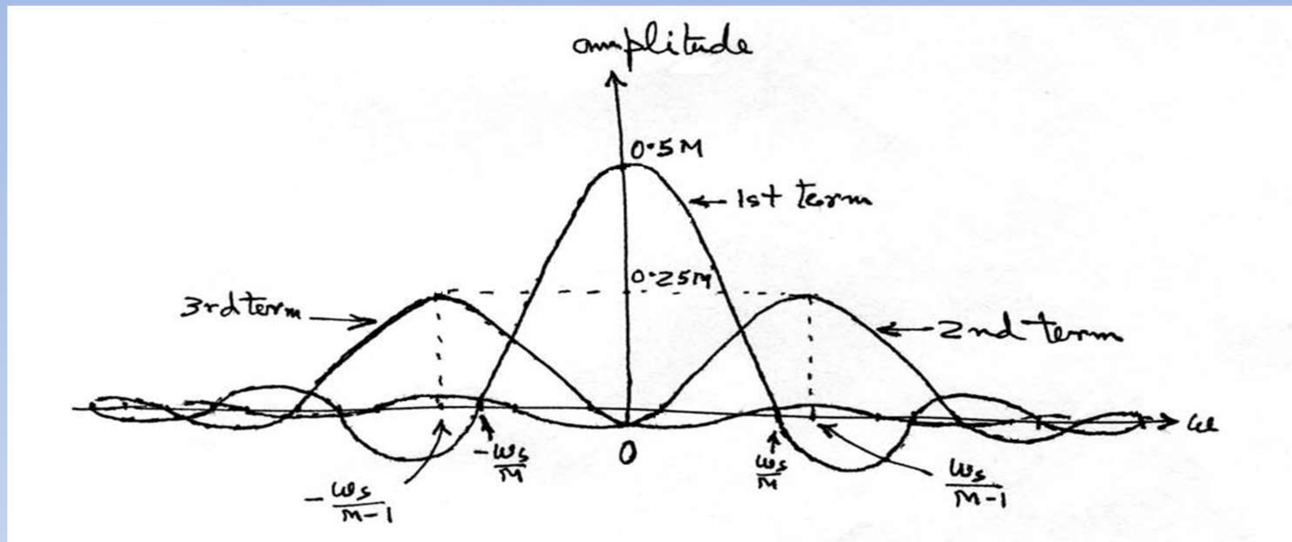


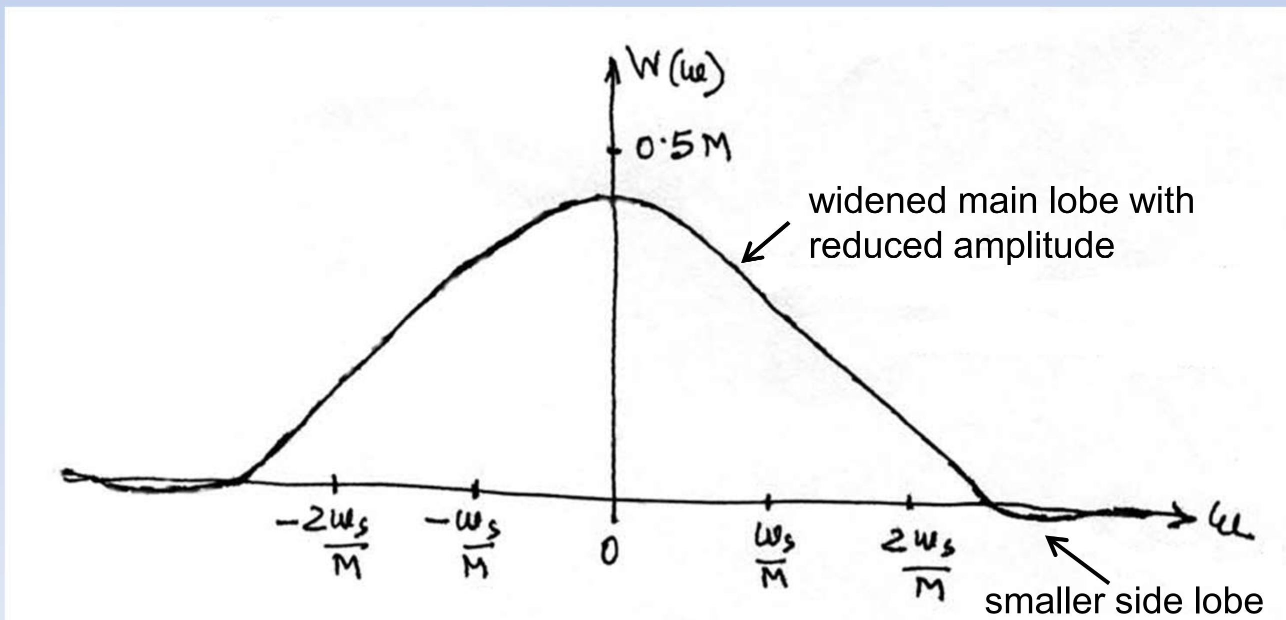
Frequency response of Hann window

$$W(\omega) = 0.5[\sin(\omega\tau M/2)/\sin(\omega\tau/2)]$$
$$+ 0.25[\sin(\omega - (\omega_s/(M-1))\tau M/2)/\sin(\omega - (\omega_s/(M-1))\tau/2)]$$
$$\boldsymbol{+ 0.25[\sin(\omega + (\omega_s/(M-1))\tau M/2)/\sin(\omega\tau + (\omega_s/(M-1))\tau/2)]}$$



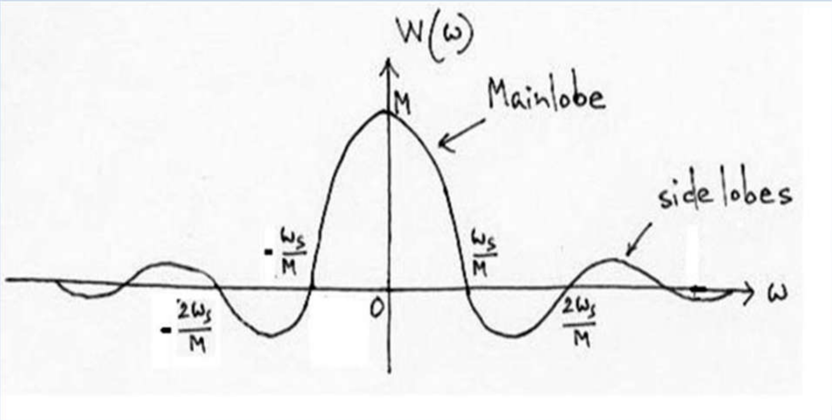
Frequency response of Hann window





Frequency domain characteristic of common window functions

Type of window	Approximate width of main lobe
Rectangular	$2\omega_s/M$
Bartlett	$4\omega_s/M$
Hamming	$4\omega_s/M$
Hann	$4\omega_s/M$
Blackman	$6\omega_s/M$



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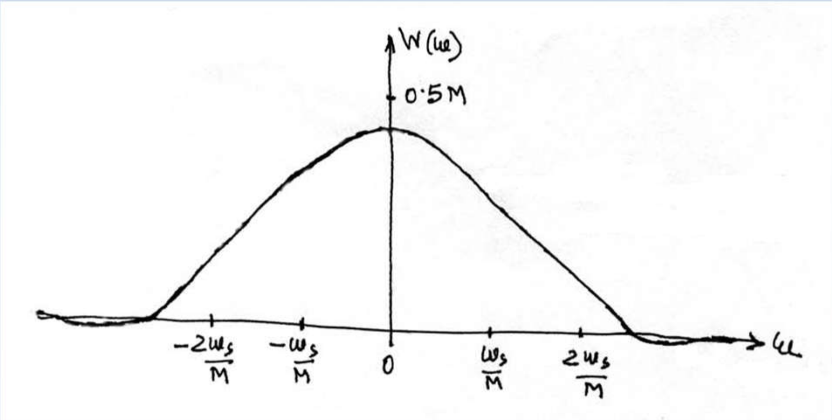
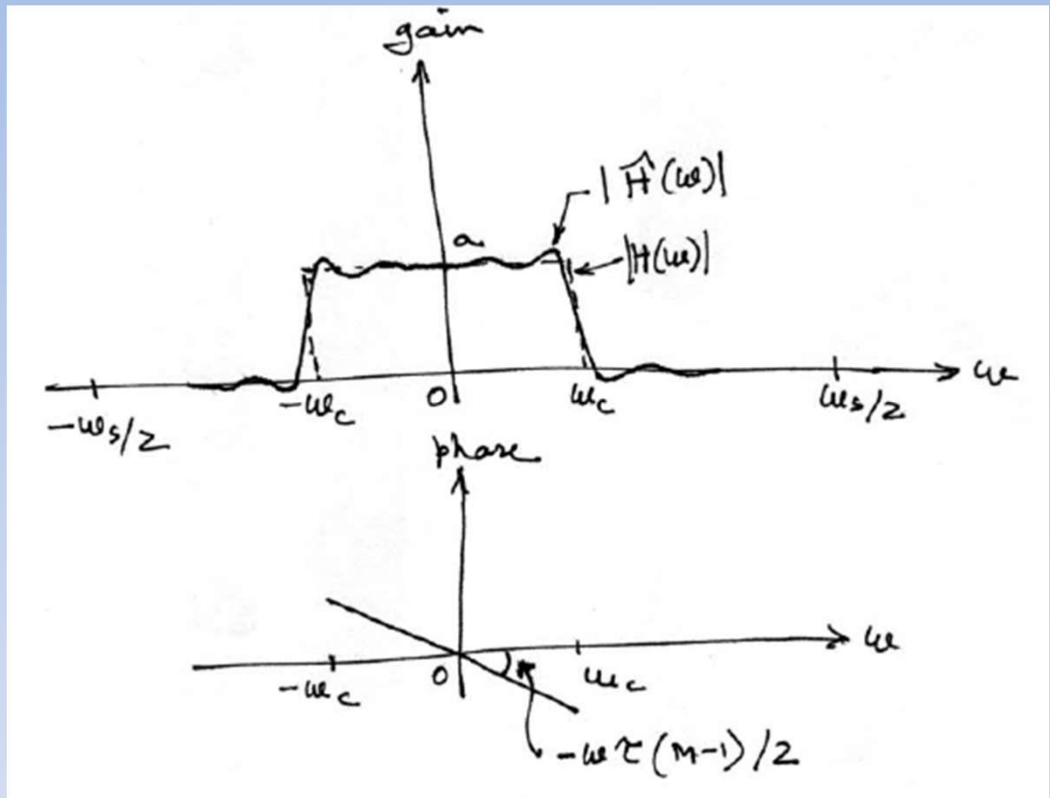


Design of brick-wall type low-pass FIR digital filter



Pass-band gain: *a*
Cut-off freq: ω_c

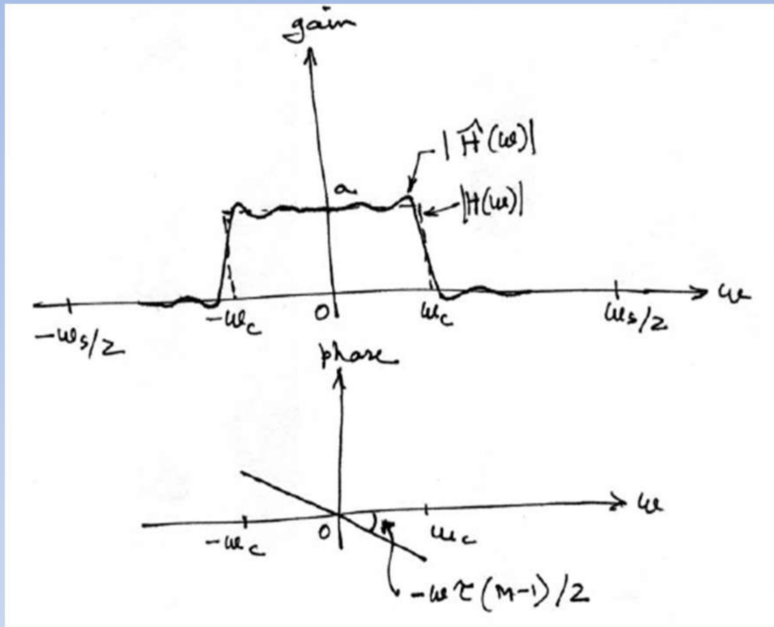
Linear-phase characteristic

$|H(\omega)|$ is the desired gain of the filter with infinite impulse sequence

$|\hat{H}(\omega)|$ is the gain of the filter with finite impulse sequence of length M

Filter coefficients = finite impulse sequence = ?

Design of brick-wall type low-pass FIR digital filter



Here,

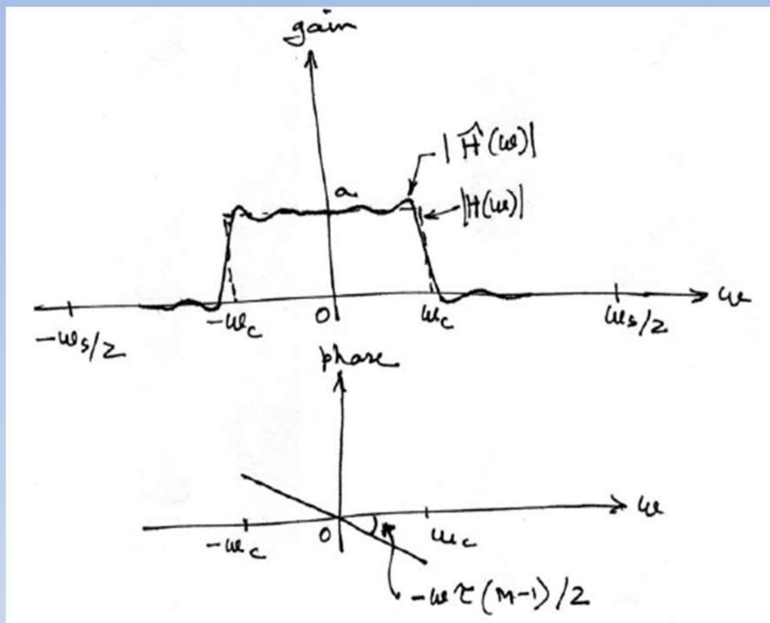
$$H(\omega) = a e^{-j\omega\tau(M-1)/2} \quad \text{for } |\omega| \le \omega_c$$

$= 0$, otherwise

considering a frequency range of - $\omega_s/2$ to $\omega_s/2$.

From relation (17) filter coefficients h_l, $l = 0,1,2,\ldots,(M-1)$ may be estimated as:

Design of brick-wall type low-pass FIR digital filter



Filter coefficients:

$$h_l = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\omega) e^{jl\tau\omega} d\omega$$

Substituting the value of *H(ω)*,

$$h_l = \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} a e^{-j\omega\tau(M-1)/2} e^{jl\tau\omega} d\omega$$

$$= \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} a e^{-j\omega\tau\left((M-1)/2-l\right)} d\omega$$

or $$h_l = \frac{a}{\omega_s}\left[\frac{e^{-j\omega\tau\left((M-1)/2-l\right)}}{-j\tau\left((M-1)/2-l\right)}\right]_{-\omega_c}^{\omega_c} = \frac{a}{\omega_s}\left[\frac{e^{-j\omega_c\tau\left((M-1)/2-l\right)} - e^{j\omega_c\tau\left((M-1)/2-l\right)}}{-j\tau\left((M-1)/2-l\right)}\right]$$

Design of brick-wall type low-pass FIR digital filter

$$h_l = \frac{a}{\omega_s}\left[\frac{e^{-j\omega_c\tau((M-1)/2-l)} - e^{j\omega_c\tau((M-1)/2-l)}}{-j\tau((M-1)/2-l)}\right]$$

$$= \frac{2a}{\omega_s}\left[\frac{e^{j\omega_c\tau((M-1)/2-l)} - e^{-j\omega_c\tau((M-1)/2-l)}}{2j\tau((M-1)/2-l)}\right]$$

$$= \frac{2a}{\omega_s\tau((M-1)/2-l)}\sin[\omega_c\tau((M-1)/2-l)]$$

$$h_l = \frac{2a\omega_c}{\omega_s}\left[\frac{\sin[\omega_c\tau((M-1)/2-l)]}{\omega_c\tau((M-1)/2-l)}\right]$$

, for $l = 0,1,2,\ldots,(M-1)$

….(35)

Design of brick-wall type low-pass FIR digital filter

$$h_l = \frac{2a\omega_c}{\omega_s}\left[\frac{\sin\left[\omega_c\tau\left((M-1)/2-l\right)\right]}{\omega_c\tau\left((M-1)/2-l\right)}\right]$$, for $l = 0,1,2,\ldots,(M-1)$

….(35)

Now, for $l = (M-1)/2$, the central coefficient $h_{(M-1)/2}$ may be estimated, using the limit theorem, as:

$$h_{(M-1)/2} = \frac{2a\omega_c}{\omega_s}$$ ….(36)

When window functions are employed to reduce Gibbs oscillations, the modified filter coefficients may be expressed as:

$$h_l^{/} = h_l \cdot w_l$$, for $l = 0,1,2,\ldots,(M-1)$ ….(37)

where w_l is the causal window sequence.

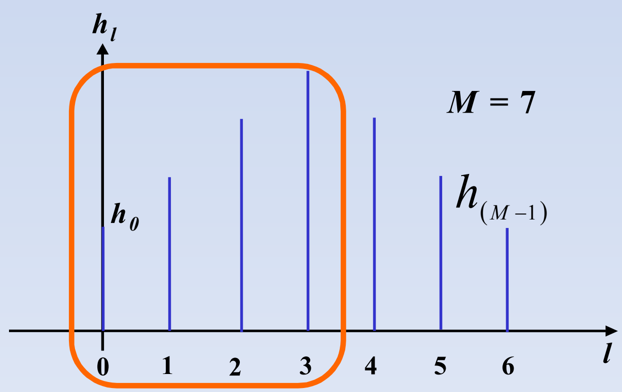
The filter input-output relation for a windowed filter may be expressed as:

$$y_k^{/} = \sum_{l=0}^{M-1} h_l^{/} x_{k-l}$$ ….(38)

Sample problem

Find filter coefficients of a 7-tap causal linear-phase FIR brick-wall type low-pass filter having a pass band gain of unity and a cut off frequency of 100 Hz, with a sampling frequency of 1 kHz. Apply Hann window for smoothing filter coefficients. Realize the filter.

Hints: only ((M-1)/2+1) i.e. 4 of h_l *need be calculated because of the symmetry property of* h_l.

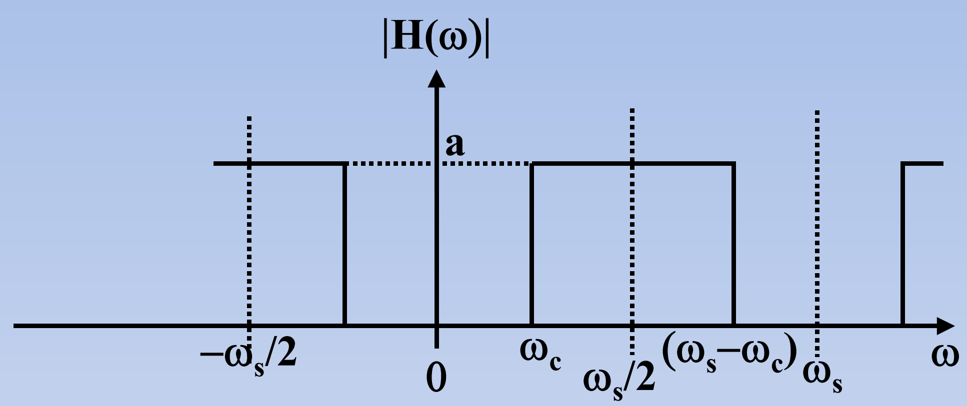


Sample problem

Find filter coefficients of a 7-tap causal linear-phase FIR brick-wall type low-pass filter having a pass band gain of unity and a cut off frequency of 100 Hz, with a sampling frequency of 1 kHz. Apply Hann window for smoothing filter coefficients. Realize the filter.

Reference: *J. R. Johnson,* Introduction to Digital Signal Processing

Design of brick-wall type high-pass FIR digital filter



For $0 \leq \omega \leq \omega_s$,

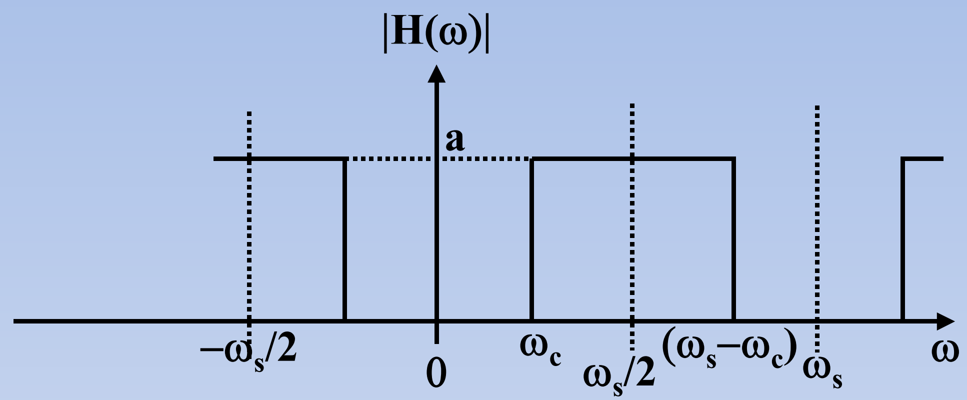
$$H(\omega) = a e^{-j\omega\tau(M-1)/2} \quad \text{, for } \omega_c \leq \omega \leq (\omega_s - \omega_c)$$

$= 0$, otherwise

Then from relation (16),

$$h_l = \frac{1}{\omega_s} \int_0^{\omega_s} H(\omega) e^{jl\tau\omega} d\omega = \frac{1}{\omega_s} \int_{\omega_c}^{\omega_s - \omega_c} a e^{-j\omega\tau(M-1)/2} . e^{jl\tau\omega} d\omega$$

Design of brick-wall type high-pass FIR digital filter



For $0 \le \omega \le \omega_s$,

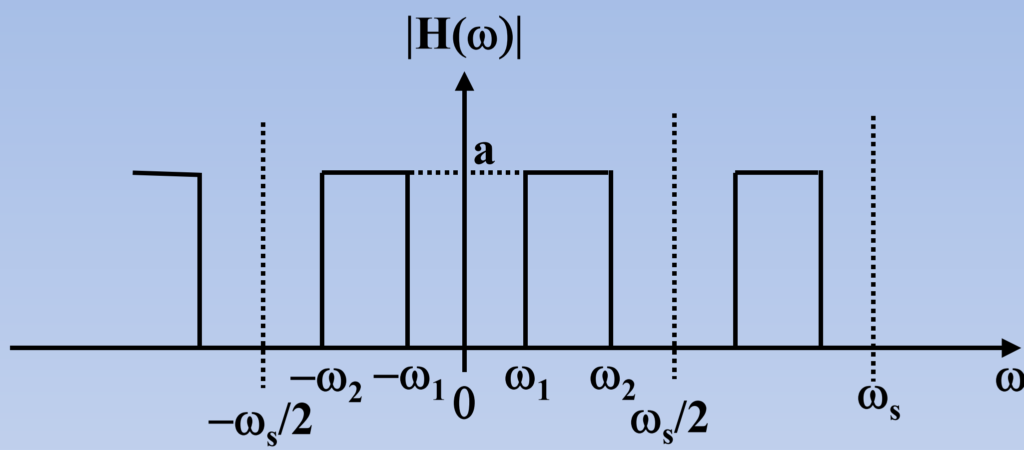
$$H(\omega) = a e^{-j\omega\tau(M-1)/2} \quad \text{, for } \omega_c \le \omega \le (\omega_s - \omega_c)$$

= 0, otherwise

Then from relation (16),

$$h_l = \frac{1}{\omega_s}\int_0^{\omega_s} H(\omega) e^{jl\tau\omega} d\omega = \frac{1}{\omega_s}\int_{\omega_c}^{\omega_s-\omega_c} a e^{-j\omega\tau((M-1)/2-l)} d\omega$$

Design of brick-wall type band-pass FIR digital filter



For $-\omega_s/2 \le \omega \le \omega_s/2,$

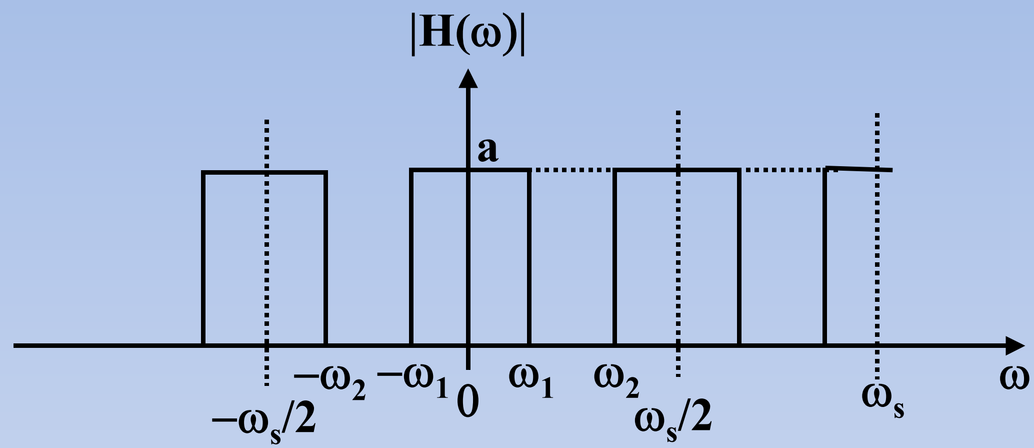
$$H(\omega) = ae^{-j\omega\tau(M-1)/2}$$, for $\omega_1 \le |\omega| \le \omega_2$

= 0, otherwise

Then from relation (17),

$$h_l = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\omega) e^{jl\tau\omega} d\omega$$

Design of brick-wall type band-stop FIR digital filter



For $-\omega_s/2 \le \omega \le \omega_s/2$,

$$H(\omega) = ae^{-j\omega\tau(M-1)/2} \quad \text{, for } |\omega| \le \omega_1$$

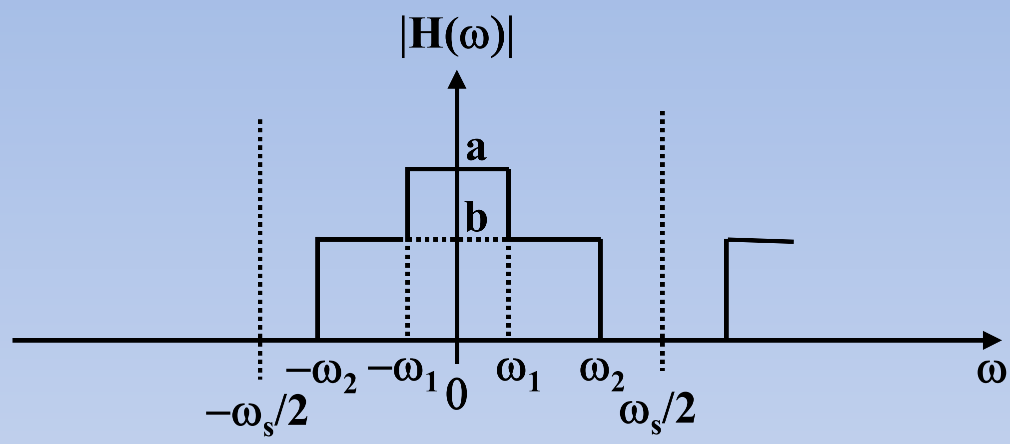
$$= ae^{-j\omega\tau(M-1)/2} \quad \text{, for } \omega_2 \le |\omega| \le \omega_s/2$$

= 0, otherwise

Then from relation (17),

$$h_l = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\omega) e^{jl\tau\omega} d\omega$$

Design of FIR digital filter with stepped characteristic



For $-\omega_s/2 \le \omega \le \omega_s/2$,

$$H(\omega) = a e^{-j\omega\tau(M-1)/2} \quad \text{, for } |\omega| \le \omega_1$$

$$= b e^{-j\omega\tau(M-1)/2} \quad \text{, for } \omega_1 < |\omega| \le \omega_2$$

$= 0$, otherwise

Then from relation (17),

$$h_l = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(\omega) e^{jl\tau\omega} d\omega$$

Thank You