Notes on

Electric Fields

by

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Fundamentals of Electric Field

Introduction

To quote legendary physicist Richard Feynman “Observation, reason, and experiment make up what we call the scientific method”. This is precisely true for electric field theory. Students often ask me that how you can discuss so many things about electric field when nobody has seen electric field. Yes, it is true that electric field cannot be directly seen. But observations on physical occurrences and related reasoning give theoretical understanding and when such understanding is validated by experimentation, then it is established that electric field does exist.

Electric Charge

Even in 17^{th} century it was known that matters exert force on each other, which varies inversely as the square of the distance between them. It is the so called long range interaction, commonly known as gravitation. But what is the nature of interaction between two things when the distance between them is very small. Is it gravitation? The answer is no. Gravitation is very weak at that dimension. Then it must be some other force. Observation shows that it is analogous to gravitational force in the sense that it also varies inversely as the square of the distance between the two things. But it is also observed that there is a big difference between gravity and this short range force. In gravitation any one matter attracts another matter. But this is not the case in short range force. Here, dissimilar things attract and similar things repel each other. In other words, there are two different types of things involved in short range interactions. This thing, which causes the short range interaction, has been named as “electric charge”. It has been found that electric charge is a basic property of matter carried by elementary particles. Typically, electric charges are of two types: positive and negative charges. When the atomic structure was properly understood, it was found that the positive charges (primarily protons) are located at the centre of the atom, i.e. nucleus, and the negative charges (electrons) revolve around the nucleus.

Experimental evidence shows that all electrons have the same amount of negative charge, which is also equal to the amount of positive charge of each proton. Consequently, it follows that charge exists in quantized unit equal to the charge of an electron or a proton (e), which is a fundamental physical constant. Thus electric charge of anything comes in integer multiples of the elementary charge, e, except for particles called quarks, which have charges that are integer multiples of $e/3$. The unit of electric charge in the SI system is coulomb (C). One coulomb consists of $6.241509324\times 10^{18}$ natural units of electric charge, such as charge of individual electrons or protons. Conversely, one electron has a negative charge of $1.60217657 \times 10^{-19}$ coulomb and one proton has a positive charge of $1.60217657 \times 10^{-19}$ coulomb. Other particles (e.g. positrons) also carry charge in multiples of the electronic charge magnitude. However, those are not going to be discussed for the sake of simplicity.

Electric charge is also conserved, i.e. in any isolated system or in any chemical or nuclear reaction; the net electric charge is constant. The algebraic sum of the elementary charges remains the same. In physical terms, it implies that if a given amount of negative charge appears in one part of an isolated system then it is always accompanied by the appearance of an equal amount of positive charge in another part of the system. In modern atomic theory, it has been proved that although fundamental particles of matter continually and spontaneously appear, disappear, and change into one another, they always obey the constraint that the net quantity of charge is preserved.

Electric Field Lines

From logical reasoning it can be stated that if a charge is present in space, then another charge will experience a force when brought into that space. In other words, the effect of the first charge, which may also be called the source charge, extends into the space around it. This is known as the electric field which is caused by the source charge. If there is no charge in space, then there will be no electric field. If the source charge is of positive polarity and the test charge is of negative polarity, then the test charge will experience an attractive force. On the other hand, if the test charge is also of positive polarity then it will experience a repulsive force. If the test charge is free to move, then it will move in accordance with the direction of the force. The loci of the movement of the test charge within the electric field are known as electric lines of force or electric field lines.

Behaviour of an electric field is conventionally analysed considering the test charge to be a unit positive charge. Hence, the test charge will experience repulsive force from a positive source charge and attractive force from a negative source charge. So the test charge will move away from the positive source charge and move towards the negative source charge. Accordingly, the directions of electric field lines are such that they originate from a positive charge and terminate on a negative charge, as shown in Fig.1.1. Fig. 1.1(a) shows the electric field lines originating from a positive source charge whose magnitude is integer (N) multiple of $+e$, while Fig.1.1(b) shows the electric field lines terminating on a negative source charge of same magnitude. The electric field lines are the directions of the force experienced by a unit positive charge $+e$ as shown in Fig.1.1.

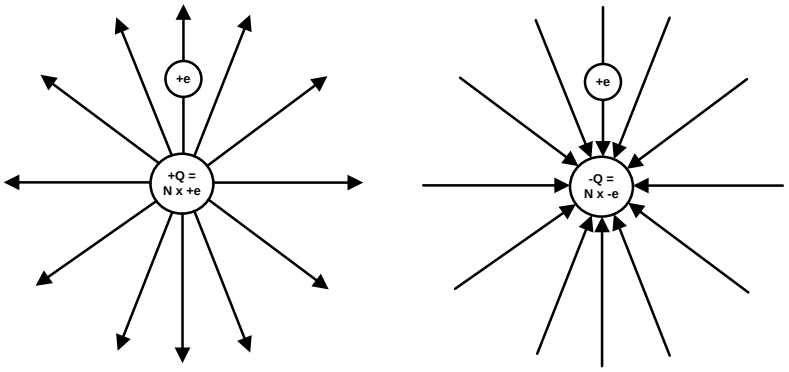


Fig.1.1 Electric field lines due to (a) positive source charge and (b) negative source charge

Fig.1.2 depicts the electric field lines due to a pair of positive and negative charges showing the field lines to be originating from the positive charge and terminating on the negative charge.

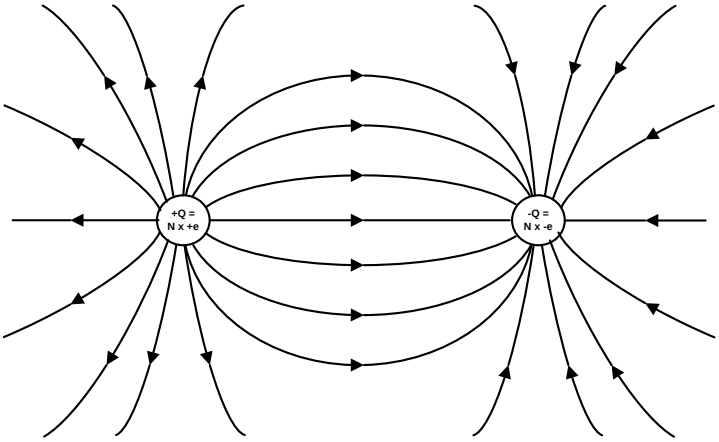


Fig.1.2 Electric field lines due to a pair of positive and negative charges

According to SI unit system, one coulomb of source charge gives rise to one coulomb of electric field lines.

Coulomb's Law

It is the law that describes the electrostatic interaction between electrically charged particles. It was published by the French physicist Charles Augustin de Coulomb in 1785. He determined the magnitude of the electric force between two point charges using a torsion balance to study the attraction and repulsion forces of charged particles. The interaction between charged particles is a non-contact force that acts over some distance of separation. There are always two charges and a separation distance between them as the three critical variables that influence the strength of the electrostatic interaction. The unit of the electrostatic force, like all forces, is Newton. Being a force, the strength of the electrostatic interaction is a vector quantity that has both magnitude and direction.

According to the statement of Coulomb's law (i) the magnitude of the electrostatic force of interaction between two point charges is directly proportional to the scalar multiplication of the magnitudes of point charges and inversely proportional to the square of the separation distance between the point charges and (ii) the electrostatic force of interaction acts along the straight line joining the two point charges. If the two point charges are of same polarity, the electrostatic force between them is repulsive; if they are of opposite polarity, the force between them is attractive.

It should be noted here that two conditions are to be fulfilled for the validity of Coulomb's law: (i) the charges involved must be point charges and (ii) the charges should be stationary with respect to each other.

Mathematically, the force between two point charges, as shown in Fig.1.3, could be written as follows:

$$\vec{F}_{21} = \frac{\pm Q_2 . Q_1}{4\pi\varepsilon_0 \left|\vec{r}_{21}\right|^2} \hat{u}_{r21} \quad \text{and} \quad \vec{F}_{12} = \frac{Q_1 . \pm Q_2}{4\pi\varepsilon_0 \left|\vec{r}_{12}\right|^2} \hat{u}_{r12} \quad \text{so that} \quad \vec{F}_{21} = -\vec{F}_{12} \qquad 1.1$$

where, ε_0 is permittivity of free space ($\approx 8.854187\times10^{-12}$ F/m) and

the unit vectors are given by $\hat{u}_{r21} = \frac{\vec{r}_{21}}{\left|\vec{r}_{21}\right|}$ and $\hat{u}_{r12} = \frac{\vec{r}_{12}}{\left|\vec{r}_{12}\right|}$.

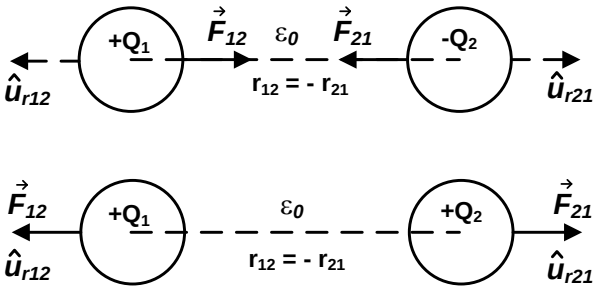


Fig.1.3 Electrostatic forces of interaction between two point charges

The distance vectors are given as $\vec{r}_{21}=\vec{r}_2-\vec{r}_1$ and $\vec{r}_{12}=\vec{r}_1-\vec{r}_2$, where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the location of the point charges Q_1 and Q_2, respectively, wrt to a defined origin.

When the scalar product of Q_1 and $\pm Q_2$ is positive the force is repulsive and when the product is negative the force is attractive.

Coulomb's Constant

The constant of proportionality k that appears in Coulomb's law is often called Coulomb's constant. In SI unit system

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987552617\times10^9 \frac{N.m^2}{C^2} \approx 9\times10^9 \frac{N.m^2}{C^2} \qquad 1.2$$

The product of Coulomb's constant and the square of the electron charge ($k.e^2$) is often convenient in describing the electric forces in atoms and nuclei, since that product appears in electric potential energy and electric force expressions.

Comparison between Electrostatic and Gravitational Forces

The expression for electrostatic force as obtained from Coulomb's law bears a strong resemblance to the expression for gravitational force given by Newton's law for universal gravitation.

$$\left|\vec{F}_{electrostatic}\right| = k.\frac{Q_1 Q_2}{r^2} \qquad \text{and} \qquad \left|\vec{F}_{gravitational}\right| = G.\frac{m_1 m_2}{r^2} \qquad 1.3$$

where, $k \approx 9\times10^9$ N.m^2/C^2 and $G \approx 6.67\times10^{-11}$ N.m^2/kg^2

Both the expressions show that the force is (i) inversely proportional to the square of the separation distance and (ii) directly proportional to the scalar product of the quantity that causes the force, i.e. electric charge in the case of electrostatic force and mass in the case of gravitational force. But, there are major differences between these two forces. Firstly, gravitational forces are only attractive, while electrical forces can be either attractive or repulsive. Secondly, a comparison of the proportionality constants reveals that the Coulomb's constant (k) is significantly greater than Newton's universal gravitational constant (G). Consequently, the electrostatic force between two electric charges of unit magnitude is significantly higher than the gravitational force between two masses of unit magnitude.

From eqn.(1.3) it may be seen that the electrostatic force between two electric charges of magnitude 1C separated by a distance of 1m will be a colossal 9×10^9 N ! On the other hand, the gravitational force between two masses of magnitude 1kg separated by a distance of 1m will be a meagre 6.67×10^{-11} N ! These values clearly show the enormous difference between magnitudes of electrostatic and gravitational forces.

The comparison can also be made between the electrostatic and gravitation forces between two electrons separated by a given distance. Considering the charge of an electron as e ($1.60217657 \times 10^{-19}$ C) and the mass of the electron as m_e ($9.10938291 \times 10^{-31}$ kg), the ratio of electrostatic to gravitational forces between two electrons is given by

$$\frac{\left|\vec{F}_{electrostatic}\right|}{\left|\vec{F}_{gravitational}\right|} = \left(\frac{e}{m_e}\right)^2 \frac{k}{G} \approx 4.174 \times 10^{42} \qquad 1.4$$

Equation 1.4 shows how strong electrostatic forces are compared to gravitational forces! If this reasoning is applied to the motion of particles in the universe, one may expect the universe to be governed entirely by electrostatic forces. However, this is not the case. The electrostatic force is enormously stronger than the gravitational force, but is usually hidden inside neutral atoms. On astronomical length-scales gravity is the dominant force, and electrostatic forces are not relevant. The key to understanding this paradox is that electric charges could be of either positive or negative polarity, whereas the masses that cause gravitational forces are only positive as there is nothing called negative mass. This means that gravitational forces are always cumulative, whereas electrical forces can cancel each other. For the sake of argument, consider that the universe starts out with randomly distributed electric charges. Initially, electrostatic forces are expected to completely dominate gravity. Because of the dominant electrostatic forces every positive charge tries to get as far away as possible from the other positive charges, and to get as close as possible to the other negative charges. After some time, the positive and negative charges come near enough ($\approx 10^{-10}$m) to form close pairs. Exactly how close the charges would come is determined by quantum mechanics. The electrostatic forces due to the charges in each pair effectively cancel one another out on length-scales that are much larger than the mutual spacing of the charge pair. If the number of positive charges in the universe is almost equal to the number of negative charges, then only the gravity becomes the dominant long-range force. For effective cancellation of long-range electrostatic forces, the relative difference in the number of positive and negative charges in the universe must be extremely small. In fact, cancellation of the effect of positive and negative charges has to be of such accuracy that most physicists believe that the net charge of the universe is exactly zero. In other words, electric charge is a conserved quantity, i.e. the net charge of the universe can neither increase nor decrease. As of today, no elementary particle reaction has been discovered that creates or destroys electric charge.

Effect of Departure from Electrical Neutrality

The fine balance of electrostatic forces due to positive and negative electric charges starts to break down on atomic scales. In fact, interatomic and intermolecular forces are all electrical in nature. But, this is electric field on the atomic scale, usually termed as quantum electromagnetism. This book is about classical electromagnetism, which is electromagnetism on length-scales much larger than the atomic scale. Classical electromagnetism generally describes phenomena in which some sort of disturbance is caused to matter, so that the close pairing of positive and negative charges is disrupted. Such disruption allows electrical forces to manifest themselves on macroscopic length-scales. Of course, very little disruption is necessary before gigantic forces are generated, which may be explained with the help of the following example.

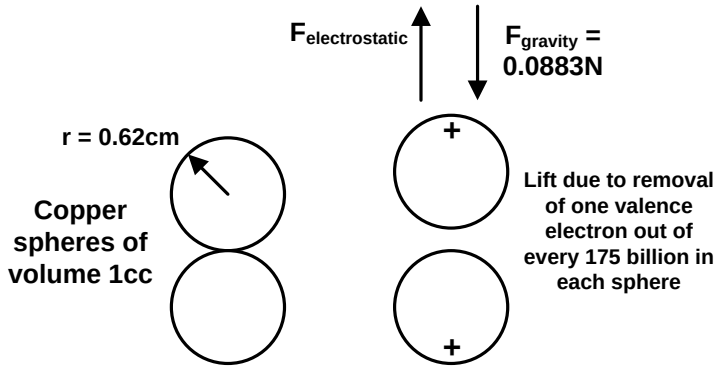


Fig.1.4 Lift of a copper sphere due to electrostatic force against gravity

Fig.1.4 shows two copper spheres of volume 1 cubic centimetre (cc) lying on one another. Copper is a good electrical conductor and has one valence electron in the outermost shell of its atom, and that electron is fairly free to move about in the volume of solid copper material. The density of metallic copper is approximately 9 g/cc and one mole of copper is 63.55g. Thus one cc of copper contains approximately 8.5×10^{22} ($\frac{9}{63.55}\times6.022\times10^{23}$) copper atoms. With one valence electron per atom, and with the electron charge of 1.6×10^{-19} Coulombs, there are about 13,600 Coulombs of potentially mobile charge within a volume of one cc of copper. How much electron charge needs to be removed from two spheres of copper so that there is enough net positive charge on them to suspend the top sphere over the bottom? The force required to lift the top sphere of copper against gravity would be its weight, i.e. 0.0883 $(9\times10^{-3}\times9.807)$N. It is fair to assume that the net charge resides at the points of the spheres most distant from each other because of the charge repulsion. The radius of a sphere of volume one cc is 0.62 cm. So the repulsive force to be considered should be that between two point charges 2.48 cm apart, i.e. twice the sphere diameter apart. From Coulomb's law

$$0.0883 = \frac{1}{4\pi\varepsilon_0}\times\frac{Q^2}{0.0248^2} \quad or, \quad Q \approx 7.75\times10^{-8}\,C \tag{1.5}$$

Compared to the total valence charge of approximately 13,600 C, this 7.75×10^{-8}C amounts to removing just one valence electron out of every 175 billion copper atoms from each sphere. In summary, the removal of just one out of every 175 billion free electrons from each copper sphere would cause enough electrostatic repulsive force on the top sphere to lift it, overcoming the gravitational pull of the entire Earth!

Force due to a System of Discrete Charges

Consider *N* charges, Q_1 through Q_N, which are located at position vectors $\vec{r}_1$ through $\vec{r}_N$, as shown in Fig.1.5. Since the electrostatic forces obey the principle of superposition, the electrostatic force acting on a test charge *q* at position vector $\vec{r}$ is simply the vector sum of all of the forces from each of the *N* charges taken in isolation. Thus, the total force acting on the test charge *q* is given by

$$\vec{F}(r) = q\sum_{i=1}^{N}\frac{q_i}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}_i}{\left|\vec{r}-\vec{r}_i\right|^3} \tag{1.6}$$

where, the distance vector $\vec{R}_i = \vec{r}-\vec{r}_i$ is directed from the *i*th charge Q_i to the test charge *q*.

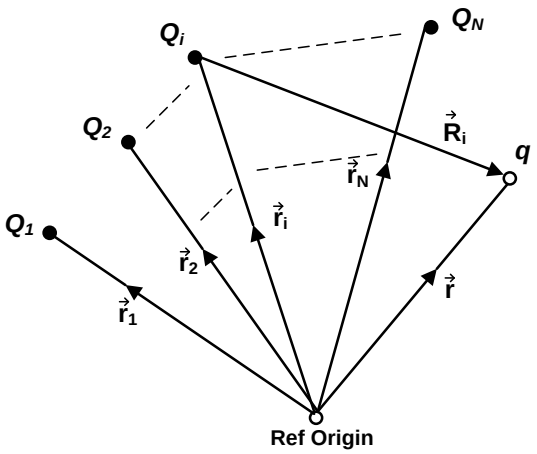


Fig.1.5 Force due to system of discrete charges

Force due to Continuous Charge Distribution

Instead of having discrete charges, consider a continuous distribution of charge represented by a charge density, which could be linear, surface or volume charge density, depending on the distribution of charge. For a continuous charge distribution, an integral over the entire region containing the charge is equivalent to a summation for infinite number of discrete charges, where each infinitesimal element of space is treated as a discrete point charge *dq*.

For linear charge distribution, e.g. charge in a wire, considering linear charge density as $\lambda(r')$ and the infinitesimal line element dl' at the position r',

$$dq = \lambda(r')\, dl'$$

For surface charge distribution, e.g. charge on a plate or disc, considering surface charge density as $\sigma(r')$ and the infinitesimal area element dA' at the position r',

$$dq = \sigma(r')dA'$$

For volume charge distribution, e.g. charge in the volume of a bulk material, considering volume charge density as $\rho(r')$ and the infinitesimal volume element dV' at the position r',

$$dq = \rho(r')\, dV'$$

The force on a test charge *q* at position *r* in free space is given by the integral over the entire continuous distribution of charge as follows:

$$\vec{F}(r) = q \int \frac{dq}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|^3} \qquad 1.7$$

In eqn.(1.7) the integration is line, surface or volume integral according to the nature of charge distribution. The integral is over all space, or, at least, over all space for which the charge density is non-zero.

Electric Field Intensity

At this juncture, it is useful to define a vector field $\vec{E}(r)$, called the electric field intensity, which is the force exerted on a unit test charge of positive polarity located at position vector $\vec{r}$. Then, the force on a test charge could be written as

$$\vec{F}(r) = q\vec{E}(r) \qquad 1.8$$

The electric field intensity could be written from eqn.(1.6) as

$$\vec{E}(r) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_i}{\left|\vec{r} - \vec{r}_i\right|^3} \qquad 1.9$$

or, from eqn.1.7 as

$$\vec{E}(r) = \int \frac{dq}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3} \qquad 1.10$$

The electric field lines from a single charge *Q* located at a given position are purely radial and are directed outwards if the charge is positive or inwards if it is negative, as shown in Fig.1.1. So the electric field intensity at any point located at a radial distance *r* from the source charge *Q* will be the force experience by a unit test charge of positive polarity at that point and is given by

$$\vec{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{u}_r \qquad 1.11$$

The unit of electric field intensity as per the above definition is N/C. However, the practical unit of electric field intensity is a different one and will be discussed in a later section.

Electric Flux and Electric Flux Density

Consider the case of air coming in through a window. The amount of air that comes through the window depends upon the speed of the air, the direction of the air and the area of the window. The air that comes through the window may be called the "air flux".

Similarly, the amount of electric field lines that pass through an area is the electric flux through that area. Consider the case of a source point charge of positive polarity as shown in Fig.1.1(a). If the source charge magnitude is *Q* Coulombs, then the total amount of electric field lines coming out of the source charge will be also *Q* Coulombs. Now, if a fictitious sphere of radius *r* is considered such that the source charge is located at the center of the sphere, then the electric flux through the surface of the sphere will be *Q* Coulombs, as the surface of the sphere completely encloses the source charge and all the electric field lines coming out radially from the source point charge passes through the spherical surface. Electric flux is typically denoted by ψ.

Electric flux density is then defined as the electric flux per unit area normal to the direction of electric flux. In the case of a point charge the electric field lines are directed radially from the source charge and hence the electric field lines are always normal to the surface of the sphere. Hence, for a point source charge of magnitude *Q* Coulombs, the electric flux that passes through the spherical surface area of magnitude $4\pi r^2$ is *Q*. Then the electric flux density ($\vec{D}$) at a radial distance *r* from the point charge is given by

$$\vec{D}(r) = \frac{Q}{4\pi r^2} \hat{u}_r \qquad 1.12$$

From eqns. (1.11) and (1.12), it may be written that

$\vec{D}(r) = \varepsilon_0 \vec{E}(r)$, when the medium is free space

or, $\vec{D}(r) = \varepsilon_0 \varepsilon_r \vec{E}(r)$, for any particular medium of relative permittivity ε_r. 1.13

Electric flux density is a vector quantity because it has a direction along the electric field lines at the position where electric flux density is being computed.

Eqn.(1.13) is known as one of the basic equations of electric field theory.

However, it is not necessary that electric flux will always be normal to the area under consideration. In such cases, the component of the area that is normal to electric flux has to be taken for computing electric flux density. Fig.1.6 shows such a case, where an electric flux of magnitude ψ passes through an area of magnitude *A*, which is not normal to the direction of electric flux.

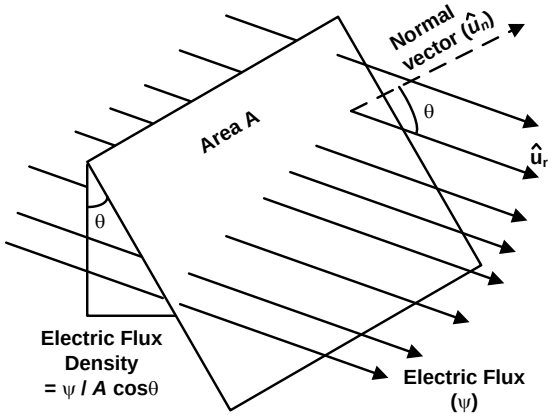


Fig.1.6 Pertaining to the area related to electric flux density

With reference to Fig.1.6, electric flux and electric flux density are related as

$$\psi = \left|\vec{D}\right| A\cos\theta \qquad 1.14$$

Again, as depicted in Fig.1.6, $\hat{u}_r . \hat{u}_n = \cos\theta$

So, from eqn.(1.14), $\psi = \left|\vec{D}\right| A\hat{u}_r . \hat{u}_n = \left|\vec{D}\right| \hat{u}_r . A\hat{u}_n = \vec{D}.\vec{A}$ 1.15

Eqn.(1.15) presents an important idea of introducing an area vector, which is a vector of magnitude equal to the scalar magnitude of the area under consideration but it has a direction normal to the area under consideration. In the case of closed surfaces, area vector is conventionally taken in the direction of the outward normal. For an open surface, any one normal direction can be taken as positive, while the opposite normal direction is to be taken as negative.

Electric Potential

Consider that a test charge of magnitude *q* is located at a given position within an electric field produced by a system of charges. The test charge will experience a force due to the source charges. If the test charge moves in the direction of the field forces, then the work is done by the field forces in moving the test charge from position-1 to position-2. In other words, energy is spent by the electric field. Hence, the potential energy of test charge at position-2 will be lower than that at position-1. On the other hand, if the charge is moved against the field forces by an external agent, then the work done by the external agent will be stored as potential energy of the test charge. Hence, the potential energy of the charge at position-2 will be higher than that at position-1. Here, it is to be noted that the force experienced by the test charge within an electric field is dependent on the magnitude of the test charge. Hence, the potential energy of the charge at any position is dependent on its magnitude and the distance by which it moves within the electric field.

The concept of electric potential is introduced to make it a property which is purely dependent on the location within an electric field and is independent of the test charge. In other words, it is a property of the electric field itself and not related to the test particle. Hence, electric potential (ϕ) at any point within an electric field is defined as potential energy

per unit charge at that point and hence it is a scalar quantity. The unit of electric potential is Volt (V), which is equivalent to Joules per Coulomb (J/C). It is interesting to note that electric field intensity is defined as force per unit charge and electric potential is defined as potential energy per unit charge.

However, it is also practically important to note that absolute values of electric potentials are not physically measurable, only difference in potential energy between two points within an electric field can be physically measured, i.e. only potential difference between two points within an electric field is measurable. The work done in moving a unit positive charge from one point to another within an electric field is equal to the difference in potential energies and hence difference in electric potentials at the two points.

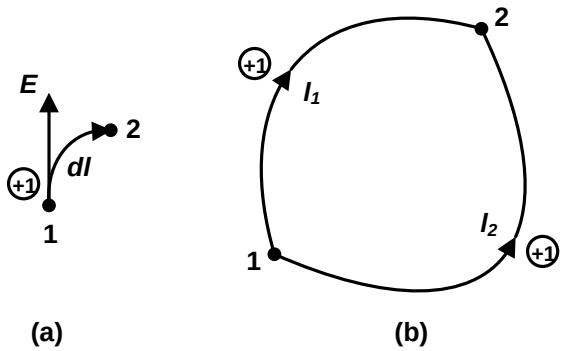


Fig.1.7 Pertaining to the definition of electric potential

As shown in Fig.1.7(a), consider that a unit positive charge is moved from point 1 to point 2 by a small distance *dl*. The force experience by the unit positive charge at point 1 is the electric field intensity $\vec{E}$. Then the potential difference between the ending point 2 and starting point 1 is given by

$$\phi_2 - \phi_1 = -\vec{E}.\vec{dl} \qquad 1.16$$

In eqn.(1.16) $\vec{E}.\vec{dl}$ is the work done by the field forces in moving a unit positive charge from point 1 to point 2. In this case the potential energy of point 2 will be lower than that of point 1 and hence the potential difference $(\phi_2 - \phi_1)$ will be negative. For this purpose, the minus sign is introduced on the right hand side of eqn.(1.16).

As shown in Fig.1.7(b), if the unit positive charge moves through a certain distance *l* within an electric field from point 1 to point 2, then the magnitude as well as direction of $\vec{E}$ may not be same at every location along the path traversed by the unit positive charge. Hence, in such a case, the potential difference between point 2 and point 1 is evaluated by integrating the RHS of eqn.(1.16) over the line *l* from point 1 to point 2 as given in eqn.(1.17).

$$\phi_2 - \phi_1 = -\int_1^2 \vec{E}.\vec{dl} \qquad 1.17$$

Eqn.(1.17) is known as the integral form of relationship between electric field intensity ($\vec{E}$) and electric potential (ϕ).

If the point 1 is chosen at an infinite distance wrt to the source charges causing the electric field, then the potential energy at point 1 due to the sources charges will be zero and hence the electric potential ϕ_1 will be zero. Then eqn.(1.17) can be rewritten as

$$\phi_2 = -\int_\infty^2 \vec{E}.\vec{dl} \qquad 1.18$$

Eqn.(1.18) shows that the electric potential at a point in an electric field can be defined as the work done in moving a unit positive charge from infinity to that point. Since work done is independent of the path traversed between the two end points, hence electric potential is a conservative field. It is important to note here that the reference point is arbitrary and is fixed as per nature and convenience of the problem. For most of the problems taking the reference point at infinity is a sound choice. However, for many others (e.g. long charged wire) a different choice may prove more useful. From eqn.(1.18) the practical unit of electric field intensity is obtained as Volt per unit length, e.g. V/m.

It is also evident from eqn.(1.18) that the work done in moving a unit positive charge from infinity to a given point within an electric field could be same for several points within that electric field depending upon the distribution of electric field intensity vectors within the field region. Hence, electric potential of all such points will be same. If all these points are joined together then one may get a line or a surface on which every point has the same electric potential. Such a line or surface is called an equipotential.

Fig.1.8 shows typical examples of equipotentials in two dimensional systems, where these will be lines. Fig.1.8(a) shows equipotential lines and electric field lines for an electric field for which $\vec{E}$ is constant everywhere, which is called uniform field. Fig.1.8(b) shows equipotentials and electric field lines for a positive polarity point charge. In this case, $\vec{E}$ varies with position and is called non-uniform field. In the case of three-dimensional systems, such equipotentials will be surfaces.

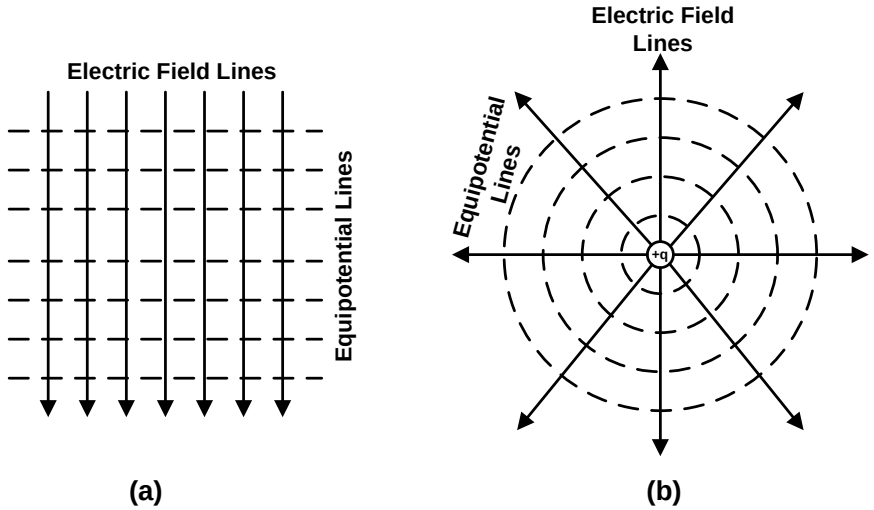


Fig.1.8 Examples of equipotentials: (a) Uniform field, (b) Non-uniform field

Equipotential vis-à-vis Electric Field Line

Consider an equipotential of any electric field as shown in Fig.1.9. At any point P on this equipotential consider that the electric field line makes an angle θ with the tangent to the equipotential at that point. If an elementary length *dl* is considered along the equipotential at the point P, then the potential difference between the two extremities of *dl* will be given by $\left|\vec{E}\right| dl \cos\theta$. But if the elementary length *dl* lies on the equipotential, then there should not be any potential difference across *dl*. Again, the magnitude of electric field intensity is not zero at P and *dl* is also a non-zero quantity. Hence, the potential difference across *dl* could only be zero if $\cos\theta$ is zero, i.e. if θ is 90^0.

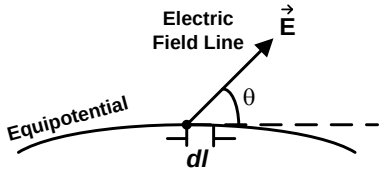


Fig.1.9 Equipotential vis-à-vis electric field line

Thus a basic constraint of electric field distribution is that the electric field lines are always normal to the equipotential surface. A practical example of this constraint is that the electric field lines will always leave or enter conductor surfaces at 90^0. This criterion is often used to check the accuracy of electric field computation by numerical techniques. The other properties of equipotential are (i) the tangential component of the electric field along the equipotential is zero and (ii) no work is required to move a charged particle along an equipotential.

Electric Potential of Earth Surface

The potential of earth surface could be determined with the help of the discussion of the previous section. Consider that a system of source charges has created an electric field over a region located in New York, USA. Now if one considers a test point on earth surface located in New Delhi, India, then the distance of this test point wrt the source charges is infinite. Hence, electric potential of the test point on earth surface in India will be zero due to the stated source charges at New York. Now, earth is an excellent conductor and in the absence of any conductive current the earth surface is an equipotential. So if the test point on the earth surface located in India is at zero potential, then all the points on the earth surface will be at zero potential. Extending the above mentioned logic, one may see that for any set of source charges located anywhere within this world, there will always be a point on the earth surface which will be at infinite distance wrt to the source charges. Hence, the earth surface potential will always be zero due to any set of source charges.

Electric Potential Gradient

It is defined as the positive rate of change of electric potential with respect to distance in the direction of greatest change. At any point in a field region it will be very difficult to comprehend the direction of greatest change. To understand it conveniently, consider that the equipotentials are known within the field region. Fig.1.10 shows three such equipotentials 1, 2 and 3. Then from the point P on the equipotential 2 having an electric potential of ϕ, if one moves to any point on the equipotential having an electric potential $\phi+\Delta\phi$, the potential difference is $+\Delta\phi$. But the minimum distance between the equipotentials 1 and 2 is the normal distance Δn. Hence, the greatest rate of change will be along the normal to the equipotential. Moreover, there are two directions of the normal to the equipotential 2 at the point P. Electric potential gradient is defined to be the greatest rate of change of potential in the positive sense. Hence, with reference to Fig.1.10, the electric potential gradient at the point P will be given by

Electric Potential Gradient ($grad\ \phi$) = ($\Delta\phi / \Delta n$) 1.19

where, $\Delta\phi$ is the potential difference and Δn is the normal distance between the two equipotentials 1 and 2 of Fig.1.10.

Since, electric potential gradient has magnitude along with a specific direction, hence is a vector quantity. It is a spatial derivative of electric potential.

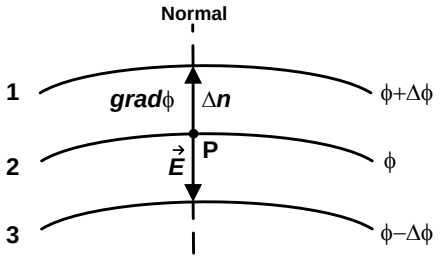


Fig.1.10 Electric potential gradient and electric field intensity

Electric Potential Gradient and Electric Field Intensity

As discussed in section 1.7.1, electric field line or $\vec{E}$ will be directed along the normal to the equipotential. But as there are two normal directions to the equipotential, the question is $\vec{E}$ will be in which direction. In this context, recall that if one moves along the direction of electric field then the potential energy decreases and hence the electric potential also decreases in the direction of electric field or $\vec{E}$. So with reference to Fig.1.10, $\vec{E}$ will act from equipotential 2 to equipotential 3 at the point P, i.e. $\vec{E}$ will act along the direction of decreasing potential. Again, when one moves from equipotential 1 to equipotential 2 along the normal distance Δn as shown in Fig.1.10, then the potential drop is $\Delta\phi$ and the work done by the field forces is given by $\left|\vec{E}\right|\Delta n$. Hence,

$$\Delta\phi = \left|\vec{E}\right| \Delta n \quad \text{or,} \quad \left|\vec{E}\right| = \frac{\Delta\phi}{\Delta n} \qquad 1.20$$

So, from eqns. (1.19) and (1.20) it may be seen that the magnitudes of $\vec{E}$ and $grad\,\phi$ at the point P are the same. But, $grad\,\phi$ acts along the direction of increasing potential and $\vec{E}$ acts along the direction of decreasing potential and both $grad\,\phi$ and $\vec{E}$ act along the normal to the equipotential at the point P. Therefore, it could be concluded that

$$\vec{E} = -\, grad\,\phi \qquad 1.21$$

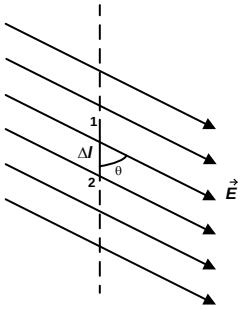


Fig.1.11 Pertaining to the relationship of electric potential gradient and electric field intensity

As shown in Fig.1.11, the potential difference ($\phi 1$-$\phi 2$) between two points 1 and 2 within a field region is given by

$$\Delta\phi = -\,\vec{E}\,.\,\overrightarrow{\Delta l} \quad \text{or,} \quad \Delta\phi = -\left|\vec{E}\right|\left|\overrightarrow{\Delta l}\right|\cos\theta \qquad 1.22$$

where, $\overrightarrow{\Delta l}$ is the distance vector from point 1 to point 2 and $\left|\vec{E}\right|\cos\theta$ is the component of $\vec{E}$ along $\overrightarrow{\Delta l}$.

Now, if $\overrightarrow{\Delta l}$ lies along the direction of x-axis, then eqn.(1.22) can be rewritten in terms of the component of $\vec{E}$ along the x-direction, i.e. E_x, and the distance Δx, as

$$E_x = -\frac{\Delta\phi}{\Delta x} \qquad 1.23$$

Instead of discrete variation, for continuous variation of electric potential in *x*-direction the eqn.(1.23) could be written in partial derivative form as

$$E_x = -\frac{\partial\phi}{\partial x} \qquad 1.24$$

Similarly, in the *y*- and *z*-directions, $E_y = -\frac{\partial\phi}{\partial y}$ and $E_z = -\frac{\partial\phi}{\partial z}$

Thus, the electric field intensity vector in terms of the three components could then be written as

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k} = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi \qquad 1.25$$

On the RHS of eqn.(1.25), the vector operator within the parenthesis is the 'del' operator, $\vec{\nabla}$.

Thus, from eqn.(1.25) $\vec{E} = -\vec{\nabla}\phi$ 1.26

The $\vec{\nabla}$ operator is an interesting operator. When it acts on a scalar quantity, the result is a vector quantity and physically it results into the spatial derivative of the scalar quantity, i.e. gradient of the scaler quantity. In this case when $\vec{\nabla}$ acts on scalar electric potential, it results into the vector quantity electric potential gradient.

Problem 1.1

The potential field in a medium having relative permittivity of 3.5 is given by $\phi = 4x^3y - 5y^3z + 3xz^3$ V. Find the electric field intensity at the point (0.1,0.5,0.2)m.

Solution:

$E_x = -\frac{\partial\phi}{\partial x} = -12x^2y - 3z^3$ Hence, $E_x\big|_{(0.1,0.5,0.2)} = -12\times0.1^2\times0.5 - 3\times0.2^3 = -0.084V/m$

$E_y = -\frac{\partial\phi}{\partial y} = -4x^3 + 15y^2z$ Hence, $E_y\big|_{(0.1,0.5,0.2)} = -4\times0.1^3 + 15\times0.5^2\times0.2 = 0.746V/m$

$E_z = -\frac{\partial\phi}{\partial z} = 5y^3 - 9xz^2$ Hence, $E_z\big|_{(0.1,0.5,0.2)} = 5\times0.5^3 - 9\times0.1\times0.2^2 = 0.589V/m$

So, $\vec{E}\big|_{(0.1,0.5,0.2)} = -0.084\hat{i} + 0.746\hat{j} + 0.589\hat{k}\ V/m$

Hence, $\left|\vec{E}\right|_{(0.1,0.5,0.2)} = 0.954V/m$

Problem 1.2

The potential field at any point in a space containing a dielectric medium of $\varepsilon_r = 5$ is given by $\phi = 7x^2y - 3y^2z - 4z^2x$ V, where *x*, *y* and *z* are in meters. Calculate the *y*-component of electric flux density at the point (1,4,2) m.

Solution:

$E_y = -\frac{\partial\phi}{\partial y} = -7x^2 + 6yz$ Hence, $E_y\big|_{(1,4,2)} = -7\times1^2 + 6\times4\times2 = 41V/m$

So, $D_y\big|_{(1,4,2)} = 41\times5\times8.854\times10^{-12} = 1.815nC/m^2$

Problem 1.3

The potential field in a space containing a dielectric medium of ε_{r1} is given by $\phi_1 = 7xy - 3yz - 4zx$ V, and another potential field in a space containing a dielectric medium of ε_{r2} is given by $\phi_2 = -2x - 7yz + 5zx$ V, where x, y and z are in meters. If the x-component of electric flux density at the point (1,2,2)m is same in both the fields, then find the ratio of ε_{r1} and ε_{r2}.

Solution:

$$E_{x1} = -\frac{\partial \phi_1}{\partial x} = -7y + 4z \qquad \text{Hence, } E_{x1}\big|_{(1,2,2)} = -7\times 2 + 4\times 2 = -6V/m$$

$$E_{x2} = -\frac{\partial \phi_2}{\partial x} = 2 - 5z \qquad \text{Hence, } E_{x2}\big|_{(1,2,2)} = 2 - 5\times 2 = -8V/m$$

Now, $D_{x1} = \varepsilon_{r1} \times \varepsilon_0 \times E_{x1} = -6\varepsilon_{r1}\varepsilon_0\, C/m^2$

and $D_{x2} = \varepsilon_{r2} \times \varepsilon_0 \times E_{x2} = -8\varepsilon_{r2}\varepsilon_0\, C/m^2$

As per the problem statement, $D_{x1} = D_{x2}$, or, $-6\varepsilon_{r1}\varepsilon_0 = -8\varepsilon_{r2}\varepsilon_0$

Hence, $\frac{\varepsilon_{r1}}{\varepsilon_{r2}} = 1.333$

Field due to Point Charge

As shown in Fig.1.12, due to a point charge $+Q_i$ the electric field intensity at any point P at a distance r from the charge is given by

$$\vec{E}_r = \frac{Q_i}{4\pi\varepsilon_0 r^2}\hat{u}_r$$

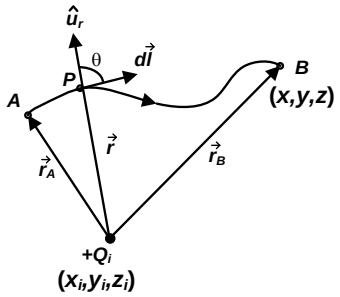


Fig.1.12 Field due to a point charge

If the point P is moved from point A to point B as shown in Fig.1.12, then the potential difference will be

$$\phi_B - \phi_A = -\int_{r_A}^{r_B} \frac{Q_i}{4\pi\varepsilon_0 r^2}\hat{u}_r . d\vec{l} = \frac{Q_i}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \qquad 1.27$$

If the point A is located at infinity, then the potential of point B will be

$$\phi_B = \frac{Q_i}{4\pi\varepsilon_0 r_B} \qquad 1.28$$

If the point charge $+Q_i$ is located at (x_i,y_i,z_i) and the point B is located at (x,y,z), then

$$\vec{r}_B = (x - x_i)\hat{i} + (y - y_i)\hat{j} + (z - z_i)\hat{k} \qquad 1.29$$

such that $\left|\vec{r}_B\right| = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$ and $\hat{u}_{rB} = \frac{\vec{r}_B}{\left|\vec{r}_B\right|}$

The electric field intensity at the point B will then be given by

$$E_B = \frac{Q_i}{4\pi\varepsilon_0\left((x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right)} \frac{(x-x_i)\hat{i}+(y-y_i)\hat{j}+(z-z_i)\hat{k}}{\left(\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}\right)} \qquad 1.30$$

Hence, the three components of electric field intensity at the point B are (dropping the suffix B for the sake of generalization)

$$E_x = \frac{(x-x_i)\,Q_i}{4\pi\varepsilon_0\left(\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}\right)^3}$$

$$E_y = \frac{(y-y_i)\,Q_i}{4\pi\varepsilon_0\left(\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}\right)^3} \qquad 1.31$$

$$E_z = \frac{(z-z_i)\,Q_i}{4\pi\varepsilon_0\left(\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}\right)^3}$$

In the presence of multiple point charges, at any point electric potential will be the scalar sum of electric potentials and electric field intensity will be the vector sum of electric field intensities at that point due to all the charges. In other words, the effect of all the charges will be superimposed at any point within the field region.

Problem 1.4

A point charge Q_1 = +1.0μC is located at (3,1,1)m and another point charge Q_2 = -0.5μC is located at (0.5,2,1.5)m. Find the magnitude and polarity of the point charge located at (1,2,2)m for which the *z*-component of electric field intensity will be zero at the origin. Medium is air.

Solution:

For the point charge-1:

Distance vector to the origin $\vec{r}_{O1} = (0-3)\hat{i}+(0-1)\hat{j}+(0-1)\hat{k} = (-3\hat{i}-\hat{j}-\hat{k})\,m$

So, $\left|\vec{r}_{O1}\right| = 3.317m$

Electric field intensity at the origin:

$$E_{O1} = \frac{10^{-6}}{4\pi\times1\times\varepsilon_0}\times\frac{-3\hat{i}-\hat{j}-\hat{k}}{3.317^3} = A\left(-0.0822\hat{i}-0.0274\hat{j}-0.0274\hat{k}\right)V/m,$$

where $A = \dfrac{10^{-6}}{4\pi\times1\times\varepsilon_0}$

For the point charge-2:

Distance vector to the origin $\vec{r}_{O2} = (0-0.5)\hat{i}+(0-2)\hat{j}+(0-1.5)\hat{k} = (-0.5\hat{i}-2\hat{j}-1.5\hat{k})\,m$

So, $\left|\vec{r}_{O2}\right| = 2.549m$

Electric field intensity at the origin:

$$E_{O2} = \frac{-0.5\times10^{-6}}{4\pi\times1\times\varepsilon_0}\times\frac{-0.5\hat{i}-2\hat{j}-1.5\hat{k}}{2.549^3} = A\left(0.015\hat{i}+0.06\hat{j}+0.045\hat{k}\right)V/m$$

So, the *z*-component of electric field intensity at the origin due to point charges 1 and 2 is (-0.0274 + 0.045)*A* = 0.0176*A* V/m

For the point charge-3: Let the magnitude of the charge be Q_3 μC.

Distance vector to the origin $\vec{r}_{O3} = (0-1)\hat{i} + (0-2)\hat{j} + (0-2)\hat{k} = (-\hat{i} - 2\hat{j} - 2\hat{k})\, m$

So, $\left|\vec{r}_{O3}\right| = 3m$

Electric field intensity at the origin:

$$E_{O3} = \frac{Q_3 \times 10^{-6}}{4\pi \times 1 \times \varepsilon_0} \times \frac{-\hat{i} - 2\hat{j} - 2\hat{k}}{3^3} = A\left(-0.037\hat{i} - 0.074\hat{j} - 0.074\hat{k}\right)Q_3\ V/m,$$

If the resultant values of *z*-component of electric field intensity at the origin is to be zero due to the three point charges, then

$-0.074AQ_3 + 0.0176A = 0, \quad or,\ Q_3 = 0.238\mu C$

Problem 1.5

A right isosceles triangle of side 1m has charges +1nC, +2 nC and −1nC arranged on its vertices, as shown in Fig.1.13. Find the magnitude and direction of electric field intensity at the point P, which is midway between the line connecting the +1nC and -1nC charges. Medium is air.

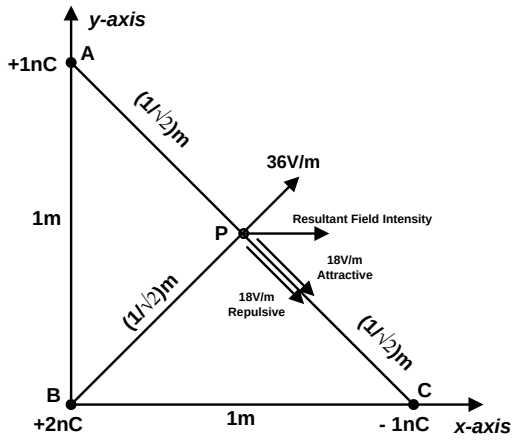


Fig.1.13 Pertaining to Problem 1.5

Solution:

Consider that a unit positive test charge is located at the point P.

The distance of this test charge from +1nC is (1/√2)m.

So electric field intensity at P due to +1nC is $\dfrac{10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times \left(\dfrac{1}{\sqrt{2}}\right)^2} = 18V/m$

and it acts along the line AC shown in Fig.1.13 from P to C, as it is a repulsive force on the test charge located at P.

The distance of the test charge from -1nC is also (1/√2)m.

So electric field intensity at P due to -1nC is also 18 V/m and it also acts along the line AC shown in Fig.1.13 from P to C, as it is an attractive force on the test charge located at P.

The distance of this test charge from +2nC is also (1/√2)m.

So electric field intensity at P due to +2nC is $\dfrac{2 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times \left(\dfrac{1}{\sqrt{2}}\right)^2} = 36V/m$

and it acts along the line BP shown in Fig.1.13 from B to P, as it is a repulsive force on the test charge located at P. The line BP is perpendicular to the line AC.

Hence, the electric field intensity acting along AC from P to C is 36V/m and that acting along BP from B to P is also 36V/m and these two are normal to each other.

So, the resultant electric field intensity at the point P is $\sqrt{36^2 + 36^2} = 50.91\,V/m$

It will make an angle of 45^0 wrt the line AC. In other words, it will act along the *x*-axis in the positive sense of the axis.

Problem 1.6

An α-particle with a kinetic energy of 1.5 MeV is projected towards a stationary nucleus of Platinum, which has 78 protons. Determine the distance of closest approach of the α-particle. Neglect the motion of the nucleus.

Solution:

The α-particle is positively charged. Hence, it experiences repulsive force from the nucleus and decelerates as it approaches the nucleus. At the closest approach point it stops before being repulsed back. At this point of closest approach all its kinetic energy is converted to potential energy.

Initial kinetic energy of the α-particle = 1.5 MeV = $1.5\times10^6\times1.602\times10^{-19}$J = 2.403×10^{-13}J

Let, the distance of closest approach be *d* meters.

An α-particle has a charge of $+3.204\times10^{-19}$C. The nucleus of Platinum has charge of $(78\times1.602\times10^{-19})$C = $+124.956\times10^{-19}$C

The electric potential at a distance of *d* from the Platinum nucleus due to all the protons is

$$\frac{78\times1.602\times10^{-19}}{4\pi\times8.854\times10^{-12}\times d} = \frac{1.123\times10^{-7}}{d}V$$

Hence, the potential energy of the α-particle at a distance *d* from Platinum nucleus will be

$$\frac{1.123\times10^{-7}}{d}\times3.204\times10^{-19}\,J = \frac{3.598\times10^{-26}}{d}J$$

Equating this potential energy to the initial kinetic energy of $2.403\times10^{-13}J$, $d = 1.497\times10^{-13}$m

Field due to Uniformly Charged Line

Fig.1.14 shows a line charge of uniform charge density λ. As a practical example, it may be considered to be a non-conducting rod which is uniformly charged. The length of the charge is *L*. For the sake of simplicity, the field is computed at a point P which located on the line perpendicular to the line charge and passing through the midpoint of the charge. The normal distance of the point P from the line charge is *y*. For the purpose of field computation, the origin is considered to be located at the mid point of the line charge as shown in Fig.1.14.

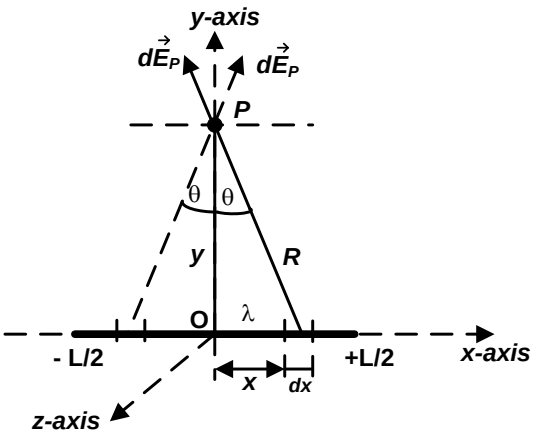


Fig.1.14 Field due to a uniformly charged line

Consider an elementary length of charge *dx* located at distance *x* from the origin. Then the charge on this elementary length is $dq = \lambda dx$ and it may be considered as a point charge. Hence, the electric potential at the point P due the elementary charge *dq* is given by

$$d\phi_P = \frac{dq}{4\pi\varepsilon_0 R} = \frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{\sqrt{\left(x^2+y^2\right)}}$$

So, the potential at P due to the entire line charge will be

$$\phi_P = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L/2}^{+L/2}\frac{dx}{\sqrt{\left(x^2+y^2\right)}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(x+\sqrt{x^2+y^2}\right)\right]_{x=-L/2}^{x=+L/2}$$

$$\text{or, } \phi_P = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\sqrt{(L/2)^2+y^2}+(L/2)}{\sqrt{(L/2)^2+y^2}-(L/2)} \qquad 1.32$$

As electric potential at the point P is dependent only on *y*, hence electric field intensity at the point P will have only the *y*-component, which will be given by

$$E_{yP} = -\frac{\partial\phi_P}{\partial y} = \frac{\lambda}{4\pi\varepsilon_0}\frac{y}{\sqrt{(L/2)^2+y^2}}\left(\frac{1}{\sqrt{(L/2)^2+y^2}-(L/2)} - \frac{1}{\sqrt{(L/2)^2+y^2}+(L/2)}\right)$$

$$\text{or, } E_{yP} = \frac{\lambda}{4\pi\varepsilon_0}\frac{L}{y\sqrt{(L/2)^2+y^2}} \qquad 1.33$$

The direction of electric field intensity at the point P along the *y*-axis can be explained with the help of Fig.1.14. For the elementary length of charge *dx* on the RHS of the origin O, another elementary length of charge can be assumed to be there at the same distance *x* from the origin on the LHS of the origin. Then the electric field intensity at the point P due to these two elementary charges will be same in magnitude, say dE_p. But they will be directed along the distance vectors from the elementary charges to the point P as shown in Fig.1.14. They will make the same angle θ with the *y*-axis. Hence the *x*-components of these two field intensity vectors dE_p at the point P will cancel each other and only the *y*-component will be present, as the *y*-components of dE_p vectors will be additive at the point P.

Field due to Uniformly Charged Ring

Consider a uniformly charged ring of radius *a*, as shown in Fig.1.15. The uniform charge density is λ. As a practical example, it may be considered to be a non-conducting annular strip of very small width which is uniformly charged. To determine the electric potential at the point P located on the axis of the ring at a height of *z* from the plane of the ring, consider an elementary arc length $ad\theta$ as shown in Fig.1.15. Then this elementary arc length will have a charge $dq=\lambda ad\theta$ and it may be considered as a point charge.

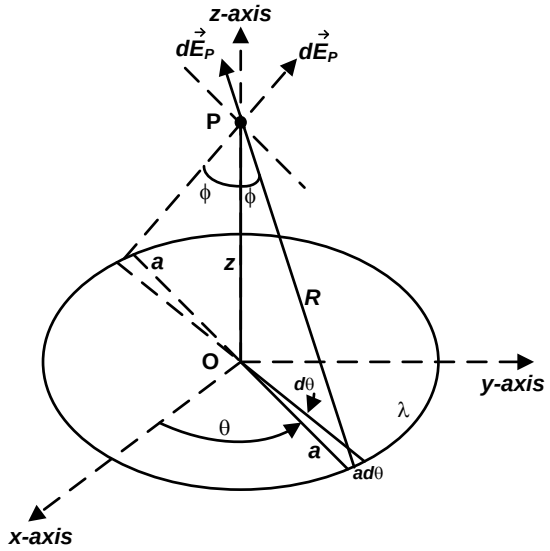


Fig.1.15 Field due a uniformly charged ring

Hence, the electric potential at the point P due the elementary charge *dq* is given by

$$d\phi_P = \frac{dq}{4\pi\varepsilon_0 R} = \frac{\lambda}{4\pi\varepsilon_0}\frac{ad\theta}{\sqrt{\left(a^2+z^2\right)}}$$

So, the potential at P due to the entire ring charge will be

$$\phi_P = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{ad\theta}{\sqrt{\left(a^2+z^2\right)}} \qquad 1.34$$

In the integral of eqn.(1.34), it may be seen that as θ is varied from 0 to 2π along the ring, the distance *R* to the point P from any point on the ring always remains the same, i.e. $\sqrt{\left(a^2+z^2\right)}$. Hence, this integral is only over θ. Hence,

$$\phi_P = \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a}{\sqrt{\left(a^2+z^2\right)}} = \frac{\lambda a}{2\varepsilon_0\sqrt{\left(a^2+z^2\right)}} \qquad 1.35$$

Eqn.(1.35) shows that electric potential at the point P is dependent only on *z*, hence electric field intensity at the point P will have only the *z*-component, which will be given by

$$E_{zP} = -\frac{\partial\phi_P}{\partial z} = \frac{\lambda}{2\varepsilon_0}\frac{a\,z}{\left(a^2+z^2\right)^{3/2}} \qquad 1.36$$

With the help of Fig.1.15, it can be explained why the resultant electric field intensity at the point P is directed along the *z*-axis. Similar to the case of uniformly charged line, for any elementary arc length $ad\theta$, another elementary arc length $ad\theta$ can be assumed to be located diametrically opposite to the first elementary arc length. Then the electric field intensity at the point P due to these two elementary arc lengths will be same in magnitude, say dE_p. But these two field intensity vectors of magnitude dE_p will act along the distance vectors from the two elementary arc lengths to the point P, as shown in Fig.1.15. It is obvious that these two vectors will make the same angle with the three axes. Hence, considering the directions of these two vectors, it can be seen that the components of electric field intensity in *x* and *y*-directions will cancel each other, while the *z*-components will be additive in nature. Therefore, the net electric field intensity at the point P due to the ring charge will have only *z*-component.

Field due to Uniformly Charged Disc

Consider a uniformly charged disc of radius *a* having a surface charge density σ, as shown in Fig.1.16. As a practical example, it may be considered to be a non-conducting disc which is uniformly charged. Point P is located on the axis of the disc at a height *z* from the plane of the disc. For computing the electric potential at the point P, consider an annular strip of radius *dr* at a radius *r*, as shown in Fig.1.16. Then take a small arc segment of angular width $d\theta$ of this annular strip. Then this elementary surface of area $dA=rd\theta dr$ will have an elementary charge $dq=\sigma dA=\sigma r d\theta dr$. This elementary charge *dq* can be considered as a point charge.
Hence, the electric potential at the point P due the elementary charge *dq* is given by

$$d\phi_P = \frac{dq}{4\pi\varepsilon_0 R} = \frac{\sigma}{4\pi\varepsilon_0}\frac{r d\theta\, dr}{\sqrt{\left(r^2+z^2\right)}}$$

So, the potential at P due to the entire annular strip of charge of radius *r* will be

$$\phi'_P = \frac{\sigma}{4\pi\varepsilon_0}\left[\int_0^{2\pi}\frac{d\theta}{\sqrt{\left(r^2+z^2\right)}}\right] r\, dr \qquad 1.37$$

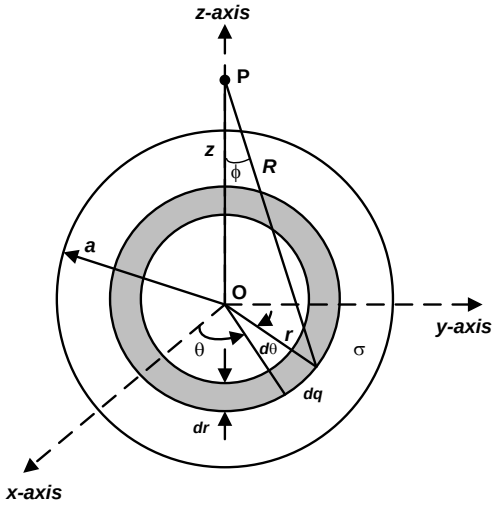


Fig.1.16 Field due to a uniformly charged disc

because, when θ varies from *0* to 2π along the annular strip of radius *r*, then the distance *R* to the point P from any point on this strip remains the same, i.e. $\sqrt{\left(r^2+z^2\right)}$, and the radial distance from the axis of the disc also remains the same, i.e. *r*. Hence the integration of eqn.(1.37) is carried out over θ only. Therefore,

$$\phi'_P = \frac{\sigma}{4\pi\varepsilon_0}\frac{2\pi r dr}{\sqrt{\left(r^2+z^2\right)}} = \frac{\sigma}{2\varepsilon_0}\frac{r dr}{\sqrt{\left(r^2+z^2\right)}}$$

Thus, the potential at P due to the entire disc of radius *a* will be

$$\phi_P = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r dr}{\sqrt{\left(r^2+z^2\right)}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{\left(r^2+z^2\right)}\right]_{r=0}^{r=a} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{\left(a^2+z^2\right)}-z\right] \qquad 1.38$$

Eqn.(1.38) shows that electric potential at the point P is dependent only on *z*, hence electric field intensity at the point P will have only the *z*-component, which will be given by

$$E_{zP} = -\frac{\partial\phi_P}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\sqrt{\left(a^2+z^2\right)}}\right] \qquad 1.39$$

The charged disc comprises large number of annular strips, as shown in Fig.1.16. Each annular strip of very small radial width *dr* can be considered as a ring charge of uniform charge density. In the previous section, it has been discussed that the electric field intensity due a ring charge at any point located on the axis of the ring has only *z*-component. Hence, the electric field intensity at the point P of Fig.1.16 due to large number of co-axial ring charges representing the annular strips of charges will also have only *z*-component as given by eqn.(1.39).

From eqn.(1.38) it may be seen that the electric potential at the centre of the disc is finite. Putting z=0 in eqn(1.38), $\phi_O = \dfrac{\sigma a}{2\varepsilon_0}$

The total charge on the entire disc is $Q = \pi a^2 \sigma$. Hence,

$$\phi_O = \frac{Q a}{2\pi a^2 \varepsilon_0} = 2\frac{Q}{4\pi\varepsilon_0 a} \qquad 1.40$$

Eqn.(1.40) shows that the electric potential at the centre of the disc is equivalent to twice the work done to bring a unit positive charge from infinity to the circumference of the disc (r=a) when the entire charge of the disc is assumed to be concentrated as a point charge at the centre.

When z is very large compared to a, then

$$\sqrt{z^2 + a^2} = z\left(1 + \frac{a^2}{z^2}\right)^{1/2} = z\left(1 + \frac{a^2}{2z^2} + ...\right) \approx z + \frac{a^2}{2z}$$

Then from eqn.(1.38),

$$\phi_P\big|_{z<<a} = \frac{\sigma}{2\varepsilon_0}\left[z + \frac{a^2}{2z} - z\right] = \frac{\sigma\pi a^2}{4\pi\varepsilon_0 z} = \frac{Q}{4\pi\varepsilon_0 z} \qquad 1.41$$

When the point P is located at a very large distance from the disc of finite radius a, then the disc could be considered to be a point charge of magnitude Q. Eqn. (1.41) is the expression for potential at a point located at distance z from a point charge of magnitude Q.

The significance of the above discussion is that the accuracy of field computation for any given charge distribution can be checked by choosing a point P which is located sufficiently far away from the charge distribution under consideration. If the charge distribution is of finite extent, then the electric field should behave as if the charge distribution is like point charge, and decreases with the square of the distance.

Problem 1.7

A circular disc of charge of radius 1m having a uniform charge density σ = +1nC/m^2 lies in the *z=0* plane, with centre at the origin. There is also a point charge of -0.4nC at the origin. Find the magnitude and polarity of uniform charge density of a circular ring of charge of radius 1m lying in the *z=0* plane, with centre at the origin, which would produce the same electric field intensity at the point (0,0,6)m as that due to the combined effect of the disc and point charges. Medium is air.

Solution:

As stated, the point P is located at (0,0,6)m. According to the problem statement, the point P lies on the axis of the disc as well as on the axis of the ring.

Hence, for the disc charge located in the *x-y* plane having centre at the origin:

σ = +1nC/m^2, a = 1m and z = 6m.

$$E_{P-disc} = \frac{10^{-9}}{2\times 1\times 8.854\times 10^{-12}}\left[1 - \frac{6}{\sqrt{\left(1^2 + 6^2\right)}}\right] = 0.768 V/m$$

Again, for the point charge located at origin: Q = -0.4nC, r = 6m

$$E_{P-Point} = \frac{-4\times10^{-10}}{4\times1\times8.854\times10^{-12}\times6^2} = -0.314V/m$$

Let the uniform charge density of the ring charge be λ C/m.

Then, for the ring charge located in x-y plane having centre at the origin: a = 1m and z = 6m

$$E_{P-Ring} = \frac{\lambda}{2\times1\times8.854\times10^{-12}}\frac{1\times6}{\left(1^2+6^2\right)^{3/2}} = 1.505\times10^9\times\lambda V/m$$

As per the statement of the problem: $E_{P\text{-}ring} = E_{P\text{-}disc} + E_{P\text{-}Point}$

or, $1.505\times10^9\times\lambda = 0.768-0.314 = 0.454$

So, λ = 0.302 nC/m of positive polarity.

Problem 1.8

Consider a ring charge of radius 10cm and uniform charge density of λ C/m and also a disc charge of radius 15cm and uniform charge density σ C/m^2. Both the two charges are placed in the x-y plane with their centre at the origin. If the electric field intensity at a point of height 20cm lying on the z-axis is same due the ring and disc charges individually, then find the ratio of λ and σ. Relative permittivity of the medium is 2.1.

Solution:

The point P is located at (0,0,0.2)m. Hence, it is located on the axis of both the ring and disc charges.

For the ring charge: a = 0.1m and z = 0.2m

$$E_{P-Ring} = \frac{\lambda}{2\times\varepsilon_r\times\varepsilon_0}\frac{0.1\times0.2}{\left(0.1^2+0.2^2\right)^{3/2}} = 0.894\frac{\lambda}{\varepsilon_r\varepsilon_0}V/m$$

For the disc charge: a = 0.15m and z = 0.2m

$$E_{P-disc} = \frac{\sigma}{2\times\varepsilon_r\times\varepsilon_r}\left[1-\frac{0.2}{\sqrt{\left(0.15^2+0.2^2\right)}}\right] = 0.1\frac{\sigma}{\varepsilon_r\varepsilon_r}V/m$$

But as per the problem: $E_{P\text{-}Ring} = E_{P\text{-}disc}$

or, $0.894\frac{\lambda}{\varepsilon_r\varepsilon_0} = 0.1\frac{\sigma}{\varepsilon_r\varepsilon_0}$

So, $\frac{\lambda}{\sigma} = 0.112$

Gauss's Law and Related Topics

Introduction

German mathematician and physicist Karl Friedrich Gauss published his famous flux theorem in 1867, which is now well known in physics as Gauss's Law. It is one of the basic laws of classical field theory. Gauss's Law is a general law that can be applied to any closed surface. With the help of Gauss's law the amount of enclosed charge could be assessed by mapping the field on a surface outside the charge distribution. By the application of Gauss' law the electric field in many practical arrangements could be evaluated by forming a symmetric surface, commonly known as Gaussian surface, surrounding a charge distribution and then determining the electric flux through that surface.

In electric field theory, distinction is made between free charge and bound charge. Free charge implies a charge that is free to move over distances large as compared to atomic scale lengths. Free charge typically comes from electrons, e.g. in the case of metals, or ions, or in the case of aqueous solutions. Bound charge implies charges of equal magnitude but opposite signs that are held close to each other and are free to move only through atomic scale lengths. Bound charges arise in the context of polarizable dielectric materials. Typical example of bound charge is the positive charge of an atomic nucleus and the negative charge of its associated electron cloud. The microscopic charge displacements in dielectric materials are not as dramatic as the rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behaviour of dielectric materials. Typically the detailed effect of bound charge is represented through electrical permittivity of dielectric materials, which will be discussed in a later chapter. The Gauss's law, as discussed in this chapter, is in terms of electric flux and the free charge only.

Useful Definitions and Integrals

Electric Flux through a Surface

Consider that the total flux through the surface *S* of area *A* needs to be evaluated, as shown in Fig.2.1. The first important point to be noted here is that the electric flux density at different locations on the surface *S* may not be the same. So the total flux through the surface *S* has to be computed by subdividing the entire surface into large number of smaller surfaces, such that over each small surface area the electric flux density vector is uniform. As shown in Fig.2.1, for any small surface area *dA*, the area vector $d\vec{A}$ and the electric flux density vector $\vec{D}$ could be along different directions. Then the electric flux through the small area *dA* is given by $\vec{D}.d\vec{A}$

Then the total flux through the surface *S* is given by the summation $\vec{D}_1.d\vec{A}_1 + \vec{D}_2.d\vec{A}_2 + ...$

When the sizes of the individual areas become infinitesimally small, then the total flux through the surface *S* is given by the integral $\int_S \vec{D}.d\vec{A} = \iint \vec{D}.d\vec{A}$ ….. 2.1

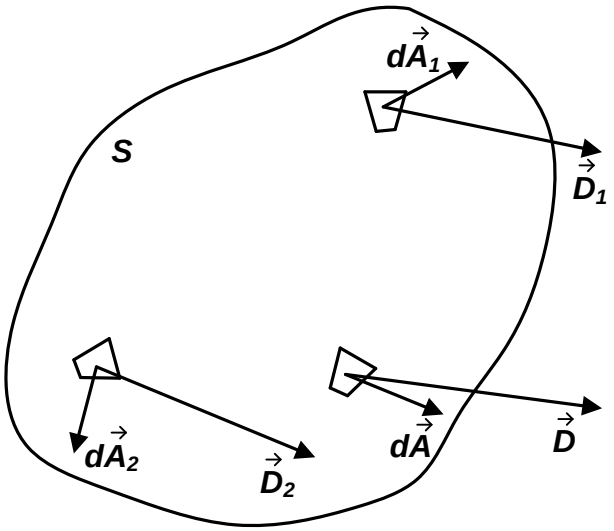


Fig.2.1 Evaluation of electric flux through a surface

Charge within a Closed Volume

Consider that the total charge within the volume *V* enclosed by the surface *S* needs to be evaluated, as shown in Fig.2.2. Here also it is to be kept in mind that the distributed charge distribution may not be uniform through out the volume *V*. Then the entire volume needs to be subdivided into large number small volumes such that within each small volume charge density is constant. Then the total charge within the volume V is given by the summation $\rho_{v1}\, dV_1 + \rho_{v2}\, dV_2 + \rho_{v3}\, dV_3 + ...$

When the sizes of the individual volumes become infinitesimally small, then the total charge within the volume *V* is given by the integral $\int_V \rho_v\, dV = \iiint \rho_v\, dV$

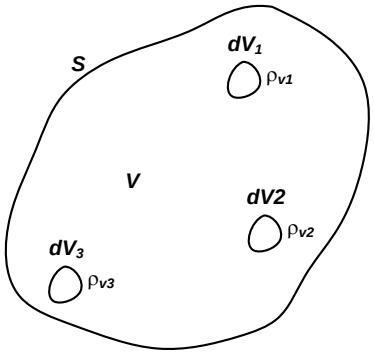


Fig.2.2 Evaluation of total charge within a volume

Solid Angle

The concept of solid angle is a natural extension of two-dimensional plane angle to three dimensions. The solid angle subtended by an area *A* at the point *O* is measured by the area Ω on the surface of the unit sphere centred at *O*, as shown in Fig.2.3. This is the area which would be cut out on the unit sphere surface by lines drawn from *O* to every point on the periphery of *A*. It is measured in terms of the unit called steradian, abbreviated as *str*.

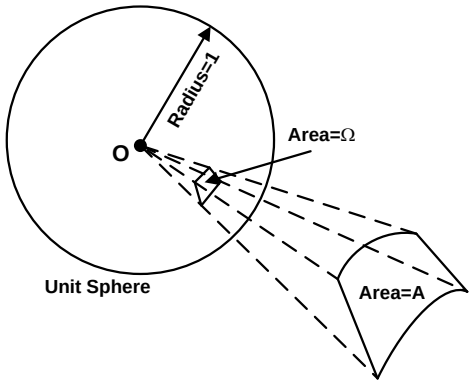


Fig.2.3 Solid angle subtended by an area at a point

Consider an elementary area *dA* as shown in Fig.2.4. If all the points on the periphery of the area *dA* are joined to the point *O*, then these lines cut out an area $d\Omega$ on the surface of the unit sphere. In other words, the area *dA* subtends a solid angle $d\Omega$ at the point *O*. Since the area is infinitesimally small, all points on *dA* could be considered to be equidistant from *O*. Let the distance of *dA* from the point *O* be *r*. The unit vector in the direction of the distance vector $\vec{r}$ is $\hat{u}_r$. Since the area *dA* is infinitesimally small, it may be taken as flat for practical purposes and hence $\hat{u}_n$ is defined as the unit vector in the direction normal to this area *dA* as shown in Fig.2.4. The angle between $\hat{u}_r$ and $\hat{u}_n$ is ϕ. Then the projection of the area *dA* on the sphere of radius *r* will be $dS = dA\cos\phi$

The areas *dS* and $d\Omega$ have the same general shape and are related to the respective radii of the spheres having the following proportionality,

$$\frac{d\Omega}{dS}=\frac{1^2}{r^2}, \quad \text{or,} \quad d\Omega=\frac{dA\cos\phi}{r^2}=\frac{\hat{u}_r.d\vec{A}}{r^2} \qquad \ldots. 2.2$$

When an area completely encloses the point *O*, then Ω due that area on the unit sphere will be 4π. Hence, the solid angle subtended at a point by an area which completely encloses that point is 4π steradians, which happens to be maximum possible value of solid angle.

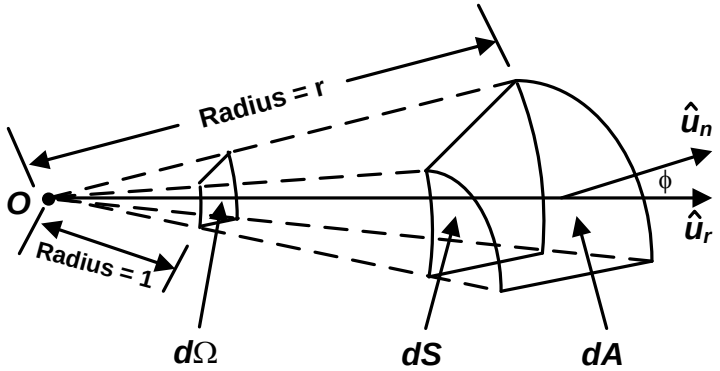


Fig.2.4 Evaluation of solid angle subtended by an elementary area

Integral Form of Gauss's Law

Gauss's law states that the net electric flux through any closed surface enclosing a homogeneous volume of material is equal to the net electric charge enclosed by that closed surface. In other words, the surface integral of electric flux density vector over a closed surface is equal to the volume integral of charge densities within the volume enclosed by that surface.

In integral form, $\int_A \vec{D}.d\vec{A} = \int_V \rho_v \, dV$ …. 2.3

Gauss's law is valid for any discrete set of point charges. Nevertheless, this law is also valid when an electric field is produced by charged objects with continuously distributed charges, since any continuously distributed charge may be viewed as a combination of discrete point charges.

Fig.2.5 shows a certain volume *V* of homogeneous material enclosed by a surface *A*. This volume is continuously charged by distributed charges. Consider these distributed charges to be represented by *N* number of discrete point charges as shown in Fig.2.5.

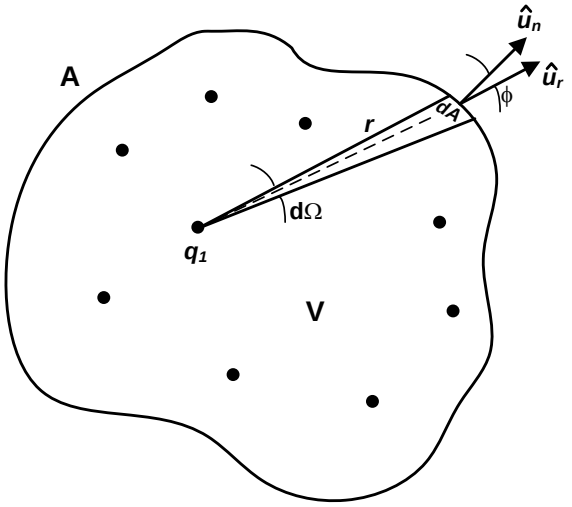


Fig.2.5 Pertaining to integral form of Gauss's law

Take a point charge q_1 located within the volume *V*. Then consider an elementary area *dA* on the surface *A* as shown in Fig.2.5. The distance of the area *dA* from q_1 is, say, *r*.

Then at a distance *r* from q_1, $\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{u}_r$ and $\vec{D}_1 = \frac{q_1}{4\pi r^2}\hat{u}_r$ …. 2.4

Then, the total flux through the area *A* due to q_1 could be obtained as

$$\int_A \vec{D}_1.d\vec{A} = \int_A \frac{q_1}{4\pi}\frac{\hat{u}_r.d\vec{A}}{r^2} = \frac{q_1}{4\pi}\int_A d\Omega, \quad \text{as } \frac{\hat{u}_r.d\vec{A}}{r^2} = d\Omega \qquad \text{…. 2.5}$$

Since the area *A* completely encloses the point charge q_1, hence the solid angle subtended by the area *A* at the location of the point charge q_1 is 4π. Thus,

$$\int_A \vec{D}_1.\, d\vec{A} = \frac{q_1}{4\pi} \times 4\pi = q_1 \qquad \text{…. 2.6}$$

Considering, *N* no of point charges within the volume *V*, total electric flux density at the location of *dA* as shown in Fig.2.5, will be

$$\vec{D} = \vec{D}_1 + \vec{D}_2 + \vec{D}_3 + ... + \vec{D}_N \qquad \text{…. 2.7}$$

So, the total flux through the surface *A* will be

$$\int_A \vec{D}.d\vec{A} = \int_A \left(\vec{D}_1 + \vec{D}_2 + \vec{D}_3 + ... + \vec{D}_N\right).d\vec{A} = \int_A \vec{D}_1.d\vec{A} + \int_A \vec{D}_2.d\vec{A} + \int_A \vec{D}_3.d\vec{A} + ... + \int_A \vec{D}_N.d\vec{A} \qquad \text{…. 2.8}$$

$$= q_1 + q_2 + q_3 + ... + q_N$$

The right hand side of eqn.(2.8) is equal to the total electric charge enclosed within the volume *V*. Considering continuously distributed charge within the volume *V*,

Total charge within the volume $V = \int_V \rho_v \, dV$

where, ρ_v is the volume charge density within the volume *V*.
Hence, eqn.(2.8) can be rewritten as $\int_A \vec{D}.d\vec{A} = \int_V \rho_v \, dV$

From a physical point of view, the above expression indicates that the sum of all sources (positive electric charges) and all sinks (negative electric charges) within a volume gives the net flux through the surface enclosing the same volume.
Gauss's law in integral form is only useful for exact solutions when the electric field has symmetry, e.g. spherical, cylindrical or planar symmetry. The symmetry of the electric field follows from the symmetry of the charge distribution that is given. So the electric field of a symmetrically charged sphere will also have the spherical symmetry.

Gaussian Surface

A closed surface in three dimensional space through which the flux of a vector field is calculated is known as Gaussian surface. It is an arbitrary closed surface over which surface integral is performed to evaluate the total amount of enclosed source quantity, e.g. electric charge. A Gaussian surface could also be used for calculating the electric field due to a given charge distribution.
Gaussian surfaces are normally carefully chosen in order to exploit symmetries to simplify the evaluation of the surface integral. If the symmetry is such that a surface can be found on which the electric field is constant, then evaluation of electric flux can be done by multiplying the value of the field with the area of the Gaussian surface.
Two commonly used Gaussian surfaces are:
(a) the spherical Gaussian surface, which is chosen to be concentric with the charge distribution. It can be used to determine the electric field or electric flux due to a point charge or a spherical shell of uniform charge density or any other charge distribution with spherical symmetry as shown in Fig.2.6.

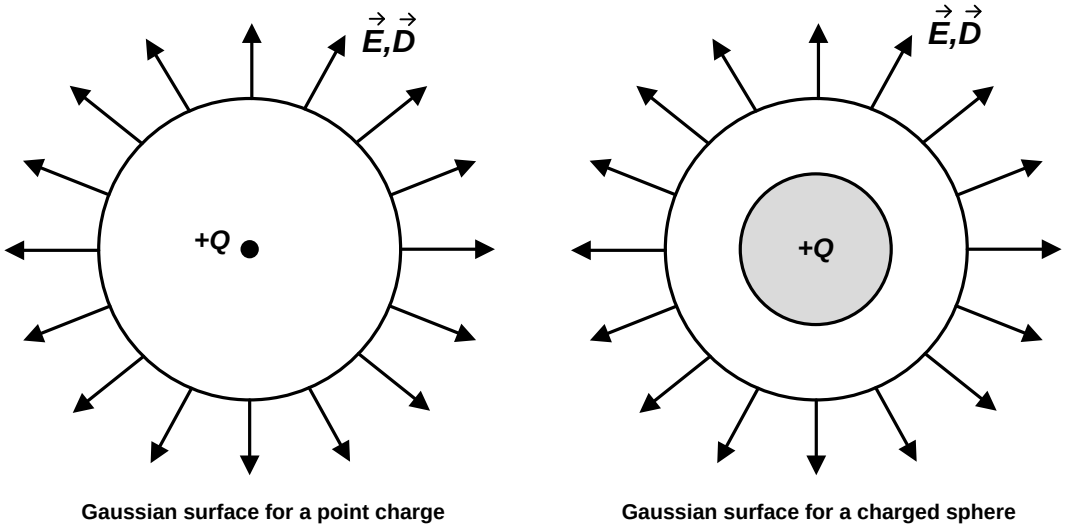


Fig. 2.6 Spherical Gaussian Surface

(b) the cylindrical Gaussian surface, which is chosen to be co-axial with the charge distribution. It can be used to determine the electric field or electric flux due to an infinitely long line of uniform charge density or an infinitely long cylinder of uniform charge density, as shown in Fig.2.7.

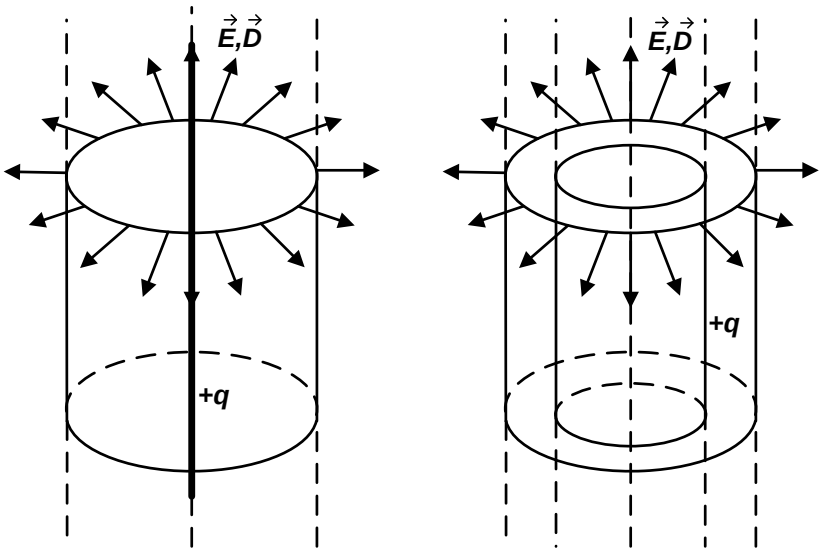


Fig. 2.7 Cylindrical Gaussian Surface

Differential Form of Gauss's Law

If the electric field is known everywhere, then with the help of integral form of Gauss's law the charge in any given field region can be deduced by integrating the electric field. However, in most of the practical cases, the electric field needs to be computed when the electric charge distribution is known. This is much more difficult because even if the total flux through a given surface is known, the information about the electric field may be unknown as electric flux could go in and out of the surface in complex patterns. In this context, the differential form of Gauss's law becomes useful.

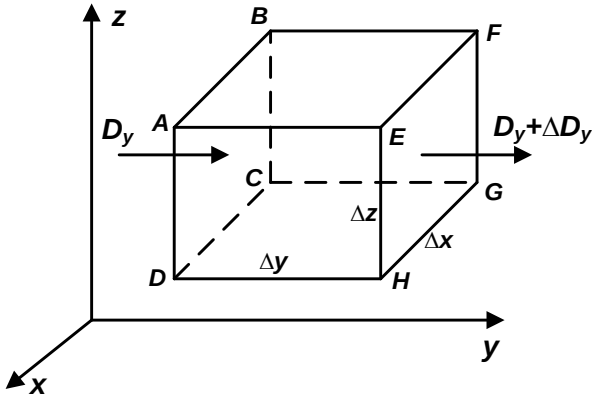


Fig.2.8 Application of Gauss's law to a differential volume

Consider that Gauss's law is to be applied for the infinitesimal parallelepiped as shown in Fig.2.8. The volume of the infinitesimal parallelepiped is $\Delta V=\Delta x\Delta y\Delta z$. Let, the component of electric flux density going normally into the surface *ABCD* is D_y and that coming out of the surface *EFGH* is $D_y+\Delta D_y$. The surfaces *ABCD* and *EFGH* are lying along the *x*-*z* plane. Then the net electric flux coming out in the *y*-direction which is normal to *x*-*z* plane is given by

$$\left(D_y + \Delta D_y\right)\Delta x\Delta z - D_y\,\Delta x\Delta z = \Delta D_y\,\Delta x\Delta z = \frac{\Delta D_y}{\Delta y}\Delta x\Delta y\Delta z \approx \frac{\partial D_y}{\partial y}\Delta V \qquad \ldots. 2.9$$

Similarly, the net electric fluxes coming out in *x*- and *z*-directions are given by $\frac{\partial D_x}{\partial x}\Delta V$ and $\frac{\partial D_z}{\partial z}\Delta V$, respectively. Hence,

Net electric flux coming of the volume ΔV $= \frac{\partial D_x}{\partial x}\Delta V + \frac{\partial D_y}{\partial y}\Delta V + \frac{\partial D_z}{\partial z}\Delta V$

$= (\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z})\Delta V$

Let, ρ_v be the volume charge density of the small volume ΔV. Then according to Gauss's law, the above expression for the net flux coming out of the volume ΔV is equal to the total charge within the volume, i.e. $\rho_v \Delta V$. Therefore,

$$(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z})\Delta V = \rho_v \Delta V$$

or, $$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v \quad \text{…. 2.10}$$

Eqn.(2.10) is known as the differential form of Gauss's law. It can be expanded as

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(D_x\hat{i} + D_y\hat{j} + D_z\hat{k}\right) = \rho_v, \text{ or, } \vec{\nabla}.\vec{D} = \rho_v \quad \text{…. 2.11}$$

In section 1.7.4, it has been mentioned that when the $\vec{\nabla}$ operator acts on a scalar quantity, it is denoted as gradient. As depicted in eqn.(2.11), when the $\vec{\nabla}$ operator acts on a vector quantity with a dot product, then it is denoted as divergence.

The physical meaning of divergence of a vector field can be explained from the left hand side of eqn.(2.10). From the above discussion it may be seen the LHS of eqn.(2.10) is the net flux coming out per unit volume at a given location. Consequently, the divergence of a vector field at any location is the net flux of that vector field coming out per unit volume at that given location. If the divergence of a vector field is positive at any location then the flux coming out of unit volume is higher than the flux going into the unit volume at that location. On the other hand, if the divergence of a vector field is negative then the flux going into the unit volume is higher than the flux coming out of the unit volume.

Divergence Theorem

Integral form of Gauss's law, $$\int_A \vec{D}.d\vec{A} = \int_V \rho_v \, dV$$

and differential form of Gauss's law, $$\vec{\nabla}.\vec{D} = \rho_v$$

By substituting ρ_v on the RHS of the integral form by the LHS of the differential form, the above two equations could be combined as follows:

$$\int_A \vec{D}.d\vec{A} = \int_V \left(\vec{\nabla}.\vec{D}\right)dV \quad \text{…. 2.12}$$

where, the volume *V* is enclosed by the surface *A*.

Eqn.(2.12) shows that the Divergence theorem could be used to convert a volume integral over the volume *V* to a surface integral over the boundary surface *A* enclosing the volume *V*.

Poisson's and Laplace's Equations

A useful approach to the calculation of electric potentials is to relate electric potential to electric charge density, which causes electric potential. Since electric potential is a scalar quantity, this approach has advantages over calculation of electric field intensity, which is a vector quantity. Once electric potential has been determined, the electric field intensity can be computed by taking the gradient of electric potential.

The relationship between electric flux density and electric charge density is given by the differential form of Gauss's law, i.e. $\vec{\nabla}.\vec{D} = \rho_v$

For homogeneous medium having uniform dielectric permittivity throughout the volume, the electric flux density and electric field intensity are related as $\vec{D} = \varepsilon \vec{E}$.

The above two equations could be combined as

$$\vec{\nabla}.\left(\varepsilon \vec{E}\right) = \rho_v, \text{ or, } \varepsilon\, \vec{\nabla}.\vec{E} = \rho_v, \text{ or, } \vec{\nabla}.\vec{E} = \frac{\rho_v}{\varepsilon} \quad \text{…. 2.13}$$

Further, electric field intensity and electric potential are related as $\vec{E} = -\vec{\nabla}\phi$. Hence, from eqn.(2.13), it may be written that

$$\vec{\nabla}.\left(-\vec{\nabla}\phi\right) = \frac{\rho_v}{\varepsilon}, \text{ or, } \vec{\nabla}.\vec{\nabla}\phi = -\frac{\rho_v}{\varepsilon}, \text{ or, } \vec{\nabla}^2\phi = -\frac{\rho_v}{\varepsilon} \quad \text{…. 2.14}$$

Eqn.(2.14) is a partial differential equation of elliptic form and is named after the French mathematician Siméon Denis Poisson. Thus, it is commonly known as Poisson's equation.

In eqn.(2.14), the mathematical operation, the divergence of gradient of a function ($\vec{\nabla}.\vec{\nabla} = \vec{\nabla}^2$), is called the Laplacian, such that

$$\vec{\nabla}^2 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\vec{k}\right).\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\vec{k}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{…. 2.15}$$

So, eqn.(2.14) can be written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho_v}{\varepsilon} \quad \text{…. 2.16}$$

Expressing the Laplacian in different coordinate systems to take advantage of the symmetry of a charge distribution simplifies the solution for the electric potential in many cases.

In electro-static field problems, the dielectric media may be considered to be ideal insulation. In such case, free charges reside only on the conductor boundaries. Hence, the volume charge density (ρ_v) within the field region is zero. Then eqn.(2.14) reduces to

$$\vec{\nabla}^2\phi = 0, \text{ or, } \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{…. 2.17}$$

Eqn.(2.17) is a second-order partial differential equation named after French Mathematician Pierre-Simon Laplace and is commonly known as Laplace's equation. There are an infinite number of functions that satisfy Laplace's equation and the proper solution is found by specifying the appropriate boundary conditions, which will be discussed in a later chapter.

In real life problems, however, the dielectric media are never ideal dielectrics. Hence, there could be volume leakage as well as surface leakage currents. Moreover, there could be discharges occurring within a particular zone of the insulation or there may be space charges that have accumulated over time within the field region of interest. In all such cases, volume charge density will be non-zero and hence Possion's equation needs to be solved to get the correct results.

Field Due to a Continuous Distribution of Charge

The electric potential at a point *p* due to a number of discrete charges could be obtained as simple algebraic superposition of the electric potentials produced at the point *p* by each of discrete charges acting in isolation. If $q_1,q_2,q_3,\ldots,q_n$ are discrete charges located at distances $r_1,r_2,r_3,\ldots,r_n$, respectively, from the point *p*, then the electric potential at *p* is given by

$$\phi_p = \frac{q_1}{4\pi\varepsilon r_1} + \frac{q_2}{4\pi\varepsilon r_2} + \ldots\ldots\ldots + \frac{q_N}{4\pi\varepsilon r_N} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{r_i} \qquad \ldots. 2.18$$

Now, if the charges are distributed continuously throughout the field region, instead of being located at discrete number of points, the field region can be divided into large number of small elements of volume ΔV, such that each element contains a charge $\rho_v \Delta V$, where ρ_v is the volume charge density of the small element ΔV. The potential at a point *p* will then be given by

$$\phi_p = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{\rho_{vi}\,\Delta V_i}{r_i} \qquad \ldots. 2.19$$

where, r_i is the distance of the i^{th} volume element from the point *p*.

As the size of volume element becomes very small, the summation becomes an integration, that is

$$\phi = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho_v\, dV}{r} \qquad \ldots. 2.20$$

The integration is performed over the volume where the volume charge density has finite value. However, it must be noted here that it is not valid for charge distribution which extends to infinity.

Eqn.(2.20) is often written in the form

$$\phi = \int_V \rho_v\, G\, dV \qquad \ldots. 2.21$$

In eqn.(2.21) the function G $(=\frac{1}{4\pi\varepsilon r})$ is the potential due to a unit point charge and is often referred as the electrostatic Green's function for a unbounded homogeneous region.

Problem 2.1

The potential field at any point in a space containing a dielectric material of relative permittivity 3.6 is given by $\phi = (3x^2y - 2y^2z + 5xyz^2)$ V, where *x,y,z* are in meters. Calculate the volume charge density at the point P(5,3,2)m.

Solution:

$$E_x = -\frac{\partial\phi}{\partial x} = (-6xy - 5yz^2)V/m, \text{ or, } D_x = \varepsilon_r\varepsilon_0 E_x\, C/m^2$$

$$E_y = -\frac{\partial\phi}{\partial y} = (-3x^2 + 4yz - 5xz^2)V/m, \text{ or, } D_y = \varepsilon_r\varepsilon_0 E_y\, C/m^2$$

$$E_z = -\frac{\partial\phi}{\partial z} = (2y^2 - 10xyz)V/m, \text{ or, } D_z = \varepsilon_r\varepsilon_0 E_z\, C/m^2$$

Therefore, $\frac{\partial D_x}{\partial x} = -6y\varepsilon_r\varepsilon_0$, $\frac{\partial D_y}{\partial y} = 4z\varepsilon_r\varepsilon_0$ and $\frac{\partial D_z}{\partial z} = -10xy\varepsilon_r\varepsilon_0$

So, $\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = (-6y + 4z - 10xy)\varepsilon_r \varepsilon_0$ C/m^3

Hence, at the point P(5,3,2), $\rho_{vP} = -160 \times 3.6 \times 8.854 \times 10^{-12} = -5.1$ nC/m^3

Problem 2.2

The electric flux density in free space is given by $\vec{D} = e^{-y}\left(\cos x\,\hat{i} - \sin x\,\hat{j}\right)$ C/m^2. Prove that the field region is charge free, i.e. no free charge is present in the field region.

Solution:

It is required to be proved that ρ_v is zero in the field region. The given expression for electric flux density vector indicates that it is a case of two-dimensional field.

As given: $D_x = e^{-y}\cos x$ and $D_y = -e^{-y}\sin x$

Hence, $\frac{\partial D_x}{\partial x} = -e^{-y}\sin x$ and $\frac{\partial D_y}{\partial y} = e^{-y}\sin x$

So, $\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = -e^{-y}\sin x + e^{-y}\sin x = 0$

Thus it is proved that no free charge is present in the field region.

Problem 2.3

A sphere of radius R carries a volume charge density $\rho_v(r) = kr$, where r is the radial distance from the centre and k is a constant. Determine the electric field intensity inside and outside the sphere.

Solution:

Inside the sphere, at any radius r_i, the amount of charge enclosed is given by

$$q(r_i) = \int_0^{r_i} kr\,\frac{4\pi}{3}r^3\,dr = \frac{4\pi k}{3}\int_0^{r_i} r^4\,dr = \frac{4\pi k\, r_i^5}{15}$$

Therefore, considering a spherical Gaussian surface of radius r_i ($<R$) within the given sphere, the electric flux is normal to the Gaussian surface and electric field intensity $E_i(r_i)$ is constant over the Gaussian surface. Thus, applying Gauss's law,

$$\varepsilon_0 \int \vec{E}_i(r_i).d\vec{A} = q(r_i), \text{ or, } \varepsilon_0\, 4\pi r_i^2\, E_i(r_i) = \frac{4\pi k\, r_i^5}{15}, \text{ or, } E_i(r_i) = \frac{k\, r_i^3}{15\varepsilon_0}$$

Since the electric field is directed radially, hence in vector form $\vec{E}_i(r_i) = \frac{k\, r_i^3}{15\varepsilon_0}\hat{u}_r$

Again, total charge within the sphere of radius R is $\frac{4\pi k R^5}{15}$

Hence, considering a spherical Gaussian surface of radius r_o ($>R$) outside the given sphere, the electric flux is again normal to the Gaussian surface and electric field intensity $E_o(r_o)$ is constant over the Gaussian surface. Thus, applying Gauss's law,

$$\varepsilon_0\, 4\pi r_o^2\, E_o(r_o) = \frac{4\pi k R^5}{15}, \text{ or, } E_o(r_o) = \frac{k R^5}{15\varepsilon_0 r_o^2}$$

Since the electric field is directed radially, hence in vector form $\vec{E}_o(r_o) = \frac{k R^5}{15\varepsilon_0 r_o^2}\hat{u}_r$

Problem 2.4

The electric flux density vector in a field region is given by $\vec{D}=(2y+z)\hat{i}+3xy\,\hat{j}+2x\hat{k}$ C/m^2. Determine the total charge enclosed by the cube defined by $0\le x\le 1$, $0\le y\le 1$ and $0\le z\le 1$.

Solution:

According to Gauss's law, the total charge enclosed by the cube will be equal to net flux through the cube.

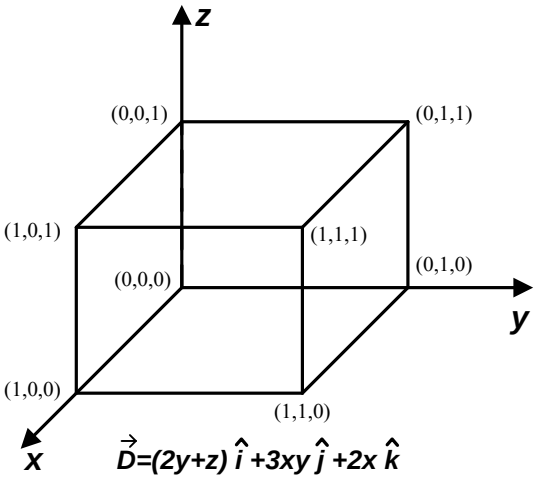


Fig.2.9 Pertaining to Problem 2.4

For *x=0*, the flux going into the cube through *y-z* plane, $\int_0^1\int_0^1(2y+z)dydz$

Again, for *x=1*, the flux coming out of the cube through *y-z* plane, $\int_0^1\int_0^1(2y+z)dydz$

Since these two integrals are same, the net flux coming out of the cube in *x*-direction, i.e. through *y-z* plane, is zero.

For *y=0*, the flux going into the cube through *x-z* plane, $\int_0^1\int_0^1 3xy\,dxdz=\left.\frac{3y}{2}\right|_{y=0}=0$

For *y=1*, the flux coming out of the cube through *x-z* plane, $\int_0^1\int_0^1 3xy\,dxdz=\left.\frac{3y}{2}\right|_{y=1}=1.5\,\text{C}$

So, the net flux coming out of the cube in *y*-direction, i.e. through *y-z* plane is 1.5C.

For *z=0*, the flux going into the cube through *x-y* plane, $\int_0^1\int_0^1 2x\,dxdy$

Again, for *z=1*, the flux coming out of the cube through *x-y* plane, $\int_0^1\int_0^1 2x\,dxdy$

Since these two integrals are same, the net flux coming out of the cube in *z*-direction, i.e. through *x-y* plane, is zero.

So, considering all three directions, net flux coming out of the cube is 1.5C.

Hence, the total charge enclosed by the given cube is also 1.5C.

Problem 2.5

A cylinder of unit volume is placed in a uniform field with its axis parallel to the direction of field. Determine the total charge enclosed by the unit cylinder.

Solution:

On the curved surface of the cylinder, the direction of the area vector is radially outward and hence is perpendicular to the electric field intensity which is directed along the axis of the

cylinder. Hence, on the curved surface $\int_A \varepsilon_0 \vec{E}.d\vec{A} = 0$, i.e. net flux through the curved surface of the cylinder will be zero.

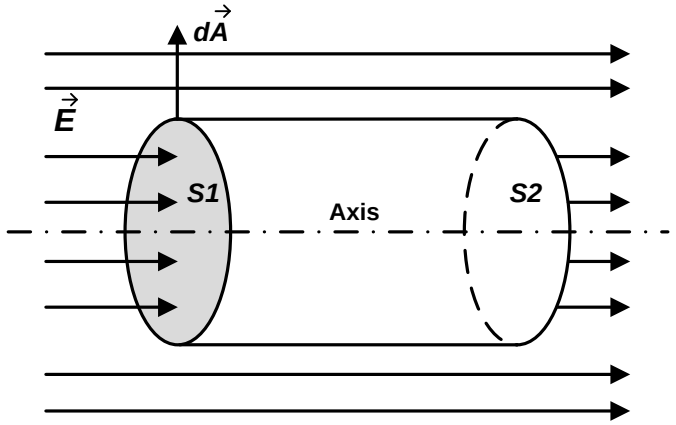


Fig.2.10 Pertaining to Problem 2.5

As shown in Fig.2.10, the two end surfaces of the cylinder *S1* and *S2* are normal to the direction of electric field intensity. Thus, the electric flux going into the cylinder through the surface *S1* on the left hand side will be $\varepsilon_0 \left|\vec{E}\right| A$, where *A* is the area of the surface *S1*. Similarly, the electric flux coming out of the cylinder through the surface *S2* on the right hand side will also be $\varepsilon_0 \left|\vec{E}\right| A$, as the area of the surface *S2* is also *A*. Since the field is stated to be uniform, hence electric field intensity is same everywhere. So the net flux through the end surfaces of the cylinder is also zero.
Thus the net flux through the cylinder considering all the surfaces of the cylinder is zero.
Hence, the total charge enclosed by the cylinder is zero.

Orthogonal Coordinate Systems

Basic Concepts

The concept that unifies the different coordinate systems is surfaces of constant coordinates. It means that an equation of the form (*coordinate=value*) defines a surface. In order to specify a point uniquely in three dimensional space three different types of surfaces are needed. The values associated with the three constant coordinate surfaces which intersect at a specific point are the three coordinates, which uniquely define that point.

Consider that three surfaces are defined by three functions f_1, f_2 and f_3, respectively, in three dimensional space. The coordinate, i.e. the value, which describes a surface could be defined by the equation $f_i=u_i$. Here, u is introduced to represent a generalized coordinate. In the stated equation, u_i refers to one of the coordinates in a particular system. Thus the three values (u_1,u_2,u_3) define a point where the three surfaces defined by eqn.(3.1) intersect. These three values (u_1,u_2,u_3) are then called the coordinates of that particular point.

$$f_1=u_1,\ f_2=u_2\ \ and\ \ f_3=u_3 \qquad \text{....} \ 3.1$$

The three functions, however, could not be chosen arbitrarily. The three functions are to be chosen in such a way that for any choice of the three coordinates (u_1,u_2,u_3) the three surfaces generated by eqn.(3.1) will intersect only at one point. It will ensure that a given set of three coordinate values (u_1,u_2,u_3) always specifies only one point in space. Moreover, the constant coordinate surfaces are always perpendicular to each other, i.e. orthogonal, at the point where they intersect. The intersection of any two constant coordinate surfaces defines a coordinate curve.

Unit Vector

An important concept associated with coordinate system is unit vector. It will be easy to understand the concept if it is explained with the help of a specific example involving Cartesian coordinate system as shown in Fig.3.1. Each value of z, e.g. z_1, z_2 or z_3, defines a plane normal to z-axis. The unit vector $\hat{k}$ is defined in such a way that it is normal to all such $z=constant$ planes and points in the direction of the planes with increasing values of z as shown in Fig.3.1. The magnitude of the vector $\hat{k}$ is taken as unity. The other two unit vectors in Cartesian coordinate system could be visualized in the same manner.

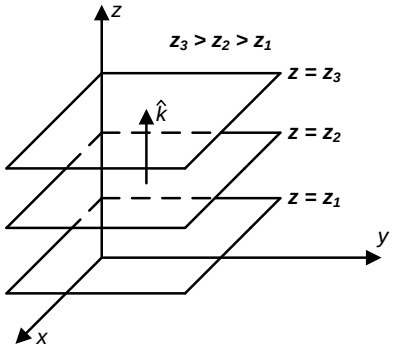


Fig.3.1 The unit vector $\hat{k}$

In a generalized manner, the properties of the unit vector associated with a constant coordinate surface defined by the function f_i are as follows: i) it is perpendicular to the surface $f_i=u_i$ at any point (u_1,u_2,u_3), ii) it points in the direction of the surfaces with increasing

value of u_i and iii) it is one unit in magnitude. Orthogonality of the constant coordinate surfaces as discussed earlier demands that the dot product of any two different unit vectors will always be zero. Conversely, the dot product of two same unit vectors, e.g. $\hat{k}.\hat{k}$, will be always 1.

Right-Handed Convention

Since three constant coordinate surfaces are involved in determining the coordinates of any point, the question that arises is in which order the three coordinates are to be mentioned while specifying a point. In this context, the choice of the first coordinate (u_1) or surface (f_1) is arbitrary. Once f_1 is chosen, then the order in which the other two constant coordinate surfaces f_2 and f_3 is to be taken is determined by the right handed system as governed by the definition of vector cross product. In other words, the cross product of the first and second unit vectors must be equal to the third unit vector.
This right handed property of coordinate system is an arbitrary convention. But once that choice is made for one coordinate system, then it has to be followed for all other coordinate systems to ensure that the expressions for the laws of electric field have the same form in all the coordinate systems.

Differential Distance and Metric Coefficient

The constant coordinate surfaces could be of many different types. The simplest of them is constant distance surfaces for which the differential change du_i in the coordinate u_i is the same as the differential distance dl_i between the surfaces $f_i = u_i$ and $f_i = u_i + du_i$. However, there are other types of surfaces like constant angle surfaces, in which case the differential distance is related to differential coordinate (angle) change as

$$dl_i = h_i\, du_i = h_i\, d\theta_i \qquad \ldots. 3.2$$

where, the scale factor h_i is the corresponding radius.
The scale factors for each of the coordinate system are called the metric coefficients of the respective coordinate system. The three metric functions of any coordinate system (h_1,h_2,h_3) constitute a unique set and hence are often known as the signature of a coordinate system. Metric coefficients are also very important in writing the vector operators in general orthogonal curvilinear coordinate system as will be explained in a later section.

Choice of Origin

A basic element of any coordinate system is the choice of origin. If any object or any arrangement is given, then any point within the object or within the whole arrangement could be chosen as origin.
From the earlier discussions, it is clear that the same point will have very different coordinates when defined in different coordinate systems. So the question that needs to be answered is that whether there is any property of the position of the point that remains the same in different coordinate systems. The answer to this question is that the distance of the point from the origin remains the same in all the coordinate systems if the origin is kept the same in all the cases. As a logical extension it may be stated that the distance between any two points remains the same no matter how their coordinates are defined in any coordinate system.

Cartesian Coordinate System

The invention of Cartesian coordinates in the 17th century by French mathematician René Descartes revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra. In this coordinate system, the three constant coordinate surfaces are defined by eqn.(3.3)

$$f_1 = x\ ,\ f_2 = y\ ,\ f_3 = z \qquad \ldots. 3.3$$

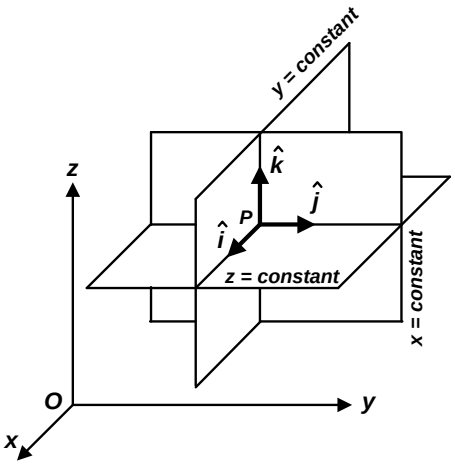


Fig.3.2 Constant coordinate surfaces in Cartesian coordinate system

Fig.3.2 shows such constant coordinate surfaces, which are three planes. Cartesian coordinate system has the unique feature that all the three constant coordinate surfaces are constant distance surfaces.

Each point *P* in space is then assigned a triplet of values (*x*,*y*,*z*), the so called Cartesian coordinates of that point. The ranges of the values of three coordinates are: $-\infty<x<\infty$, $-\infty<y<\infty$ and $-\infty<z<\infty$. The coordinates can be defined as the positions of the normal projections of the point onto three mutually perpendicular constant coordinate surfaces passing through the origin, expressed as signed distances from the origin.

In Cartesian coordinate system, three mutually perpendicular axes are commonly defined as follows. The intersection of two mutually perpendicular planes *y=constant* and *z=constant* passing through the origin gives a coordinate curve, which is a straight line. This line is then defined as the *x-axis*. Similarly, the intersections of other two pairs of mutually perpendicular constant coordinate planes passing through origin define the *y-axis* and *z-axis*.

The unit vector $\hat{i}$ is perpendicular to all the surfaces described by *x=constant* and points in the direction of increasing value of *x*-coordinate. In the same manner, unit vectors $\hat{j}$ and $\hat{k}$ are defined in *y*- and *z*-directions, respectively. Another unique feature of Cartesian coordinate system is that the directions of the three unit vectors remain the same irrespective of the location of the point in space. The orthogonality of Cartesian coordinate system is defined by eqn.(3.4)

$$\hat{i}.\hat{j}=0\ ,\ \hat{j}.\hat{k}=0\ ,\ \hat{k}.\hat{i}=0 \qquad \ldots. 3.4$$

and the right handedness is defined by eqn.(3.5)

$$\hat{i}\times\hat{j}=\hat{k}\ ,\ \hat{j}\times\hat{k}=\hat{i}\ ,\ \hat{k}\times\hat{i}=\hat{j} \qquad \ldots. 3.5$$

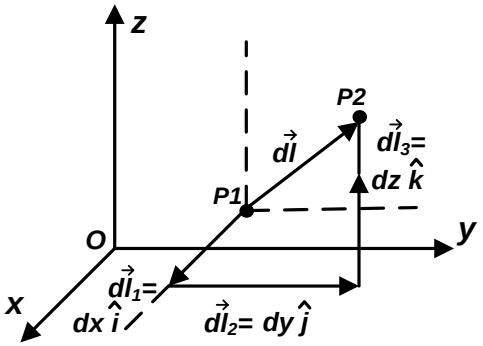


Fig.3.3 Differential line element in Cartesian coordinate system

As shown in Fig.3.3, the differential line element in Cartesian coordinate system is given by eqn.(3.6)

$$\vec{dl} = dl_1\,\hat{i} + dl_2\,\hat{j} + dl_3\,\hat{k} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \qquad \text{…. 3.6}$$

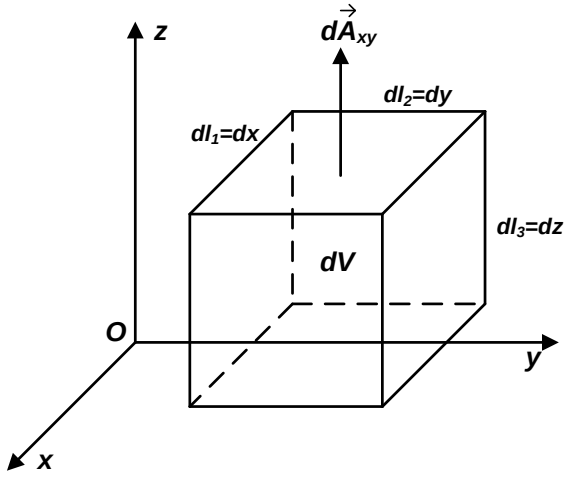


Fig.3.4 Differential area and volume elements in Cartesian coordinate system

As shown in Fig.3.4, the differential area element corresponding to the surface of a differential cubical volume is given by eqn.(3.7)

$$\vec{dA}_{xy} = dl_1\,dl_2\,\hat{k} = dx\,dy\,\hat{k} \qquad \text{…. 3.7}$$

As shown in Fig.3.4, the differential volume element is given by eqn.(3.8)

$$dV = dl_1\,dl_2\,dl_3 = dx\;dy\;dz \qquad \text{…. 3.8}$$

As depicted in eqn.(3.6), the differential distance in *x*-direction is $dl_1=dx=h_1dx$, i.e. $h_1=1$. Similarly, in *y*- and *z*-directions $h_2=h_3=1$. The metric coefficients of Cartesian coordinate system are therefore $h_1=1$, $h_2=1$ and $h_3=1$.

Cylindrical Coordinate System

Cylindrical coordinates are an alternate way of describing points in three dimensional space. In this coordinate system, one of the rectangular coordinate planes, viz. the *x-y* plane as described by Cartesian coordinate system, is replaced by a polar plane. In the cylindrical coordinate system, everything is measured with respect to a fixed point called the pole and an axis called the polar axis. The pole is the equivalent to the origin in the Cartesian coordinate system and the polar axis corresponds to the positive direction of *x*-axis. The cylindrical coordinates of a point are then the ordered triplet (r,θ,z) as defined in Fig.3.5.

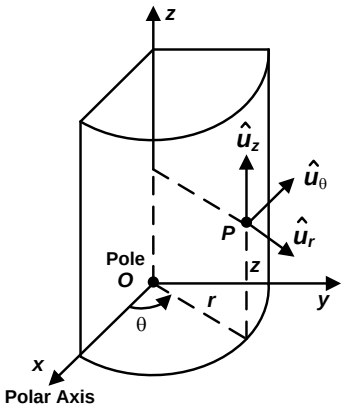


Fig.3.5 Depiction of cylindrical coordinates of a point

As shown in Fig.3.5, *r* is the distance from the pole to the projection of the point *P* on the polar plane, i.e. the *x-y* plane passing through the pole, θ is the azimuthal angle, i.e. the angle from the polar axis spinning around the *z*-axis in counter-clockwise direction, and *z* is the vertical height from the polar plane. The ranges of the values of the three coordinates are: $0 \le r < \infty$, $0 \le \theta \le 2\pi$ and $-\infty < z < \infty$.

In cylindrical coordinate system, the three constant coordinate surfaces are defined by eqn.(3.9)

$$f_1 = r \ , \ f_2 = \theta \ , \ f_3 = z \qquad \text{…. 3.9}$$

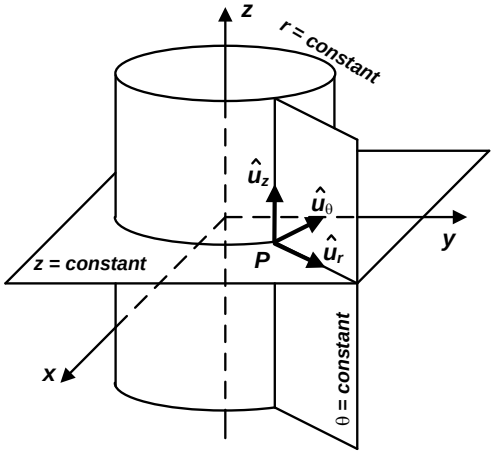


Fig.3.6 Constant coordinate surfaces in cylindrical coordinate system

Fig.3.6 shows the three constant coordinate surfaces in cylindrical coordinate system. Out of these three surfaces, the first and the third surfaces, viz. $f_1 = r$ *and* $f_3 = z$, are constant distance surfaces, while the second one, i.e. $f_2 = \theta$, is a constant angle surface. As shown in Fig.3.6, the surfaces θ*=constant* and *z=constant* are planes, while the surface *r=constant* is a cylindrical surface.

In this coordinate system, two unit vectors are defined on the *x-y* plane. The unit vector $\hat{u}_r$ points in the direction of increasing *r*, i.e. radially outward from the *z*-axis and the unit vector $\hat{u}_\theta$ points in the direction of increasing θ, i.e. it points in the direction of the tangent to the circle of radius *r* in the counter-clockwise sense. The third unit vector $\hat{u}_z$ points in the direction of increasing *z*, i.e. vertically upward from the *x-y* plane. The unit vectors are shown in Fig.3.5. The orthogonality of cylindrical coordinate system is defined by eqn.(3.10)

$\hat{u}_r . \hat{u}_\theta = 0$, $\hat{u}_\theta . \hat{u}_z = 0$, $\hat{u}_z . \hat{u}_r = 0$ …. 3.10

and the right handedness is defined by eqn.(3.11)

$\hat{u}_r \times \hat{u}_\theta = \hat{u}_z$, $\hat{u}_\theta \times \hat{u}_z = \hat{u}_r$, $\hat{u}_z \times \hat{u}_r = \hat{u}_\theta$ …. 3.11

Unlike Cartesian system, in cylindrical coordinate system the directions of all the three unit vectors do not remain the same as one move around the space. As shown in Fig.3.7, the directions of the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ get changed at different points in space keeping $\hat{u}_z$ unchanged.

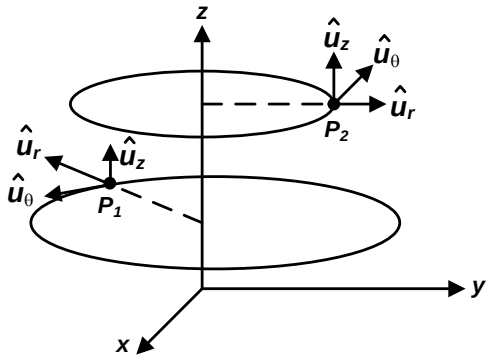


Fig.3.7 Unit vectors at different points in cylindrical coordinate system

As shown in Fig.3.8, the differential line element in cylindrical coordinate system is given by eqn.(3.12)

$d\vec{l} = dl_1 \hat{u}_r + dl_2 \hat{u}_\theta + dl_3 \hat{u}_z = dr\,\hat{u}_r + rd\theta\,\hat{u}_\theta + dz\,\hat{u}_z$ …. 3.12

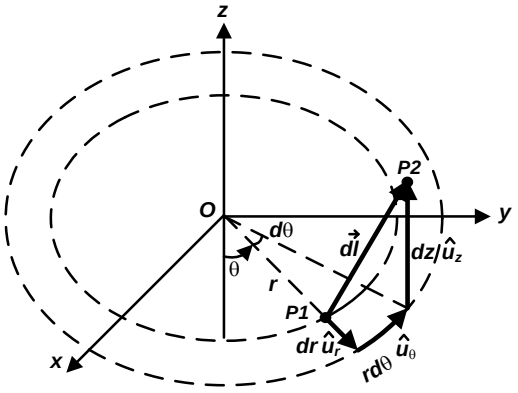


Fig.3.8 Differential line element in cylindrical coordinate system

As shown in Fig.3.9, in cylindrical coordinate system the differential area element on the surface of a cylinder of radius *r* is given by eqn.(3.13)

$d\vec{A}_{\theta z} = dl_2\, dl_3 \hat{u}_r = rd\theta\, dz\, \hat{u}_r$ …. 3.13

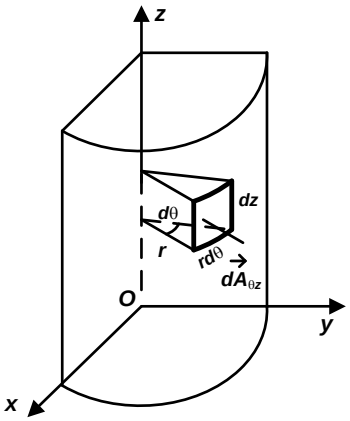


Fig.3.9 Differential area element on the cylinder surface in cylindrical coordinate system

and the differential area element on the surface of a disc in the *x-y* plane as shown in Fig.3.10 is given by eqn.(3.14)

$$d\vec{A}_{r\theta} = dl_1\, dl_2\, \hat{u}_z = dr\ rd\theta\, \hat{u}_z \qquad \text{…. 3.14}$$

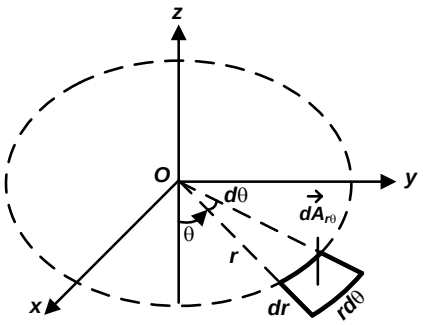


Fig.3.10 Differential area element on the disc surface in cylindrical coordinate system

As shown in Fig.3.11, the differential volume element is given by eqn.(3.15)

$$dV = dl_1\, dl_2\, dl_3 = dr\ rd\theta\ \ dz \qquad \text{…. 3.15}$$

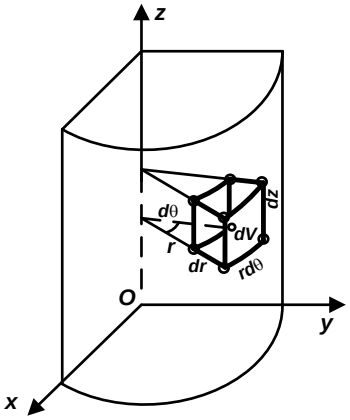


Fig.3.11 Differential volume element of a cylinder in cylindrical coordinate system

As depicted in eqn.(3.12), the differential distance in *r*-direction is $dl_1=dr=h_1dr$, i.e. $h_1=1$, in θ-direction it is $dl_2=rd\theta=h_2d\theta$, i.e. $h_2=r$, and in *z*-direction $dl_3=dz=h_3dz$, i.e. $h_3=1$. The metric coefficients of cylindrical coordinate system are therefore $h_1=1$, $h_2=r$ and $h_3=1$.

Spherical Coordinate System

Spherical coordinates are particularly useful for analysing fields having spherical symmetry. In spherical coordinate system the coordinates of any point in space are the ordered triplet (r,θ,ϕ) as shown in Fig.3.12. The coordinate *r* measures the radial distance from the origin to the point *P*. The coordinate θ is the angle that the *r* vector makes with the positive direction of *z*-axis, while the coordinate ϕ is the azimuthal angle with respect to the positive direction of *x*-axis spinning around the *z*-axis in counter-clockwise sense. The angle ϕ is defined on the *x-y* plane. The ranges of the values of the three coordinates are: $0\le r<\infty$, $0\le\theta\le\pi$ and $0\le\phi\le2\pi$.

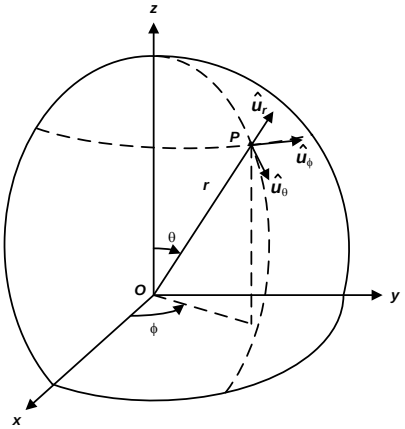


Fig.3.12 Depiction of spherical coordinates of a point

In spherical coordinate system, the three constant coordinate surfaces are defined by eqn.(3.16)

$$f_1=r\ ,\ f_2=\theta\ ,\ f_3=\phi \qquad \text{…. 3.16}$$

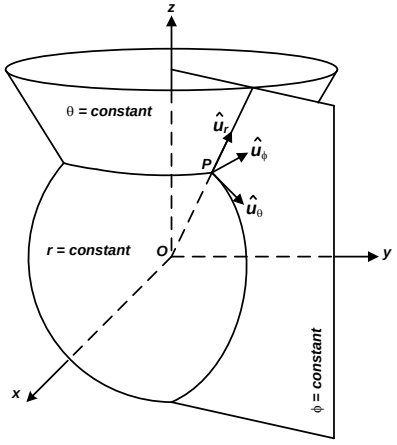


Fig.3.13 Constant coordinate surfaces in spherical coordinate system

Fig.3.13 shows the three constant coordinate surfaces in spherical coordinate system. Out of these three surfaces, the first surface, viz. $f_1=r$, is a constant distance surface, while the other two surfaces, i.e. $f_2=\theta$ *and* $f_3=\phi$, are constant angle surfaces. As shown in Fig.3.13, *r=constant* is a spherical surface, θ*=constant* is a conical surface and ϕ*=constant* is a plane.

In this coordinate system, the unit vector $\hat{u}_r$ points in the direction of increasing *r*, i.e. radially outward from the origin. The unit vector $\hat{u}_\theta$ points in the direction of increasing θ along the tangent to a circle of radius *r* in the plane containing the *z*-axis and the *r* vector. The unit vector $\hat{u}_\phi$ points in the direction of increasing ϕ along the tangent to a circle in the *x*-*y* plane which is centered on the *z*-axis. The unit vectors are shown in Fig.3.12. The orthogonality of spherical coordinate system is defined by eqn.(3.17)

$$\hat{u}_r.\hat{u}_\theta = 0\ ,\ \hat{u}_\theta.\hat{u}_\phi = 0\ ,\ \hat{u}_\phi.\hat{u}_r = 0 \qquad \text{…. 3.17}$$

and the right handedness is defined by eqn.(3.18)

$$\hat{u}_r \times \hat{u}_\theta = \hat{u}_\phi\ ,\ \hat{u}_\theta \times \hat{u}_\phi = \hat{u}_r\ ,\ \hat{u}_\phi \times \hat{u}_r = \hat{u}_\theta \qquad \text{…. 3.18}$$

Similar to cylindrical coordinate system, in spherical coordinate system the directions of all the three unit vectors do not remain the same as one move around the space. As shown in Fig.3.14, the directions of all the three unit vectors $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_\phi$ get changed at different points in space.

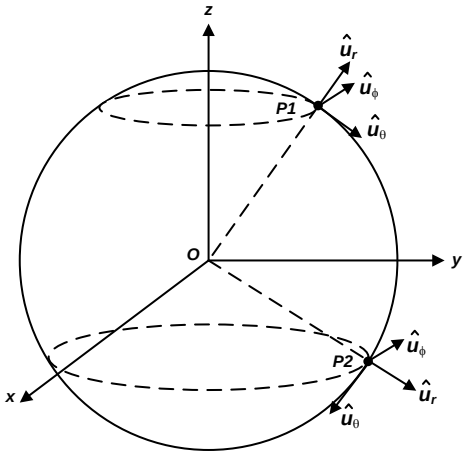


Fig.3.14 Unit vectors at different points in spherical coordinate system

As shown in Fig.3.15, the differential line element in spherical coordinate system is given by eqn.(3.19)

$$d\vec{l} = dl_1\hat{u}_r + dl_2\hat{u}_\theta + dl_3\hat{u}_\phi = dr\,\hat{u}_r + rd\theta\,\hat{u}_\theta + r\sin\theta\,d\phi\,\hat{u}_\phi \qquad \text{…. 3.19}$$

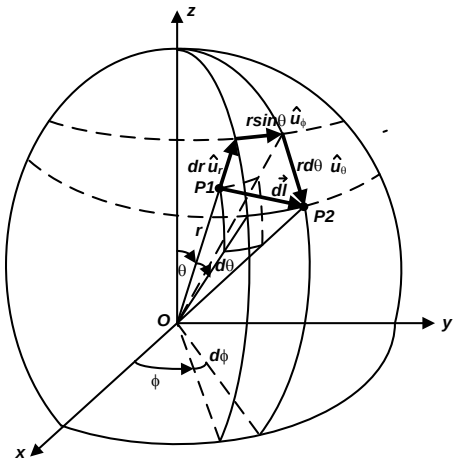


Fig.3.15 Differential line element in spherical coordinate system

As shown in Fig.3.16, in spherical coordinate system the differential area element on the surface of a sphere of radius r is given by eqn.(3.20)

$$d\vec{A}_{\theta\phi} = dl_2\, dl_3\, \hat{u}_r = (r d\theta)(r\sin\theta\, d\phi)\hat{u}_r = r^2 \sin\theta\, d\theta\, d\phi\, \hat{u}_r \quad \text{…. 3.20}$$

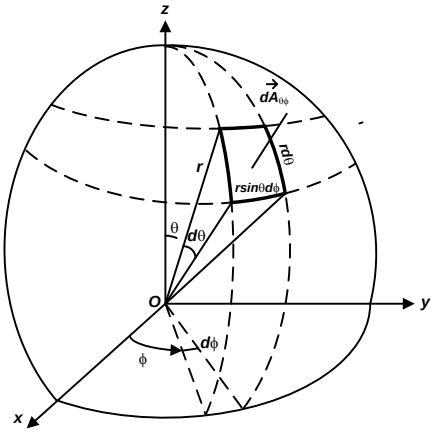


Fig.3.16 Differential area element on the sphere surface in spherical coordinate system

As shown in Fig.3.17, the differential volume element is given by eqn.(3.21)

$$dV = dl_1\, dl_2\, dl_3 = (dr)(rd\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi \quad \text{…. 3.21}$$

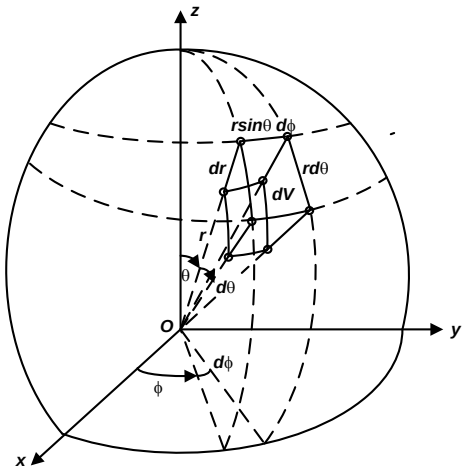


Fig.3.17 Differential volume element of a sphere in spherical coordinate system

As depicted in eqn.(3.19), the differential distance in r-direction is $dl_1=dr=h_1dr$, i.e. $h_1=1$, in θ-direction it is $dl_2=rd\theta=h_2d\theta$, i.e. $h_2=r$, and in ϕ-direction $dl_3=rsin\theta d\phi=h_3d\phi$, i.e. $h_3=rsin\theta$. Therefore, the metric coefficients of spherical coordinate system are $h_1=1$, $h_2=r$ and $h_3=rsin\theta$.

Generalized Orthogonal Curvilinear Coordinate System

From the discussions of the previous sections, it may be seen that a major hindrance in treating the three commonly used orthogonal coordinate systems, viz. Cartesian, cylindrical and spherical coordinate systems, on an equal footing is that the unit vectors are not the best way to visualize these three coordinate systems. In this context, a generalized orthogonal curvilinear coordinate system provides insight for useful linkage between such orthogonal coordinate systems. Generalized orthogonal curvilinear coordinate system emphasizes the

similarities between different orthogonal coordinate systems rather than highlighting their differences. Thus the definition and utilization of generalized orthogonal curvilinear coordinate system is very important for proper understanding of the formulation and solution of electric field problems.

Consider that $f(x,y,z)=u$ is a function of three independent space variables x,y,z in the Cartesian coordinate system that specify a surface characterized by the constant parameter u. Setting such function equal to three different constant parameters u_1, u_2 and u_3 defines three different surfaces as follows

$$f_1(x,y,z)=u_1,\ \ f_2(x,y,z)=u_2 \ and \ \ f_3(x,y,z)=u_3 \qquad \text{…. 3.22}$$

Considering that these three surfaces are orthogonal, they intersect in space only at one point. In other words, any point in space could be uniquely defined by a set of values of the three parameters (u_1,u_2,u_3), which are then called the orthogonal curvilinear coordinates of the point being defined. The constant coordinate surfaces for a generalized orthogonal curvilinear coordinate system are shown in Fig.3.18.

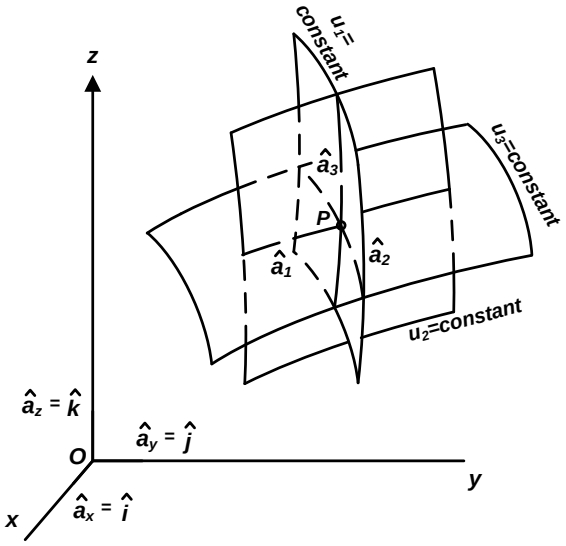


Fig.3.18 Constant coordinate surfaces in generalized orthogonal curvilinear coordinate system

As shown in Fig.3.18, the unit vectors $\hat{a}_1, \hat{a}_2$ and $\hat{a}_3$ are normal to the surfaces u_1=*constant*, u_2=*constant* and u_3=*constant*, respectively, and point towards the increasing values of the coordinates u_1, u_2 and u_3, respectively. The orthogonality of the generalized curvilinear coordinate system is defined by eqn.(3.23)

$$\hat{a}_1.\hat{a}_2=0\ ,\ \hat{a}_2.\hat{a}_3=0,\ \hat{a}_3.\hat{a}_1=0 \qquad \text{…. 3.23}$$

and the right handedness is defined by eqn.(3.24)

$$\hat{a}_1\times\hat{a}_2=\hat{a}_3\ ,\ \hat{a}_2\times\hat{a}_3=\hat{a}_1,\ \hat{a}_3\times\hat{a}_1=\hat{a}_2 \qquad \text{…. 3.24}$$

As shown in Fig.3.19, the surfaces $u=u_1$ and $u=u_1+du_1$ are separated by a differential length element dl_1 which is normal to the surface $u=u_1$, where $dl_1=h_1(u_1,u_2,u_3)du_1$. The scale factor or the metric coefficient h_1 is a function of the three curvilinear coordinates (u_1,u_2,u_3). The other two metric coefficients, i.e. $h_2(u_1,u_2,u_3)$ and $h_3(u_1,u_2,u_3)$, are also similarly defined.

As shown in Fig.3.19, the differential line element in generalized orthogonal curvilinear coordinate system is given by eqn.(3.25)

$$d\vec{l}=dl_1\,\hat{a}_1+dl_2\,\hat{a}_2+dl_3\,\hat{a}_3=h_1du_1\hat{a}_1+h_2du_2\hat{a}_2+h_3du_3\hat{a}_3 \qquad \text{…. 3.25}$$

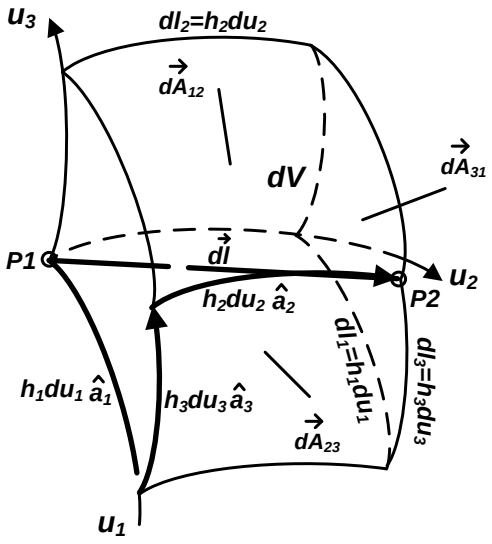


Fig.3.19 Differential line, area and volume elements in generalized orthogonal curvilinear coordinate system

As shown in Fig.3.19, the differential area elements in generalized orthogonal curvilinear coordinate system are given by eqn.(3.26)

$$d\vec{A}_{12} = (h_1 du_1)(h_2 du_2)\hat{a}_3 = h_1 h_2 du_1 du_2 \hat{a}_3$$

$$d\vec{A}_{23} = (h_2 du_2)(h_3 du_3)\hat{a}_1 = h_2 h_3 du_2 du_3 \hat{a}_1$$

$$d\vec{A}_{31} = (h_3 du_3)(h_1 du_1)\hat{a}_2 = h_3 h_1 du_3 du_1 \hat{a}_2 \quad \text{…. 3.26}$$

and the differential volume element in generalized orthogonal curvilinear coordinate system is given by eqn.(3.27)

$$dV = (h_1 du_1)(h_2 du_2)(h_3 du_3) = h_1 h_2 h_3 du_1 du_2 du_3 \quad \text{…. 3.27}$$

In the light of the generalized orthogonal curvilinear coordinate system, the three commonly used orthogonal coordinate systems could be summarized as in Table3.1.

Table 3.1 Characterization of orthogonal coordinate systems

Orthogonal Coordinate System	Coordinates	Unit vectors	Metric Coefficients
Generalized	(u_1,u_2,u_3)	$(\hat{a}_1,\hat{a}_2,\hat{a}_3)$	(h_1,h_2,h_3)
Cartesian	(x,y,z)	$(\hat{i},\hat{j},\hat{k})$	$(1,1,1)$
Cylindrical	(r,θ,z)	$(\hat{u}_r,\hat{u}_\theta,\hat{u}_z)$	$(1,r,1)$
Spherical	(r,θ,ϕ)	$(\hat{u}_r,\hat{u}_\theta,\hat{u}_\phi)$	$(1,r,r\sin\theta)$

Recall that the metric coefficients were earlier stated to be functions of the coordinates (u_1,u_2,u_3). Table 3.1 clearly shows such dependence of the metric coefficients on the respective coordinates in cylindrical and spherical coordinate systems. Cartesian coordinate system is unique in the sense that its metric coefficients are constant.

Vector Operations

In field analysis it is often required to perform vector operations to get vector and scalar derivatives of scalar and vector fields, which are scalar and vector functions of position,

respectively. Most commonly used vector operations are: gradient, divergence, curl and Laplacian operation. It is possible to get expressions for all these vector operations in generalized orthogonal curvilinear coordinate system. In order to get such expressions, consider an arbitrary scalar field like electric potential or temperature as $U(u_1,u_2,u_3)$ and an arbitrary vector field like electric flux density or force as

$$\vec{S} = S_1(u_1,u_2,u_3)\,\hat{a}_1 + S_2(u_1,u_2,u_3)\,\hat{a}_2 + S_3(u_1,u_2,u_3)\,\hat{a}_3 \quad \text{…. 3.28}$$

Gradient

In electric field analysis, a common application of gradient operation is to determine electric field intensity from the knowledge of electric potential. In order to understand this concept clearly, a change *dU* in the scalar function *U* is expressed in terms of the changes in differential distances dl_1, dl_2 and dl_3 in generalized orthogonal curvilinear coordinate system, as follows:

$$dU = \frac{\partial U}{\partial l_1}dl_1 + \frac{\partial U}{\partial l_2}dl_2 + \frac{\partial U}{\partial l_3}dl_3 \quad \text{…. 3.29}$$

Introducing the differential distance as $d\vec{l} = dl_1\,\hat{a}_1 + dl_2\,\hat{a}_2 + dl_3\,\hat{a}_3$, eqn 3.29 could be rewritten as

$$dU = \left(\frac{\partial U}{\partial l_1}\hat{a}_1 + \frac{\partial U}{\partial l_2}\hat{a}_2 + \frac{\partial U}{\partial l_3}\hat{a}_3\right).\left(dl_1\,\hat{a}_1 + dl_2\,\hat{a}_2 + dl_3\,\hat{a}_3\right) = \vec{\nabla}U.d\vec{l} \quad \text{…. 3.30}$$

where, $$\vec{\nabla}U = \frac{\partial U}{\partial l_1}\hat{a}_1 + \frac{\partial U}{\partial l_2}\hat{a}_2 + \frac{\partial U}{\partial l_3}\hat{a}_3 = \frac{1}{h_1}\frac{\partial U}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial U}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial U}{\partial u_3}\hat{a}_3 \quad \text{…. 3.31}$$

Eqn.(3.30) indicates that the change *dU* in the scalar function *U* over the differential distance *dl* at any point in space is maximum when $\vec{\nabla}U$ and $d\vec{l}$ are in the same direction. In other words, the magnitude of the vector $\vec{\nabla}U$, which is the spatial derivative of the scalar function *U* and is called the gradient of *U*, is equal to the maximum value of *dU/dl* and it is in the direction in which *dU/dl* is maximum.

With the help of the metric coefficients and other parameters mentioned in Table 3.1, the gradient function can be written in different orthogonal coordinate systems as detailed below:

Cartesian coordinate system:

$$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k} \quad \text{…. 3.32}$$

Cylindrical coordinate system:

$$\vec{\nabla}U = \frac{\partial U}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{u}_\theta + \frac{\partial U}{\partial z}\hat{u}_z \quad \text{…. 3.33}$$

Spherical coordinate system:

$$\vec{\nabla}U = \frac{\partial U}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{u}_\phi \quad \text{…. 3.34}$$

Del Operator

Eqn.(3.30) introduces a vector differential operator $\vec{\nabla}$ such that when $\vec{\nabla}$ is applied on a scalar function it results into a vector function. Such vector operation is called Gradient.

From eqn.(3.31)

$$\vec{\nabla}U = \frac{1}{h_1}\frac{\partial U}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial U}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial U}{\partial u_3}\hat{a}_3 = \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}\hat{a}_3\right)U$$

so that $\vec{\nabla} = \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}\hat{a}_3\right)$ …. 3.35

It has been discussed in Chapter-2 that when $\vec{\nabla}$ is applied on a vector function as scalar or dot product, then it results into a scalar function and such operation is called Divergence. On the other hand when $\vec{\nabla}$ is operated on a vector function with vector or cross product then it results into another vector function and this operation is called Curl.

Divergence

In electric field analysis divergence is used to relate electric field with the source, i.e. the charges, as given by the differential form of Gauss's law.

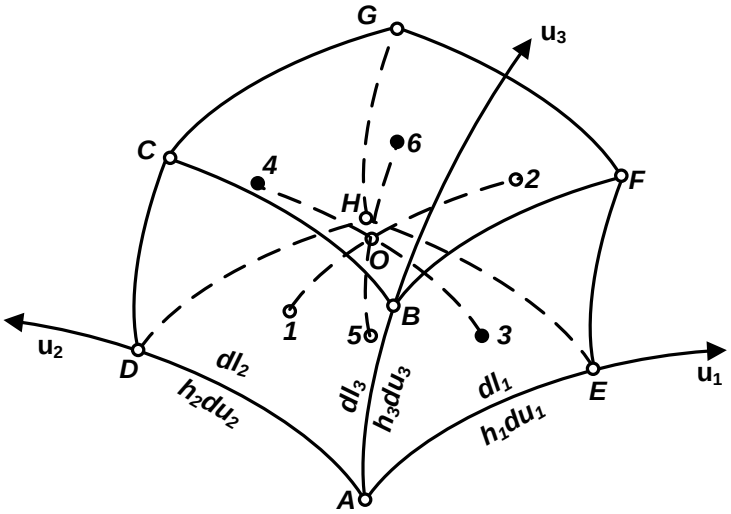


Fig.3.20 Divergence in generalized orthogonal curvilinear coordinate system

It has already been discussed in Chapter-2 that the divergence of a vector field $\vec{S}$ could be evaluated by finding the net flux coming out per unit volume. In this context, consider the differential volume element $dV(=dl_1dl_2dl_3)$ as shown in Fig.3.20. Consider that the point *O* is located at the center of the volume element *dV* and also that the vector $\vec{S}$ is known at the point *O*. The points *1* and *2* are the midpoints of the surfaces *ABCD* and *EFGH*, respectively, which are normal to u_1 direction. So along the line *1-O-2* the derivatives of S_1, which is the u_1 component of $\vec{S}$, wrt u_1 will be finite, while the derivative of S_1 wrt u_2 and u_3 will be zero as S_1 is orthogonal to u_2 and u_3. Considering the differential distance dl_1 between the two surfaces under reference, viz. *ABCD* and *EFGH*, S_1 at the points *1* and *2* can be evaluated from the knowledge of S_1 at the point *O* with the help of Taylor series. Noting that the distances between the point *O* and both the points *1* and *2* are $dl_1/2$, from Taylor Series expansion it can be written neglecting higher order terms that

$S_1\big|_1 = S_1 - \frac{\partial S_1}{\partial l_1}\frac{dl_1}{2}$ (as the distance from *O* to *1* is in the negative sense of u_1)

$S_1\big|_2 = S_1 + \frac{\partial S_1}{\partial l_1}\frac{dl_1}{2}$ (as the distance from *O* to 2 is in the positive sense of u_1)

where, S_1 stands for the value of S_1 at the point O.

Therefore, $S_1\big|_1 = S_1 - \frac{\partial S_1}{h_1 \partial u_1}\frac{h_1 du_1}{2} = S_1 - \frac{\partial S_1}{\partial u_1}\frac{du_1}{2}$

and $S_1\big|_2 = S_1 + \frac{\partial S_1}{h_1 \partial u_1}\frac{h_1 du_1}{2} = S_1 + \frac{\partial S_1}{\partial u_1}\frac{du_1}{2}$

The area of the surface *ABCD* is given by $dl_2 dl_3 = h_2 h_3 du_2 du_3$. So the flux of $\vec{S}$ going into the differential volume through the surface *ABCD* in the u_1 direction is given by

$$d\psi_{ABCD} = \left(S_1 h_2 h_3 du_2 du_3\right) - \frac{\partial\left(S_1 h_2 h_3 du_2 du_3\right)}{\partial u_1}\frac{du_1}{2} = \left(S_1 h_2 h_3 du_2 du_3\right) - \frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1}\frac{du_1 du_2 du_3}{2}$$

as (h_1, h_2, h_3) are functions of (u_1, u_2, u_3), the product $h_2 h_3$ is kept within the partial derivative term.

Similarly, the flux of $\vec{S}$ coming out of the surface *EFGH* in the u_1 direction is given by

$$d\psi_{EFGH} = \left(S_1 h_2 h_3 du_2 du_3\right) + \frac{\partial\left(S_1 h_2 h_3 du_2 du_3\right)}{\partial u_1}\frac{du_1}{2} = \left(S_1 h_2 h_3 du_2 du_3\right) + \frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1}\frac{du_1 du_2 du_3}{2}$$

Hence, the net flux coming out of the differential volume in the u_1 direction is given by

$$d\psi_1 = d\psi_{EFGH} - d\psi_{ABCD} = \frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1} du_1 du_2 du_3 \qquad \text{…. 3.36}$$

So considering all the three directions, i.e. u_1, u_2 and u_3 directions, the net flux coming out of the volume dV is

$$\psi_{dV} = \left(\frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1} + \frac{\partial\left(S_2 h_3 h_1\right)}{\partial u_2} + \frac{\partial\left(S_3 h_1 h_2\right)}{\partial u_3}\right) du_1 du_2 du_3 \qquad \text{…. 3.37}$$

Therefore, the divergence of $\vec{S}$, which is the net flux coming out per unit volume, is given by

$$div\,\vec{S} = \frac{\psi_{dV}}{dV} = \frac{1}{dl_1 dl_2 dl_3}\left(\frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1} + \frac{\partial\left(S_2 h_3 h_1\right)}{\partial u_2} + \frac{\partial\left(S_3 h_1 h_2\right)}{\partial u_3}\right) du_1 du_2 du_3$$

$$or,\ div\,\vec{S} = \frac{1}{h_1 h_2 h_3 du_1 du_2 du_3}\left(\frac{\partial\left(S_1 h_2 h_3\right)}{\partial u_1} + \frac{\partial\left(S_2 h_3 h_1\right)}{\partial u_2} + \frac{\partial\left(S_3 h_1 h_2\right)}{\partial u_3}\right) du_1 du_2 du_3$$

$$or,\ \vec{\nabla}.\vec{S} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(h_2 h_3 S_1\right) + \frac{\partial}{\partial u_2}\left(h_3 h_1 S_2\right) + \frac{\partial}{\partial u_3}\left(h_1 h_2 S_3\right)\right) \qquad \text{…. 3.38}$$

The divergence function can be written in different orthogonal coordinate systems as detailed below:

Cartesian coordinate system:

$$\vec{\nabla}.\vec{S} = \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right) \qquad \text{…. 3.39}$$

Cylindrical coordinate system:

$$\vec{\nabla}.\vec{S} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r S_r\right) + \frac{\partial S_\theta}{\partial \theta} + \frac{\partial}{\partial z}\left(r S_z\right)\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r S_r\right) + \frac{1}{r}\frac{\partial S_\theta}{\partial \theta} + \frac{\partial S_z}{\partial z} \qquad \text{…. 3.40}$$

Spherical coordinate system:

$$\vec{\nabla}.\vec{S} = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta\, S_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta\, S_\theta\right) + \frac{\partial}{\partial \phi}\left(r S_\phi\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, S_\theta\right) + \frac{1}{r \sin\theta}\frac{\partial S_\phi}{\partial \phi} \qquad \text{…. 3.41}$$

Laplacian

Let, $\vec{S} = \vec{\nabla} U = \frac{1}{h_1}\frac{\partial U}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial U}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial U}{\partial u_3}\hat{a}_3 = S_1\hat{a}_1 + S_2\hat{a}_2 + S_3\hat{a}_3$

so that $S_1 = \frac{1}{h_1}\frac{\partial U}{\partial u_1}$, $S_2 = \frac{1}{h_2}\frac{\partial U}{\partial u_2}$ and $S_3 = \frac{1}{h_3}\frac{\partial U}{\partial u_3}$

Then $\vec{\nabla}^2 U = \vec{\nabla}.\vec{\nabla} U = \vec{\nabla}.\vec{S}$

So from eqn.(3.38)

$$\vec{\nabla}^2 U = \vec{\nabla}.\vec{S} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 S_1) + \frac{\partial}{\partial u_2}(h_3 h_1 S_2) + \frac{\partial}{\partial u_3}(h_1 h_2 S_3)\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial U}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial U}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial U}{\partial u_3}\right)\right) \quad \text{…. 3.42}$$

The Laplacian can be written in different orthogonal coordinate systems as detailed below:

Cartesian coordinate system:

$$\vec{\nabla}^2 U = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) \quad \text{…. 3.43}$$

Cylindrical coordinate system:

$$\vec{\nabla}^2 U = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial U}{\partial z}\right)\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

…. 3.44

Spherical coordinate system:

$$\vec{\nabla}^2 U = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial \phi}\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

…. 3.45

Curl

The curl is evaluated as closed line integral per unit area. The closed line integral of a vector is also known as circulation. The circulation of a vector is obtained by multiplying the component of that vector parallel to the specified closed path at each point along it by the differential path length and summing the results of multiplication as the differential lengths approach zero. Since there are three planes which are normal to the three components of a vector, hence the circulation of a vector is to be computed separately for these three planes, each one of which will give one component of the curl. The curl of any vector thus results into another vector. Any component of the curl is given by the quotient of the closed line integral of the vector about a small path and the area enclosed by that path, as the path shrinks to zero. The small path is to be chosen in a plane normal to the desired component of the curl.

With reference to Fig.3.21, consider that the values of three components of $\vec{S}$, viz. S_1, S_2 and S_3, are known at the midpoint O of the differential areas as shown.

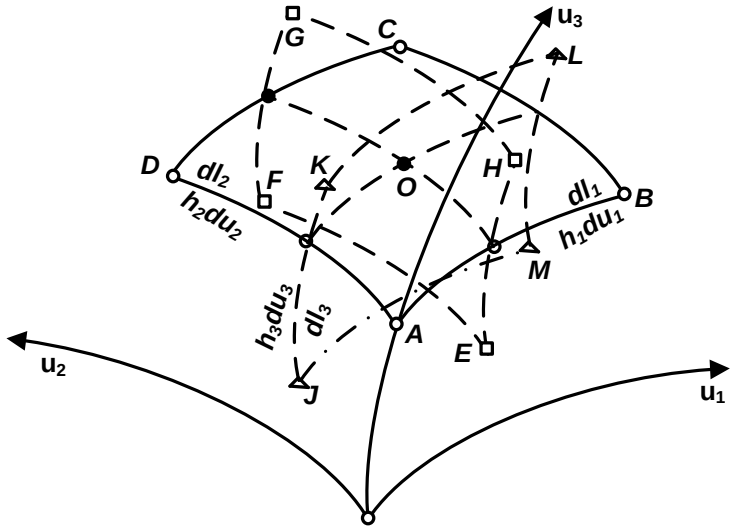


Fig.3.21 Curl in generalized orthogonal curvilinear coordinate system

Considering the differential path length *AB*, the value of S_1 at the midpoint of *AB* is $\left(S_1 - \frac{\partial S_1}{\partial l_2}\frac{dl_2}{2}\right)$

So, $\int_A^B \vec{S}.d\vec{l} = \left(S_1 dl_1 - \frac{\partial(S_1 dl_1)}{\partial l_2}\frac{dl_2}{2}\right) = \left((S_1 h_1 du_1) - \frac{\partial(S_1 h_1 du_1)}{\partial u_2}\frac{du_2}{2}\right)$ …. 3.46

Again, considering the differential path length *CD*, the value of S_1 at the midpoint of *CD* is $\left(S_1 + \frac{\partial S_1}{\partial l_2}\frac{dl_2}{2}\right)$

So, $\int_C^D \vec{S}.d\vec{l} = \left(S_1(-dl_1) + \frac{\partial(-S_1 dl_1)}{\partial l_2}\frac{dl_2}{2}\right) = -\left((S_1 h_1 du_1) + \frac{\partial(S_1 h_1 du_1)}{\partial u_2}\frac{du_2}{2}\right)$ …. 3.47

Similarly, considering the differential path length *BC* the value of S_2 at the midpoint of *BC* is $\left(S_2 + \frac{\partial S_2}{\partial l_1}\frac{dl_1}{2}\right)$

So, $\int_B^C \vec{S}.d\vec{l} = \left(S_2 dl_2 + \frac{\partial(S_2 dl_2)}{\partial l_1}\frac{dl_1}{2}\right)dl_2 = \left((S_2 h_2 du_2) + \frac{\partial(S_2 h_2 du_2)}{\partial u_1}\frac{du_1}{2}\right)$ …. 3.48

Again, considering the differential path length *DA* the value of S_2 at the midpoint of *DA* is $\left(S_2 - \frac{\partial S_2}{\partial l_1}\frac{dl_1}{2}\right)$

So, $\int_D^A \vec{S}.d\vec{l} = \left(S_2(-dl_2) - \frac{\partial(-S_2 dl_2)}{\partial l_1}\frac{dl_1}{2}\right) = \left(-(S_2 h_2 du_2) + \frac{\partial(S_2 h_2 du_2)}{\partial u_1}\frac{du_1}{2}\right)$ …. 3.49

From eqns. (3.46) through (3.49) $\oint_{ABCDA} \vec{S}.d\vec{l} = \int_A^B \vec{S}.d\vec{l} + \int_B^C \vec{S}.d\vec{l} + \int_C^D \vec{S}.d\vec{l} + \int_D^A \vec{S}.d\vec{l}$

or, $\oint_{ABCDA} \vec{S}.d\vec{l} = \frac{\partial(h_2 S_2)}{\partial u_1}du_1 du_2 - \frac{\partial(h_1 S_1)}{\partial u_2}du_1 du_2 = \left(\frac{\partial(h_2 S_2)}{\partial u_1} - \frac{\partial(h_1 S_1)}{\partial u_2}\right)du_1 du_2$ …. 3.50

Dividing the eqn.(3.50) by the area of the integration, i.e. $dl_1 dl_2$,

$$\frac{1}{dl_1 dl_2}\oint_{ABCDA}\vec{S}.d\vec{l} = \frac{1}{h_1 h_2 du_1 du_2}\left(\frac{\partial(h_2 S_2)}{\partial u_1} - \frac{\partial(h_1 S_1)}{\partial u_2}\right)du_1 du_2 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 S_2)}{\partial u_1} - \frac{\partial(h_1 S_1)}{\partial u_2}\right) \quad .. 3.51$$

Noting that the area enclosed by the path *ABCDA* is an area normal to u_3 direction, eqn.(3.51) gives the u_3 component of curl $\vec{S}$.

Considering the closed path *EFGHE* normal to u_1 direction, the u_1 component of curl $\vec{S}$ can be evaluated as

$$\frac{1}{dl_2 dl_3}\oint_{EFGHE}\vec{S}.d\vec{l} = \frac{1}{h_2 h_3 du_2 du_3}\left(\frac{\partial(h_3 S_3)}{\partial u_2} - \frac{\partial(h_2 S_2)}{\partial u_3}\right)du_2 du_3 = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 S_3)}{\partial u_2} - \frac{\partial(h_2 S_2)}{\partial u_3}\right) \quad .. 3.52$$

and considering the closed path *JKLMJ* normal to u_2 direction, the u_2 component of curl $\vec{S}$ can be evaluated as

$$\frac{1}{dl_3 dl_1}\oint_{JKLMJ}\vec{S}.d\vec{l} = \frac{1}{h_3 h_1 du_3 du_1}\left(\frac{\partial(h_1 S_1)}{\partial u_3} - \frac{\partial(h_3 S_3)}{\partial u_1}\right)du_3 du_1 = \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 S_1)}{\partial u_3} - \frac{\partial(h_3 S_3)}{\partial u_1}\right) \quad .... 3.53$$

Here, it is to be mentioned that for evaluating the integral the closed paths have to be traversed in a manner that if a right handed screw is rotated in that direction then the screw will move in the positive direction of the coordinate to which the plane containing that path is perpendicular, i.e. *ABCDA* for the u_1 direction, *EFGHE* for the u_2 direction and *JKLMJ* for the u_3 direction with reference to Fig.3.21.

From eqns. (3.51), (3.52) and (3.53)

$$curl\,\vec{S} = \vec{\nabla}\times\vec{S} =$$

$$\frac{1}{h_2 h_3}\left(\frac{\partial(h_3 S_3)}{\partial u_2} - \frac{\partial(h_2 S_2)}{\partial u_3}\right)\hat{a}_1 + \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 S_1)}{\partial u_3} - \frac{\partial(h_3 S_3)}{\partial u_1}\right)\hat{a}_2 + \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 S_2)}{\partial u_1} - \frac{\partial(h_1 S_1)}{\partial u_2}\right)\hat{a}_3 \quad .... 3.54$$

Eqn.(3.54) could be written in the following form

$$\vec{\nabla}\times\vec{S} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 S_1 & h_2 S_2 & h_3 S_3 \end{vmatrix} \quad .... 3.55$$

The Laplacian can be written in different orthogonal coordinate systems as detailed below:

Cartesian coordinate system:

$$\vec{\nabla}\times\vec{S} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ S_x & S_y & S_z \end{vmatrix} \quad .... 3.56$$

Cylindrical coordinate system:

$$\vec{\nabla}\times\vec{S} = \frac{1}{r}\begin{vmatrix} \hat{u}_r & r\hat{u}_\theta & \hat{u}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ S_r & rS_\theta & S_z \end{vmatrix} \quad .... 3.57$$

Spherical coordinate system:

$$\vec{\nabla}\times\vec{S}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{u}_r & r\hat{u}_\theta & r\sin\theta\hat{u}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ S_r & rS_\theta & r\sin\theta S_\phi\end{vmatrix} \quad \ldots. 3.58$$

Curl of Electric Field

With reference to the derivation of section 3.6.5, consider that the vector quantity be electric field intensity $\vec{E}$. Then the closed line integral $\oint\vec{E}.d\vec{l}$ along any of the three paths as shown in Fig.3.21, viz. *ABCDA*, *EFGHE* and *JKLMJ*, will be zero, as the work done over a closed path is zero. Hence, the results of the integrals as given by eqns. (3.51), (3.52) and (3.53) will be zero. In other words all the three components of curl $\vec{E}$ will be zero. Consequently,

$$curl\,\vec{E}=\vec{\nabla}\times\vec{E}=0 \quad \ldots. 3.59$$

which holds good at every point within an electric field. It shows that an electric field is path independent and irrotational.

Alternately, it can be proved by considering $\vec{E}=-\vec{\nabla}U$, where U is electric potential, which is a scalar quantity. Then

$$\vec{\nabla}\times\vec{E}=\vec{\nabla}\times-\vec{\nabla}U=-\left(\frac{\partial}{\partial y}\frac{\partial U}{\partial z}-\frac{\partial}{\partial z}\frac{\partial U}{\partial y}\right)\hat{i}-\left(\frac{\partial}{\partial z}\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\frac{\partial U}{\partial z}\right)\hat{j}-\left(\frac{\partial}{\partial x}\frac{\partial U}{\partial y}-\frac{\partial}{\partial y}\frac{\partial U}{\partial x}\right)\hat{k}=0$$

Eqn.(3.59) is commonly used to prove that a valid electric field is conservative.

Problem 3.1

Determine whether $\vec{E}=x\hat{i}+y\,\hat{j}+z\,\hat{k}$ is a valid form of electric field or not.

Solution

In order to find whether an electric field is valid or not, the curl of the given electric field needs to be evaluated. So for the given electric field

$$\vec{\nabla}\times\vec{E}=\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)\hat{i}+\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)\hat{j}+\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\hat{k}=0$$

So the given electric field is a valid one.

Single Dielectric Configurations

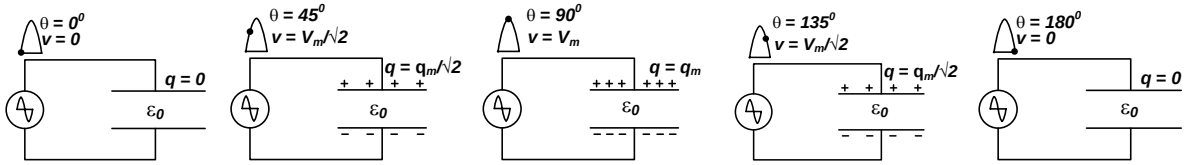
Introduction

Mother Nature has presented the mankind an insulation as a gift which is available in abundance and also is free. This dielectric is nothing but the atmospheric air. It is not difficult to conceive that no electrical system would have worked had air been not an insulating medium. Since air is present everywhere, hence most of the real life arrangements are multi dielectric configurations where the equipment insulation, whether it is solid, liquid, or gaseous or a combination of two or more of these insulating media, is commonly surrounded by atmospheric air. However, there are specific cases where the configuration could be single dielectric one, e.g. a co-axial cable having only one dielectric medium. Analysis of single dielectric arrangement is necessary to grasp the theoretical aspects of electric field in a meaningful way. Moreover, there are certain aspects like displacement current and energy stored in an electric field, which need to be understood from a single dielectric point of view to start with.

Displacement Current

Consider a capacitor with a perfect dielectric or, say vacuum, in between its two plates. In such a case no charge can move through the perfect dielectric or vacuum. In other words, there could be no current flowing between the two plates within the capacitor. Now if a dc voltage is applied between its two plates, then a charging current flows from the source to supply the charges to the plates of the capacitor for the short duration of charging. When the capacitor is fully charged, the charging current ceases to flow. It is to be noted here that this charging current is measurable only in the circuit between the source and the capacitor. If one tries to measure the current between the plates within the capacitor, then the result of the current measurement will be zero. It means that this charging current does not flow through the entire closed loop formed by the source and the capacitor. But why these charges are to be supplied by the source to the capacitor plates? The answer is that the capacitor with a perfect dielectric or vacuum is an open circuit and hence the potential difference between the capacitor plates will be equal to the source voltage. Accordingly there will be an electric field established between the capacitor plates. In order to establish this electric field, charges have to be supplied from the source to the capacitor plates.

Depending upon the area and separation distance between the plates, there is a finite capacitance of the capacitor. Hence, if an ac voltage is applied between the plates of the capacitor, a capacitive current will flow in the circuit, which will be a continuous current varying with time that could be measured by an ammeter placed in between the source and the capacitor. However, due to perfect dielectric or vacuum, as stated earlier, there could be no movement of charge between the plates within the capacitor. Hence, this continuous capacitive current is measurable only in the circuit between the ac source and the capacitor plates. Then the question is how this continuous capacitive current is appearing in a section of the circuit external to the capacitor plates and not through the whole circuit? The answer to this question lies in the fact that this current is not conduction current. It appears due to displacement of charges between the source and the capacitor plates, even when there is no charge movement between the plates within the capacitor. Consequently, this current is termed as displacement current. The concept of displacement current may be explained as detailed below.



(a) (b) (c) (d) (e)

Fig.4.1 Physical explanation of displacement current of a capacitor

Take the example of a capacitor with vacuum between its plates as shown in Fig.4.1. If the separation distance between the plates is much smaller than the plate length/diameter and if fringing of flux at the edges is neglected, then the electric field between the capacitor plates will be uniform. If the potential difference between its plates at any time instant is *v* and the separation distance between the plates is *d*, then the magnitude of electric field intensity between the capacitor plates at that time instant will be (*v/d*). Thus a unit positive charge placed in vacuum between the plates within the capacitor will experience this force on it due to the charges present on the capacitor plates. Due to application of alternating voltage, the electric field intensity within the capacitor (= *v/d*) will vary in direct proportion to the potential difference between the plates. Consequently, for this variation in electric field intensity, the amount of charge on the capacitor plates will again have to vary in direct proportion to the potential difference between the plates. Let the amount of charge on the capacitor plates corresponding to the peak value (V_m) of the sinusoidal applied voltage be q_m. At $\theta = 0^o$, the potential difference across the capacitor plates is zero and hence the charge on the plates is also zero, as shown in Fig. 4.1(a). Then the charge on the plates increases with time as the sinusoidal voltage magnitude increases with time from to $\theta = 0^o$ to $\theta = 90^o$ as shown in Figs.4.1(a), (b) and (c). These charges are supplied by the ac source to the capacitor plates and hence in the circuit between the source and the capacitor plates, there is a change in charge with time ($\frac{dq}{dt}$). Whenever there is such change in charge with time, there is a measurable current. So if an ammeter is placed in between the source and the capacitor, this current can be measured. The magnitude of this current will be determined by the rate of change of charge, which is the same as the rate of change of voltage with time as explained earlier in this section. As shown in Fig.4.2, the rate of change of sinusoidal voltage is maximum at the zero crossing of the voltage waveform, i.e. at $\theta = 0^o$, and is zero at the voltage peak, i.e. at $\theta = 90^o$. In between this time span, this current will also have a sinusoidal variation with time.

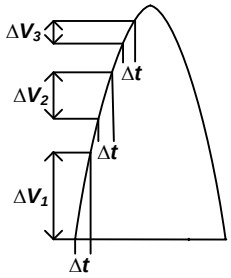


Fig. 4.2 Slope of sinusoidal voltage waveform

From $\theta = 90^o$ to $\theta = 180^o$, the negative slope of sinusoidal voltage waveform increases from zero to maximum at the voltage zero crossing at $\theta = 180^o$. The magnitude of voltage starts decreasing from $\theta = 90^o$ and hence the charges on the capacitor plates also start decreasing from q_m corresponding to $\theta = 90^o$. In other words, the charges start to move from the capacitor plate back to the source. Hence, the ammeter between the source and the capacitor

will record a current in the opposite direction which will increase in magnitude from $\theta = 90^o$ and will become maximum at $\theta = 180^o$, in this opposite direction. The same sequential process will be repeated from $\theta = 180^o$ to $\theta = 360^o$ albeit the direction will be just the opposite of that between $\theta = 0^o$ and $\theta = 90^o$.

So it could be seen that although there is no charge movement between the plates within the capacitor itself, but there is continuous displacement of charges between the source and the capacitor plates in the external circuit. As a result, the Displacement Current flows in the external circuit. From the above discussion, it can be noted that (a) the current is positive maximum when $\frac{dq}{dt}$ and consequently $\frac{dv}{dt}$ is positive maximum at $\theta = 0^o$, (b) the current is zero when $\frac{dq}{dt}$ and consequently $\frac{dv}{dt}$ is zero when $\theta = 90^o$, (c) the current is at its negative maximum when $\frac{dq}{dt}$ and consequently $\frac{dv}{dt}$ is at its negative maximum at $\theta = 180^o$ and so on.

Considering the facts from (a) to (c), it can be concluded that there is a phase difference of 90^o between the voltage across the capacitor plates and the displacement current and this displacement current leads the voltage by 90^o.

In real life, the dielectric between the capacitor plates is never a perfect one and hence a small amount of conduction current flows between the capacitor plates. Consequently, the current through a real life capacitor leads the voltage by an angle slightly less than 90^o.

Parallel Plate Capacitor

A capacitor is an arrangement of conductors along with dielectrics that is used primarily to store electric charge. A very simple capacitor is a system consisting of two parallel metallic plates with free space or any dielectric in between as shown in Fig.4.3.

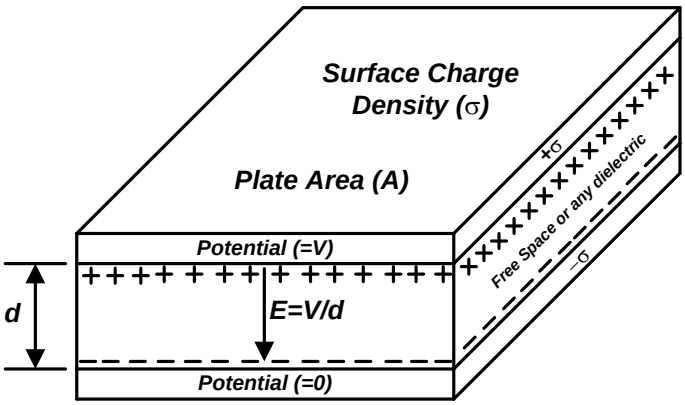


Fig. 4.3 Parallel Plate Capacitor Arrangement

In order to understand the electric field within a parallel plate capacitor, it is necessary to know the electric field due to infinitely long charged planar sheet having uniform surface charge density (σ). As an infinitely long plane possesses a planar symmetry and the charge is uniformly distributed on the planar surface, the electric field intensity acts in the direction of the normal to the plane. In other words, the magnitude of the electric field intensity is constant on all the planes that are parallel to the charged sheet.

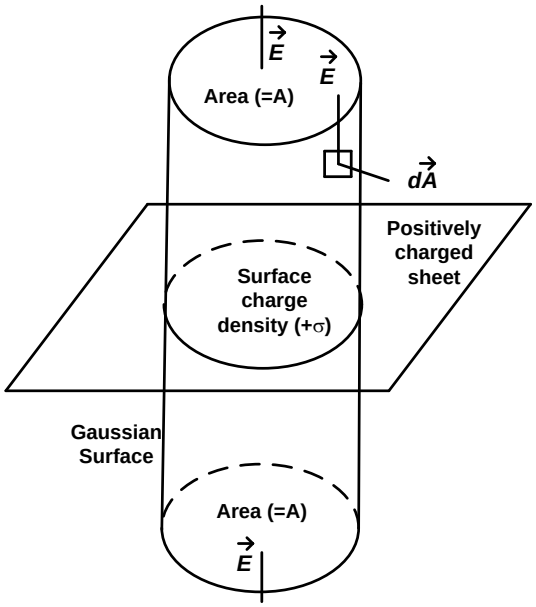


Fig.4.4 Field due to infinitely long charged sheet

To analyze the field due to such a charged sheet, the best choice for a Gaussian surface is a cylinder whose axis is perpendicular to the plane, as shown in Fig.4.4. As the surface charge density is assumed to be uniform, the charge enclosed by the Gaussian surface is given by (σA). As shown in Fig.4.4, the electric field intensity vector is parallel to the area vector at the two end surfaces and is normal to the area vector on the curved wall surfaces. Hence, applying Gauss's law on the cylindrical Gaussian surface

$$\int_{\substack{Cylinder\\ Surface}} \varepsilon_0 \vec{E}.d\vec{A} = \sigma A$$

or, $\varepsilon_0 \int_{walls} \left|\vec{E}\right| \cos(90^0)\, dA + 2\varepsilon_0 \int_{ends} \left|\vec{E}\right| \cos(0^0)\, dA = \sigma A$

or, $0 + 2\varepsilon_0 \left|\vec{E}\right| A = \sigma A$ or, $\left|\vec{E}\right| = \dfrac{\sigma}{2\varepsilon_0}$ …. 4.1

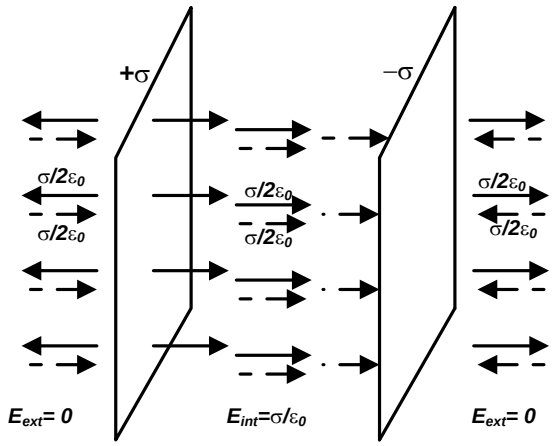


Fig.4.5 E-field within a parallel plate capacitor

When two plates of equal and opposite charge density are placed near and parallel to each other with free space between them, the electric field intensities due to the two plates add

between the plates while they cancel each other outside the plates as shown in Fig.4.5. Thus the electric field intensity between the two plates is given by $E=\frac{\sigma}{\varepsilon_0}$

When the separation distance between the two plates is small compared to the sides of the plate, electric field intensity between the plates is constant through out the interior of the capacitor. The flux lines are parallel to each other near the centre of the capacitor, while concentration of flux lines occurs at the edges, which is known as fringing of flux as shown in Fig.4.6. Neglecting fringing of flux and considering the potential difference between the two plates to be *V*, the electric field intensity can also be written as $E=\frac{V}{d}$

Noting that the charge on the plates $Q=\sigma A$, it may be written as

$$\frac{\sigma}{\varepsilon_0}=\frac{V}{d} \quad \text{or,} \quad \frac{QA}{\varepsilon_0}=\frac{V}{d} \quad \text{or,} \quad \frac{Q}{V}=\frac{\varepsilon_0 A}{d} \quad \text{or,} \quad C=\frac{\varepsilon_0 A}{d} \qquad \ldots. 4.2$$

where, *C* is the capacitance of the parallel plate capacitor. Eqn.(4.2) shows that the capacitance of a parallel capacitor is independent of the charge on the plates or the potential difference across the capacitor plates.

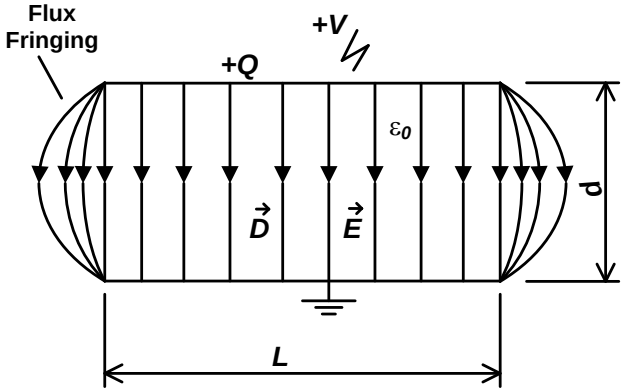


Fig.4.6 Flux lines in a parallel plate capacitor

If any dielectric medium of relative permittivity (ε_r) is inserted between the plates, then the electric field intensity within the capacitor remains unchanged ($=\frac{V}{d}$), but the capacitance (*C*) is changed to $\frac{\varepsilon_r \varepsilon_0 A}{d}$.

Energy Stored in a Parallel Plate Capacitor

Consider that a parallel plate capacitor has no charge at the beginning. Now, if a voltage source is connected across its plates so that the potential difference between the two plates becomes *V*, then the capacitor gets charged and these charges are supplied by the source to the capacitor plates. In order to supply these charges to the capacitor plates, some work have to be done by the external source which is then stored as electrostatic energy in the capacitor. Let at any instant the charge on the capacitor plates be $+q$ and $-q$, respectively and the potential difference between the plates is *v*. At this instant consider that an additional amount of charge *dq* is supplied by the source to the plates. Then the energy spent by the source to deliver this charge is *vdq* and this energy is stored in the electric field of the capacitor.

So if the total charge on the plates are $+Q$ and $-Q$, respectively and the corresponding potential difference between the plates is V, then the total energy stored in the electric field is given by

$$W = \int_0^Q v\,dq = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{C} = \frac{C^2V^2}{2C} = \frac{1}{2}CV^2 \qquad \text{…. 4.3}$$

Now, surface charge density on the plates (σ) = (Q/A) = D. So,

$$W = \frac{Q^2}{2C} = \frac{D^2A^2}{2C} = \frac{\varepsilon^2E^2A^2}{2\frac{\varepsilon A}{d}} = \frac{1}{2}\varepsilon E^2(Ad)$$

(Ad) being the volume of the parallel plate capacitor, the total energy stored in electric field per unit volume, i.e. energy density, is given by

$$W_E = \frac{1}{2}\varepsilon E^2 \qquad \text{…. 4.4}$$

However, such a derivation for energy stored in electric field is only valid for a parallel plate capacitor. A more generalized way of obtaining the energy stored in the electric field is discussed in the next section.

Problem 4.1

Find the maximum electrostatic energy that can be stored in a parallel plate capacitor of which the plates are in the form of discs of radius 10cm and the plate separation distance is 1cm. The dielectric is air.

Solution

Breakdown strength of air is 30kV/cm. So the electric field intensity within the capacitor cannot exceed 30kV/cm or in other words the maximum value of electric field intensity (E_m) is 30kV/cm or 3×10^6V/m.

So, maximum energy density of the capacitor = $\frac{1}{2}\times8.854\times10^{-12}\times(3\times10^6)^2 = 39.843$ J/m^3

Volume of the capacitor = $\pi\times0.1^2\times0.01 = 3.141\times10^{-4}$ m^3

So, Maximum electrostatic energy stored in the capacitor = $39.843\times3.141\times10^{-4} = 12.51$ mJ

Energy Stored in Electric Field

Consider that an electric field is established by an assembly of charges. To obtain the energy stored in this electric field, the work done to assemble these charges need to be determined. Assume that all the charges are point charges, which are at infinity initially, and an external agent brings these charges one by one and places them at the respective positions, as shown in Fig.4.7. It is obvious that no work is to be done to bring the first charge q_1 from infinity to its location at P_1, as there is no existing electric field created by another charge. Then the second charge q_2 is brought from infinity to P_2 within the electric field created by q_1. So the work done in bringing the charge q_2 will be given by

$$W_2 = q_2\phi_{21} = q_2\frac{q_1}{4\pi\varepsilon_0 r_{21}} \qquad \text{…. 4.5}$$

where, ϕ_{21} is the potential at the location P_2 due to the charge q_1 located at P_1.

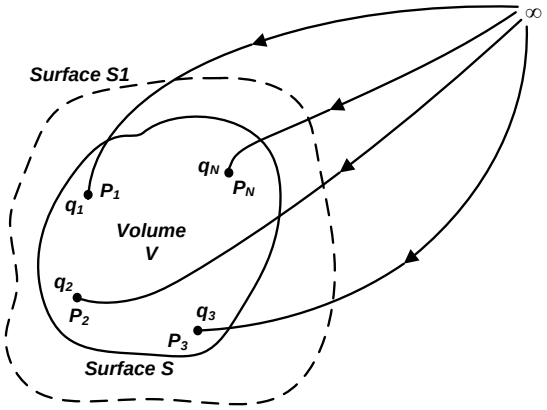


Fig.4.7 Assembling a set of charges

Then the third charge q_3 is brought from infinity to P_3 within the electric field created by q_1 and q_2. So the work done in bringing the charge q_2 will be given by

$$W_3 = q_3\phi_{31} + q_3\phi_{32} = q_3\frac{q_1}{4\pi\varepsilon_0 r_{31}} + q_3\frac{q_2}{4\pi\varepsilon_0 r_{32}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_3}{r_{31}} + \frac{q_2q_3}{r_{32}}\right) \quad \text{…. 4.6}$$

So the total work done in bringing q_1, q_2 and q_3 is

$$0 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{21}} + \frac{q_1q_3}{r_{31}} + \frac{q_2q_3}{r_{32}}\right) \quad \text{…. 4.7}$$

Denoting the charge that is being brought in by the suffix *i* and the charges which created the field within which this *i*th charge is being brought in by the suffix *j*, the work done in assembling *N* no of charges $q_1, q_2, \ldots\ldots, q_N$ can be written as

$$W = \sum_{i=2}^{N} W_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{q_i\,q_j}{r_{ij}} \quad \text{…. 4.8}$$

Fig.4.8 gives a pictorial representation of the numbers over which the summation of eqn.(4.8) is being carried out. This summation is clearly over the triangular region marked *I* in Fig.4.8. Since the quantity being summed is symmetric in *i* and *j*, the same energy *W* would be obtained by a summation over the triangular region marked *II* in Fig.4.8. The summation over the triangular regions *I* and *II* must then give 2*W*. Thus *W* may also be obtained as follows

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i\,q_j}{r_{ij}},\ i \neq j \quad \text{…. 4.9}$$

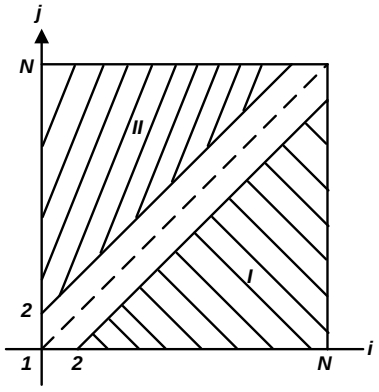


Fig.4.8 Representation of summation to obtain energy stored

Instead of considering discrete point charges, if the charge distribution is taken as continuous throughout the volume *V* of Fig.4.7, then the potential at any point can be found by summing

the contributions from individual differential volume elements of charge. Thus by writing $\rho(r)dV$ in place of q_i and $\rho(r')dV'$ in place of q_j, the summation may be replaced by integrations over volume, which must be large enough to contain all the charges present. Thus the work done to form a continuous charge distribution is given by

$$W = \frac{1}{8\pi\varepsilon_0}\int_V\int_{V'}\frac{\rho(r)\,\rho(r')}{R_{rr'}}dV'\,dV \text{ , where } R_{rr'} = |\vec{r} - \vec{r}'| \qquad \text{…. 4.10}$$

The potential at the location r due to all the charges located at the respective positions r' is given by

$$\phi(r) = \int_{V'}\frac{\rho(r')}{4\pi\varepsilon_0 R_{rr'}}dV' \qquad \text{…. 4.11}$$

Substitution of eqn.(4.11) in eqn.(4.10) yields

$$W = \frac{1}{2}\int_V \rho(r)\phi(r)\,dV \qquad \text{…. 4.12}$$

Since eqn.(4.12) deals with quantities related to only one location r, hence the parameter r is omitted hereafter. Noting that $\vec{\nabla}.\vec{D} = \rho$, eqn.(4.12) could be rewritten as

$$W = \frac{1}{2}\int_V \phi\,(\vec{\nabla}.\vec{D})\,dV \qquad \text{…. 4.13}$$

For any vector $\vec{D}$ and any scalar ϕ, the following vector identity could be written

$$\vec{\nabla}.(\phi\vec{D}) = \vec{D}.\vec{\nabla}\phi + \phi(\vec{\nabla}.\vec{D}) \qquad \text{…. 4.14}$$

or, $\phi(\vec{\nabla}.\vec{D}) = \vec{\nabla}.(\phi\vec{D}) - \vec{D}.\vec{\nabla}\phi$

So, from eqn.(4.13) and 4.14, $$W = \frac{1}{2}\int_V \vec{\nabla}.(\phi\vec{D})\,dV - \frac{1}{2}\int_V (\vec{D}.\vec{\nabla}\phi)\,dV \qquad \text{…. 4.15}$$

Applying Divergence theorem to the first term on the RHS of eqn.(4.15),

$$W = \frac{1}{2}\oint_S \phi\vec{D}\,dS - \frac{1}{2}\int_V (\vec{D}.\vec{\nabla}\phi)\,dV \qquad \text{…. 4.16}$$

As $\vec{E} = -\vec{\nabla}\phi$ and $\vec{D} = \varepsilon_0\vec{E}$, eqn. 4.16 is rewritten as

$$W = \frac{1}{2}\oint_S \phi\vec{D}\,dS + \frac{1}{2}\int_V \varepsilon_0(\vec{E}.\vec{E})\,dV \qquad \text{…. 4.17}$$

In the first term on the RHS of eqn.(4.17), ϕ varies as $\frac{1}{r}$, while $\vec{D}$ varies as $\frac{1}{r^2}$ and surface term varies as r^2. So as a whole the first term varies as $\frac{1}{r}$. If the bounding surface of volume V as shown in Fig.4.7 is expanded from S to $S1$, then the region between S and $S1$ does not contribute to the energy integral as there is no charge located in this region. However, although this region between S and $S1$ is charge free, the electric field intensity is not zero in this region. Hence, as the bounding surface is expanded, the second term of eqn.(4.17) will increase. Then in order to keep the energy integral unchanged, the first term should decrease. Thus if the bounding surface is taken to be infinitely large, then the first term on the RHS of eqn.(4.17) becomes zero, keeping the total energy same. Therefore, the total work done in forming the continuous charge distribution, i.e. the total energy stored in the electric field, is given by

$$W = \frac{1}{2}\int_V \varepsilon_0 E^2\,dV \qquad \text{…. 4.18}$$

Hence, energy stored per unit volume, i.e. energy density, is given by

$$W_E = \frac{1}{2}\varepsilon_0 E^2 \qquad \text{…. 4.19}$$

Considering the relative permittivity of the dielectric medium to be ε_r, energy density of an electric field is given by

$$W_E = \frac{1}{2}\varepsilon_r \varepsilon_0 E^2 \qquad \text{…. 4.20}$$

Eqn.(4.20) is very useful in computing the energy stored in a complex dielectric arrangement where electric field intensity is non-uniform. In such cases the electric field distribution is first determined using suitable method and then the entire volume under consideration is divided into smaller volume elements so that each volume element contains only one dielectric medium and the electric field intensity remains constant within the volume element. As a result the energy stored in each element can be computed using eqn.(4.20). The energy stored in all the volume elements could then be summed up to get the total energy stored in any real life dielectric arrangement having complex geometries and several dielectric media.

Problem 4.2

Three point charges of magnitude -2nC, -3nC and 1nC are located in free space at (0,0,0)m, (0,2,0)m and (2,0,0)m, respectively. Find the energy stored in the system of charges.

Solution

Denoting the charges as q_1= -2nC, q_2= -3nC and q_3=1nC, no work is done to bring in q_1.

Electric potential at the location of q_2 due to q_1 is

$$\phi_{21} = \frac{-2\times10^{-9}}{4\times\pi\times8.854\times10^{-12}\times r_{21}}, \text{ where } r_{21} = \sqrt{0+2^2+0} = 2m$$

Hence, ϕ_{21} = - 8.987 V

So, work done to bring in q_2 = $-3\times10^{-9}\times-8.987$ = 26.96 nJ.

Again, electric potential at the location of q_3 due to q_1 is

$$\phi_{31} = \frac{-2\times10^{-9}}{4\times\pi\times8.854\times10^{-12}\times r_{31}}, \text{ where } r_{31} = \sqrt{0+0+2^2} = 2m$$

Hence, ϕ_{31} = - 8.987 V.

and electric potential at the location of q_3 due to q_2 is

$$\phi_{32} = \frac{-3\times10^{-9}}{4\times\pi\times8.854\times10^{-12}\times r_{32}}, \text{ where } r_{32} = \sqrt{0+2^2+2^2} = 2\sqrt{2}m$$

Hence, ϕ_{32} = - 6.354 V.

So, work done to bring in q_3 = $1\times10^{-9}\times(-8.987 - 6.354)$ = - 15.34 nJ.

Hence, the total energy stored in the charge system = (26.96 – 15.34) = 11.62 nJ.

Two Concentric Spheres with Homogeneous Dielectric

A simple single dielectric arrangement that have spherical symmetry is two concentric spheres with a homogeneous dielectric medium in between the two spheres as shown in Fig.4.9. The inner sphere is charged to an electric potential *+V*, while the outer sphere is earthed.

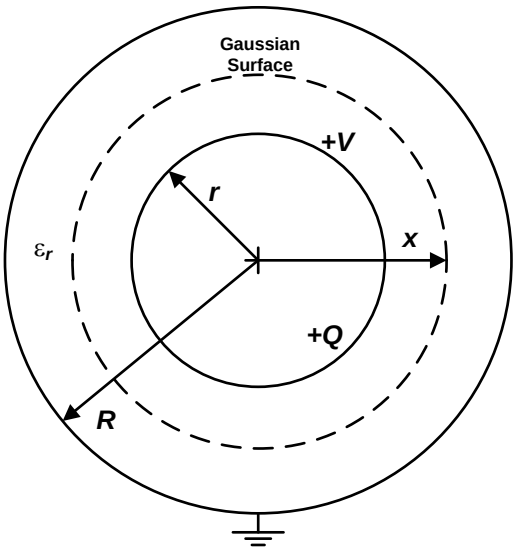


Fig.4.9 Two concentric spheres with homogeneous dielectric

Since this configuration has spherical symmetry, the best choice for a Gaussian surface is a concentric sphere of radius *x*, as shown in Fig.4.9.

Let +*Q* be total charge on the inner sphere. According to Gauss's law, the total flux leaving the Gaussian surface of radius *x* is equal is the total charge enclosed, i.e. the charge on the inner conductor surface +*Q*. As the flux lines are symmetrically distributed and are directed radially outwards, the electric flux density at a radial distance of *x* is given by

$$D_x = \frac{+Q}{4\pi x^2}$$

Hence, electric field intensity at a radial distance of *x* is $E_x = \frac{+Q}{4\pi\varepsilon x^2}$

Then the potential difference between the two spheres of radii *r* and *R* could be obtained as

$$V = -\int_R^r E_x\,dx = -\int_R^r \frac{+Q}{4\pi\varepsilon x^2}dx = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{r}-\frac{1}{R}\right) \quad \text{…. 4.21}$$

Since electric potential is readily measurable quantity, the charge on the inner sphere could be obtained from the knowledge of electric potential as follows

$$Q = \frac{4\pi\varepsilon_r\varepsilon_0 V}{\left(\frac{1}{r}-\frac{1}{R}\right)} \quad \text{…. 4.22}$$

The capacitance of the system could be obtained as

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_r\varepsilon_0}{\left(\frac{1}{r}-\frac{1}{R}\right)} \quad \text{…. 4.23}$$

From a practical point of view, electric field intensity is a significant quantity that needs to be determined. Hence, electric field intensity at any radius *x* as expressed in terms of the potential difference between the two spheres is given by

$$E_x = \frac{V}{x^2\left(\frac{1}{r}-\frac{1}{R}\right)} \quad \text{…. 4.24}$$

Eqn.(4.24) shows that electric field intensity varies with radial distance in a non-linear way. Highest value of electric field intensity occurs at a radial distance *r*, which is given by

$$E_{\max} = \frac{V}{r^2\left(\frac{1}{r} - \frac{1}{R}\right)} \quad \text{…. 4.25}$$

For the above mentioned concentric spherical system, a value of r for a fixed outer radius R could be obtained that gives the lowest possible value of electric field intensity on the inner conductor. From eqn.(4.25), it may be seen that for E_{max} to be minimum, the denominator has to be maximum for a given value of V, i.e.

$$\frac{d}{dr}\left[r^2\left(\frac{1}{r} - \frac{1}{R}\right)\right] = 0$$

or, $r = \frac{R}{2}$ …. 4.26

However, it may be noted here that concentric spherical system with a charged inner sphere completely enclosed by an earthed outer sphere is more of theoretical interest, as it is very difficult to realize such a system in practice.

Two Co-axial Cylinders with Homogeneous Dielectric

A simple single dielectric arrangement that have cylindrical symmetry is two co-axial cylinders with a homogeneous dielectric medium in between the two cylinders as shown in Fig.4.10. The inner cylinder is charged to an electric potential $+V$, while the outer cylinder is earthed. In real-life a single-core cable having one dielectric medium is typical example of such a system. In this case electric field varies with location over the cross-sectional area of the cable. But electric field does not vary along the length of the cable. Hence, the configuration as shown in Fig.4.10 is represented as two dimensional system in Cartesian coordinates, where the cross-sectional area is taken on x-y plane. The field is then independent of z-axis, where z-direction is along the length of the cylinder.

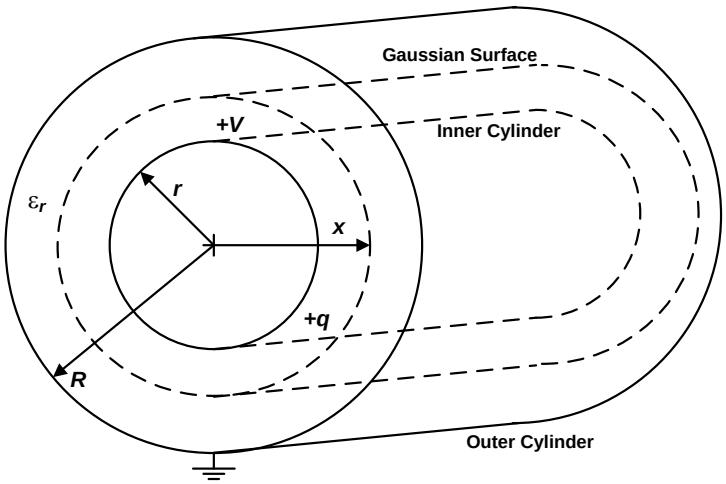


Fig.4.10 Two co-axial cylinders with homogeneous dielectric

Since this configuration has cylindrical symmetry, the best choice for a Gaussian surface is a co-axial cylinder of radius x, as shown in Fig.4.10.

Let $+q$ be total charge per unit length on the inner cylinder. According to Gauss's law, the total flux leaving the Gaussian surface of radius x is equal is the total charge enclosed, i.e. the charge on the inner conductor surface $+q$. As the flux lines are symmetrically distributed and are directed radially outwards, the electric flux density at a radial distance of x is given by

$$D_x = \frac{+q}{2\pi x.1}$$

Hence, electric field intensity at a radial distance of x is $E_x = \frac{+q}{2\pi\varepsilon x}$

Then the potential difference between the two cylinders of radii r and R could be obtained as

$$V = -\int_R^r E_x\,dx = -\int_R^r \frac{+q}{2\pi\varepsilon x}dx = \frac{q}{2\pi\varepsilon_r\varepsilon_0}\ln\frac{R}{r} \quad \text{…. 4.27}$$

The charge per unit length on the inner cylinder could be obtained from the knowledge of potential difference between the two cylinders as follows

$$q = \frac{2\pi\varepsilon_r\varepsilon_0 V}{\ln\frac{R}{r}} \quad \text{…. 4.28}$$

The capacitance per unit length of the system could be obtained as

$$C = \frac{q}{V} = \frac{2\pi\varepsilon_r\varepsilon_0}{\ln\frac{R}{r}} \quad \text{…. 4.29}$$

Electric field intensity at any radius x as expressed in terms of the potential difference between the two cylinders is given by

$$E_x = \frac{V}{x\ln\frac{R}{r}} \quad \text{…. 4.30}$$

From eqn.(4.30) it becomes clear that highest value of electric field intensity occurs at a radial distance r, which is given by

$$E_{\max} = \frac{V}{r\ln\frac{R}{r}} \quad \text{…. 4.31}$$

For the above mentioned co-axial cylindrical system, a value of r for a fixed outer radius R could be obtained that gives the lowest possible value of electric field intensity on the inner conductor. From eqn.(4.31), it may be seen that for E_{max} to be minimum, the denominator has to be maximum for a given value of V, i.e.

$$\frac{d}{dr}\left[r\ln R - r\ln r\right] = 0$$

or, $\frac{R}{r} = e$, or, $r = \frac{R}{e}$ …. 4.32

Then from eqn.(4.31), the highest electric field intensity on the inner cylinder becomes

$$E_{\max}\big|_{lowest} = \frac{V}{r} \quad \text{…. 4.33}$$

A practical use of eqn.(4.32) in real life is in the finalization of the dimensions of the inner and outer conductors of Gas Insulated Transmission Line (GIL). The typical configuration of a GIL, which is a co-axial cylindrical arrangement where the primary insulation is SF_6 gas or a mixture of SF_6 and N_2, is shown in Fig.4.11.

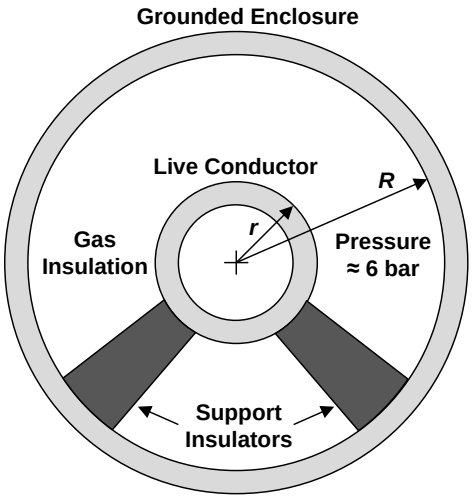


Fig.4.11 Typical GIL configuration

If for a given value of *V*, i.e. the potential of the live conductor, values of *R* and *r* are chosen as per eqn.(4.32) and the value of *r* is so chosen the $E_{\max}\big|_{lowest}$ is equal to the dielectric strength of the gaseous insulation at the designed pressure, then the gaseous insulation is utilized in the most economical way. However, in practical design adequate safety margin is taken so that unwanted breakdown does not take place.

Problem 4.3
Find the most economical dimensions of a single-core metal sheathed cable for a working voltage of 76 kV_{rms}, if the maximum electric field intensity that can be allowed within the cable insulation is 5kVrms/mm.
Solution

For the most economical design, $r = \frac{R}{e}$

But, electric field intensity on the conductor surface should not exceed the maximum allowable limit, i.e. 5kVrms/mm.

So, electric field intensity on the conductor surface $\frac{V}{r} = 5$, where V = 76kV_{rms}.

Hence, the radius of inner conductor (r) = $\frac{76}{5} = 15.2mm$

Therefore, the radius of the outer conductor (R) = 15.2 × 2.718 = 41.3*mm*

Field Factor

For practical configurations, electric field factor (f) is defined as

$$f = \frac{E_{\max}}{E_{av}} \qquad \text{…. 4.34}$$

where, E_{max} is the maximum value of electric field intensity in the system.
E_{av} is defined as

$$E_{av} = \frac{V}{d_{\min}} \qquad \text{…. 4.35}$$

where, d_{min} is the minimum distance between the two conductors having a potential difference of *V*.

For a parallel plate capacitor as discussed in section 4.3, E_{max} is $\frac{V}{d}$, neglecting fringing of flux at the edges, and the E_{av} is also $\frac{V}{d}$. Hence,

for parallel plate capacitor, $f = \frac{E_{\max}}{E_{av}} = 1.0$ …. 4.36

For two concentric spheres with one dielectric medium as discussed in section 4.5,

$$E_{\max} = \frac{V}{r^2\left(\frac{1}{r} - \frac{1}{R}\right)}$$

and $E_{av} = \frac{V}{R - r}$

Hence, for two concentric spheres with single dielectric $f = \frac{E_{\max}}{E_{av}} = \frac{R}{r}$ …. 4.37

For two co-axial cylinders with one dielectric medium as discussed in section 4.6,

$$E_{\max} = \frac{V}{r \ln \frac{R}{r}}$$

and $E_{av} = \frac{V}{R - r}$

Hence, for two concentric spheres with single dielectric $f = \frac{E_{\max}}{E_{av}} = \frac{\frac{R}{r} - 1}{\ln \frac{R}{r}}$ …. 4.38

In fact for any uniform field system, field factor is unity. The degree of non-uniformity of an electric field is represented by the value of field factor. The higher the field factor the higher is the non-uniformity of the field distribution. In some references, field utilization factor (u) is also used, which is simply defined as the reciprocal of field factor (f), such that $u = \frac{1}{f} = \frac{E_{av}}{E_{\max}}$

Dielectric Polarization

Introduction

Inside any material the electric field varies rapidly with distance in a scale corresponding to the spacing between the atoms or molecules. The local electric field at the site of an atom is significantly different from the macroscopic electric field. It is because of the fact that the local electric field acting on an atom is strongly affected by nearest atoms, while the macroscopic field is averaged over a large number of atoms or molecules. Therefore, determination of electric field is extremely complicated, if not impossible, at every "mathematical point" in a given space. As a result the average value of electric field over a finite volume is commonly determined, particularly in the case of power engineering. This finite volume should be such that it is practically small enough to be considered as a point, but large enough to accommodate enough numbers of atoms or molecules to give smoothly varying average value of electric field.

While conductors have large number of free charges that can move in response to an external electric field, the materials known as dielectrics do not have free charges inside them. So it may be argued that the dielectrics cannot have any effect of the electrostatic field. But this argument is incorrect as the mechanism by which dielectric materials affect the electrostatic field is quite different than the mechanism in the case of conductors. Moreover, in reality there is no electrical equipment or device without conductors as well as dielectrics. Hence, it is important from a practical point of view to analyze the behavior of dielectrics in electrostatic field.

Atoms of all dielectric materials consist of charged constituents like electrons and nucleus, which could be displaced, albeit through a small distance, by an external electric field. In the process an electric dipole will be induced by the external electric field in a symmetrical atom or molecule, which originally had zero dipole moment. On the other hand, there are large numbers of dielectric materials containing molecules having permanent dipole moment. In those cases, the permanent dipoles will be aligned by the external electric field in its direction. The degree of alignment of permanent dipoles is higher for stronger external field. The net dipole moment of a dielectric piece is typically zero, when not influenced by external electric field, because the atoms have zero dipole moment and the permanent dipoles are randomly oriented. But due to induction or alignment of dipoles under the action of external electric field, a dielectric piece may be considered as arrays of oriented electric dipoles. As a result a dielectric piece acquires a net dipole moment and the dielectric is said to be polarized. The process by which a dielectric material gets polarized is known as polarization.

Field due to an Electric Dipole and Polarization Vector

A polarized dielectric can be assumed to be a collection of oriented electric dipoles situated in vacuum. If the charges of the electric dipoles and the distances between them are known, then it is possible to determine the electric potential and electric field intensity at any external location due to the polarized dielectric. But this is practically very difficult due to immensely large number of such dipoles in a polarized dielectric. Because of this reason, a kind of average dipole density is defined in the form of a vector quantity known as polarization vector for the ease of analysis.

Electric Dipole and Dipole Moment

When two point charges of equal magnitude but of opposite polarities are separated by a small distance, then the arrangement is known as an electric dipole as shown in Fig.5.1. For field analysis it is required that a single dipole be characterized by a vector quantity. As depicted in Fig.5.1, let the magnitudes of the charges be $+Q$ and $-Q$, respectively and the distance between them is d. The distance vector $\vec{d}$ between the two point charges is considered to be directed from the negative charge to the positive charge. Then the dipole moment of the electric dipole is defined as a vector

$$\vec{p} = Q\vec{d} \qquad \text{…. 5.1}$$

The unit of dipole moment is C.m

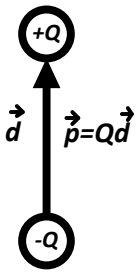


Fig.5.1 The dipole moment of an electric dipole

Field due to an Electric Dipole

The field due to a single electric dipole can be evaluated as the superposition of the field due to two point charges $+Q$ and $-Q$, as shown in Fig.5.2. Then the electric potential at the point P due to the electric dipole is given by

$$V_P = \frac{Q}{4\pi\varepsilon_0 r_1} + \frac{-Q}{4\pi\varepsilon_0 r_2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{…. 5.2}$$

The distance d between the dipole charges is always much smaller than the distances of the point P from the two charges. Hence, the line segments r_1 and r_2 will be parallel for all practical purposes. Hence,

$$r_1 = r - \frac{d}{2}\cos\theta \qquad \text{and} \qquad r_2 = r + \frac{d}{2}\cos\theta \qquad \text{…. 5.3}$$

where, r = distance from the center of the electric dipole to the point P

and θ = the angle between the distance vectors $\vec{d}$ and $\vec{r}$.

Thus, assuming $r>>d$, electric potential at a distance r from the electric dipole may be written as

$$V_P = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}.\hat{u}_r}{4\pi\varepsilon_0 r^2} \qquad \text{…. 5.4}$$

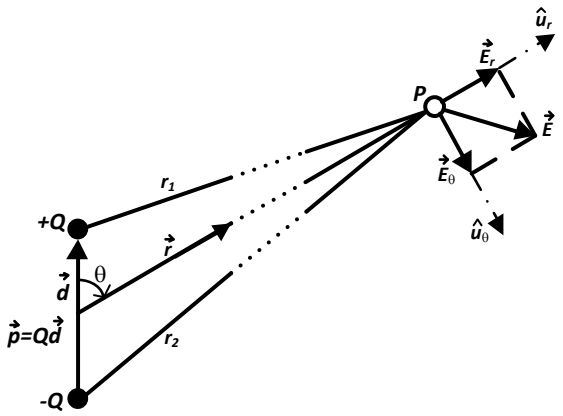


Fig.5.2 Field due to an electric dipole

From eqn.(5.4) it may be seen that the field due to an electric dipole is two dimensional in nature when represented in spherical coordinate system, as the field depends on r and θ coordinates and not on ϕ coordinate.

Then electric field intensity at the point P can be expressed as

$$\vec{E}_P = -\vec{\nabla} V_P = -\frac{\partial V_P}{\partial r}\hat{u}_r - \frac{1}{r}\frac{\partial V_P}{\partial \theta}\hat{u}_\theta = \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{u}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{u}_\theta \quad \text{…. 5.5}$$

Thus the r and θ components of electric field intensity are

$$E_r = \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3} \quad \text{and} \quad E_\theta = \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3} \quad \text{…. 5.6}$$

Eqns.(5.4) and (5.5) show that electric potential and electric field intensity due to an electric dipole depend on dipole moment of the electric dipole and not on the magnitude of the charges and their separation distance separately.

Polarization Vector

Consider a small volume Δv of a polarized dielectric. If there are N number of molecules per unit volume in Δv and $\vec{p}$ is the average dipole moment per molecule, then as a measure of intensity of the polarization, the polarization vector, $\vec{P}$, at a point inside Δv is defined as follows

$$\vec{P} = \frac{\sum_{\Delta v} \vec{p}}{\Delta v} = N\vec{p} \quad \text{…. 5.7}$$

Eqn.(5.7) is valid for a rarified medium, e.g. a gas. The relationship needs to be written in a different way for a dense medium such as a liquid or a solid.

In a generalized manner, the net dipole moment in the small volume Δv is given by

$$Q_1\vec{d}_1 + Q_2\vec{d}_2 + \ldots\ldots\ldots + Q_{N1}\vec{d}_{N1} = \sum_{i=1}^{N1} Q_i\vec{d}_i \quad \text{…. 5.8}$$

where, $N1 = N \times \Delta v$

Then, $\vec{P} = \dfrac{\sum_{i=1}^{N1} Q_i\vec{d}_i}{\Delta v}$ …. 5.9

The unit of dipole moment is *C.m* and hence the unit of polarization vector is C/m^2, which is the same as that of surface charge density.

In other words, if the polarization vector $\vec{P}$ is known at a point, then a small volume Δv, which encloses that point and contains large number of dipoles, can be replaced by a single dipole of moment

$$\vec{p} = \vec{P}\,\Delta v \qquad \ldots. 5.10$$

With the help of eqn.(5.10), electric potential and electric field intensity due to a polarized dielectric can be evaluated by an integral.

Polarizability

The molecules of dielectric materials, which are the basic building blocks of the material, either have zero dipole moment or have some permanent dipole moments depending on their structure. When an external electric field is applied, then the opposite polarity charges are pulled apart and/or the permanent dipoles get aligned under the action of the external field. In this way the dielectric material becomes polarized and this property of dielectric materials is known as polarizability.

Non-Polar and Polar Molecules

The molecules of a dielectric material are classified into two categories, viz. non-polar and polar. Symmetrical molecules such as CO_2, as shown in Fig. 5.3(c), fall in this category. In non-polar molecules the "centers of gravity" of positive and negative charge distribution usually coincide at one point and hence the molecules have zero dipole moment. On the other hand, a polar molecule such as H_2O and CO, as shown in Fig.5.3(a) and (b), respectively, have permanent dipoles even in the absence of any external electric field. However, in the absence of any external field, a macroscopic piece of polar dielectric is not polarized, i.e. it does not contain any dipole moment, because the molecules are randomly oriented due to thermal agitation. When the polar dielectric is subjected to an external electric field, then the individual permanent dipoles within the polar dielectric experiences torques, which tend to align these dipoles in the direction of the external field, and the dielectric gets polarized.

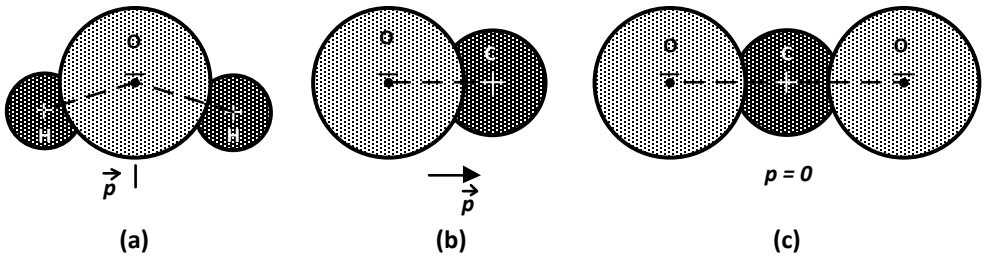


Fig.5.3 Non-polar and polar molecules: (a) Polar-H_2O, (b) Polar-CO and (c) Non-polar-CO_2

Electronic Polarizability of an Atom

A simplified model of an atom is a uniformly charged electron cloud, which is spherical in shape having radius *R*, surrounding the total positive charge located at the point nucleus. The centers of gravity of the total negative charge ($-q$) of the electron cloud and the total positive charge ($+q$) coincides at the same point, as shown in Fig.5.4, in the absence of any external electric field.

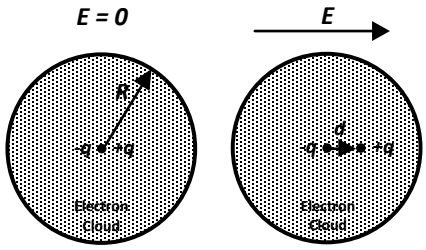


Fig.5.4 Electronic polarizability of an atom

When an external electric field is applied, the electron cloud is displaced by a small distance *d* until the mutual attractive force between the negatively charged electron cloud and the positive charged point nucleus balances the force due to the external electric field.
The attractive force between the electron cloud and the point nucleus is given by

$$F_{\text{int}} = q \times \frac{\frac{d^3}{R^3} q}{4\pi\varepsilon_0 d^2} \quad \text{…. 5.11}$$

The force due to the external electric field is given by

$$F_{ext} = qE \quad \text{…. 5.12}$$

At equilibrium, $F_{\text{int}} = F_{ext}$, or, $\frac{d\,q^2}{4\pi\varepsilon_0 R^3} = qE$

or, $d = \frac{4\pi\varepsilon_0 R^3 E}{q}$ …. 5.13

Hence, the magnitude of dipole moment induced by the external electric field

$$p = q\,d = 4\pi\varepsilon_0 R^3 E = \alpha_e E \quad \text{…. 5.14}$$

where, $\alpha_e = 4\pi\varepsilon_0 R^3$ = Electronic polarizability of the atom.

Types of Polarizability

The physical processes that give rise to polarizability can be subdivided into four categories: i) Electronic polarizability, ii) Ionic polarizability, iii) Orientational or dipolar polarizability and iv) Interfacial polarizability.

Electronic Polarizability

Electronic polarizability arises due to displacement of negatively charged electron cloud wrt the positively charged nucleus under the influence of external electric field, as discussed in section 5.3.2. Electronic polarizability is present in all types of dielectric materials. It is an elastic process without any power loss and is an extremely fast process which takes place within 10^{-16} to 10^{-13}s.

Ionic Polarizability

In the case of dielectric molecules that contain ionic bonds, the lengths of the bonds get stretched under the influence of the external electric field. Consider the case of NaCl as shown in Fig.5.5. The external electric field displaces the positive Na^+ ion towards the right, while the negative Cl^- ion is displaced towards the left. Thus the forces due to external field stretch the length of the ionic bond between Na^+ and Cl^- ions. As a result of the change in the

length of the bond, a net dipole moment appears in the unit cell of NaCl, which does not have any dipole moment in the absence of the external electric field. Since the dipole moment in such cases arises due to displacement of oppositely charged ions, this process is known as ionic polarizability. Ionic polarizability exists in all dielectric materials which contain ionic bonds. Similar to electronic polarizability, ionic polarizability is also an elastic property involving no power loss. But unlike electronic polarizability, ionic polarizability is slower as ions are heavier than electrons. However, it is still a very fast process and occurs within 10^{-13} to 10^{-9}s.

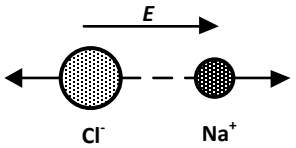


Fig.5.5 Ionic Polarization in NaCl

Orientational or Dipolar Polarizability

Although the individual molecules of a polar dielectric material have permanent dipoles, the net dipole moment becomes zero in a macroscopic piece of such dielectric material in the absence of external electric field because of random orientation of molecular dipole moments caused by thermal perturbations. Such random orientation of molecular dipoles results in near complete cancellation of dipole moment in any given direction in a macroscopic piece of polar dielectric material. But, when an external electric field is applied, then the molecular dipoles tend to align in the direction of the applied field as shown in Fig.5.6. It is because of the fact that the energy of a dipole $\vec{p}$ placed in a local electric field $\vec{E}_{loc}$ is $W = -\vec{p}.\vec{E}_{loc}$. This energy is minimum when the dipole is oriented parallel to the applied electric field. As a result of such alignment of molecular dipoles, the net dipole moment in a macroscopic piece of polar dielectric material gets a non-zero value. This mechanism through which a non-zero dipole moment arises in a polar dielectric is known as orientational or dipolar polarizability. Dipolar polarizability is much slower than electronic or ionic polarizability as it involves rotation of molecular dipoles that causes molecular friction. It is an inelastic process associated with power loss due to molecular friction and occurs within 10^{-9} to 10^{-4}s.

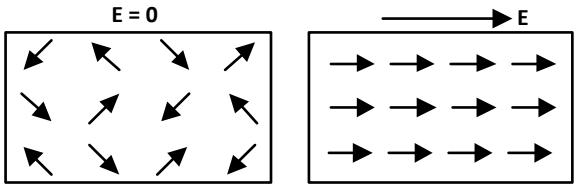


Fig. 5.6 Dipolar Polarization in polar dielectric materials

Electronic polarizability is present in all dielectrics, but the presence of ionic and dipolar polarizabilities depend on the molecular structure of dielectric materials. The relative magnitudes of the three polarizabilities, as discussed in section 5.3.3.1, 5.3.3.2 and 5.3.3.3, are such that in no-polar, ionic dielectric materials electronic polarizability is of the same order as ionic polarizability. On the other hand, in polar dielectric materials, the dipolar polarizability is much larger than both electronic and ionic polarizabilities.

Interfacial Polarizability

It occurs mainly in insulation system composed of different dielectric materials, e.g. oil impregnated paper/pressboard. Under the influence of an external electric field, small numbers of positive and negative charges, which are free to move within the bulk of the dielectric, get trapped at the interfaces of different materials, as shown in Fig.5.7, and thus produces separation of charges at the dielectric interfaces causing polarization. This mechanism known as interfacial polarizability is very slow and in general takes hours to complete.

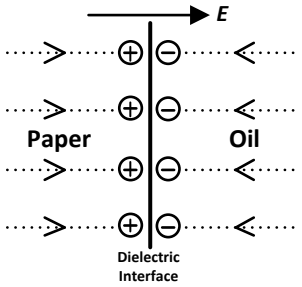


Fig.5.7 Interfacial polarization at dielectric interface

Field due to a Polarized Dielectric

Consider a block of polarized dielectric material, as shown in Fig.5.8, containing a polarization vector $\vec{P}$ that varies with position. According to eqn.(5.10), the small volume dV' located at $r'(x', y', z')$ can be replaced by a single dipole moment $\vec{p} = \vec{P}\,dV'$. Then the electric potential at any point *P* outside the volume of the dielectric and located at $r(x, y, z)$ is given by

$$d\phi_P = \frac{\vec{P}.\hat{u}_R}{4\pi\varepsilon_0 R^2} dV' \qquad \text{…. 5.15}$$

where, $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ = The distance between the small volume dV' and the external field point *P*.

In eqn.(5.15), $\frac{\vec{P}.\hat{u}_R}{R^2} = \vec{P}.\vec{\nabla}'\left(\frac{1}{R}\right)$ …. 5.16

Thus, $d\phi_P = \frac{1}{4\pi\varepsilon_0}\left[\vec{P}.\vec{\nabla}'\left(\frac{1}{R}\right)\right] dV'$ …. 5.17

where, $\vec{\nabla}'$ is the gradient with respect to the primed quantities, i.e. wrt the position of the small volume dV' within the dielectric volume.

So, the potential at the external field point *P* due to the entire volume *V* of the polarized dielectric

$$\phi_P = \int_V \frac{1}{4\pi\varepsilon_0}\left[\vec{P}.\vec{\nabla}'\left(\frac{1}{R}\right)\right] dV' \qquad \text{…. 5.18}$$

Consider a scalar quantity *a*, a vector quantity $\vec{A}$ and the vector identity as follows

$$\vec{\nabla}'.a\vec{A} = a\,\vec{\nabla}'.\vec{A} + \vec{A}.\vec{\nabla}'a \qquad \text{…. 5.19}$$

Putting $a = \frac{1}{R}$ and $\vec{A} = \vec{P}$ in eqn.(5.19)

$$\vec{P}.\vec{\nabla}'\left(\frac{1}{R}\right) = \vec{\nabla}'.\frac{\vec{P}}{R} - \frac{1}{R}\vec{\nabla}'.\vec{P} \quad \text{…. 5.20}$$

So, from eqns.(5.18) and (5.20)

$$\phi_P = \int_V \frac{1}{4\pi\varepsilon_0}\left[\vec{\nabla}'.\frac{\vec{P}}{R} - \frac{1}{R}\vec{\nabla}'.\vec{P}\right]dV' \quad \text{…. 5.21}$$

Applying Divergence Theorem to the first term of eqn.(5.21),

$$\phi_P = \int_S \frac{\vec{P}.\hat{u}_n'}{4\pi\varepsilon_0 R}dS' + \int_V \frac{-\vec{\nabla}'.\vec{P}}{4\pi\varepsilon_0 R}dV' \quad \text{…. 5.22}$$

where, $\hat{u}_n'$ is the outward unit normal vector to the surface dS' of the small volume dV' of the dielectric.

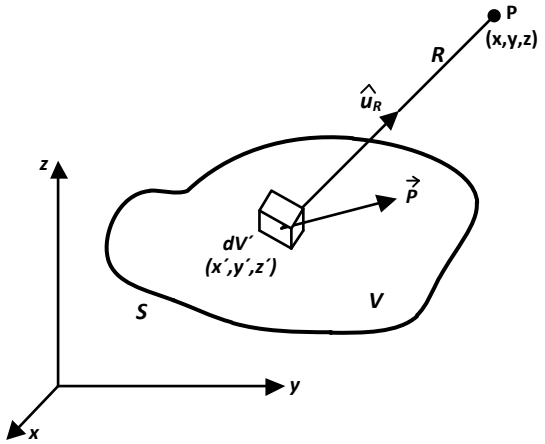


Fig.5.8 Field at an external point due to a polarized dielectric

The two terms on the RHS of eqn.(5.22) can be re-written as follows

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_{sb}(r')}{R}dS' + \int_V \frac{\rho_{vb}(r')}{R}dV'\right] \quad \text{…. 5.23}$$

where, $\sigma_{sb}(r') = \vec{P}.\hat{u}_n'$

$$\rho_{vb}(r') = -\vec{\nabla}'.\vec{P} \quad \text{…. 5.24}$$

and r' denotes the location within the polarized dielectric volume.

Then electric field intensity at the field point *P* is given by

$$\vec{E}_P = -\vec{\nabla}\phi_P = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_{sb}(r')\hat{u}_R}{R^2}dS' + \int_V \frac{\rho_{vb}(r')\hat{u}_R}{R^2}dV'\right] \quad \text{…. 5.25}$$

Bound Charge Densities of Polarized Dielectric

Eqns.(5.23) and (5.25) show that the field at an external point due to a polarized dielectric is superposition of the field due a volume charge density and a surface charge density. In other words, a polarized dielectric can be replaced by an equivalent volume charge density (ρ_{vb}) and an equivalent surface charge density (σ_{sb}). Both volume and surface charge densities can be considered to be in vacuum, as rest of the dielectric does not produce any field at an external point. These charge densities are called bound volume charge density and bound surface charge density, respectively, as these charges appearing due to polarization are not free to move within the dielectric material. These charges are caused by displacement or

rotation occurring in molecular scale during polarization. Such equivalent charge distribution is very useful because the problem of finding the field due to a polarized dielectric is converted to the problem of finding the field due to distribution of charges in vacuum, which is easier to solve. If the polarization vector is known at all the points within the polarized dielectric, then both bound volume and surface charge densities could be found from the polarization vector. Hence, the problem comes down to the determination of polarization vector within the volume of the polarized dielectric, which now-a-days is done with the help of numerical techniques.

Bound Volume Charge Density

From eqn.(5.24), bound volume charge density is $\rho_{vb}(r') = -\vec{\nabla}'.\vec{P}$, i.e. if the divergence of polarization vector $\vec{P}$ is non-zero, then the bound volume charge density will exist within the volume of the polarized dielectric. For uniform polarization, divergence of $\vec{P}$ is zero and hence there could be no bound volume charge density. But for non-uniform polarization, there can be net increase or decrease of charge within a given volume. For inhomogeneous dielectrics, there will be some net volume charge, because all the molecular dipoles are not identical and hence, their effect does not cancel out on average. In such cases, the divergence of $\vec{P}$ will be non-zero and hence bound volume charge density will be finite and non-zero. This fact can be understood from Fig.5.9(a), where at the center of the volume the negative ends of the dipoles are concentrated and hence there will be an excess of negative charges at that location giving rise to non-zero polarization vector $\vec{P}$.

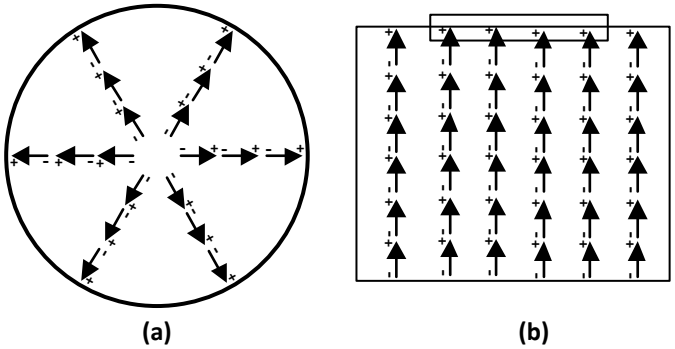


Fig.5.9 Equivalent charge distribution of polarized dielectric: (a) volume charge density due to non-uniform polarization, (b) surface charge density due to uniform polarization

Bound Surface Charge Density

From eqn.(5.24), bound surface charge density is $\sigma_{sb}(r') = \vec{P}.\hat{u}'_n$ Such surface charge densities are present for both uniform and non-uniform polarization. As shown in Fig.5.9(b), in the case of uniform polarization, for a macroscopic volume of the dielectric there will be equal amount of positive and negative charges and the net charge within the volume will be zero. But if a small volume is considered that includes the upper boundary perpendicular to the direction of polarization as shown in Fig.5.9(b), there will be net positive charge within the volume, no matter how thin the volume is made. In the limiting case, the thickness tends to zero, but there will still be excess positive charges on the surface. Therefore, bound charge density appears on the surface of the polarized dielectric due to uncompensated charges on the surface.

The Macroscopic Field

The expressions for electric potential and electric field intensity due to polarized dielectric only are given by eqns.(5.23) and (5.25), respectively. But the effects of the external charge distribution which causes the polarization must be added to these to get the resultant field. The effects are simply additive because the bound surface and volume charge densities due to polarization are considered to be in vacuum. Therefore, the complete expressions for electric potential and electric field intensity at a point outside the polarized dielectric due to the external charge distribution and the equivalent bound charge distributions of the polarized dielectric are given by

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\{\sigma_{ext} + \sigma_{sb}(r')\}}{R} dS' + \int_V \frac{\{\rho_{ext} + \rho_{vb}(r')\}}{R} dV'\right] \quad \text{…. 5.26}$$

$$\vec{E}_P = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\{\sigma_{ext} + \sigma_{sb}(r')\}\hat{u}_R}{R^2} dS' + \int_V \frac{\{\rho_{ext} + \rho_{vb}(r')\}\hat{u}_R}{R^2} dV'\right] \quad \text{…. 5.27}$$

Field due to a Narrow Column of Uniformly Polarized Dielectric

Consider a narrow column of polarized dielectric of cross-sectional area *dS* with a polarization vector of magnitude *P* directed along the axis of the column as shown in Fig.5.10.

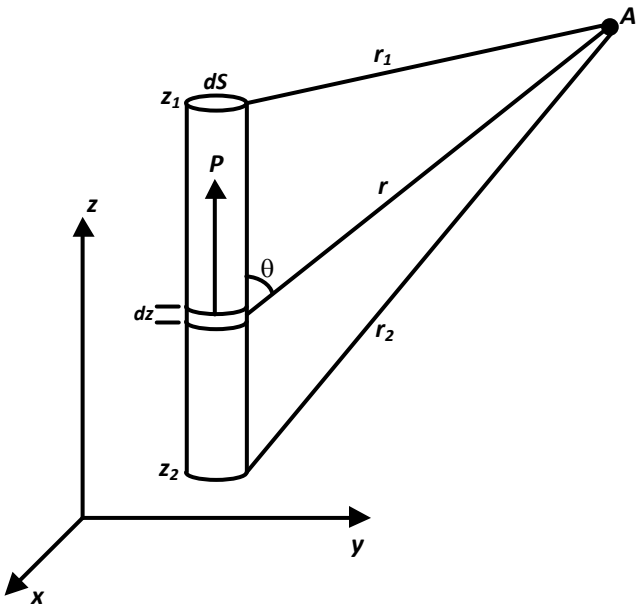


Fig.5.10 Field due to a narrow column of uniformly polarized dielectric

Electric potential at the external point *A* due to the small volume *dSdz* of the polarized column can be expressed according to eqn.(5.15) as follows

$$d\phi_A = \frac{P\,dS\,dz\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{P\,dS\,dr}{4\pi\varepsilon_0 r^2} \quad \text{…. 5.28}$$

Hence, the electric potential at *A* due to the entire column is

$$\phi_A = \frac{P\,dS}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2}$$

or, $$\phi_A = \frac{P\,dS}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = \frac{P\,dS}{4\pi\varepsilon_0 r_1} + \frac{-P\,dS}{4\pi\varepsilon_0 r_2} \quad \text{…. 5.29}$$

For the area *dS* at z_1, $\vec{P}.\hat{u}_n$ is positive and that at z_2 is negative. Thus eqn.(5.29) shows that the field due to the narrow column of polarized dielectric is the same as that due to positive charges (+*PdS*) on the surface *dS* at z_1 and the negative charges (-*PdS*) on the surface *dS* at z_2. In other words, the bound charges due to polarization appear at the two surfaces as it is a case of uniform polarization.

Field within a Sphere having Uniformly Polarized Dielectric

According to the discussions in section 5.4, the eqns.(5.23) and (5.25) are valid if the field observation point is located outside the polarized dielectric. However, these two equations can also be applied when the field observation point is inside the polarized dielectric, provided that average electric field is determined.

Consider a sphere of radius *R* containing a dielectric medium of uniform polarization $\vec{P}$. Such a uniformly polarized dielectric sphere could be equivalently constructed by superimposing two uniformly charged spheres of opposite polarity with centers displaced by a small distance *d*, as shown in Fig.5.11.

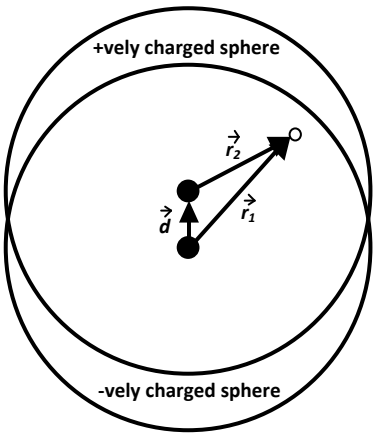


Fig.5.11 Field within a uniformly polarized dielectric sphere

Electric field intensity at a radial distance *r* within a charged sphere of radius *R* is given by

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\frac{r^3}{R^3}\hat{u}_r = \frac{q\vec{r}}{4\pi\varepsilon_0 R^3} \quad \text{…. 5.30}$$

where, $\vec{r}$ points outward from the center of the sphere and *q* is the total charge within the sphere.

In the case of two overlapping but slightly displaced spheres, as shown in Fig.5.11, for *r*<*R*

$$\vec{E} = \frac{q\vec{r}_2}{4\pi\varepsilon_0 R^3} + \frac{-q\vec{r}_1}{4\pi\varepsilon_0 R^3} = \frac{q(\vec{r}_2-\vec{r}_1)}{4\pi\varepsilon_0 R^3} = \frac{-q\vec{d}}{4\pi\varepsilon_0 R^3} \quad \text{…. 5.31}$$

Considering uniform polarization, net dipole moment within the dielectric sphere is $\vec{p} = q\vec{d}$.

Therefore, average polarization vector, $\vec{P} = \frac{3\vec{p}}{4\pi R^3} = \frac{3q\vec{d}}{4\pi R^3}$

So, from eqn.(5.31), $\vec{E} = -\frac{\vec{P}}{3\varepsilon_0}$ …. 5.32

Electric field intensity within the sphere is the average value determined by representing the sphere as a continuum medium with uniform polarization $\vec{P}$.
Outside the uniformly polarized sphere, electric field intensity is equivalent to that due to two point charges of opposite polarity located at the centers of the two slightly displaced spheres.

Sphere having Constant Radial Distribution of Polarization

Consider a sphere of radius *R* with constant radial distribution of polarization as shown in Fig.5.9(a). Thus the magnitude of polarization (*P*) is constant but the direction changes with position within the sphere. Hence, the polarization vector within the sphere is given by $\vec{P} = P\,\vec{r}$.
The polarization gives rise to bound charge density over the surface of the sphere which is given by

$\sigma_{sb} = \vec{P}.\hat{u}_n = P$ …. 5.33

Hence, total surface polarization charge is given by

$q_{sb} = 4\pi R^2 P$ …. 5.34

The volume charges due to polarization are distributed over the entire volume of the sphere and diverge at the center of the sphere. The bound volume charge density can be found as

$$\rho_{vb} = -\vec{\nabla}.\vec{P} = -P\vec{\nabla}.\vec{r} = -P\frac{1}{r^2}\frac{\partial r^2}{\partial r} = -\frac{2P}{r} \quad \text{…. 5.35}$$

Total volume charge due to polarization is

$$q_{vb} = \int_V \rho_{vb}\, dV = \int_0^R \left(-\frac{2P}{r}\right) 4\pi r^2\, dr = -4\pi R^2 P \quad \text{…. 5.36}$$

Hence, the total polarization charge, which is the sum of the bound volume charges (q_{vb}) and bound surface charges (q_{sb}), is zero.

Problem 5.1
A thin dielectric rod of cross-sectional area *S* extends along the *z*-axis from *z=0* to *z=H*. The polarization of the dielectric rod is along the *z*-axis and is given by $\vec{P} = (a_1 z + a_2)\hat{k}$. Calculate the bound surface charge density at each end, bound volume charge density and the total bound charge within the rod.
Solution
For the surface at z=*H*, $\hat{u}_n = \hat{k}$ and for the surface at *z*=0, $\hat{u}_n = -\hat{k}$
Hence, the bound surface charge density at *z=H*, $\sigma_{sb}\big|_{z=H} = \vec{P}.\hat{k}\Big|_{z=H} = (a_1 z + a_2)\big|_{z=H} = a_1 H + a_2$
and the bound surface charge density at *z=0*, $\sigma_{sb}\big|_{z=0} = \vec{P}.-\hat{k}\Big|_{z=0} = -(a_1 z + a_2)\big|_{z=0} = -a_2$
So, total bound surface charge $q_{sb} = (a_1 H + a_2)S - a_2 S = a_1 HS$
The bound volume charge density $\rho_{vb} = -\vec{\nabla}.\vec{P} = -a_1$
So, total bound volume charge $q_{vb} = -a_1 HS$
Therefore, total bound charge due to polarization $= q_{sb} + q_{vb} = 0$

Problem 5.2

A dielectric cube of side 2m and center at origin has a radial polarization given by $\vec{P}=4\vec{r}\ nC/m^2$ and $\vec{r}=2x\hat{i}+2y\hat{j}+2z\hat{k}$ m. Find bound surface charge density, bound volume charge density and the total bound charge due to polarization.

Solution

Given, $\vec{P}=8x\hat{i}+8y\hat{j}+8z\hat{k}\ nC/m^2$

For each of the six faces of the cube, there is a surface charge density σ_{sb}. For the face at x=1m,

$\sigma_{sb}=\vec{P}.\hat{i}\Big|_{x=1}=8x\Big|_{x=1}=8\,nC/m^2$

The magnitude of σ_{sb} is same for all the six faces. Therefore, considering all the six faces of the cube, total bound surface charge

$q_{sb}=6\times8\times2^2=192\,nC$

The bound volume charge densitdy is

$\rho_{vb}=-\vec{\nabla}.\vec{P}=-(8+8+8)=-24\,nC/m^3$

Hence, total bound volume charge $q_{vb}=-24\times2^3=-192\,nC$

Thus, total bound charge within the cube due to polarization = 192-192 = 0.

Electric Displacement Vector

According to Gauss's law, $\varepsilon_0\int_S\vec{E}.dS=\varepsilon_0\int_V\vec{\nabla}.\vec{E}\,dV=q_t$ …. 5.37

where, q_t is the total charge enclosed by the volume V, which for dielectric materials include free as well as bound volume charges.

Thus, $q_t=\int_V(\rho_f+\rho_{vb})dV$. Hence, from eqn.(5.37)

$\varepsilon_0\int_V\vec{\nabla}.\vec{E}\,dV=\int_V(\rho_f+\rho_{vb})dV$

or, $\vec{\nabla}.\vec{E}=\dfrac{(\rho_f+\rho_{vb})}{\varepsilon_0}=\dfrac{\rho_t}{\varepsilon_0}$ …. 5.38

But, according to eqn.(5.24), $\rho_{vb}=-\vec{\nabla}.\vec{P}$

Therefore, $\vec{\nabla}.\vec{E}=\dfrac{1}{\varepsilon_0}(\rho_f-\vec{\nabla}.\vec{P})$

or, $\vec{\nabla}.(\varepsilon_0\vec{E}+\vec{P})=\rho_f$ …. 5.39

Eqn.(5.39) is very important in the sense that the divergence of the vector $(\varepsilon_0\vec{E}+\vec{P})$ through any volume is equal to the free charge density in that volume. This form of Gauss' law is more convenient because the only charges that can be influenced externally are the free charges.

This vector $(\varepsilon_0\vec{E}+\vec{P})$ is called Electric Displacement Vector ($\vec{D}$), so that

$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$ …. 5.40

and from eqn.(5.39) $\vec{\nabla}.\vec{D}=\rho_f$ …. 5.41

The integral form eqn.(5.41) is $\int_S \vec{D}.dS = \int_V \rho_f \, dV$ …. 5.42

It should be noted here that both $\vec{\nabla}.\vec{D}$ and $\int_S \vec{D}.dS$ are related to free charges only and are unaffected by bound charges due to polarization.

From eqn.(5.38), it may be seen that both free and bound charges are sources of $\vec{E}$, while eqn.(5.41) shows that only free charges are sources of $\vec{D}$. In other words lines of $\vec{D}$ begin and end on free charges only, but the lines of $\vec{E}$ begin and end on either free or bound charges.

From eqn.(5.40), it may be written that $\vec{E}=\frac{\vec{D}}{\varepsilon_0}-\frac{\vec{P}}{\varepsilon_0}$ …. 5.43

It means that the E-field within a dielectric is resultant of two fields, viz. D-field and P-field. D-field is associated with free charges, while P-field is associated with bound charges due to polarization. Moreover, P-field acts in opposition to D-field. Thus, in the presence of P-field, i.e. in the presence of bound charges due to polarization, the E-field within a dielectric becomes less than the D-field.

Electric Susceptibility

For linear dielectric materials, polarization vector $\vec{P}$ varies directly with applied electric field $\vec{E}$. Hence, $\vec{P}$ and $\vec{E}$ are related as follows

$\vec{P}=\varepsilon_0 \chi_e \vec{E}$ …. 5.44

where, χ_e is known as electric susceptibility of the dielectric. It is a dimensionless quantity and is a measure of how susceptible a dielectric material is to applied electric field. In other words, it indicates the relative ease of polarization of the dielectric.

Dielectric Permittivity

From eqns. (5.40) and (5.44), for linear dielectric materials

$\vec{D} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$ …. 5.45

or, $\vec{D}=\varepsilon \vec{E}$ …. 5.46

where, $\varepsilon=\varepsilon_0 \varepsilon_r$ …. 5.47

and $\varepsilon_r = 1+\chi_e = \frac{\varepsilon}{\varepsilon_0}$ …. 5.48

Here, ε is called the permittivity and ε_r is called the dielectric constant or relative permittivity of linear dielectric. ε_r is also a dimensionless quantity. For any material in which polarization vector is non-zero, electric susceptibility is greater than unity and hence relative permittivity is always greater than unity. For the majority of dielectric materials, ε_r varies with the frequency of the applied electric field. The value of ε_r that is relevant to electrostatics is the value in steady electric field or at low frequencies (<1000Hz).

Relationship between free charge density and bound volume charge density

As discussed above, for linear dielectric media, $\vec{P}=\vec{D}-\varepsilon_0\vec{E}=\left(1-\frac{1}{\varepsilon_r}\right)\vec{D}$, as $\vec{D}=\varepsilon_r\varepsilon_0\vec{E}$

Therefore, $\vec{\nabla}.\vec{P}=\left(1-\frac{1}{\varepsilon_r}\right)\vec{\nabla}.\vec{D}=\left(1-\frac{1}{\varepsilon_r}\right)\rho_f$, as $\vec{\nabla}.\vec{D}=\rho_f$

or, $-\rho_{vb}=\left(1-\frac{1}{\varepsilon_r}\right)\rho_f$, as $\rho_{vb}=-\vec{\nabla}.\vec{P}$

Hence, total charge density $\rho_t=\rho_f+\rho_{vb}=\frac{\rho_f}{\varepsilon_r}$

Thus for $\varepsilon_r>1$, $\rho_t<\rho_f$, because ρ_f and ρ_{vb} are of opposite polarities.

Problem 5.3

In a dielectric material E_y=10V/m and polarization vector $\vec{P}=(400\hat{i}+300\hat{j}-200\hat{k})\,pC/m^2$. Calculate a) Electric susceptibility (χ_e), b) $\vec{E}$ and c) $\vec{D}$.

Solution

$P_y=\varepsilon_0\chi_e E_y$, or, $300\times10^{-12}=8.854\times10^{-12}\times\chi_e\times10$, or, $\chi_e=3.39$

$$E_x=\frac{P_x}{\varepsilon_0\chi_e}=\frac{400\times10^{-12}}{8.854\times10^{-12}\times3.39}=13.33V/m$$

and $$E_z=\frac{P_z}{\varepsilon_0\chi_e}=\frac{-200\times10^{-12}}{8.854\times10^{-12}\times3.39}=-6.67V/m$$

So, $\vec{E}=(13.33\hat{i}+10\hat{j}-6.67\hat{k})V/m$

Now, $\vec{D}=\varepsilon_0\vec{E}+\vec{P}$

So, $\vec{D}=\left\lfloor 8.854\times(13.33\hat{i}+10\hat{j}-6.67\hat{k})+(400\hat{i}+300\hat{j}-200\hat{k})\right\rfloor\times10^{-12}C/m^2$

or, $\vec{D}=(518.02\hat{i}+388.54\hat{j}-259.05\hat{k})\,pC/m^2$

Classification of Dielectrics

Eqns.(5.44) to (5.48) are not applicable to dielectric materials in general. These equations are valid for a sub-class of dielectric materials known as linear, isotropic and homogeneous (LIH) materials. LIH dielectrics exhibit the following properties:

i) Linearity: A dielectric is said to be linear if $\vec{D}$ varies linearly with $\vec{E}$. For such materials, permittivity is constant and independent of applied electric field. For non-linear dielectrics, $\vec{D}$ and $\vec{E}$ have non-linear relationship.

ii) Isotropy: A dielectric is said to be isotropic if $\vec{D}$ and $\vec{E}$ are in the same direction. Isotropic dielectrics have same permittivity in all directions. For anisotropic materials, $\vec{D}$, $\vec{E}$ and $\vec{P}$ are not parallel and hence permittivity varies with direction. Crystalline dielectrics are mostly anisotropic.

iii) Homogeneity: Dielectric materials for which properties are same at all points within the volume of the material are called homogenous. For inhomogeneous dielectrics, properties like permittivity vary with space coordinates. A typical example of inhomogeneous dielectric is atmosphere air, as the permittivity of air varies with altitude.

Molecular Polarizability of Linear Dielectric

Eqn.(5.44) relates polarization vector $\vec{P}$ and macroscopic electric field $\vec{E}$ through electric susceptibility (χ_e) in the case of linear dielectrics. The electric field that causes polarization of a molecule of a dielectric is known as molecular field $\vec{E}_{mol}$. Molecular field $\vec{E}_{mol}$ is different from the macroscopic field $\vec{E}$ because the polarization of neighbouring molecules gives rise to an internal field $\vec{E}_{\text{int}}$.

Hence, $\vec{E}_{mol} = \vec{E} + \vec{E}_{\text{int}}$ …. 5.49

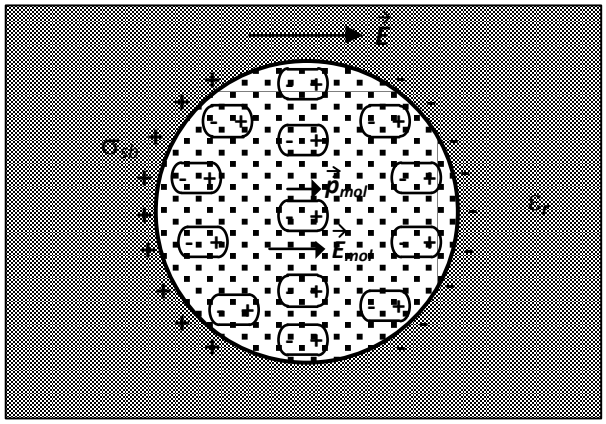


Fig.5.12 Molecular and macroscopic fields

As shown in Fig.5.12, consider an imaginary sphere which contains the neighbouring molecules. This sphere is much larger in dimension compared to the molecules, but is infinitesimally small in macroscopic scale. The dielectric outside the sphere is replaced by the system of bound charges due to polarization (σ_{sb}). Then the internal field can be resolved into two components:

$\vec{E}_{\text{int}} = \vec{E}_{near} + \vec{E}_{far}$ …. 5.50

where, $\vec{E}_{near}$ is the field due to neighbouring molecules which are located close to the given molecule and $\vec{E}_{far}$ is the field due to all other molecules, which arises from the bound charge density (σ_{sb}) on the sphere surface. $\vec{E}_{far}$ can be expressed in spherical coordinates as follows

$$\vec{E}_{far} = \frac{1}{4\pi\varepsilon_0}\int_S \sigma_{sb}\frac{-\vec{r}}{r^3}dS = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}(P\cos\theta)\frac{\vec{r}}{r^3}(r\,d\theta)(r\sin\theta\,d\phi) \quad \text{…. 5.51}$$

Considering the Cartesian coordinates, the *x*-component vanishes as it involves the integral $\int_0^{2\pi}\cos\phi\,d\phi$, which evaluates to zero, and the *y*-component vanishes as it involves the integral $\int_0^{2\pi}\sin\phi\,d\phi$, which is also zero. Therefore,

$$\vec{E}_{far} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi} P\hat{u}_z\cos^2\theta\sin\theta\,d\theta\,d\phi = \frac{\vec{P}}{3\varepsilon_0} \quad \text{…. 5.52}$$

If the neighbouring molecules are randomly distributed in location, which is the case in most linear dielectrics, then $\vec{E}_{near} = 0$. So, $\vec{E}_{\text{int}} = \dfrac{\vec{P}}{3\varepsilon_0}$

Then, $\vec{E}_{mol} = \vec{E} + \dfrac{\vec{P}}{3\varepsilon_0}$ …. 5.53

Let, N be the number of molecules per unit volume and $\vec{p}_{mol}$ be the dipole moment of each molecule, then the polarization vector $\vec{P}$ is given by

$\vec{P} = N\,\vec{p}_{mol}$ …. 5.54

Molecular dipole moment and molecular field can be related with the help of molecular polarizability (α_{mol}) as follows

$\vec{p}_{mol} = \varepsilon_0\,\alpha_{mol}\,\vec{E}_{mol}$ …. 5.55

Hence, from eqns.(5.53), (5.54) and (5.55),

$\vec{P} = N\,\varepsilon_0\,\alpha_{mol}\,\vec{E}_{mol} = N\,\alpha_{mol}\left(\varepsilon_0\vec{E} + \dfrac{\vec{P}}{3}\right)$ …. 5.56

Putting $\vec{P} = \varepsilon_0\,\chi_e\,\vec{E}$ in eqn.(5.56),

$\chi_e = \dfrac{3N\,\alpha_{mol}}{3 - N\,\alpha_{mol}}$ …. 5.57

Using $\varepsilon_r = 1 + \chi_e$ in eqn.(5.57)

$\varepsilon_r = 1 + \dfrac{3N\,\alpha_{mol}}{3 - N\,\alpha_{mol}}$, or, $\alpha_{mol} = \dfrac{3}{N}\dfrac{(\varepsilon_r - 1)}{(\varepsilon_r + 2)}$ …. 5.58

Eqn.(5.58) is known as Clausius-Mossotti relation.

In the discussion of this section a simple molecular model is used to understand the linear behaviour of dielectric that is characteristics of large number of dielectric materials. A detailed treatment, however, will necessitate quantum mechanical consideration.

For a non-polar dielectric, dipole moment induced by the external field is $\vec{p} = \alpha_e\,\vec{E}$, where, α_e = Electronic polarizability of the atom. Then, polarization vector

$\vec{P} = N\,\vec{p}$, where N is the number of dipoles per unit volume.

But, $\vec{P} = \varepsilon_0\,\chi_e\,\vec{E}$

Hence, $\varepsilon_0\,\chi_e\,\vec{E} = N\,\alpha_e\,\vec{E}$, or, $\chi_e = \dfrac{N\,\alpha_e}{\varepsilon_0}$, or, $\varepsilon_r = 1 + \chi_e = 1 + \dfrac{N\,\alpha_e}{\varepsilon_0}$ …. 5.59

Piezoelectric Materials

The term piezoelectricity refers to the fact that when a dielectric is mechanically stressed, then an electric field is produced within the dielectric. As a result of this electric field, measurable quantity of electric potential difference appears across the dielectric sample, which can be measured to find the mechanical strain on the dielectric. This principle is commonly used in piezoelectric strain transducers. The inverse effect also exists, that is mechanical strain is produced in a dielectric due to the application of electric field. Piezoelectric effect is mostly reversible.

Piezoelectric materials could be natural or synthetic. The most commonly used natural piezoelectric material is quartz (SiO_2). But synthetic piezoelectric materials, e.g. ceramics and polymers, are more efficient. The piezoelectric materials used in practice are, Berlinite

($AlPO_4$), Gallium Orthophosphate ($GaPO_4$), Barium Titanate ($BaTiO_3$), Lead Zirconate Titanate (PZT: $PbZr_{1-x}Ti_xO_3$), Aluminum Nitride (AlN), Polyvinylidine Fluoride (PVDF) to name a few. In recent years, piezoceramics and piezopolymers are widely used in smart structures. Very recently, breakthrough in single crystal growth technique has enabled the development of high strain and high electric breakdown piezoceramics.
The nature of piezoelectric effect is strongly related to the large number of electric dipoles present in the piezoelectric materials. These dipoles can either be due to ions on crystal lattice sites with asymmetric charge distribution or due to certain molecular groups having asymmetric configurations. When a mechanical stress is applied on a piezoelectric material, the crystalline structure is disturbed and it changes the direction of the polarization vector due to the electric dipoles. If the dipole is due to the ions, then the change in polarization is caused by a re-configuration of ions within the crystalline structure. On the other hand, if the dipole is due to molecular groups, then re-orientation of molecular groups causes the change in the polarization. The electric field developed because of the change in net polarization gives rise to piezoelectric effect.

Ferroelectric Materials

A ferroelectric material is a dielectric with at least two discrete stable or metastable states of different non-zero electric polarization under zero applied electric field, referred as "spontaneous polarization". For a material to be considered ferroelectric, it must be possible to switch between these states by the application and removal of an applied electric field. In the case of conventional ferroelectrics, the spontaneous polarization is produced by the atomic arrangement of ions in the crystal structure, depending on their positions, and in electronic ferroelectrics the spontaneous polarization is produced by charge ordering of multiple valences. A non-zero spontaneous polarization can be present only in a crystal with a polar space group. Numerical values of spontaneous polarization are customarily given in units of $\mu C/m^2$. All ferroelectric materials are necessarily piezoelectric.
In the ferroelectric state the plot of polarization versus applied electric field shows a hysteresis loop, as shown in Fig.5.13, similar to ferromagnetic materials. A significant feature of ferroelectrics is that the spontaneous polarization can be reversed by an appropriately strong electric field applied in the opposite direction. Dielectric permittivity, which is dependent on the slope of *P-E* curve, is therefore dependent on the applied field, in the ferroelectric state.
In most ferroelectrics, there is a phase transition from the ferroelectric state, to a non-polar paraelectric phase, with increasing temperature. The phase transition temperature is known as Curie Point (T_C) at which the ferroelectric material changes from the low temperature polarized state to high temperature unpolarized state. The spontaneous polarization in most ferroelectric crystals is greatest at temperatures well below T_C and decreases to zero at T_C. Some ferroelectric materials have no Curie Point as these materials melt before the phase transition temperature is reached. The range of phase transition temperature for different ferroelectric materials is very wide varying from very low (50~100K) to very high (over 1000K) temperature.

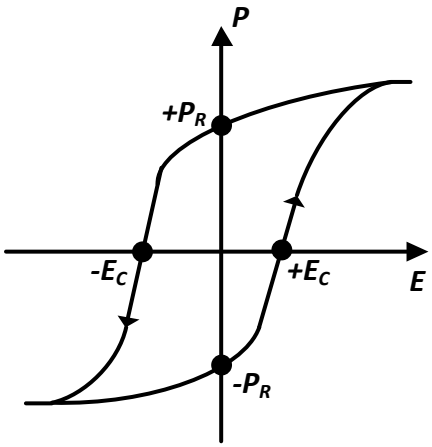


Fig.5.13 Hysteresis loop in ferroelectric state

While ferroelectricity was discovered in hydrogen-bonded material Rochelle salt ($NaKC_4H_4O_6,4H_2O$), dramatic change in the understanding of this phenomenon came through the discovery of ferroelectricity in the much simpler, non-hydrogen containing, perovskite oxide $BaTiO_3$. $BaTiO_3$ is the typical example of the very large and extensively studied and used perovskite oxide family, which not only includes perovskite compounds, but also includes ordered and disordered solid solutions.
Ferrolecetrics are used in variety of applications like non-volatile memories, capacitors having tunable capacitance, varactors for RF/Microwave Circuits, Electro-Optic Modulators, high permittivity applications, pyroelectric detectors and so on.

Electrets

Electrets are unique, man-made materials that can hold an electrical charge after being polarized in an electric field. It is a piece of dielectric material which has been specially prepared to possess an overall fixed dipolarity. It is the electrical analogy of a permanent magnet. Instead of opposite magnetic poles, the electret has two electrical poles of trapped opposite polarity charges. Therefore, a fixed “static” potential exists between the two opposite poles of the dipolar electret.
Electrets can be prepared from different dielectric materials depending on their structures and properties. The very first electrets were made of carnauba wax (Brazilian palm gum) and its mixtures with rosin, beeswax, ethylcellulose and other components. When a polar dielectric is placed in an electric field, the applied field causes the dipoles to be oriented in such a way that the dipole moments are directed parallel to the applied field. The degree of alignment achieved depends upon the freedom with which a dipole can turn around its axis. In the case of a material like carnauba wax, this freedom is greater when the wax is in molten state, but is almost zero when the wax is in the solid state. Hence, it is possible to melt the wax, keep it in an electric field so that the dipoles align themselves. Afterwards, while still under the influence of the electric field, the wax is allowed to cool and solidify. Then the molecular dipoles get set in the aligned position, and the wax piece becomes an electret. Electrets prepared in this way are known as Thermo-electrets. It has been reported in published literature that effective surface charge density of the order of 4~6nC/m^2 has been preserved practically unchanged in carnauba wax based electrets for more than 35 years.
However, there are several other methods of preparation of electrets: (a) Photo-electrets are produced using light as heat source, (b) Radio-electrets are prepared by exposing the

dielectric to a beam of charged particles, (c) Corona-electret is produced by placing the dielectric in a corona discharge field and is now-a-days preferred for industrial production of electrets, and (d) Mechano-electrets are formed by mechanical compression of dielectric between heated plates. Many modern electrets have only space or surface charge (like Teflon or Polypropylene electrets) and no dipole polarization.
Electret materials include ceramics, non-polar/polar semi-crystalline and amorphous synthetic polymers, biopolymers and ferro-electric ceramic/polymer composites. The most common applications of electret are microphones, electro-acoustic devices, infrared detection and photocopying machines. Electrets are also being useful as novel devices in bio-medical and high-energy charge storage applications.

Frequency Dependence of Polarizabilities

If the behavior of dielectric polarizability is studied under alternating field, important distinctions between various polarizabilities emerge. Typical dependence of polarizabilities on frequency is depicted in Fig.5.14 over a wide range starting from static field to frequencies above ultraviolet region.

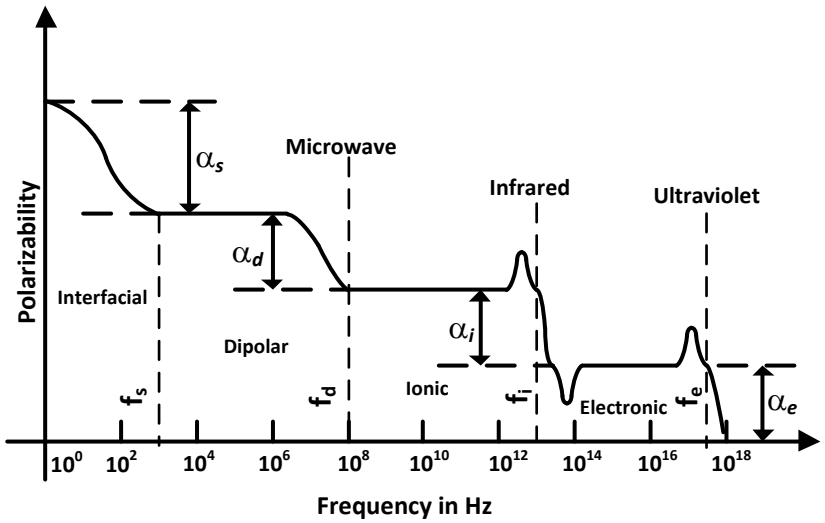


Fig.5.14 Frequency dependence of polarizabilities

It may be seen that between f=0 to $f=f_s$, where f_s is typical around 1kHz, the polarizability gradually increases as frequency is decreased. Such increasing polarizability at lower frequencies arises because of interfacial polarization mechanism. From $f=f_s$ to $f=f_d$, polarizability remains more or less constant, where f_d is typically in microwave region. In this frequency span dipolar polarization mechanism is predominant. For $f>f_d$ the polarizability decreases by a significant amount and the amount of decrease corresponds to the dipolar polarizability. The reason for disappearance of dipolar polarizability for $f>f_d$ is that the field at these frequencies oscillate at a rapid rate which the dipoles are unable to follow and hence the dipolar polarizability vanishes. Further the polarizability remains nearly constant in the frequency range $f=f_s$ to $f=f_i$, where f_i is in infrared region. The drop in polarizability for $f>f_i$ is due to the absence of ionic polarizability at such higher frequencies, because ions being heavy are unable to follow the very rapidly varying ac field for $f>f_i$. As a result for $f>f_i$ only electronic polarizability is active, because electrons being lighter are still able to follow the oscillating field at these frequencies. At extremely high frequencies for $f>f_e$, where f_e is in

ultraviolet region, even the electronic polarizability vanishes, as the electrons are also unable to follow such extremely rapid oscillating field.
Typically the frequencies f_e, f_i, f_d and f_s characterize electronic, ionic, dipolar and interfacial polarizabilities, respectively. These frequencies depend on the dielectric materials and vary from dielectric to dielectric and also on the condition of the dielectric. Various polarizabilities can be determined by measuring dielectric properties at various frequencies of appropriate value. In fact this principle is the foundation of frequency domain spectroscopy, which is a major techniques used for condition monitoring of high voltage insulation system.

Electrostatic Boundary Conditions

Introduction

In real life any conductor is always surrounded by at least one dielectric. It should be kept in mind that air is also a dielectric which is present almost everywhere. So even if there is no solid or liquid or any other gaseous dielectric around a conductor, it will in all probability be surrounded by air. Therefore, there will be boundaries between a conductor and a dielectric in practice. Moreover, except for very few cases like single core cables having only one dielectric or transmission line conductors surrounded by air at mid-span, dielectric materials are arranged either in series or in parallel between two conductors having a particular potential difference. For example, if one takes the case of an outdoor porcelain insulator, it may appear that there is only one dielectric, i.e. porcelain, involved. But the porcelain insulator will be surrounded by air and hence it becomes a two dielectric configuration. As a result there will be boundaries between two different dielectric media in practical configurations. Changes in some electric field quantities in direction and/or magnitude occur at such boundaries. The equations which describe such field behaviors by relating electric field quantities on two sides of a boundary surface are known as boundary conditions. The transition of electric field from one medium to another medium through a boundary surface is governed by the boundary conditions.

Boundary Conditions between a Perfect Conductor and a Dielectric

A perfect conductor is defined as a material within which the charges are able to move freely. In electrostatics it is considered that the charges have attained the equilibrium positions and are fixed in space. Theoretically consider that the charges are initially distributed uniformly throughout the volume of a perfect conductor. Such distributed charges should be of same polarity within a conductor of one particular value of electric potential, because if there are charges of opposite polarity within the volume of the conductor, then such charges will immediately recombine with each other as they are free to move. Hence, the charges of same polarity those are present in the volume of the conductor will exert repulsive forces on each other. Since the charges are able to move without any hindrance, hence the charges will disperse in such a direction so that the distance between the charges will increase. In the process all the charges will arrive at the surface of the conductor. But the conductor being surrounded by a dielectric, the charges are unable to move further and the charges will be fixed in space on the surface of the conductor. Consequently, any Gaussian surface within a perfect conductor will enclose zero charge and hence, electric field within a perfect conductor will be zero.

Boundary Condition for Normal Component of Electric Flux Density

Consider a coin-like closed volume of cylindrical shape as shown in Fig.6.1. Such a volume is often termed as a “Gaussian pillbox”. The pillbox has finite surface area ΔA and an infinitesimally small height δ, such that half of the pillbox is within the conductor and the other half is within the dielectric medium as shown. The top and bottom surfaces of the pillbox are parallel to the conductor-dielectric interface.

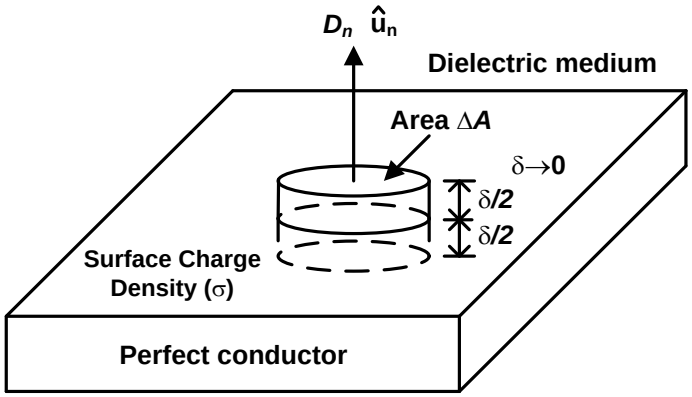


Fig.6.1 Pertaining to boundary condition for electric flux density at conductor-dielectric boundary

Application of Gauss's law to this pillbox yields

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \text{Charge enclosed by the pillbox} \quad \text{…. 6.1}$$

For the RHS of eqn.(6.1) the following need to be considered: a) the volume of the pillbox is infinitesimally small and b) half of the volume of the pillbox within the perfect conductor does not contain any volume charge density, as the charges reside on the surface of the conductor and the other half of the volume of the pillbox within the dielectric also does not contain any volume charge density, if ideal dielectric is assumed. But the surface charge density on the conductor has a finite value and the area of the pillbox is also finite.

Hence, the RHS of eqn.(6.1) = $\sigma \Delta A$ …. 6.2

where, σ = surface charge density on the conductor.

The integral on the LHS of eqn.(6.1) could be expanded as follows

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \int_{Top\ surface} \vec{D}.d\vec{s} + \int_{Bottom\ surface} \vec{D}.d\vec{s} + \int_{Wall\ surfaces} \vec{D}.d\vec{s} \quad \text{…. 6.3}$$

As the height of the pillbox is infinitesimally small, hence the integral over the wall surfaces is negligible. The integral over the bottom surface is also zero as the field within the perfect conductor is zero. Hence, eqn.(6.3) could be rewritten as

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \int_{Top\ surface} \vec{D}.d\vec{s} = D_n\ \Delta A \quad \text{…. 6.4}$$

where, D_n is the normal component of electric flux density.

Hence, from eqns.(6.2) and (6.4),

$D_n \Delta A = \sigma \Delta A$, or, $D_n = \sigma$ …. 6.5

Bringing in the unit normal vector, eqn.(6.5) could be written as $\hat{u}_n . \vec{D} = \sigma$ …. 6.6

Boundary Condition for Tangential Component of Electric Field Intensity

Consider an infinitesimally small closed rectangular loop "*abcda*" as shown in Fig.6.2, of which the length segments *ab* and *cd* are parallel to the conductor-dielectric boundary and the length segments *bc* and *da* are normal to the boundary. The length of the loop parallel the boundary is Δl and is finite, but the length of the loop normal to the boundary is δ, which is negligibly small.

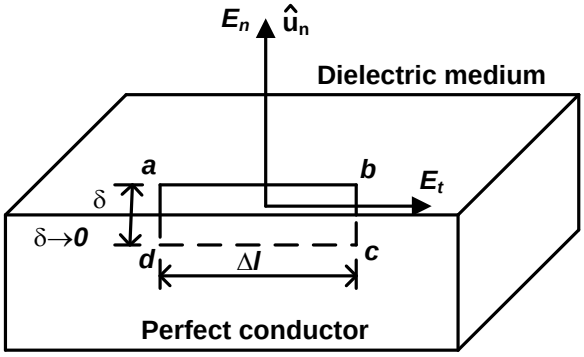


Fig.6.2 Pertaining to boundary condition for electric field intensity at conductor-dielectric boundary

As *E*-field is conservative in nature, hence the integral of $\vec{E}.d\vec{l}$ over the loop contour *abcda* will be zero, i.e.

$$\oint_{abcda} \vec{E}.d\vec{l} = \int_a^b \vec{E}.d\vec{l} + \int_b^c \vec{E}.d\vec{l} + \int_c^d \vec{E}.d\vec{l} + \int_d^a \vec{E}.d\vec{l} = 0 \qquad \text{…. 6.7}$$

As stated the lengths *bc* and *da* are negligibly small. Hence, $\int_b^c \vec{E}.d\vec{l} = \int_d^a \vec{E}.d\vec{l} \approx 0$

Again the field within the perfect conductor is zero. Hence, $\int_c^d \vec{E}.d\vec{l} = 0$

Therefore, from eqn.(6.7), $\int_a^b \vec{E}.d\vec{l} = 0$

The length *ab* is Δl, which is small but finite. Then considering $\vec{E}$ to be constant over the small length Δl, it may be written that

$E_t \Delta l = 0$, or, $E_t = 0$ …. 6.8

where, E_t is the component of electric field intensity along the length *ab*, which is the tangential component of electric field intensity.

Bringing in the unit normal vector, eqn.(6.8) could be written as $\hat{u}_n \times \vec{E} = 0$ …. 6.9

Field just off the Conductor Surface

From eqn.(6.8), on the conductor surface $E_t = 0$. In other words, the electric field acts in the direction normal to the conductor surface. Then, from eqn.(6.5), the normal component of electric field intensity could be written as

$$E_n = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_r \varepsilon_0} \qquad \text{…. 6.10}$$

As discussed in section 6.2.1, the electric field intensity as given by eqn.(6.10) is for the top surface of the pillbox. The height of the pillbox is infinitesimally small, but is not zero. Hence, the top surface of the pillbox, as shown in Fig.6.1, is not exactly on the conductor surface. As a result the value of electric field intensity as obtained from eqn.(6.10) is stated to be electric field intensity within the dielectric medium just off the conductor surface.

Problem 6.1

A charged conductor is surrounded by air. Calculate

i) the maximum charge density that the conductor can hold at STP and

ii) the mechanical pressure acting on the conductor surface at that charge density.

Solution:

The conductor can hold the maximum charge density for which the electric field intensity just off the surface is equal to the breakdown strength of air at STP. This is due to the fact that any further increase in the value of charge density on the conductor surface will cause breakdown of air and the charges will be drained from the conductor.

Breakdown strength of air at STP = 30 kV/cm = $30\times10^3\times10^2$ V/m = 3×10^6 V/m

Let, the maximum charge density that the conductor can hold be σ_m.

Now, ε_r for air = 1

So, $\sigma_m / \varepsilon_0 = 3\times10^6$

or, $\sigma_m = 3\times10^6 \times 8.854 \times 10^{-12} = 26.5\ \mu C/m^2$

Mechanical pressure acting on the conductor for $\sigma_m = (3\times10^6\times26.5\times10^{-6}/2) = 39.75\ N/m^2$

Boundary Conditions between Two Different Dielectric Media

In the case of practical configurations comprising multiple homogeneous dielectric media, there could be many dielectric-dielectric boundaries. On such dielectric-dielectric boundaries there could be charges depending on the dissimilarities of the dielectric media that are present on the two sides of the boundaries. These surface charges will serve as source of electric field acting in opposite directions on the two sides of the boundary. Consequently, electric field quantities get changed in direction as well as magnitude on the two sides of the boundary.

6.3.1 Boundary Condition for Normal Component of Electric Flux Density

For the boundary between two dielectric media having permittivities ε_1 and ε_2, consider a cylindrical coin-like Gaussian pillbox as shown in Fig.6.3. The height of the pill box is infinitesimally small. As in eqn.(6.1) application of Gauss's law to this pillbox yields

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \textit{Charge enclosed by the pillbox}$$

Subdividing the integral on the LHS into contributions from the top, bottom and wall surfaces of the pillbox and subdividing the wall surface into two halves, one in dielectric 1 and the other in dielectric 2,

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \int_{Top\ surface\ Dielectric-2} \vec{D}_2.d\vec{s} + \int_{Bottom\ surface\ Dielectric-1} \vec{D}_1.d\vec{s} + \int_{Wall\ surface\ Dielectric-1} \vec{D}_1.d\vec{s} + \int_{Wall\ surface\ Dielectric-2} \vec{D}_2.d\vec{s}$$

As the height of the pillbox is vanishingly small, hence the contribution of the integral over the wall surfaces will be negligible. Thus

$$\int_{Surface\ of\ pillbox} \vec{D}.d\vec{s} = \int_{Top\ surface\ Dielectric-2} \vec{D}_2.d\vec{s} + \int_{Bottom\ surface\ Dielectric-1} \vec{D}_1.d\vec{s} \qquad \ldots. 6.11$$

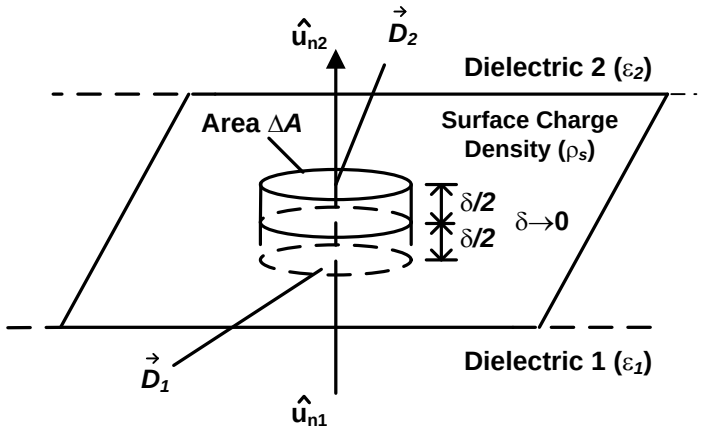


Fig.6.3 Pertaining to boundary condition for electric flux density at dielectric-dielectric boundary

As the top and bottom surfaces are small, $\vec{D}_1$ and $\vec{D}_2$ could be assumed to be constant over the bottom and top surfaces, respectively. Hence,

$$\int_{Top\ surface\ Dielectric-2} \vec{D}_2 . d\vec{s} \ = \vec{D}_2 \ . \Delta\vec{A} = (\vec{D}_2 . \hat{u}_{n2}) \Delta A \qquad \text{…. 6.12}$$

and $$\int_{Top\ surface\ Dielectric-1} \vec{D}_1 . d\vec{s} \ = \vec{D}_1 \ . \Delta\vec{A} = (\vec{D}_1 . \hat{u}_{n1}) \Delta A \qquad \text{…. 6.13}$$

Now considering the unit normal to be directed into the dielectric-2, $\hat{u}_n = \hat{u}_{n2} = -\hat{u}_{n1}$

So, from eqn.(6.11), $$\int_{Surface\ of\ pillbox} \vec{D} . d\vec{s} = \left(\vec{D}_2 . \hat{u}_n - \vec{D}_1 . \hat{u}_n\right) \Delta A \qquad \text{…. 6.14}$$

Let, the surface charge density on the boundary be ρ_s and the volume charge densities in the two halves of the pillbox be ρ_{v1} and ρ_{v2}. Then the net charge enclosed by the pillbox is given by

$$\textit{Charge enclosed by the pillbox} = \rho_s \Delta A + \rho_{v1} \Delta A \frac{\delta}{2} + \rho_{v2} \Delta A \frac{\delta}{2} \qquad \text{…. 6.15}$$

As the height of the pillbox, δ, is vanishingly small, the contribution of the terms involving volume charge densities will be negligible. Hence, eqn.(6.15) becomes

$$\textit{Charge enclosed by the pillbox} = \rho_s \Delta A \qquad \text{…. 6.16}$$

So, from eqns.(6.14) and (6.16), $\left(\vec{D}_2 . \hat{u}_n - \vec{D}_1 . \hat{u}_n\right) = \rho_s$

or, $$D_{2n} - D_{1n} = \rho_s \qquad \text{…. 6.17}$$

as, $\vec{D}_2 . \hat{u}_n = D_{2n}$ and $\vec{D}_1 . \hat{u}_n = D_{1n}$, which are normal components of electric flux density in dielectric 2 and 1, respectively.

Eqn.(6.17) is valid in general. For example, consider that the medium 1 is a perfect conductor. Then D_1 is zero. Then D_{2n} is equal to the surface charge density, which is accordance with eqn.(6.5).

Boundary Condition for Tangential Component of Electric Field Intensity

Considering the infinitesimally small closed rectangular contour *abcdefa* as shown in Fig.6.4 and applying the principle of conservative *E*-field along this closed contour

$$\oint_{abcdefa} \vec{E}.d\vec{l} = \int_a^b \vec{E}_2.d\vec{l} + \int_b^c \vec{E}_2.d\vec{l} + \int_c^d \vec{E}_1.d\vec{l} + \int_d^e \vec{E}_1.d\vec{l} + \int_e^f \vec{E}_1.d\vec{l} + \int_f^a \vec{E}_2.d\vec{l} = 0 \qquad \text{…. 6.18}$$

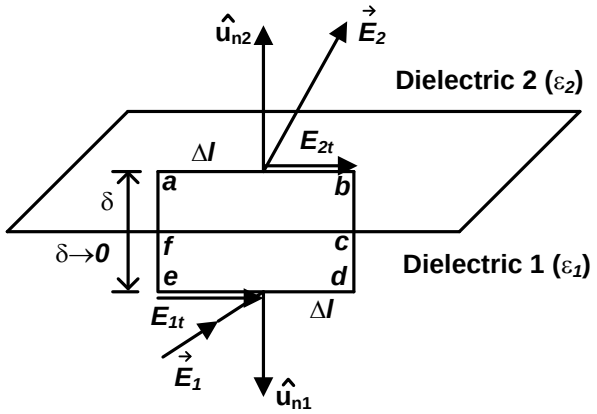


Fig.6.4 Pertaining to boundary condition for electric field intensity at dielectric-dielectric boundary

But the length of the segments *bc*, *cd*, *ef* and *fa* are negligibly small and hence the integrals over these length elements contribute insignificantly. Thus from eqn.(6.18)

$$\int_a^b \vec{E}_2.d\vec{l} + \int_d^e \vec{E}_1.d\vec{l} = 0$$

Then considering $\vec{E}_2$ and $\vec{E}_1$ to be constant over the small lengths *ab* and *de* and noting that $d\vec{l}_{ab} = \Delta l = -d\vec{l}_{de}$, it may be written that

$$\int_a^b \vec{E}_2.d\vec{l} + \int_d^e \vec{E}_1.d\vec{l} = E_{2t}\,\Delta l - E_{1t}\,\Delta l = 0$$

or, $E_{2t} = E_{1t}$ …. 6.19

where, E_{2t} and E_{1t} are tangential components of E_2 and E_1, respectively, along the boundary. Eqn.(6.19) is also valid in general. For example, if the medium 1 is considered to be a perfect conductor, then E_1 is zero. Then E_{2t} is also zero, which is accordance with eqn.(6.8).

Problem 6.2

For a two dielectric arrangement comprising transformer oil (ε_{r1}=2) as dielectric-1 and mica (ε_{r2}=6) as dielectric-2, it is given that $\vec{E}_1 = 6\hat{i} + 4\hat{j} + 9\hat{k}$ kV/cm. Find $\vec{E}_2$: (A) considering the boundary to be charge free and (B) considering a surface charge density of +70pC/cm^2 on the boundary. Given that *x*-*y* plane is the boundary between the two media.

Solution:

Since *x*-*y* plane is the boundary between transformer oil and mica, hence *x*- and *y*-components of $\vec{E}$ are the tangential components along the boundary and are equal on both sides of the boundary. The *z*-component of $\vec{D}$ is the normal component of electric flux density on the boundary.

Application of boundary condition for tangential component of electric field intensity yields $E_{1x} = E_{2x} = 6$ kV/cm and $E_{1y} = E_{2y} = 4$ kV/cm.

z-component of $\vec{E}$ need to be determined from the boundary condition of D_n separately for Case-A and Case-B.

Case-A: Boundary is charge free.

As ρ_s=0, $D_{2n} = D_{1n}$, or, $\varepsilon_{r2}\,\varepsilon_0\,E_{2z} = \varepsilon_{r1}\,\varepsilon_0\,E_{1z}$

As E_{1z}=9kV/cm, $E_{2z} = \frac{2}{6}\times 9 = 3$ kV/cm

Therefore, $\vec{E}_2 = 6\hat{i} + 4\hat{j} + 3\hat{k}$ kV/cm.

Case-B: Surface charge density (ρ_s) on the boundary is +70pC/cm^2.

From the boundary condition of D_{2n},

$\varepsilon_{r2}\,\varepsilon_0\,E_{2z} - \varepsilon_{r1}\,\varepsilon_0\,E_{1z} = \rho_s$

or, $E_{2z} = \frac{\varepsilon_{r1}\,\varepsilon_0\,E_{1z} + \rho_s}{\varepsilon_{r2}\,\varepsilon_0} = \frac{\varepsilon_{r1}\,E_{1z}}{\varepsilon_{r2}} + \frac{\rho_s}{\varepsilon_{r2}\,\varepsilon_0} = \frac{2\times 9\times 10^3\times 10^2}{6} + \frac{70\times 10^{-12}\times 10^4}{6\times 8.854\times 10^{-12}}$

= 313176 V/m = 3.131 kV/cm

Therefore, $\vec{E}_2 = 6\hat{i} + 4\hat{j} + 3.131\hat{k}$ kV/cm.

Boundary Condition for Charge free Dielectric-Dielectric Interface

Consider an interface between two different dielectric media having permittivities ε_1 and ε_2, respectively, having no surface charge density on the boundary as shown in Fig.6.5. Typically if the dielectric media are ideal in nature so that there is no volume as well as surface conduction and also if there is no discharge taking place within the dielectric media or on the interface between the dielectric media, then the surface charge density on the interface is zero.

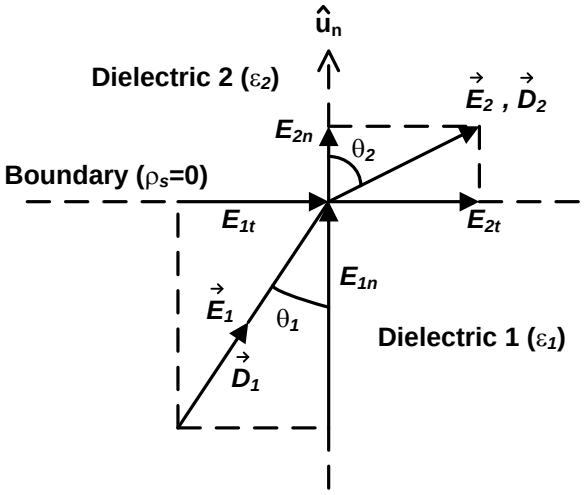


Fig.6.5 Pertaining to boundary condition for charge free dielectric-dielectric boundary

From eqn.(6.17), $D_{2n} - D_{1n} = 0$, as $\rho_s = 0$

or, $D_{2n} = D_{1n}$, or, $\varepsilon_{r2}\,\varepsilon_0\,E_{2n} = \varepsilon_{r1}\,\varepsilon_0\,E_{1n}$, or, $\varepsilon_{r2}\,E_2\cos\theta_2 = \varepsilon_{r1}\,E_1\cos\theta_1$ …. 6.20

Again, from eqn.(6.19), $E_{2t} = E_{1t}$, or, $E_2\sin\theta_2 = E_1\sin\theta_1$ …. 6.21

From eqns.(6.20) and (6.21), $\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$ …. 6.22

Problem 6.3

The flux lines of an electric field pass from air into glass, making an angle 30° with the normal to the plane surface separating air and glass at the air-side of the surface. The relative permittivity of glass is 5.0. The field intensity in air is 200 V/m. Calculate the electric flux density in glass and also the angle, which the flux lines make with the normal on the glass side.

Solution:

Given, ε_{r1}=1.0, ε_{r2}=5.0 and θ_1=30^0

So, $\frac{\tan\theta_2}{\tan\theta_1}=\frac{\tan\theta_2}{\tan 30^0}=\frac{\varepsilon_{r2}}{\varepsilon_{r1}}=\frac{5}{1}$, or, $\tan\theta_2$ = 2.887

or, θ_2 = 70.9^0

Again, E_1 = 200 V/m and $E_1\sin\theta_1=E_2\sin\theta_2$ so that $E_2\sin 70.9^0=200\times\sin 30^0$

or, E_2 = 105.8 V/m

Hence, $D_2=\varepsilon_{r2}\,\varepsilon_0\,E_2=5\times 8.854\times 10^{-12}\times 105.8=4.68\,\text{nC/m}^2$

Multi-Dielectric Configurations

Introduction

Although single dielectric arrangements are employed in real life, but the number of such applications is small. In most of the cases, multiple dielectric media are used in various combinations between the electrodes or conductors. In such multi-dielectric arrangements, contrary to common belief, the location of maximum electric field intensity may not be just off the live conductor in all the cases. Thus it becomes imperative that not only the magnitude but also the location of maximum electric field intensity should be determined accurately. This is to ensure that the maximum electric field intensity remains well below the dielectric strength of the material within which such maximum electric field intensity occurs. In the case of porous solid dielectric, the small pores are normally filled with air or another gaseous or liquid dielectric. Due to mismatch of the permittivity of the solid dielectric and the dielectric within the pores, electric field intensification may take place in the pores, which can also give rise to unwanted discharges within the equipment if appropriate measures are not taken to eliminate such field intensification. Thus electric field analysis in multi-dielectric arrangements is not only important from the design point of view but also is significant from the point of view of life extension of equipment that have such multi-dielectric arrangement.

Parallel Plate Capacitor

Multiple dielectrics within a parallel plate capacitor can be either in series or in parallel between the plates. Such series and parallel dielectric arrangements need to be analysed separately.

Dielectrics in Parallel between the Plates

A parallel plate capacitor with two dielectrics in parallel in between the plates is shown in Fig.7.1, such that the dielectric-dielectric boundary is perpendicular to the plates. If the separation distance between the plates is considered much smaller compared to the length and breadth of the plates, then the flux lines may be considered to be parallel to each other between the plates and also perpendicular to the plates within the dielectrics, neglecting the fringing of flux at the edges of the plates. Then along the paths of integration in both the dielectric media as shown in Fig.7.1, the flux lines will be tangential. Hence,

in dielectric-1, $\left|\vec{E}_1\right| l = V$, or, $\left|\vec{E}_1\right| = \frac{V}{l}$

and in dielectric-2, $\left|\vec{E}_2\right| l = V$, or, $\left|\vec{E}_2\right| = \frac{V}{l} = \left|\vec{E}_1\right|$

where, V is the potential difference between the plates of the capacitor.

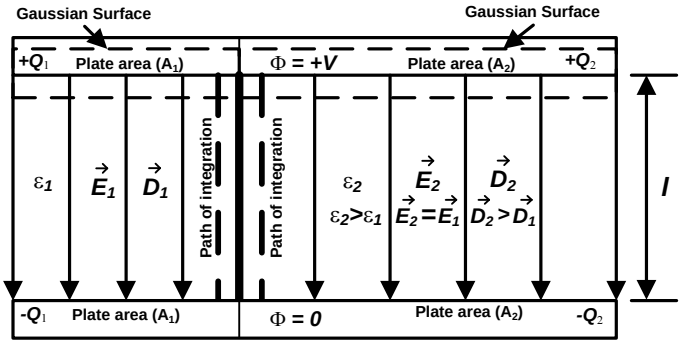


Fig.7.1 Two dielectrics in parallel within a parallel plate capacitor

Applying Gauss's law considering the Gaussian surface in dielectric-1 as shown in Fig.7.1,

$\left|\vec{D}_1\right| \times A_1 = Q_1$, or, $\left|\vec{D}_1\right| = \frac{Q_1}{A_1}$, or, $\left|\vec{E}_1\right| = \frac{Q_1}{\varepsilon_1 A_1}$, or, $\frac{V}{l} = \frac{Q_1}{\varepsilon_1 A_1}$ …. 7.1

So, the capacitance between the plates comprising dielectric-1, $C_1 = \frac{Q_1}{V} = \frac{\varepsilon_1 A_1}{l}$ …. 7.2

Again, applying Gauss's law considering the Gaussian surface in dielectric-2 as shown in Fig.7.1,

$\left|\vec{D}_2\right| \times A_2 = Q_2$, or, $\left|\vec{D}_2\right| = \frac{Q_2}{A_2}$, or, $\left|\vec{E}_2\right| = \frac{Q_2}{\varepsilon_2 A_2}$, or, $\frac{V}{l} = \frac{Q_2}{\varepsilon_2 A_2}$ …. 7.3

So, the capacitance between the plates comprising dielectric-2, $C_2 = \frac{Q_2}{V} = \frac{\varepsilon_2 A_2}{l}$ …. 7.4

Hence, the total capacitance of the parallel plate capacitor

$C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} = C_1 + C_2 = \frac{1}{l}\left(\varepsilon_1 A_1 + \varepsilon_2 A_2\right)$ …. 7.5

Dielectrics in Series between the Plates

A parallel plate capacitor with two dielectrics in series in between the plates is shown in Fig.7.2, such that the dielectric-dielectric boundary is parallel to the plates. As in section 7.2.1, if the separation distance between the plates is considered much smaller compared to the length and breadth of the plates, then the flux lines may be considered to be parallel to each other between the plates and also perpendicular to the plates as well as the dielectric-dielectric boundary within the capacitor, neglecting the fringing of flux at the edges of the plates.

Applying Gauss's law considering the Gaussian surface as shown in Fig.7.2,

$\left|\vec{D}\right| \times A = Q$, or, $\left|\vec{D}\right| = \frac{Q}{A}$

According to the boundary condition on the dielectric-dielectric boundary of Fig.7.2, $D_{1n} = D_{2n}$. Since, the flux lines are perpendicular to the plates and hence to the dielectric-dielectric boundary, $\left|\vec{D}\right| = D_n$. In other words, electric flux density is the same in both the dielectric media.

So, in dielectric-1, $\left|\vec{E}_1\right| = \frac{Q}{\varepsilon_1 A}$ and in dielectric-2, $\left|\vec{E}_2\right| = \frac{Q}{\varepsilon_2 A}$

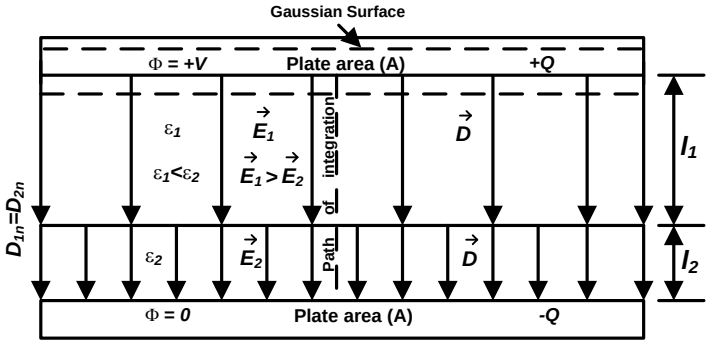


Fig.7.2 Two dielectrics in series within a parallel plate capacitor

Again, along the path of integration through the two dielectric media as shown in Fig.7.2, the flux lines will be tangential. Hence,

in dielectric-1, $\left|\vec{E}_1\right| l_1 = V_1$, or, $\left|\vec{E}_1\right| = \frac{V_1}{l_1} = \frac{Q}{\varepsilon_1 A}$ …. 7.6

where, V_1 = potential difference across the dielectric-1

and in dielectric-2, $\left|\vec{E}_2\right| l_2 = V_2$, or, $\left|\vec{E}_2\right| = \frac{V_2}{l_2} = \frac{Q}{\varepsilon_2 A}$ …. 7.7

where, V_2 = potential difference across the dielectric-2

If the potential difference between the plates of the capacitor is V, then

$$V = V_1 + V_2 = \left|\vec{E}_1\right| l_1 + \left|\vec{E}_2\right| l_2 = \frac{Q l_1}{\varepsilon_1 A_1} + \frac{Q l_2}{\varepsilon_2 A_2} \quad \text{…. 7.8}$$

Let, the capacitance of the parallel plate capacitor be C, the capacitance of the part comprising dielecric-1 be C_1 and the capacitance of the part comprising dielecric-2 be C_2.

Then from eqn.(7.6), $C_1 = \frac{Q}{V_1} = \frac{\varepsilon_1 A}{l_1}$

and from eqn.(7.7), $C_2 = \frac{Q}{V_2} = \frac{\varepsilon_2 A}{l_2}$

Hence, from eqn.(7.8), $\frac{1}{C} = \frac{V}{Q} = \frac{l_1}{\varepsilon_1 A_1} + \frac{l_2}{\varepsilon_2 A_2} = \frac{1}{C_1} + \frac{1}{C_2}$ …. 7.9

Further, from eqns.(7.6) and (7.7), $\frac{\left|\vec{E}_1\right|}{\left|\vec{E}_2\right|} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$ …. 7.10

It may be seen from eqn.(7.10), that if the dielectric within a capacitor is gas ($\varepsilon_{r1} \approx 1$) then partial filling of the capacitor by a solid or liquid dielectric is actually detrimental to the capacitor. Because in that case the electric field intensity in the gaseous dielectric will increase by a factor of ε_{r2} ($\varepsilon_{r2} > \varepsilon_{r1}$) and may even exceed the dielectric strength of the gaseous dielectric resulting in discharge within the capacitor.

Void in Insulation

Voids could be present in insulation in many different shapes and due to several reasons. Voids could be in the form of a gas bubble in liquid insulation such as transformer oil, a gas bubble in moulded epoxy resin or a small gap in solid insulation, e.g. gap formed due to delamination of pressboard insulation as a result of aging. Such voids are typically filled with air at a pressure slightly less than atmospheric pressure. A void in an insulation is

schematically shown in Fig.7.3. Considering the void dimensions to be small in comparison to the insulation surrounding it, it may be reasonably argued that the presence or absence of the void will not alter electric flux density at the location of the void. Then according to the discussions of section 7.2.2 and eqn.(7.10),

$$\frac{\left|\vec{E}_{void}\right|}{\left|\vec{E}_{insulation}\right|} = \varepsilon_{r-insulation} \quad (\varepsilon_{r-insulation} > 1) \qquad \ldots. 7.11$$

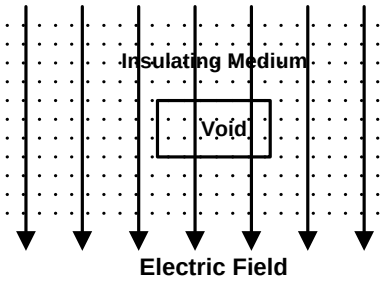


Fig.7.3 Void in insulating medium

Thus from eqn.(7.11) it is evident that electric field intensity within the void will be higher than that in the insulating medium. On the other hand the dielectric strength of air filling the void will be lower than that of the insulating medium. Consequently, there is every possibility that electric field intensity in the void may be in excess of the dielectric strength of air at the pressure and temperature within the void. This could result in discharge within the void. But there will be no discharge within the insulating medium around the void as electric field intensity there is lower and the dielectric strength of the insulation is higher. Such localized discharge within the void is called partial discharge (PD) and is highly detrimental to high voltage equipment as PD reduces the life of the equipment significantly.

Impregnation of Porous Solid Insulation

Solid insulation that are porous in nature, e.g. paper or pressboard, are very commonly used in high voltage equipment. In such cases the solid insulation is mainly cellulose, while the gas in the pores is air. The relative permittivity of cellulose insulation is of the order of 3~4 and the dielectric strength of cellulose is about 16kV/mm. On the other hand relative permittivity of air is ~1 and the dielectric strength of air is 3kV/mm. In order that the solid insulation does not fail due to discharges within the pores, electric field intensity within the solid insulation has to be kept around 9~12kV/mm, so that electric field intensity in the pores remains below 3kV/mm. But in that case the capacity of the paper insulation is not utilized to the full extent. Therefore, in practice such porous solid insulation is always impregnated with a suitable liquid dielectric. Due to impregnation the pores in the solid insulation will be filled by the liquid insulation which will have higher dielectric strength compared to air. But here it is important to note that the relative permittivity of the liquid dielectric used for impregnation must be close to the relative permittivity of the solid insulation being impregnated. Otherwise, there will be again a mismatch of the electric field intensity values within the liquid and solid dielectrics and the full benefit of impregnation could not be obtained.

Co-axial Cylindrical Configurations

Co-axial multi-dielectric arrangements, which are commonly used in power engineering, are cables and bushings. A single-core cable having three different dielectrics in between the core

and the earthed metallic screen is shown in Fig.7.4. Bushings are special components used in high voltage system when a high voltage conductor has to pass through an earthed barrier, e.g. earthed tank of a transformer or a wall etc. Fig.7.5 shows a solid type oil impregnated paper bushing used in transformers typically for voltage ratings below 90kV. If a cross-section is taken at AA' as shown in Fig.7.5, then the cross-sectional view will be same as that shown in Fig.7.4. Hence, the radial field distribution at the critical zone AA', where the distance between the high voltage conductor and the earthed metallic flange is minimum, could be determined with reasonable degree of accuracy using the same analysis as in the case of cables.

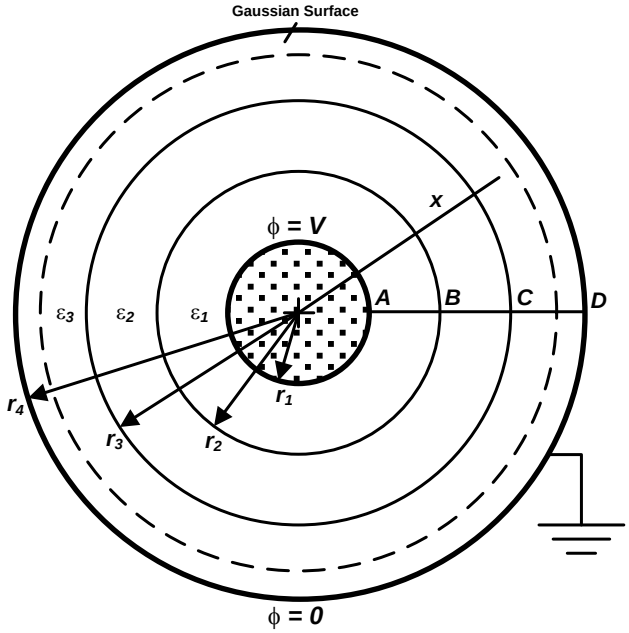


Fig.7.4. A single-core cable having three different dielectric media

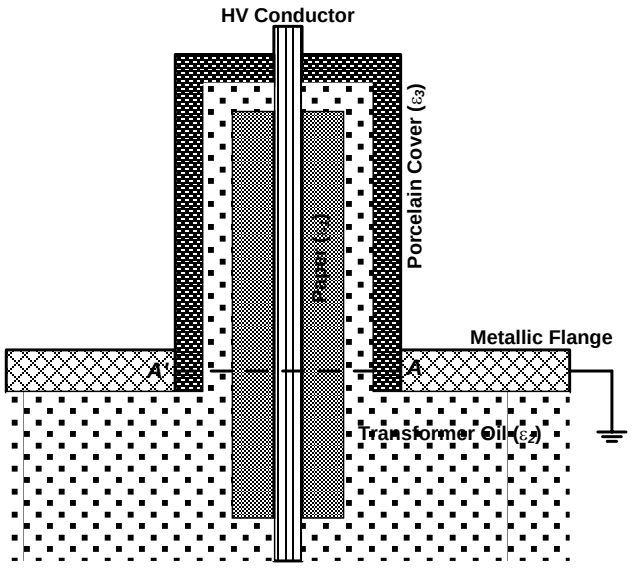


Fig.7.5 Solid type oil impregnated paper bushing used in transformers

Electric field in the co-axial arrangement of Fig.7.4 is analysed assuming that electric field does not vary along the length of the cylinder normal to the plane of the paper. For cables this assumption is perfectly valid, while for bushings it is not valid as such, but the nature of

radial field distribution thus obtained gives a fair idea about the field concentration in the critical zone AA' that may cause eventual failure of a bushing.

Let the charge per unit length on the inner conductor be q. Then for the Gaussian surface as shown in Fig.7.4, electric flux density at a radial distance of x can be obtained from the surface area of a cylinder of radius x and axial length unity, i.e.

$$D_x = \frac{q}{2\pi x \times 1}$$

Hence, electric field intensity at any radius x is given by

$$E_x = \frac{q}{2\pi \varepsilon_x x} \qquad \text{…. 7.12}$$

where, ε_x is the permittivity of the dielectric at the radial distance x.

Then the potential difference V between the high voltage conductor and the earthed enclosure could be obtained from the line integral of electric field intensity. Considering the path of integral from D to A,

$$V = -\int_D^A E_x \, dx = -\int_D^A \frac{q}{2\pi \varepsilon_x x} dx \qquad \text{…. 7.13}$$

But the integral of eqn.(7.13) needs to be evaluated section-wise where the permittivity remains constant. Thus

$$V = -\int_B^A \frac{q}{2\pi \varepsilon_1 x} dx - \int_C^B \frac{q}{2\pi \varepsilon_2 x} dx - \int_D^C \frac{q}{2\pi \varepsilon_3 x} dx$$

or, $$V = -\int_{r_2}^{r_1} \frac{q}{2\pi \varepsilon_1 x} dx - \int_{r_3}^{r_2} \frac{q}{2\pi \varepsilon_2 x} dx - \int_{r_4}^{r_3} \frac{q}{2\pi \varepsilon_3 x} dx = \frac{q}{2\pi}\left(\frac{1}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{1}{\varepsilon_2}\ln\frac{r_3}{r_2} + \frac{1}{\varepsilon_3}\ln\frac{r_4}{r_3}\right) \qquad \text{…. 7.14}$$

In eqn.(7.14) it may be seen that the expression within the parenthesis on the RHS is a constant for a given multi-dielectric arrangement. Hence, the charge per unit length on the inner conductor can be expressed in terms of the potential difference and the computed constant of eqn.(7.14) as follows

$$q = \frac{2\pi V}{K}, \text{ where } K = \left(\frac{1}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{1}{\varepsilon_2}\ln\frac{r_3}{r_2} + \frac{1}{\varepsilon_3}\ln\frac{r_4}{r_3}\right) \qquad \text{…. 7.15}$$

The capacitance per unit length of the multi-dielectric arrangement is

$$C = \frac{q}{V} = \frac{2\pi}{K} \qquad \text{…. 7.16}$$

Electric field intensity at any radius x is then given by

$$E_x = \frac{q}{2\pi \varepsilon_x x} = \frac{V}{K \varepsilon_x x} \qquad \text{…. 7.17}$$

From eqn.(7.17) it is clear that electric field intensity varies inversely with radial distance within each section of the arrangement comprising one particular dielectric. Thus the variation of electric field intensity can be plotted with radial distance as shown in Fig.7.6. From Fig.7.6, it may be seen that there is a discontinuity in electric field intensity at the boundary between two different dielectric media. The value of electric field intensity may increase or decrease at the boundary depending upon the relative values of the permittivity of the two dielectric media at the boundary. Thus for a multi-dielectric arrangement as shown in Fig.7.4, it cannot be stated that the maximum value of electric field intensity will occur just off the surface of the inner conductor in all the cases. On the contrary, it has to be ascertained case by case considering the physical dimensions and the dielectric arrangement.

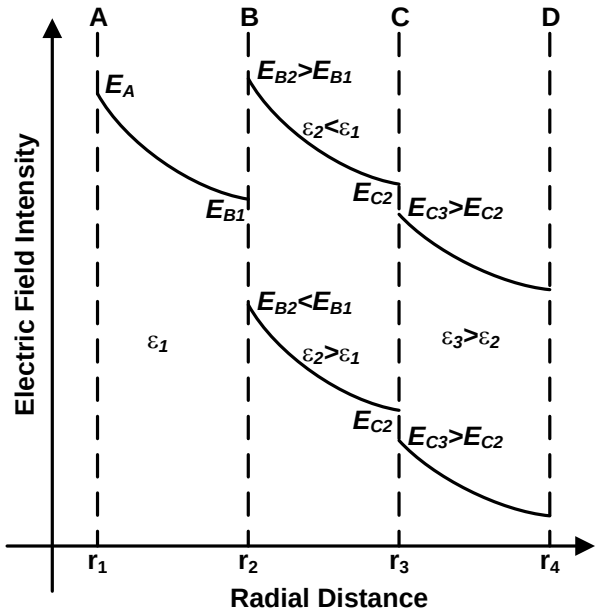


Fig.7.6 Radial variation of electric field intensity in co-axial multi-dielectric arrangement

Problem 7.1

A single-core, lead sheathed cable joint has a conductor of 10mm diameter and two layers of different insulating materials, each 10mm thick. The relative permittivities are 3 and 2.5 for inner and outer dielectric, respectively. Calculate the potential gradient just off the conductor surface, when the potential difference between the conductor and lead sheath is 33 kV_{rms}.

Solution

Given: ε_{r1}=3 and ε_{r2}=2.5, r_1=(10/2)=5mm, r_2=5+10=15mm and r_3=15+10=25mm

So according to eqn.(7.15), $K=\frac{1}{\varepsilon_0}\left(\frac{1}{3}\ln\frac{15}{5}+\frac{1}{2.5}\ln\frac{25}{15}\right)=\frac{0.57}{\varepsilon_0}$

Hence, potential gradient just off the conductor surface

$$E_{r1}=\frac{33}{\frac{0.57}{\varepsilon_0}\times 3\times\varepsilon_0\times 5}=3.85\,kV_{rms}/mm$$

Problem 7.2

A transformer bushing for 36kV_{rms} consists of the following:

Component	Outside diameter	ε_r
Copper rod	4 cm	
Treated paper	5 cm	3
Transformer oil	10 cm	2.1
Porcelain	15 cm	5

Find the magnitudes and locations of maximum and minimum electric field intensities within the bushing.

Solution

This problem is solved in accordance with Fig.7.4 and eqns.(7.15) & (7.17).

Given: ε_{r1}=3 and ε_{r2}=2.1 and ε_{r3}=5, r_1=2cm, r_2=2.5cm, r_3=5cm and r_4=7.5cm

So according to eqn.(7.15), $K=\frac{1}{\varepsilon_0}\left(\frac{1}{3}\ln\frac{2.5}{2}+\frac{1}{2.1}\ln\frac{5}{2.5}+\frac{1}{5}\ln\frac{7.5}{5}\right)=\frac{0.4855}{\varepsilon_0}$

So according to Fig.7.4, within paper insulation just off the conductor surface

$$E_A = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 3 \times \varepsilon_0 \times 2} = 12.36\, kV_{rms}/cm$$

At the paper-oil boundary, on the paper side $E_B\big|_{paper} = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 3 \times \varepsilon_0 \times 2.5} = 9.88\, kV_{rms}/cm$

At the paper-oil boundary, on the oil side $E_B\big|_{oil} = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 2.1 \times \varepsilon_0 \times 2.5} = 14.12\, kV_{rms}/cm$

At the oil-porcelain boundary, on the oil side $E_C\big|_{oil} = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 2.1 \times \varepsilon_0 \times 5} = 7.06\, kV_{rms}/cm$

At the oil-porcelain boundary, on porcelain side $E_C\big|_{Porcelain} = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 5 \times \varepsilon_0 \times 5} = 2.97\, kV_{rms}/cm$

Within porcelain just off the earth surface $E_D = \frac{36}{\frac{0.4855}{\varepsilon_0} \times 5 \times \varepsilon_0 \times 7.5} = 1.98\, kV_{rms}/cm$

So the maximum electric field intensity is 14.12 kV_{rms}/cm at the oil side of the oil-paper boundary and the minimum electric field intensity is 1.98 kV_{rms}/cm within porcelain just off the earth surface.

Problem 7.3

A conductor 2.8 cm in diameter is passed centrally through a porcelain bushing (ε_r=5) having internal and external diameters of 3cm and 9cm, respectively. The potential difference between the conductor and the earthed metallic flange around the porcelain is 12kV_{rms}. Determine whether or not partial discharges will be present in the air-space around the conductor. Also find the electric stress just off the conductor surface, if the air-space is filled with transformer oil (ε_r=2.1).

Solution

The arrangement as per the statement of the problem is shown in Fig.7.7. With reference to Fig.7.7, the given quantities are as follows:

ε_{r2}=5, r_1=1.4cm, r_2=1.5cm and r_3=4.5cm

Case-I: When the small space between the copper rod and porcelain cover is filled with air. Then ε_{r1}=1. So, according to eqn.(7.15) in this case

$$K1 = \frac{1}{\varepsilon_0}\left(\frac{1}{1}\ln\frac{1.5}{1.4} + \frac{1}{5}\ln\frac{4.5}{1.5}\right) = \frac{0.2887}{\varepsilon_0}$$

Within this air space the maximum electric field intensity will occur just of the conductor surface and will be

$$E_{cond}\big|_{air} = \frac{12}{\frac{0.2887}{\varepsilon_0} \times \varepsilon_0 \times 1.4} = 29.69\, kV_{rms}/cm$$

Breakdown strength of air is $30kV_p/cm$ or $21.21kV_{rms}/cm$ at STP. So partial discharges will be present in the air space.

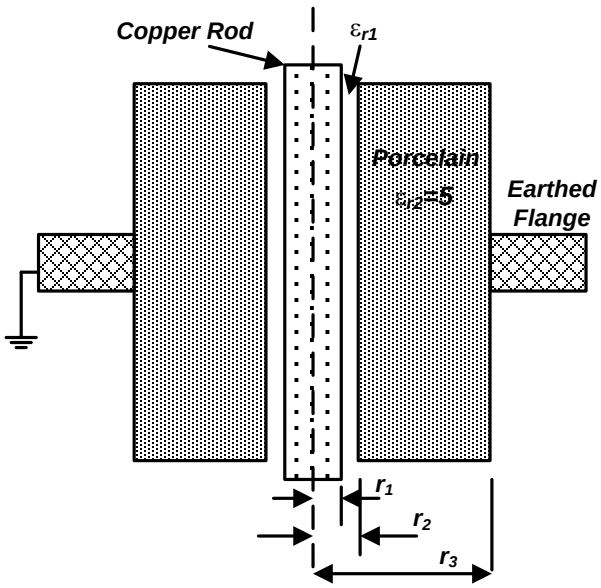


Fig.7.7 Pertaining to Problem 7.3

Case-II: When the small space between the copper rod and porcelain cover is filled with transformer oil instead of air. Then ε_{r1}=2.1. So, according to eqn.(7.15) in this case

$$K2=\frac{1}{\varepsilon_0}\left(\frac{1}{2.1}\ln\frac{1.5}{1.4}+\frac{1}{5}\ln\frac{4.5}{1.5}\right)=\frac{0.2525}{\varepsilon_0}$$

Then the maximum electric field intensity just off the conductor surface is

$$E_{cond}\big|_{oil}=\frac{12}{\frac{0.2525}{\varepsilon_0}\times 2.1\times\varepsilon_0\times 1.4}=16.16\,kV_{rms}/cm$$

It may be seen that filling up the small space by transformer oil brings down the electric field intensity considerably. Moreover, transformer oil has higher breakdown strength than air. So, partial discharges will not occur in the small gap.

This problem brings out an important aspect of transformer bushing design. Since copper and porcelain are both solid, it is very difficult to get a perfect contact between them. So intentionally a small gap is kept between these two for easy operation. But if this small space is kept filled with air then partial discharge is inevitable, which is undesirable. Hence, the solid type bushings are always so designed that this small space is filled with transformer oil in stead of air.

Uniqueness Theorem

It states that once any method of solving Poisson's or Laplace's equations subject to given boundary conditions has been found, the problem has been solved once and for all. No other method can ever give a different solution.

Proof:
Consider a volume *V* bounded by a surface *S*. Also consider that there is a charge density ρ_v throughout the volume *V*, and the value of the scalar electric potential on the surface *S* is ϕ_s. Assume that there are two solutions of Poisson's equation, viz. ϕ_1 and ϕ_2. Then

$$\vec{\nabla}^2\phi_1 = -\frac{\rho_v}{\varepsilon} \quad \text{and} \quad \vec{\nabla}^2\phi_2 = -\frac{\rho_v}{\varepsilon}$$

So, $\vec{\nabla}^2(\phi_1 - \phi_2) = 0$ …. 13.4

Now, each solution must also satisfy the boundary conditions. It is to be noted here that one particular point can not have two different electric potentials, as the work done to move a unit positive charge from infinity to that point is unique. Let, the value of ϕ_1 on the boundary is ϕ_{1s} and the value of ϕ_2 on the boundary is ϕ_{2s} and they must be identical to ϕ_s.

Therefore, $\phi_{1s} = \phi_{2s} = \phi_s$

or, $\phi_{1s} - \phi_{2s} = 0$

For any scalar ϕ and any vector $\vec{D}$, the following vector identity can be written.

$$\vec{\nabla}(\phi\vec{D}) \equiv \phi(\vec{\nabla}.\vec{D}) + \vec{\nabla}\phi.\vec{D} \quad \text{…. 13.5}$$

Consider the scalar as $(\phi_1 - \phi_2)$ and the vector as $\vec{\nabla}(\phi_1 - \phi_2)$. Then from identity (13.5),

$$\vec{\nabla}.\left[(\phi_1 - \phi_2)\vec{\nabla}(\phi_1 - \phi_2)\right] \equiv (\phi_1 - \phi_2)\left[\vec{\nabla}.\vec{\nabla}(\phi_1 - \phi_2)\right] + \vec{\nabla}(\phi_1 - \phi_2).\vec{\nabla}(\phi_1 - \phi_2) \quad \text{…. 13.6}$$

Now, integrating throughout the volume *V* enclosed by the boundary surface *S*,

$$\int_V \vec{\nabla}.\left[(\phi_1 - \phi_2)\vec{\nabla}(\phi_1 - \phi_2)\right]dv \equiv \int_V (\phi_1 - \phi_2)\left[\vec{\nabla}.\vec{\nabla}(\phi_1 - \phi_2)\right]dv + \int_V \vec{\nabla}(\phi_1 - \phi_2).\vec{\nabla}(\phi_1 - \phi_2)dv$$

$$\equiv \int_V (\phi_1 - \phi_2)\left[\vec{\nabla}^2(\phi_1 - \phi_2)\right]dv + \int_V \left[\vec{\nabla}(\phi_1 - \phi_2)\right]^2 dv \quad \text{…. 13.7}$$

Applying divergence theorem to the L.H.S of identity (13.7),

$$\int_V \vec{\nabla}.\left[(\phi_1 - \phi_2)\vec{\nabla}(\phi_1 - \phi_2)\right]dv = \int_S (\phi_{1S} - \phi_{2S})\vec{\nabla}(\phi_1 - \phi_2)ds = 0 \quad \text{…. 13.8}$$

as $\phi_{1s} = \phi_{2s}$ on the specified surface *S*.

On the RHS of identity (13.7), $\vec{\nabla}^2(\phi_1 - \phi_2) = 0$ from eqn.(13.4). Hence, identity (13.7) reduces to

$$\int_V \left[\vec{\nabla}(\phi_1 - \phi_2)\right]^2 dv = 0 \quad \text{…. 13.9}$$

Since, $\left[\vec{\nabla}(\phi_1 - \phi_2)\right]^2$ cannot be negative, hence the integrand must be zero everywhere so that the integral may be zero.

Hence,

$$\left[\vec{\nabla}(\phi_1 - \phi_2)\right]^2 = 0 \quad \text{or,} \quad \vec{\nabla}(\phi_1 - \phi_2) = 0 \quad \text{…. 13.10}$$

Again, if the gradient of $(\phi_1 - \phi_2)$ is zero everywhere, then

$$\phi_1 - \phi_2 = Constant \quad \text{…. 13.11}$$

This constant may be evaluated by considering a point on the boundary surface *S*. So that,

$$\phi_1 - \phi_2 = \phi_{1s} - \phi_{2s} = 0$$

or, $\phi_1 = \phi_2$

which means that the two solutions are identical.

However, in practice if the same problem is solved by using different numerical techniques the results are not exactly the same. This is due to the fact that the errors in a particular numerical method are often problem dependent and hence the results are not exactly same in all the methods. So, this is not a violation of the Uniqueness theorem.

Method of Images

Introduction

To explain the idea behind the method of images, consider two distinctly different electrostatic problems. The real problem is the one in which a charge density is given in a finite domain V bounded by its surface S with specific boundary conditions on the surface S. The other is a fictitious problem in which the charge density with the finite domain V is the same as that for the real problem, but the boundary surfaces are replaced by suitable fictitious charge distribution located outside the domain V. If the fictitious charge distribution is so chosen that the solution to the fictitious problem satisfies the boundary conditions specified in the real problem, then the solution to the fictitious problem is also the solution to the real problem. The fictitious charge distribution so determined is called the image of the true charge distribution for the real problem.
In other words, the idea is to convert an electrostatic problem involving conducting objects, which are spatially extended, in such a way that conducting surfaces having given boundary conditions are replaced by a finite number of appropriately chosen and suitably placed discrete charges known as image charges. Image charges are always located outside the region where the field is to be determined. While doing this replacement of boundary surfaces by image charges, the original boundary conditions of the real problem are retained. Thus the more complicated original problem could then be solved as a relatively simpler problem having known charge configuration.

Image of a Point Charge wrt an Infinitely Long Conducting Plane

Consider a point charge of positive polarity and magnitude Q located at a height z_1 from an infinitely long conducting plane present at $z=0$ as shown in Fig.9.1, such that the location of the point charge is given by (x_1,y_1,z_1). As discussed in section 1.7.2, an infinitely long conducting plane, which is also an equipotential surface, will have zero electric potential. The practical example of such a plane is earth surface. In fact in any electric field distribution earth surface having zero potential plays a significant role and hence taking the image of a point charge wrt an infinitely long conducting plane takes into account the effect of grounded earth surface in electric field calculation.
Considering the conducting plane to be the x-y plane, the problem is to find a solution for electric field in the region $z>0$. This solution can be obtained by solving Laplace's equation with the following conditions:

i) The solution $\phi(x,y,z)$ will be valid at every point for $z>0$ except the location of the point charge.
ii) The potential at all the points on the conducting plane is zero, i.e $\phi(x,y,0)=0$.
iii) The potential approaches zero as the distance from the charge approaches infinity.

It is not easy to find a solution to Laplace's equation that satisfies all the above conditions.

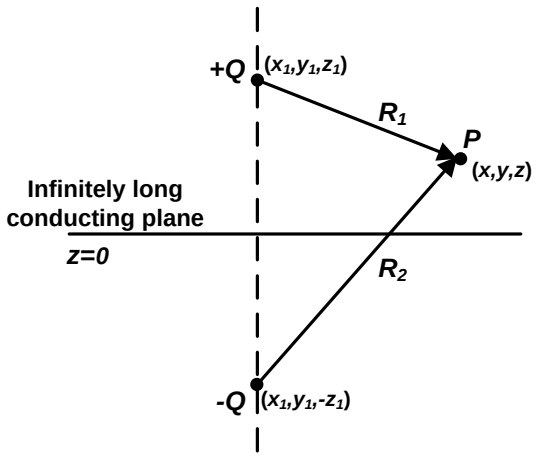


Fig.9.1 Point charge near an infinitely long conducting plane

The problem can also be viewed from another angle. The positive polarity point charge at $z=z_1$ will induce negative polarity charges on the surface of the conducting plane. Say, the induced surface charge density is $-\sigma_s$. Then the electric field at any point in the region $z>0$ will be due to the point charge and the induced surface charges. But the difficulty is that σ_s needs to be determined using the boundary condition $\phi(x,y,0)=0$, before the solution to electric field in the region $z>0$ could be obtained. Furthermore, the evaluation of the surface integral to obtain the field due to the induced charges is also not an easy task. From all these considerations, taking the image of the point charge wrt to the infinitely long conducting plane to replace the plane by the image charge is an easy and practical methodology.

The electric potential at a distance r from a point charge is given by $\frac{Q}{4\pi\varepsilon_0 r}$. Considering the spherical symmetry of the field due to a point charge, it may be seen that the electric field due to a point charge is dependent only on r-coordinate in spherical coordinate system and is independent of θ and ϕ coordinates. Hence, this electric potential satisfies Laplace's equation in spherical coordinates as follows.

$$\vec{\nabla}^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times-\frac{Q}{4\pi\varepsilon_0 r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{Q}{4\pi\varepsilon_0}\right)=0$$

Consider that the infinitely long conducting plane is replaced by a fictitious charge of magnitude Q_1 located at $(x_1,y_1,-z_2)$. Since the field due a point charge satisfies Laplace's equation, then the field due to the real point charge and the image point charge will also satisfy Laplace's equation at all points for $z>0$ except at the location of the real point charge. Therefore, the electric potential at any point $P(x,y,z)$ will be given by

$$\phi_P=\frac{Q}{4\pi\varepsilon_0 R_1}+\frac{Q_1}{4\pi\varepsilon_0 R_2} \qquad \ldots. 9.1$$

where, $R_1=\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right]^{1/2}$

and $R_2=\left[(x-x_1)^2+(y-y_1)^2+(z+z_2)^2\right]^{1/2}$

From eqn.(9.1) it is evident that the potential at a very large distance from the point charge Q will be zero.

So two of the above-mentioned three conditions are satisfied by eqn.(9.1). In order to satisfy the third condition, ϕ_P at any point on the conducting plane has to be zero. From eqn.(9.1), it is obvious that the polarity of Q_1 has to be negative to satisfy this condition. Furthermore, the following has to be satisfied at all the points on the conducting plane

$$\frac{Q}{R_1}=\frac{Q_1}{R_2} \qquad \text{…. 9.2}$$

The best solution for eqn.(9.2) is obtained when the magnitude of the image charge (Q_1) is equal to the magnitude of the real charge (Q) and location of the image charge is such that the magnitudes of R_1 and R_2 are equal at all the points on the conducting plane. From the expressions of R_1 and R_2 it is clear that for $z=0$, R_1 will be equal to R_2 if $z_1=z_2$. So the third condition will also be satisfied if $Q_1=-Q$ and the location of the image charge is at $(x_1,y_1,-z_1)$.
Thus the combination of the real charge Q located at (x_1,y_1,z_1) and the image charge $-Q$ located at $(x_1,y_1,-z_1)$ satisfies all the three conditions stated above. Hence, the electric field in the region $z>0$ due to the real point charge $+Q$ and the infinitely long conducting plane will be the same as that due to the real point charge $+Q$ and its image point charge $-Q$ located as mentioned above.
Therefore, the electric potential at any point $P(x,y,z)$ for $z>0$ will be given by

$$\phi_P=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \text{…. 9.3}$$

where, $R_1=\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right]^{1/2}$

and $\quad R_2=\left[(x-x_1)^2+(y-y_1)^2+(z+z_1)^2\right]^{1/2}$

In Cartesian coordinates, electric field intensity at any point $P(x,y,z)$ for $z>0$ will be given by

$$\vec{E}_P=-\frac{\partial\phi_P}{\partial x}\hat{i}-\frac{\partial\phi_P}{\partial y}\hat{j}-\frac{\partial\phi_P}{\partial z}\hat{k} \qquad \text{…. 9.4}$$

where, $E_{xP}=-\dfrac{\partial\phi_P}{\partial x}=\dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{x-x_1}{R_1^3}-\dfrac{x-x_1}{R_2^3}\right)$

$$E_{yP}=-\frac{\partial\phi_P}{\partial y}=\frac{Q}{4\pi\varepsilon_0}\left(\frac{y-y_1}{R_1^3}-\frac{y-y_1}{R_2^3}\right)$$

$$E_{zP}=-\frac{\partial\phi_P}{\partial z}=\frac{Q}{4\pi\varepsilon_0}\left(\frac{z-z_1}{R_1^3}-\frac{z+z_1}{R_2^3}\right)$$

For any point very near to the conducting plane, i.e. $z\to 0$, $R_1\approx R_2$. Hence, E_{xP} and E_{yP} will be zero. Thus

$$\lim_{z\to 0}\vec{E}_{zP}=\frac{\sigma_s}{\varepsilon_0}\hat{k}=-\frac{Q z_1}{2\pi\varepsilon_0\left[(x-x_1)^2+(y-y_1)^2+z_1^2\right]^{3/2}}\hat{k}$$

or, $$\sigma_s=-\frac{Q z_1}{2\pi\left[(x-x_1)^2+(y-y_1)^2+z_1^2\right]^{3/2}} \qquad \text{…. 9.5}$$

Eqn.(9.5) gives the negative surface charge density induced on the conducting plane by the positive polarity real point charge.
This problem can also be viewed from a different perspective. The field due to the real point charge and the image point charge is the same as that due to a spatially extended electric dipole. Considering spherical coordinates and assuming that the electric dipole to be oriented along the axis of symmetry as shown in Fig.9.2, electric potential and electric field intensity can be expressed in spherical coordinates.
With reference to Fig.9.2,

$r_1=\left(r^2+h^2-2rh\cos\theta\right)^{1/2}$ and

$$r_2=\left(r^2+h^2-2rh\cos(\pi-\theta)\right)^{1/2}=\left(r^2+h^2+2rh\cos\theta\right)^{1/2} \qquad \text{…. 9.6}$$

Then, $\phi_P = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\left(r^2+h^2-2rh\cos\theta\right)^{1/2}} - \frac{1}{\left(r^2+h^2+2rh\cos\theta\right)^{1/2}}\right)$ …. 9.7

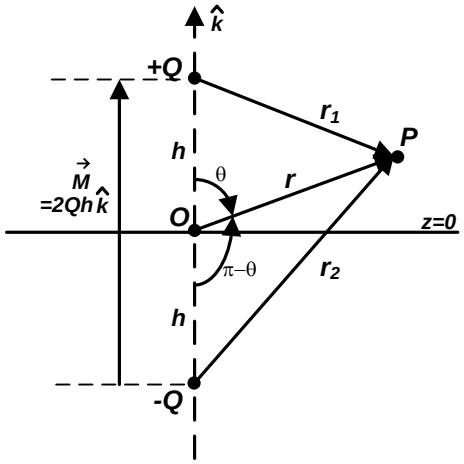


Fig.9.2 Electric dipole formed by the real and image charges

$$\vec{E}_P = -\frac{\partial\phi_P}{\partial r}\hat{u}_r - \frac{1}{r}\frac{\partial\phi_P}{\partial\theta}\hat{u}_\theta - \frac{1}{r\sin\theta}\frac{\partial\phi_P}{\partial\phi}\hat{u}_\phi \quad \text{…. 9.8}$$

where, $E_{rP} = -\frac{\partial\phi_P}{\partial r} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{r-h\cos\theta}{r_1^3} - \frac{r+h\cos\theta}{r_2^3}\right)$

$$E_{\theta P} = -\frac{1}{r}\frac{\partial\phi_P}{\partial\theta} = \frac{Qh\sin\theta}{4\pi\varepsilon_0}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

and $\vec{E}_{\phi P} = -\frac{1}{r\sin\theta}\frac{\partial\phi_P}{\partial\phi} = 0$

Eqns.(9.7) and (9.8) are the expressions for the electric field due to an electric dipole of dipole moment $\vec{M} = 2Qh\hat{k}$, as shown in Fig.9.2.

However, it is to be noted here that the solution obtained with the help of image charge is valid for the region *z>0* as for *z<0* the region is below an infinitely long conducting plane whose potential is zero and hence for *z<0* the electric field is zero. The image charge does not exist in reality as it is a fictitious charge.

Image of a Point Charge wrt a Grounded Conducting Sphere

Image of a charge is not necessarily to be taken wrt infinitely long plane only. It can also be taken wrt curved surfaces like sphere, cylinder etc. To elaborate this issue, consider a point charge *+Q* located at distance *d* from the center (*O*) of the sphere of radius *a* (*a<d*), as shown in Fig.9.4. Consider also that the electric potential of the sphere is zero. The field due to the point charge and the grounded sphere in the region outside the sphere could be determined by replacing the grounded sphere by an image charge. From the symmetry of the system it is evident that the image charge *q* will of negative polarity and will be located inside the sphere on the line joining the center of the sphere and the point charge, as shown in Fig.9.4. However, in this case the magnitude of *q* will not be equal to *Q* because such a pair of charges will not result into a zero potential spherical surface of radius *a* as required by the boundary condition.

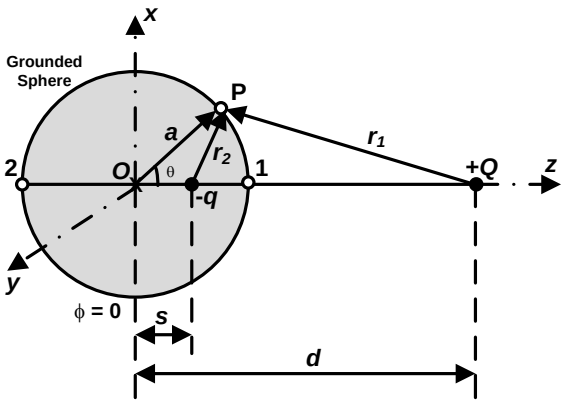


Fig.9.4 Point charge near a grounded sphere

Consider that the image charge is located at a distance *s* from the center of the sphere as shown in Fig.9.4. Now the problem is to determine the magnitude as well as the location of the image charge that satisfies the zero potential boundary condition for the spherical surface. With reference to Fig.9.4, the potential at the point *P* due to the point charge and its image is given by

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r_1} - \frac{q}{r_2}\right) \qquad \text{…. 9.9}$$

Imposition of the boundary condition $\phi_P = 0$ leads to

$$\frac{q}{Q} = \frac{r_2}{r_1} = \alpha \qquad \text{…. 9.10}$$

If α is kept constant, then $\frac{r_2}{r_1}$ =*constant* is the equation of a sphere. Hence, the problem now is to find the constant α.

For the point 1 as shown in Fig.9.4 $\alpha = \frac{r_2}{r_1} = \frac{a-s}{d-a}$

and for the point 2 as shown in Fig.9.4 $\alpha = \frac{r_2}{r_1} = \frac{a+s}{d+a}$

So, $\alpha = \frac{r_2}{r_1} = \frac{a-s}{d-a} = \frac{a+s}{d+a} = \frac{a}{d} = \frac{s}{a}$ (by componendo-dividendo) …. 9.11

Since, the radius of the sphere and the location of the point charge are known, hence the constant α can be computed from the ratio of *a* and *d*, as given by eqn.(9.11).

Therefore, the magnitude of the image charge is given by $q = \frac{a}{d}Q$ …. 9.12

and the location of the image charge is given by $s = \frac{a^2}{d}$ …. 9.13

Considering the line joining the point charge and its image passing through the center of the sphere to be along the *z*-axis and the center of the sphere to be the origin as shown in Fig.9.4, and also taking into account the spherical symmetry of the configuration, the field can be expressed in spherical coordinates as follows.

With reference to the point *P* of Fig.9.4,

$r_1 = \left|\vec{r}_P - \vec{r}_{+Q}\right| = \left(a^2 + d^2 - 2ad\cos\theta\right)^{1/2}$, where θ is the angle between *a* and *d* at the point *P*

$r_2 = \left|\vec{r}_P - \vec{r}_{-q}\right| = \left(a^2 + s^2 - 2as\cos\theta\right)^{1/2}$

Similarly, for any point in the field region for which r is the distance of the point from the origin, i.e. the center of the sphere, and θ is the angle between r and d

$r_1 = \left|\vec{r} - \vec{r}_{+Q}\right| = \left(r^2 + d^2 - 2rd\cos\theta\right)^{1/2}$ and

$$r_2 = \left|\vec{r} - \vec{r}_{-q}\right| = \left(r^2 + s^2 - 2rs\cos\theta\right)^{1/2} = \left(r^2 + \frac{a^4}{d^2} - 2r\frac{a^2}{d}\cos\theta\right)^{1/2} = \left(\frac{r^2d^2 + a^4 - 2rda^2\cos\theta}{d^2}\right)^{1/2}$$

So, electric potential at any point due to the point charge and its image is given by

$$\phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left(r^2 + d^2 - 2rd\cos\theta\right)^{1/2}} - \frac{a}{\left(r^2d^2 + a^4 - 2rda^2\cos\theta\right)^{1/2}}\right] \quad \text{…. 9.14}$$

So, $\vec{E}_r(a,\theta) = -\left.\frac{\partial\phi(r,\theta)}{\partial r}\right|_{r=a} = -\frac{Q}{4\pi\varepsilon_0 a}\frac{d^2 - a^2}{\left(a^2 + d^2 - 2ad\cos\theta\right)^{3/2}}$ …. 9.15

Now, r-component of electric field intensity is the normal component on the sphere surface. So, assuming the induced surface charge density on the sphere surface to be σ_s, the normal component of electric field intensity is equal to $\frac{\sigma_s}{\varepsilon_0}$ just off the sphere surface. Equating this expression with the one given by eqn.(9.15), the induced surface charge density on the grounded sphere surface is given by

$$\sigma_s = -\frac{Q}{4\pi a}\frac{d^2 - a^2}{\left(a^2 + d^2 - 2ad\cos\theta\right)^{3/2}} \quad \text{…. 9.16}$$

Method of Successive Images

Sphere gap arrangements are very commonly used in high voltage system for voltage measurement. As shown in Fig.9.5, in this arrangement two spheres of identical radii are separated by a specific distance s, where one sphere is charged while the other is earthed. The field within the sphere gap due to the two spheres could be analyzed with the help of image charges as described in section 9.3. The live sphere of potential V is at first replaced by a charge of magnitude $Q_A = 4\pi\varepsilon_0 aV$ located at the center of the live sphere. Then to keep the potential of the grounded sphere at zero, $-q_1$ is introduced within the grounded sphere which is the image of Q_A, as shown in Fig.9.5. The magnitude and location of q_1 are given by

$$q_1 = \frac{a}{d}Q_A \qquad \text{and} \qquad s_1 = \frac{a^2}{d} \quad \text{…. 9.17}$$

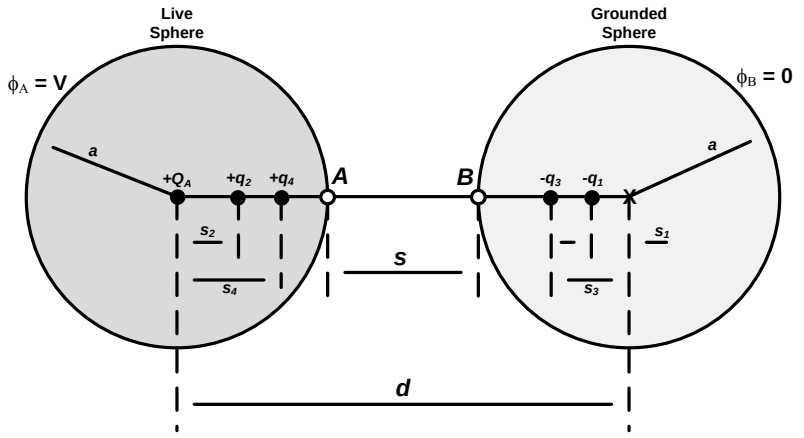


Fig.9.5 Method of successive imaging as applied to sphere gap arrangement

But the introduction of $-q_1$ will make the potential of live sphere different from V. So to keep the potential of live sphere equal to V, $+q_2$, which is the image of $-q_1$, is introduced within the live sphere such that the potential of live sphere due to $+q_2$ and $-q_1$ will be zero. As a result, the potential of live sphere due to $+Q_A$, $-q_1$ and $+q_2$ will be again V. The magnitude and location of q_2 are given by

$$q_2 = \frac{a}{d - s_1} q_1 = \frac{a^2}{d\left(d - s_{1`}\right)} Q_A \quad \text{and} \qquad s_1 = \frac{a^2}{d - s_1} \qquad \text{…. 9.18}$$

But the introduction of $+q_2$ will make the potential of grounded sphere different from zero. So $-q_3$ is introduced within the grounded sphere as the image of $+q_2$ to make the potential of the grounded sphere equal to zero. Further, introduction of $-q_3$ warrants introduction of $+q_4$ within the live sphere and so on. In this way there will be an infinite series of charges within the two spheres: positive charges such as $+Q_A$, $+q_2$, $+q_4$ within the live sphere and negative charges such as $-q_1$, $-q_3$ within the grounded sphere. This method of taking successive image charges within the two spheres is known as method successive imaging. It may be seen from eqns.(9.17) and (9.18) that each successive image charge is smaller in magnitude and gradually shifts towards the surface of the sphere within which it is located. In all practicality it is adequate to take the first few images within the two spheres to achieve reasonably good accuracy in the computation of electric field. In the sphere gap arrangement, maximum value of electric field intensity occurs at the so called sparking tips of the spheres, viz. points A and B as shown in Fig.9.5. This maximum electric field intensity can be obtained as

$$E_{\max} = \frac{V}{s}\left[\frac{\frac{s}{a} + 1 + \sqrt{\left(\frac{s}{a} + 1\right)^2 + 8}}{4}\right] \qquad \text{…. 9.19}$$

As discussed in section 4.7, $E_{av} = \dfrac{V}{s}$

So, field factor (f) for sphere gap arrangement = $\dfrac{E_{\max}}{E_{av}} = \left[\dfrac{\frac{s}{a} + 1 + \sqrt{\left(\frac{s}{a} + 1\right)^2 + 8}}{4}\right]$ …. 9.20

Variation of field factor (f) with gap distance (s) in the case of sphere gap arrangement is presented in Table 9.1. It may be seen from Table 9.1 that the deviation from uniform field (f=1) for s/a=0.2 is 6.8% while that for s/a=1.0 is 36.6%. Accuracy of voltage measurement by sphere gap depends significantly on the degree of field non-uniformity between the two spheres. Hence, it is recommended in practice that the gap distance should not be made more than the radius of the spheres.

Table 9.1
Variation of field factor with gap distance for sphere gap

s/a	0.2	0.4	0.6	0.8	1.0
f	1.068	1.139	1.212	1.288	1.366

Problem 9.1
Two spheres of 25cm diameter have a gap distance of 2.5cm between them. Determine the breakdown voltage of the sphere gap in air at STP.
Solution
Given, s = 2.5cm and a = (25/2) = 12.5cm

So, $\frac{s}{a}=\frac{2.5}{12.5}=0.2$ Correspondingly, field factor (f) = 1.068

E_{max} corresponding to breakdown of air at STP is 30kV$_p$/cm.

So, $E_{av}=\frac{E_{\max}}{f}=\frac{30}{1.068}=28.09\,\text{kV}_p/\text{cm}$

But, $E_{av}=\frac{V}{s}$ Hence, $\frac{V}{2.5}=28.09$, or, V = 70.22 kV$_p$.

Image of an Infinitely Long Line Charge wrt an Infinitely Long Conducting Plane

Consider an infinitely long line charge of positive polarity and uniform line charge density $+\lambda_l$ is located parallel to an infinitely long conducting plane present at $y=0$ and is at a height y_1 from the plane as shown in Fig.9.7, such that the location of the point charge is given by (x_1,y_1). The configuration as shown in Fig.9.7 depicts the view on the cross-sectional plane perpendicular to the length of the charge and the plane. As discussed earlier, the practical example of such a plane is earth surface having zero potential. The problem is therefore a two-dimensional one in Cartesian coordinates, where the field varies only on the x-y plane, i.e. the cross-sectional plane.

So, the problem is to find a solution for electric field in the region $y>0$. As discussed in section 9.2, this solution can be obtained by solving Laplace's equation with the following conditions:

i) The solution $\phi(x,y)$ will be valid at every point for y>0 except the location of the line charge.

ii) The potential at all the points on the conducting plane is zero, i.e $\phi(x,0)=0$.

iii) The potential approaches zero as the distance from the charge approaches infinity.

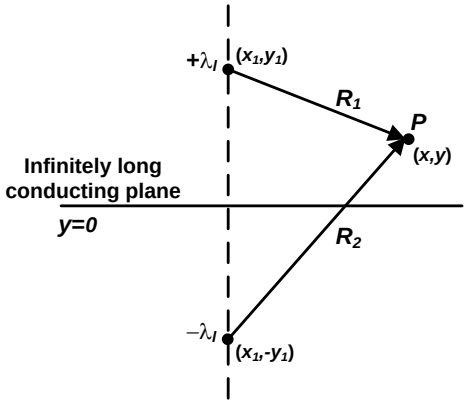


Fig.9.7 Infinitely long line charge near an infinitely long conducting plane

The electric potential at a distance r from an infinitely long line charge is given by $\frac{Q}{2\pi\varepsilon_0}\ln\frac{R}{r}$, where R$\to\infty$. Considering the cylindrical symmetry of the field due to an infinitely long line charge, it may be seen that the electric field due to an infinitely long line charge is dependent only on r-coordinate in cylindrical coordinate system and is independent of θ and z coordinates. Hence, this electric potential satisfies Laplace's equation in cylindrical coordinates as follows.

$$\vec{\nabla}^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\times-\frac{Q}{2\pi\varepsilon_0 r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{Q}{2\pi\varepsilon_0}\right)=0 \qquad \ldots. 9.24$$

Similar to the discussion of section 9.2, the infinitely long conducting plane is replaced by a fictitious line charge of uniform line charge density $-\lambda_l$ located at $(x_1,-y_1)$, which is the image of the real line charge. Since the field due an infinitely long line charge satisfies Laplace's equation, then the field due to the real line charge and the image line charge will also satisfy Laplace's equation at all points for $y>0$ except at the location of the real line charge.
Therefore, the electric potential at any point $P(x,y)$ will be given by

$$\phi_P = \frac{\lambda_l}{2\pi\varepsilon_0}\ln\frac{R_2}{R_1} \qquad \text{…. 9.25}$$

where, $R_1 = \left[(x-x_1)^2 + (y-y_1)^2\right]^{1/2}$

and $R_2 = \left[(x-x_1)^2 + (y+y_1)^2\right]^{1/2}$

From eqn.(9.1) it is evident that the potential at a very large distance from the line charge will be zero and also that the potential at all the points on the conducting plane at $y=0$ will be zero, because at every point on the plane $R_1=R_2$.
In Cartesian coordinates for two-dimensional system, electric field intensity at any point $P(x,y)$ for $y>0$ will be given by

$$\vec{E}_P = -\frac{\partial\phi_P}{\partial x}\hat{i} - \frac{\partial\phi_P}{\partial y}\hat{j} \qquad \text{…. 9.26}$$

where, $E_{xP} = -\frac{\partial\phi_P}{\partial x} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{x-x_1}{R_1^2} - \frac{x-x_1}{R_2^2}\right)$

$$E_{yP} = -\frac{\partial\phi_P}{\partial y} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{y-y_1}{R_1^2} - \frac{y+y_1}{R_2^2}\right)$$

For any point very near to the conducting plane, i.e. $y\to 0$, $R_1\approx R_2$. Hence, E_{xP} will be zero. Thus

$$\lim_{y\to 0}\vec{E}_{yP} = \frac{\sigma_s}{\varepsilon_0}\hat{k} = -\frac{Q\,y_1}{\pi\varepsilon_0\left[(x-x_1)^2 + y_1^2\right]^{3/2}}\hat{k}$$

or, $\sigma_s = -\frac{Q\,y_1}{\pi\left[(x-x_1)^2 + y_1^2\right]^{3/2}}$ …. 9.27

Eqn.(9.27) gives the negative surface charge density (σ_s) induced on the conducting plane by the positive polarity real line charge.
From eqn.(9.25), an expression for the equipotential surface can be obtained as

$\frac{R_2}{R_1} = \alpha$, where α is a constant

or, $(x-x_1)^2 + (y+y_1)^2 = \alpha^2\left[(x-x_1)^2 + (y-y_1)^2\right]$

or, $(x-x_1)^2 + \left(y - y_1\frac{\alpha^2+1}{\alpha^2-1}\right)^2 = \left(\frac{2\alpha\,y_1}{\alpha^2-1}\right)^2$ …. 9.28

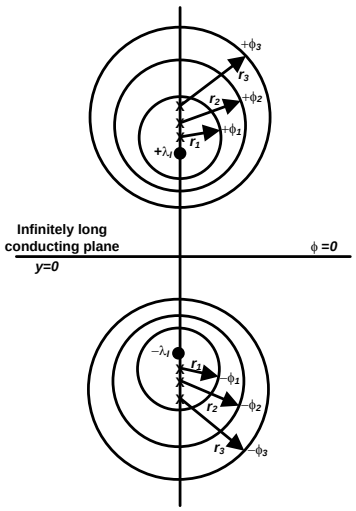


Fig. 9.8 Equipotentials for an infinitely long charge and its image wrt an infinitely long conducting plane

Eqn.(9.28) shows that the equipotentials are circles of radius $\left|\frac{2\alpha y_1}{\alpha^2-1}\right|$ having center at $x_1, \frac{\alpha^2+1}{\alpha^2-1} y_1$ on the cross-sectional plane. For the infinite conducting plane of zero potential α=1. For equipotentials above the infinite conducting plane α>1 and those below the infinite conducting plane α<1. In physical terms, the equipotentials are circular cylinders with axes parallel to the two line charges as shown in Fig.9.8. As α increases the radius of the cylinder increases and the axis of the cylinder shifts further away from the line charge.

The equipotentials below the conducting planes does have any physical meaning as the electric field is zero below the infinitely long conducting plane. But the equipotentials as shown in Fig.9.8 gives an important idea that two parallel cylinders having electric potentials $+V$ and $-V$ could be replaced by two infinitely long line charges. However, the line charges will not be located at the axes of the cylinders.

Two Infinitely Long Parallel Cylinders

Electric field due to two parallel cylindrical transmission line conductors is the same as the field due to two infinitely long parallel cylinders. The cross-sectional view of the arrangement is shown in Fig.9.9. Electric field for this arrangement is two-dimensional in Cartesian coordinates, because the field does not vary along the *z*-axis, which is along the length of the cylinders. Electric field varies only on the cross-sectional plane which is taken as the *x*-*y* plane. As discussed in section 9.4, these two parallel cylinders having potential $+V$ and $-V$ could be replaced by two infinitely long line charges of uniform line charge density $+\lambda_l$ and $-\lambda_l$ located within the respective cylinders as shown in Fig.9.9. These two line charges together will create two cylindrical equipotential surfaces of radius *a* having the prescribed potentials $+V$ and $-V$. The charges will be located at a distance *s* from the axis of the respective cylinders. So the problem is to find the location of these charges.

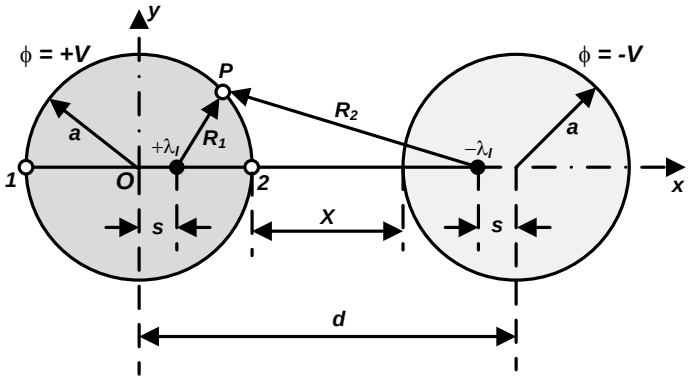


Fig.9.9 Two infinitely long parallel cylinders replaced by two infinitely long line charges

With reference to Fig.9.9, the potential of the point *P* on the surface of the cylinder is

$$\phi_P = \frac{\lambda_l}{2\pi\varepsilon_0}\ln\frac{R_2}{R_1}$$

For the cylinder surface to be equipotential the ratio of R_2 to R_1 must be constant.
Considering the point *1* on the cylinder surface as shown in Fig.9.9,

$$\phi_1 = \frac{\lambda_l}{2\pi\varepsilon_0}\ln\frac{d+a-s}{a+s} \qquad \text{…. 9.29}$$

and for the point *2* on the cylinder surface as shown in Fig.9.9,

$$\phi_2 = \frac{\lambda_l}{2\pi\varepsilon_0}\ln\frac{d-a-s}{a-s} \qquad \text{…. 9.30}$$

But, $\phi_1=\phi_2$. Hence, $\frac{d+a-s}{a+s}=\frac{d-a-s}{a-s}=\frac{d-s}{a}=\frac{a}{s}$ (by componendo-dividendo)

Therefore, $s^2 - sd + a^2 = 0$

or, $s = \frac{d}{2} \pm \frac{\sqrt{d^2-4a^2}}{2}$ …. 9.31

In the solution of *s* as given by eqn.(9.31), the additive expression has to be neglected, because in that case the image charge will be located outside the cylinder. Therefore,

$$s = \frac{d}{2} - \frac{\sqrt{d^2-4a^2}}{2} \qquad \text{…. 9.32}$$

For transmission lines, $d>>a$ and hence $s\approx 0$, i.e. the line charges are placed on the axes of the two cylinders.
Now, the potential at the point *2* on the cylinder surface, as shown in Fig.9.9, is *+V*. Hence,

$$\phi_2 = \frac{\lambda_l}{2\pi\varepsilon_0}\ln\frac{d-a-s}{a-s} = V$$

or, $\lambda_l = \frac{2\pi\varepsilon_0 V}{\ln\frac{d-a-s}{a-s}}$ …. 9.33

Eqn.(9.33) gives the magnitude of the uniform line charge density.
In the arrangement shown in Fig.9.9, maximum electric field intensity (E_{max}) occurs at the point *2*, which is given by

$$E_{\max} = \frac{\lambda_l}{2\pi\varepsilon_0}\left(\frac{1}{a-s}+\frac{1}{d-a-s}\right) = \frac{V}{\ln\frac{d-a-s}{a-s}}\left(\frac{1}{a-s}+\frac{1}{d-a-s}\right) \qquad \text{…. 9.34}$$

RHS of eqn.(9.34) is in terms of the physical dimensions of the arrangement and the electric potential of the cylinders and hence can be computed in a straightforward manner.

Again, for the physical arrangement of Fig.9.9, $E_{av} = \frac{2V}{d-2a}$

Therefore, field factor (f) = $\frac{E_{\max}}{E_{av}} = \frac{(d-2a)}{2\ln\frac{d-a-s}{a-s}}\left(\frac{1}{a-s}+\frac{1}{d-a-s}\right)$ …. 9.35

Putting the value of s from eqn.(9.32) in eqn.(9.35) and upon simplification it may be written that

$$f = \frac{\sqrt{\left(\frac{d}{2a}\right)^2 - 1}}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]} \quad \text{…. 9.36}$$

For transmission lines, eqn.(9.36) is often modified by putting $d=X+2a$, which yields

$$f = \frac{\sqrt{\left(\frac{X}{a}\right)^2 + 4\frac{X}{a}}}{2\ln\left[\frac{X}{2a}+1+\frac{1}{2}\sqrt{\left(\frac{X}{a}\right)^2+4\frac{X}{a}}\right]} \quad \text{…. 9.37}$$

Eqn.(9.37) represents the field factor as a function of the ratio of the gap distance between the two transmission line conductors (X) and the radius of the conductors (a).

For high voltage transmission lines, $d>>a$. As a result the field factor as given by eqn.(9.36) reduces to

$$f = \frac{d}{2a\ \ln\frac{d}{a}} \quad \text{…. 9.38}$$

and $E_{av} = \frac{2V}{d-2a} \approx \frac{2V}{d}$

Hence, $E_{\max} = E_{av} \times f = \frac{2V}{d} \times \frac{d}{2a\ln\frac{d}{a}} = \frac{V}{a\ln\frac{d}{a}}$ …. 9.39

Capacitance per unit length between the two parallel cylinders can be obtained from eqn.(9.32) and (9.33) as follows

$$C = \frac{\lambda_l}{2V} = \frac{\pi\varepsilon_0}{\ln\frac{d-a-s}{a-s}} = \frac{\pi\varepsilon_0}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]} \quad \text{…. 9.40}$$

Since, $\ln\left(x+\sqrt{x^2-1}\right) = \cosh^{-1}x$, hence for $x>1$ eqn.(9.40) can be written as

$$C = \frac{\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d}{2a}\right)} \quad \text{…. 9.41}$$

Problem 9.2

A long conductor of negligible radius is at a height 5m from earth surface and is parallel to it. It has a uniform line charge density of +1nC/m. Find the electric potential and field intensity at a point 3m below the line.

Solution

The arrangement of the problem is shown in Fig.9.10. Since the conductor is considered to have negligible radius, hence the line charge is located on the axis of the conductor.

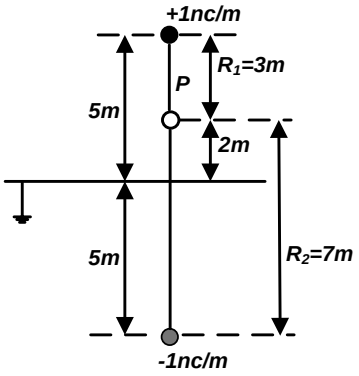


Fig.9.10 Pertaining to Problem 9.2

With reference to eqn.(9.25), λ_l=1nc/m, R_1=3m and R_2=7m.

So, $\phi_P = \frac{10^{-9}}{2\pi \times 8.854 \times 10^{-12}} \ln \frac{7}{3} = 15.23\, V$

Electric field intensity components at the point P are obtained from eqn.(9.26) as follows:

$E_{xP} = 0$

$$E_{yP} = \frac{10^{-9}}{2\pi \times 8.854 \times 10^{-12}} \left(\frac{2-5}{3^2} - \frac{2+5}{7^2} \right) = -8.56\, V/m$$

Problem 9.3

Determine the breakdown voltage in air at STP of a 20cm diameter cylindrical electrode placed horizontally with its axis 20cm above earth surface.

Solution

The arrangement of the problem is shown in Fig.9.11.

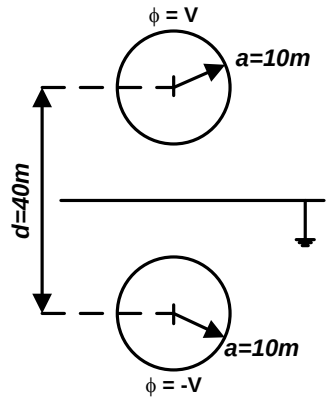


Fig.9.11 Pertaining to Problem 9.3

With reference to eqn.(9.36)

d = 40cm and a = 10cm. Hence, (d/2a) = 2

So, $f = \frac{\sqrt{2^2 - 1}}{\ln(2 + \sqrt{2^2 - 1})} = 1.315$

E_{max} corresponding to breakdown of air at STP is 30kV$_p$/cm.

So, $E_{av} = \frac{E_{\max}}{f} = \frac{30}{1.315} = 22.81\,\text{kV}_p/\text{cm}$

But, $E_{av} = \frac{2V}{d - 2a}$ Hence, $\frac{2V}{20} = 22.81$, or, V = 228.1 kV$_p$.

Salient Features of Method of Images

a) Image charges are always located outside the region where the field is to be determined.
b) Depending on the type of problem, magnitude of image charge may or may not be the same as that of the physical charge.
c) Depending on the type of problem, image charges may or may not be of polarity opposite to that of the physical charge.
d) In the image charge system electric field is finite on both sides of the imaging surface. But in the physical system electric field is non zero and finite only on one side of the imaging surface. Since energy in electric field is proportional to volume integral of E^2, the electrostatic energy in the physical charge system is half of the electrostatic energy of the image charge system.

Numerical Field Calculation

The analytical methods can only be applied to the cases, where the electrode or dielectric boundaries are of simple geometrical forms such as cylinders, spheres etc. In other words, in this method the boundaries are required to be defined exclusively by known mathematical functions. The results obtained are very accurate. But, as it is obvious, this method cannot be applied to complex problems. However, the results obtained by analytical methods for standard configurations are used still today to validate the results obtained by some other approximate methods such as numerical methods.

Earlier the experimental as well as the graphical methods were used to get a fair idea about the nature of field distribution in some practical cases. However, these methods are greatly limited in their areas of usage and the errors involved are usually very high for any complex problem to be taken directly for design purposes.

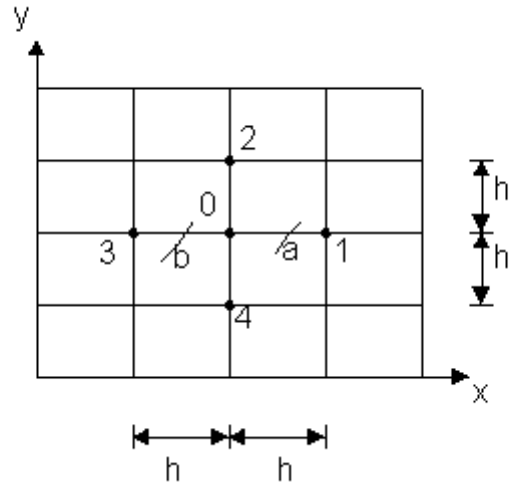
As most of practical field problems are so complex that graphical, experimental or analytical methods of solution are very difficult, if not impossible, numerical methods of field calculation have been developed. There are several numerical methods. For each practical field problem, depending upon the dielectric properties, complexity of contours and boundary conditions, one or the other numerical method is more suited.

Finite Difference Method

Two-Dimensional System with Equal Nodal Distance

In a 2-D system, V is independent of one of the axis directions, e.g. in the case of cable V is taken to be independent of z-direction, where z is along the length of the cable. Then the *Laplace's Equation* is written as –

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



In FDM, V_o is expressed in terms of the potentials of the connected nodes, i.e. V_1, V_2, V_3 and V_4 as shown in the figure above, such that Laplace's equation is satisfied at the point "0". Let, 'a' and 'b' be the midpoints between 0 & 1 and 3 & 0 respectively. Then according to mean value theorem:

$$\left(\frac{\partial V}{\partial x}\right)_a = \frac{V_1 - V_o}{h}$$

$$\text{and } \left(\frac{\partial V}{\partial x}\right)_b = \frac{V_o - V_3}{h}$$

Again, '0' being the mid-point between 'a' and 'b',

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_o = \frac{\left(\frac{\partial V}{\partial x}\right)_a - \left(\frac{\partial V}{\partial x}\right)_b}{h}$$

$$= \frac{V_1 + V_3 - 2V_o}{h^2}$$

$$\text{similarly, } \left(\frac{\partial^2 V}{\partial y^2}\right)_o = \frac{V_2 + V_4 - 2V_o}{h^2}$$

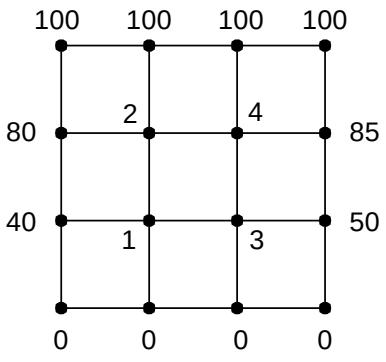
Now, satisfying Laplace's equation at "0",

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_o + \left(\frac{\partial^2 V}{\partial y^2}\right)_o = 0$$

$$\text{or, } \frac{V_1 + V_3 - 2V_o}{h^2} + \frac{V_2 + V_4 - 2V_o}{h^2} = 0$$

$$\text{or } V_o = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

Problem:



Let, the boundary values of the potentials for any given system are as indicated in the figure above.
It is required to obtain the values of the potentials at the nodes 1, 2, 3 and 4.

Solution:

For the given problem, at first the FDM equations for the potentials at the nodes 1, 2, 3 and 4 are to be written. Then the obtained FDM equations are solved, usually by an iterative method, e.g. Gauss-Seidel Method using over-relaxation for accelerated convergence.

The boundary values of node potentials are as indicated in the figure. The values of the potential at the nodes 1, 2, 3 and 4 can be obtained by the following FDM equations:

$$V_1 = (V_2 + V_3 + 80 + 0) / 4$$

$$V_2 = (V_1 + V_4 + 100 + 80) / 4$$

$$V_3 = (V_1 + V_4 + 50 + 0) / 4$$

$$V_4 = (V_2 + V_3 + 100 + 85) / 4$$