Linear Programming

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Introduction

- A linear programming problem may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints*.
- The constraints may be equalities or inequalities.
- For example
  - Find numbers $x1$ and $x2$ that maximize the sum $x1 + x2$ subject to the constraints $x1 \geq 0$, $x2 \geq 0$, and

$x1 + 2 * x2 \leq 4$

$4 * x1 + 2 * x2 \leq 12$

$-x1 + x2 \leq 1$

  - Two unknowns and five constraints
  - First one is called *non-negativity constraint*
  - Other constraints are called *main constraints*
  - The function $x1 + x2$ is called *objective function*

Introduction

- Simple problems with small number of unknowns (here only two) can be solved graphically
- In this example a family of parallel lines with slope -1 presents x1 + x2 = c
- Here the maximum occurs at the intersection of the lines *x*1 + 2 * *x*2 = 4 and

  4 * *x*1 + 2 * *x*2 = 12,

i.e. (*x*1, *x*2) = (8/3, 2/3)

- However, not all linear programming problems are so easily solved

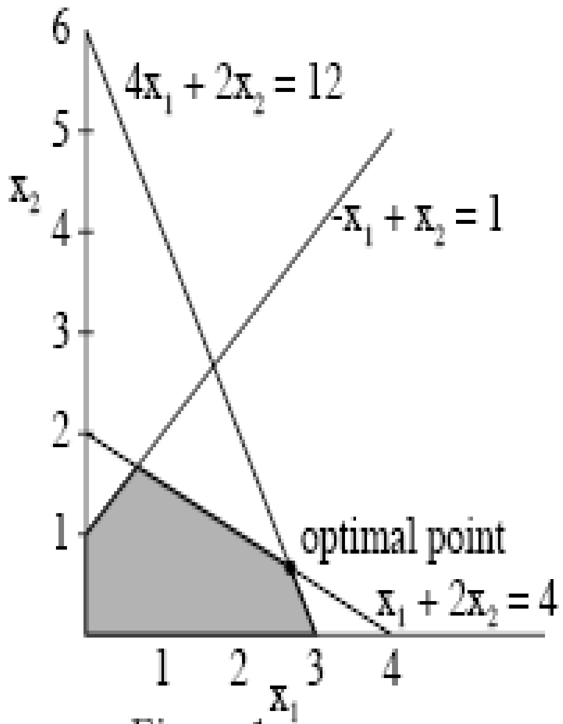


Figure 1.

General Linear Programming Problems

- Input
  - A set of real variables *X* with arbitrary bounds
  - A set of real constraints on *X*
  - A linear objective function on *X*
- Output
  - An assignment to *X* with optimal value of the objective function
- Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m$$

$$x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n$$

A small real life problem

- You are allowed to share your time between two companies.
  - company $C1$ pays 1 dollar per hour;
  - company $C2$ pays 10 dollars per hour.
- Knowing that you can only work up to 8 hours per day, what schedule should you go for?
- The solution is that you will work full-time at company $C2$.
- Linear formulation: $x1$ is the time spent at $C1$ and $x2$ the time spent at $C2$
  - Constraints: $x1 \geq 0$, $x2 \geq 0$, $x1 + x2 \leq 8$
  - Objective function: maximize $x1 + 10 * x2$
  - Solution: $x1 = 0$, $x2 = 8$.

Standard form

- given an m-vector, $\boldsymbol{b} = (b_1, \ldots, b_m)^T$, an n-vector, $\boldsymbol{c} = (c_1, \ldots, c_n)^T$, and an $m \times n$ matrix of real numbers

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **The Standard Maximum Problem** aims at finding an n-vector, $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, to maximize $\boldsymbol{c}^T\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$ subject to the constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$
...
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$

(or $\boldsymbol{Ax} \le \boldsymbol{b}$) and

$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$ (or $\boldsymbol{x} \ge \boldsymbol{0}$).

Standard form

The Standard Minimum Problem aims at finding an m-vector, $\boldsymbol{y}$ = ($y_1, . . . , y_m$), to minimize $\boldsymbol{y}^T\boldsymbol{b} = y_1b_1 + \cdot\cdot\cdot + y_mb_m$ subject to the constraints

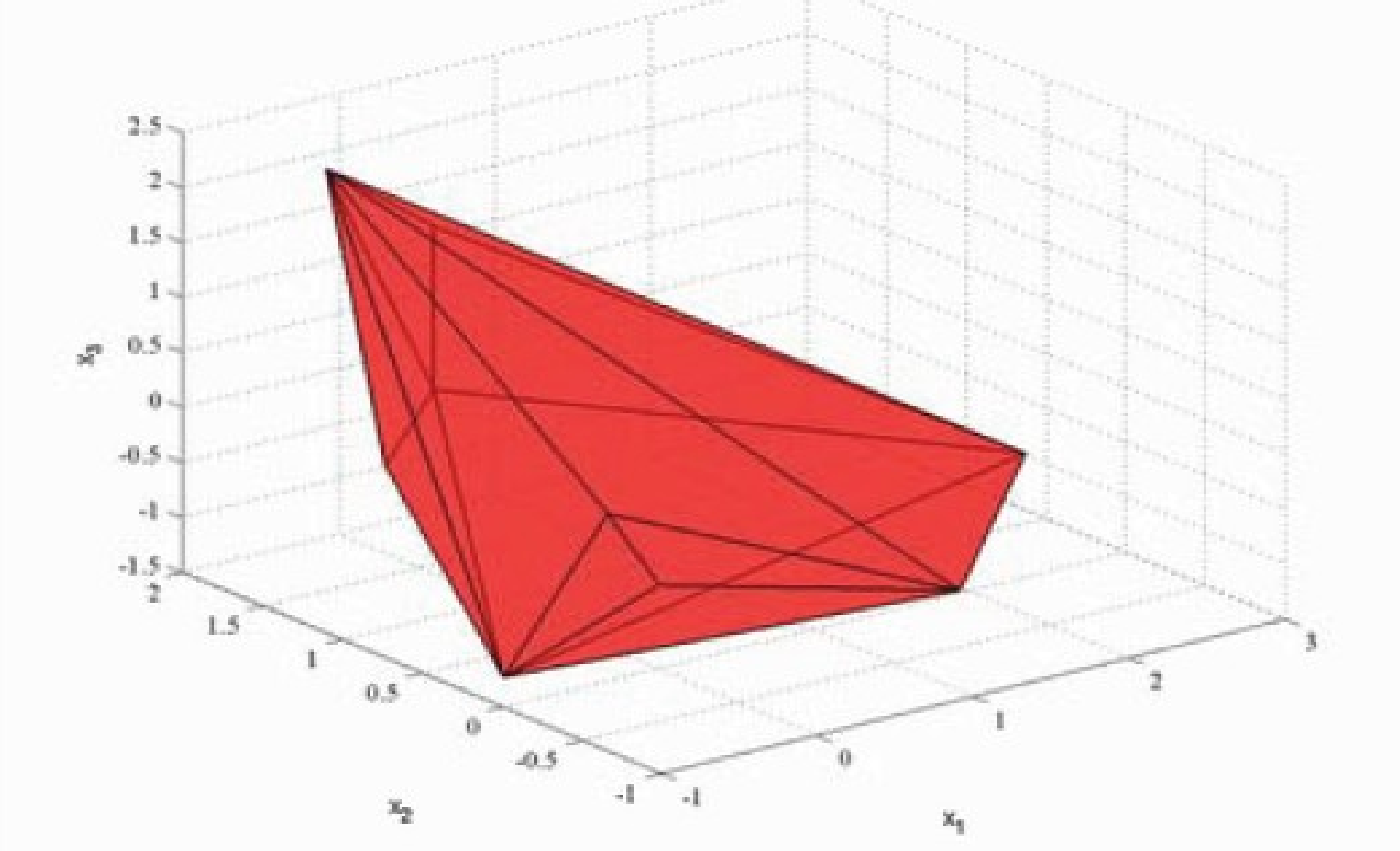
$y_1a_{11} + y_2a_{21} + \cdot\cdot\cdot + y_ma_{m1} \geq c_1$
$y_1a_{12} + y_2a_{22} + \cdot\cdot\cdot + y_ma_{m2} \geq c_2$
...
$y_1a_{1n} + y_2a_{2n} + \cdot\cdot\cdot + y_ma_{mn} \geq c_n$

(or $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T$) and

$y_1 \geq 0, y_2 \geq 0, . . . , y_m \geq 0$ (or $\boldsymbol{y} \geq \boldsymbol{0}$).

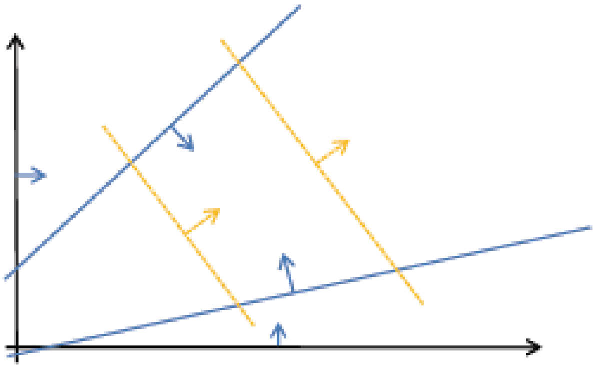
More than two variables

Source: http://control.ee.ethz.ch/~mpt/downloads/25/gradcolor_off.png

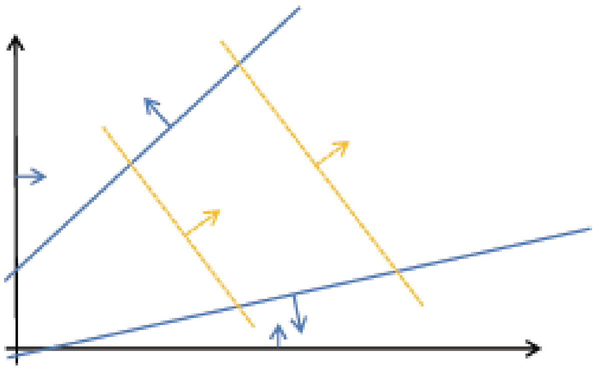


Unbounded and infeasible problems

Unbound



Infeasible



Unbounded and infeasible problems

- A vector, ***x***, is said to be **feasible** if it satisfies the corresponding constraints
- The set of feasible vectors is called the **constraint set**
- A linear programming problem is said to be **feasible** if the constraint set is not empty; otherwise it is said to be **infeasible**
- A feasible maximum (or minimum) problem is said to be **unbounded** if the objective function can assume arbitrarily large positive (or negative) values at feasible vectors; otherwise, it is said to be **bounded**
- The **value** of a bounded feasible maximum (or minimum) problem is the maximum (or minimum) value of the objective function as the variables range over the constraint set
- A feasible vector at which the objective function achieves the value is called **optimal**

Duality

- To every linear program there is a dual linear program with which it is intimately connected
- e.g. the dual of the standard maximum problem

maximize $\boldsymbol{c}^T\boldsymbol{x}$

subject to the constraints $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ *and* $\boldsymbol{x} \geq 0$

can be defined as a minimum problem

minimize $\boldsymbol{y}^T\boldsymbol{b}$

subject to the constraints $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T$ *and* $\boldsymbol{y} \geq 0$

Duality

e.g. dual of the following problem:

Find numbers $x1$ and $x2$ that maximize the sum $x1 + x2$ subject to the constraints $x1 \geq 0$, $x2 \geq 0$, and

$x1 + 2 * x2 \leq 4$

$4 * x1 + 2 * x2 \leq 12$

$-x1 + x2 \leq 1$

is

Find $y1$, $y2$, and $y3$ to minimize $4 * y1 + 12 * y2 + y3$ subject to the constraints $y1 \geq 0$, $y2 \geq 0$,

$y3 \geq 0$, and

$y1 + 4 * y2 - y3 \geq 1$

$2 * y1 + 2 * y2 + y3 \geq 1$.

The general standard maximum problem and the dual standard minimum problem

	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$
	$\geq c_1$	$\geq c_2$	$\cdots$	$\geq c_n$	

	x_1	x_2	
y_1	1	2	≤ 4
y_2	4	2	≤ 12
y_3	-1	1	≤ 1
	≥ 1	≥ 1	

Our example

The Diet Problem

m different types of food, $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients, $N_1, \ldots, N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j. Let b_i be the price per unit of food, F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i.

The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m \ldots .(1)$$

The amount of nutrient N_j contained in this diet is

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m \text{ for } j = 1, \ldots, n$$

We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m \geq c_j \text{ for } j = 1, \ldots, n \ldots .(2)$$

We cannot purchase a negative amount of food, so

$$y_1 \geq 0,\ y_2 \geq 0, \ldots, y_m \geq 0 \quad \ldots .\ (3)$$

The Transportation Problem

There are I *ports*, $P_1, \ldots, P_I$, that supply a certain commodity, and there are J markets, $M_1, \ldots, M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity ($i = 1, 2, \ldots, I$), and market M_j must receive the amount r_j of the commodity ($j = 1, \ldots, J$). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is

The Transportation Problem

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is given by 1.

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij} b_{ij}. \quad \ldots.1$$

The amount sent from port P_i is Y and since the amount available at port P_i is s_i, we must satisfy inequality 2.

$$Y = \sum_{j=1}^{J} y_{ij}$$

$$\sum_{j=1}^{J} y_{ij} \leq s_i \quad \text{for } i = 1, \ldots, I. \quad \ldots.2$$

The amount sent to market M_j is W and since the amount required there is r_j, we must satisfy inequality 3.

$$W = \sum_{i=1}^{I} y_{ij},$$

$$\sum_{i=1}^{I} y_{ij} \geq r_j \quad \text{for } j = 1, \ldots, J. \quad \ldots.3$$

It is assumed that we cannot send a negative amount from P_I to M_j, thus, inequality 4 is to be satisfied.

$$y_{ij} \geq 0 \quad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \quad \ldots.4$$

Our problem is: minimize (1) subject to (2), (3) and (4).

All Linear Programming Problems Can be Converted to Standard Form

- A minimum problem can be changed to a maximum problem by multiplying the objective function by −1
- Constraints with ≥ can be changed to constraints with ≤ by multiplying both sides of the inequality with -1
- An equality constraint $\Sigma a_{ij}x_j = b_i$ may be removed, by solving this constraint for some x_j for which $a_{ij} \neq 0$ and substituting this solution into the other constraints and into the objective function wherever x_j appears
  - This removes one constraint and one variable from the problem.
- If a variable may not be restricted to be nonnegative, the unrestricted variable, x_j, may be replaced by the difference of two nonnegative variables, $x_j = u_j - v_j$, where $u_j \geq 0$ and $v_j \geq 0$.
  - This adds one variable and two non-negativity constraints to the problem.

Duality

- To every linear program there is a dual linear program with which it is intimately connected
- e.g. the dual of the standard maximum problem

maximize $\boldsymbol{c}^T\boldsymbol{x}$

subject to the constraints $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ *and* $\boldsymbol{x} \geq 0$

can be defined as a minimum problem

minimize $\boldsymbol{y}^T\boldsymbol{b}$

subject to the constraints $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T$ *and* $\boldsymbol{y} \geq 0$

Duality

Theorem 1: *If* $\boldsymbol{x}$ *is feasible for the standard maximum problem and if* $\boldsymbol{y}$ *is feasible for its dual, then* $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{b}$

Proof: $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{b}$

The first inequality follows from $\boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{c}^T \leq \boldsymbol{y}^T\boldsymbol{A}$. The second inequality follows from $\boldsymbol{y} \geq \mathbf{0}$ and $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.

Corollary 1: *If a standard problem and its dual are both feasible, then both are bounded feasible.*

Proof. If $\boldsymbol{y}$ is feasible for the minimum problem, then theorem 1 shows that $\boldsymbol{y}^T\boldsymbol{b}$ is an upper bound for the values of $\boldsymbol{c}^T\boldsymbol{x}$ for $\boldsymbol{x}$ feasible for the maximum problem. Similarly for the converse.

Duality

Corollary 2: *If there exists feasible* $\mathbf{x}*$ *and* $\mathbf{y}*$ *for a standard maximum problem and its dual such that* $\mathbf{c}^T\mathbf{x}* = \mathbf{y}*^T\mathbf{b}$, *then both are optimal for their respective problems.*

Proof: If $\mathbf{x}$ is any feasible vector for the maximum problem, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{y}*^T\mathbf{b} = \mathbf{c}^T\mathbf{x}*$. which shows that $\mathbf{x}*$ is optimal. A similar argument works for $\mathbf{y}*$.

The Duality Theorem: *If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exists optimal vectors for both problems.*

Duality

- Three possibilities for a linear program:
  - feasible bounded (f.b.),
  - feasible unbounded (f.u.), or
  - infeasible (i).
- As per corollary 1 if a problem and its dual are both feasible, then both must be bounded feasible
- As per Duality Theorem if a program is feasible bounded, its dual cannot be infeasible
- The remaining four possibilities can occur

		Standard Maximum Problem		
		f.b.	f.u.	i.
	f.b.		x	x
Dual	f.u.	x	x	
	i.	x		

Example of Corollary 2

- A maximum problem:

Find x_1 , x_2 , x_3 , x_4 to maximize $2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints $x_j \geq 0$ for all j, and

$x_1 + 3x_2 + x_4 \leq 4$

$2x_1 + x_2 \leq 3$

$x_2 + 4x_3 + x_4 \leq 3$.

- The dual problem:

Find y_1 , y_2 , y_3 to minimize $4y_1 + 3y_2 + 3y_3$ subject to the constraints $y_i \geq 0$ for all i, and

$y_1 + 2y_2 \geq 2$

$3y_1 + y_2 + y_3 \geq 4$

$4y_3 \geq 1$

$y_1 + y_3 \geq 1$.

- The vector $(x_1, x_2, x_3, x_4) = (1, 1, 1/2, 0)$ satisfies the constraints of the maximum problem; value of the objective function being 13/2.
- The vector $(y_1, y_2, y_3) = (11/10, 9/20, 1/4)$ satisfies the constraints of the minimum problem and has the same value 13/2 also.

The Simplex Algorithm

Lecture notes of ´Eric Schost