Dimensional Analysis.

The subject deals with the process where all the important parameters involved in a physical phenomena are systematically organized into dimensionless no.
(Ex: Re - $\frac{\rho V d}{\mu}$)

Adv:- ① No. of unknown quantities are reduced.
② The problem is generalised and need for specifying a particular system of units is eleminated. The eqns are true for any system
③ of units.

Importance:- These eqns are independent of geometry and thus it is useful for determining the performance of a prototype (A full scale ~~model~~ structure) from data obtained by tests on a model (a reduced-scale structure).

~~System of Dimensions:-~~

START

Engg problems are complex in nature. They are determined from experiment. Due to economic or other problem, the experiments can't be performed ~~it~~ under identical condition.

Lab tests are carried out in altered condition of operating parameters from the actual one.
(i) The test results are applied to actual problem (How??)
(ii) Different variables are ~~ele~~ reduced to less no. of variables.

The soln. lies in physical ~~sy~~ similarity. By this the above two problems are solved.

So various test can be carried out by a scale model. [Example aerodplane, ship, performance of fluid machines like turbines, pump & propellors etc]

Test with one fluid & result applied with another fluid

Type of physical similarity ??

<u>Prototype</u> – Full sized structure of actual engg design. This operate under the

<u>Objective & importance of model studies</u> – involves model design & fabrication .. and performance test on models and analysis.

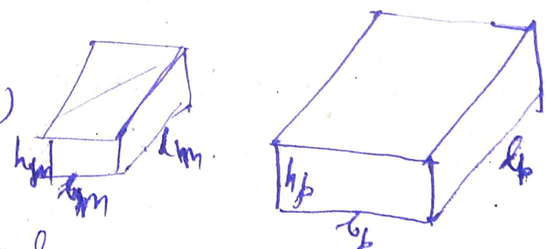
~~Sim~~

<u>Similitude</u> → represents the theory of and art of predicting prototype conditions from model observation.

The result obtained from experiments on the models can be applied to the prototype only if ~~a~~ ~~complete~~ both ~~flow~~ follow same physics. And two systems are geometrically, kinematically & dynamically similar.

① Geometric similarity

$$\frac{l_m}{l_p} = \frac{b_m}{b_p} = \frac{h_m}{h_p} = l_r \text{ (scale ratio)}$$



Area ratio $A_r = \frac{A_m}{A_p} = \frac{l_m \cdot b_m}{l_p \, b_p} = l_r^2$

Volume ratio $V_r = l_r^3$.

* Sometimes distorted model is used to keep the physics same. Example model of a river. Scale factor is different for length, width and height

Kinematic similarity :-

Vel ratio $V_r = \frac{V_m}{V_p} = \frac{L_m/T_m}{L_p/T_p} = \frac{L_r}{T_r}$ T_r = Time ratio.

Acc^n $\frac{a_m}{a_p} = a_r = \frac{L_r}{T_r^2}$

$\frac{Q_m}{Q_p} = \frac{L_m^3/T_m}{L_p^3/T_p} = \frac{L_r^3}{T_r} = Q_r$

Dynamic similarity → Similarity of forces.
Ratio of particular forces are similar for both model and prototype.

The forces are viscous force F_v, Pressure force F_p, Gravity force F_g, Capillary force (due to surface tension) F_c, Compressibility force (force due to elasticity) F_e

Resultant force $F_R = F_v + F_p + F_g + F_c + F_e$

F_I → inertia force equal & opposite to resultant force

$F_R + F_I = 0$ or $\left(\frac{F_R}{F_I}\right)_r = 0$

$F_I \propto$ mass × acc^n $\propto \rho l^3 \times \frac{V}{T}$

$\Rightarrow F_I \propto \rho l^3 \cdot \frac{V}{l} \cdot V \propto \rho l^2 V^2 \Rightarrow F_I \propto \rho l^2 V^2$

$V = \frac{l}{T}$
$T = \frac{l}{V}$

Viscous force F_v = shear stress × area
$\mu \times$ vel grad $\quad l^2$

$F_v \propto \mu \cdot \frac{V}{l} \cdot l^2$
$\propto \mu V l$

Pressure force $F_p \propto (\Delta P) \cdot l^2$

Gravity force $F_g \propto mg$
$\propto \rho l^3 g$

Capillary or surface tension force $F_c \propto \sigma l$.

Elastic force $F_e \propto E \cdot l^2$

All the # ratios are with respect to inertia force which in common:

(1) $\therefore \frac{F_I}{F_v} = \frac{\rho V^2 l^2}{\mu V l} = \frac{\rho l V}{\mu} = Re$. Reynold's no.

(2) $\frac{F_p}{F_I} = \frac{\Delta p l^2}{\rho V^2 l^2} = \frac{\Delta p}{\rho V^2} = Eu$ Euler no.

These are dynamic similarity governed by viscous pressure and inertia forces.

(3) $\frac{F_g}{F_I} = \frac{\rho l^3 g}{\rho l^2 V^2} = \frac{lg}{V^2}$ } for flow governed by gravity pressure and Inertia forces.

$\frac{V}{\sqrt{lg}}$ is known as Froude no. $Fr = \sqrt{\frac{F_I}{F_g}}$.

(4)

$\frac{\text{Surface Tension force}}{\text{Inertia force}} = \frac{F_e}{F_I} = \frac{\sigma}{\rho V^2 l} = We$ (Weber No.)

Applicable where surface tension is dominant force.

(i) Capillary flow where capillary wave appear.

(ii) Flow of small jets

(iii) Flow of thin sheet of liquid over solid surface.

(5) ~~Mach No~~ $= \frac{\text{Inertia force}}{\text{Elastic Force}} = \frac{\rho V^2 l^2}{E_s l^2} = \frac{\rho V^2}{E_s}$ = Cauchy number. E_s = isentropic

Austrian Physicist Earnst Mach.

velocity of sound $= (a) = \sqrt{E_s/\rho}$

$\therefore \frac{\rho v^2}{E_s} = \frac{v^2}{E_s/\rho} = \frac{v^2}{a^2}$ $\frac{V}{a}$ = Mach No $= \frac{V}{\sqrt{E_s/\rho}} = Ma$

Dynamic similarity of flows with elastic forces.

E = modulus of elasticity E_s = isentropic bulk modulus of elasticity.

Flow become compressible if $Ma > 0.3$

Situation:- Flow of air past high-speed aircraft, missiles, propellers and rotary compressors. In these cases Mach no. is a condition for dynamic similarity $\frac{V_p}{a_p} = \frac{V_m}{a_m}$

Prob: - A ship has a length of 150 m and wetted area of 3000 m^2. A model of this ship 5 m in length when towed in fresh water at 2 m/s produce resistance of 40 N. Calculate:- (i) the corresponding speed of the ship (ii) The shaft power required to propel the ship at this speed through sea water (1030) take propeller $\eta = 75\%$.

The resistance R to the motion of ship is given ~~by~~
$R = \rho v^2 l^2 f\left(\frac{\rho v l}{\mu}\right)$. For dynamic similarity the non-dimensional terms $R/\rho v^2 l^2$ and $\frac{\rho v l}{\mu}$ should be equal for model & prototype.

(i) $\frac{V_m \cdot l_m}{\nu_m} = \frac{V_p l_p}{\nu_p}$ $\therefore V_p = V_m \cdot \frac{l_m}{l_p} = 2 \times \frac{5}{150} = \frac{1}{15}$ m/s.

(ii) $\left(R/\rho v^2 l^2\right)_m = \left(\frac{R}{\rho v^2 l^2}\right)_p$

$\therefore$ Ratio of the drag forces / resistance

$\frac{R_p}{R_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{l_p}{l_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2 = \frac{1030}{1000} \times \left(\frac{150}{50}\right)^2 \cdot \left(\frac{1}{15}/2\right)^2 = 1.03$

$\therefore$ Drag force on the prototype $R_p = 1.03 \times R_m$
$= 1.03 \times 40 = 41.2$ N

Prob: A model of torpedo is tested in a towing tank at 26 m/s vel whilst the prototype is to run at 6.5 m/sec. (a) What model scale has been used? for water $\nu = 1.13 \times 10^{-6}$ m^2/sec.
(b) What would be model speed if tested in wind tunnel under a pressure of 2000 kPa and a constant temp of 27°c. Absolute ~~vel.~~ viscosity of air under these condition is 1.85×10^{-4} poise and Gas constant R = 287 J/kgk.

Soln:- Viscous effect predominant here,
So Re is significant ratio. $\therefore \frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$

$\therefore \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p} \times \frac{V_p}{V_m} = \frac{6.5}{26} = \frac{1}{4}$.

$\therefore$ model scale ratio = 1:4

(b) For air 2000 kPa and 27°c. $\rho = \frac{P}{RT} = \frac{2000 \times 10^3}{287 \times (273+27)}$
$= 23.23$ kg/m³

$\mu = 1.85 \times 10^{-4}$ poise. $= 1.85 \times 10^{-5}$ Ns/m²

$\nu = \frac{\mu}{\rho} = \frac{1.85 \times 10^{-5}}{23.23} = 0.796 \times 10^{-6}$ m²/s

$V_m = V_p \times \frac{L_p}{L_m} \times \frac{\nu_m}{\nu_p} = 6.5 \times 4 \times \frac{0.796 \times 10^{-6}}{1.13 \times 10^{-6}} = 18.32$ m/s.

P.T.O

Resistance R to the motion of a completely submerged body is given by $R = \rho V^2 l^2 f\left(\frac{Vl}{\nu}\right)$ where ρ and ν are the kinematic viscosity of the fluid, l = length of body & V = flow velocity. If the resistance of $\frac{1}{8}$ th scale air ship model when tested in water at 12 m/s in 200 N, what will be the resistance of the air ship at the corresponding speed? Kinematic of viscosity of air is 13 times that of water is 810 times that of air.

Ans:- From similarity consideration, the non-dimensional terms $R/\rho V^2 l^2$ and $\frac{Vl}{\nu}$ should be same for prototype and its model.

$$\left(\frac{Vl}{\nu}\right)_m = \left(\frac{Vl}{\nu}\right)_p \Rightarrow V_p = V_m \times \frac{l_m}{l_p} \times \frac{\nu_p}{\nu_m} = 12 \times \frac{1}{8} \times 13 = 19.5 \text{ m/sec.}$$

$$\left(\frac{R}{\rho V^2 l^2}\right)_m = \left(\frac{R}{\rho V^2 l^2}\right)_p \qquad R_p = R_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{V_p}{V_m}\right)^2 \cdot \left(\frac{l_p}{l_m}\right)^2$$

$$= 200 \times \frac{1}{810} \times \left(\frac{19.5}{12}\right)^2 \times 8^2 = 41.73 \text{ N.}$$

The application of dynamic similarity and dimensional analysis.

~~Expl~~ [Explain importance of ~~model~~ ratio of forces and non-dimentional number like Re, Eu, Fr, We and Ma. etc.] ~~These are Ob'~~

To change Re we can change any of the variables.

as $Re = \frac{\rho V D}{\mu}$.

The above relations has been obtained by straight forward method. But sometimes it is ~~very~~ very difficult. Hence an alternative method of determining these ~~dimensional~~ dimensionless parameters are dimensional analysis

Explain the dimension of physical quantilities.

Physical Quantity	Dimension	Dynamic Quantities: Physical Quantity	Dimension
Mass (m)	M	Density	ML^{-3}
Length (L)	L	Sp wt	$ML^{-2}T^{-2}$
Time (T)	T	Surface Tension σ	MT^{-2}
Force (F)	MLT^{-2}	Pressure p	$ML^{-1}T^{-2}$
Temp	θ	Mod. of Elasticity (E)	$ML^{-1}T^{-2}$
Area (A)	L^2	μ	$ML^{-1}T^{-1}$
Volume ($\forall$)	L^3	Resisting force	MLT^{-2}
Linear Vel (V)	LT^{-1}	Thrust	MLT^{-2}
Angular " (ω)	T^{-1}	Torque/work/Energy	ML^2T^{-2}
Accn (a)	LT^{-2}	Power	ML^2T^{-3}
Discharge (Q)	L^3T^{-1}		
Gravity (g)	LT^{-2}		
Kinematic viscosity (ν)	L^2T^{-1}		

The fundamental theory of ~~dimensionless~~ analysis is based on the following axiom.
'Equations describing a physical phenomenon must be dimensionally homogeneous and the units therein must be consistent.'

i.e dimension ~~of~~ on the two sides of the eqn is identical.

$$T = 2\pi\sqrt{L/g} \quad [T] = [1]\left[\frac{L}{LT^{-2}}\right]^{1/2} = T$$

Dimensional Analysis - Buckingham's Pi-Theorem.

If there are n variables in dimensionally homogeneous equation and if these variables contain m primary dimensions, then the variables can be grouped into (n-m) non-dimensional parameters".

The non-dimensional groups are called pi terms.

Mathematically $f(x_1, x_2, x_3, \ldots\ldots x_n) = 0$
where x's are physical quantities (such as velocity, density, viscosity, pressure etc) pertinent to a physical phenomenon, then the same ~~phenom~~ phenomenon can be described by (n-m) dimensionless pi terms.

$$\phi\,[\pi_1, \pi_2, \ldots \pi_{n-m}] = 0$$

Here m represents the fundamental dimensions such as mass, length and time. [or Force, length & time]

Experience shows that suitable non-dimensional groups results when a geometric property (e.g, length) a fluid property (e.g, density) and a flow characteristic (e.g velocity) are chosen to represent such variables.

Ex:- Show by use of Buckingham; Pi-theorem that vel through an orifice is given by

$$V = \sqrt{2gH}\, f\left(\frac{D}{H}, \frac{\mu}{\rho V H} ; \frac{\sigma}{\rho V^2 H}\right).$$

H = head, D = dia of orifice, μ = coeff of viscosity, ρ = density, σ = surface tension & g = accn due to gravity.

Soln:- It can be recognised that the functional relationship is

$$f(V, D, H, g, \rho, \mu, \sigma) = 0$$

There are 7 physical quantities and 3 fundamental units, hence $(7-3) = 4$ pi terms. Choosing density ρ, velocity V and head H as repeating variables with unknown exponents, the non-dimensional π terms can be as follows.

(i) $\pi_1 = \rho^{a_1} V^{b_1} H^{c_1} D$

$M^0 L^0 T^0 = [ML^{-3}]^{a_1} \cdot [LT^{-1}]^{b_1} [L]^{c_1} [L]$

Equating exponents we get $a_1 = 0$, $b_1 = 0$

$-3a_1 + b_1 + c_1 + 1 = 0$ $c_1 = -1$

$\therefore \pi_1 = H^{-1} \cdot D = \frac{D}{H}$

(ii) $\pi_2 = \rho^{a_2} \cdot V^{b_2} \cdot H^{c_2} g$

$M^0 L^0 T^0 = [ML^{-3}]^{a_2}, [LT^{-1}]^{b_2} [L]^{c_2} LT^{-2}$

$a_2 = 0$ $-3a_2 + b_2 + c_2 + 1 = 0$ $-b_2 - 2 = 0$

$\Rightarrow b_2 = -2$

$\Downarrow$

$c_2 = 1$

$\therefore \pi_2 = V^{-2} \cdot H \cdot g = gH/V^2$

(iii) $\pi_3 = \rho^{a_3} V^{b_3} H^{c_3} \mu$

$M^0 L^0 T^0 = [ML^{-3}]^{a_3} \cdot [LT^{-1}]^{b_3} \cdot [L]^{c_3} \cdot [ML^{-1}T^{-1}]$

$a_3 + 1 = 0$ $-3a_3 + b_3 + c_3 - 1 = 0$ $-b_3 - 1 = 0$

$a_3 = -1$ $b_3 = -1$ $c_3 = -1$

$\pi_3 = \rho^{-1} V^{-1} H^{-1} \mu = \frac{\mu}{\rho V H}$

iv) $\pi_4 = \rho^{a_4} V^{b_4} H^{c_4} \cdot \sigma$

$[M^0 \cdot L^0 \cdot T^0] = [ML^{-3}]^{a_4} \cdot [LT^{-1}]^{b_4} L^{c_4} [MT^{-2}]$

$a_4 + 1 = 0$. $-3a_4 + b_4 + c_4 = 0$ $-b_4 - 2 = 0$

$a_4 = -1$ $b_4 = -2$, $c_4 = -1$

$\pi_4 = \rho^{-1} V^{-2} H^{-1} \sigma = \frac{\sigma}{\rho V^2 H}$

∴ Functional relationship can then be written as

$$\phi\left[\frac{D}{H}, \frac{gH}{V^2}, \frac{\mu}{\rho V H}, \frac{\sigma}{\rho V^2 H}\right] = 0$$

or $$\frac{V^2}{gH} = \phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}, \frac{\sigma}{\rho V^2 H}\right]$$

$$V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}, \frac{\sigma}{\rho V^2 H}\right]$$

Problem:- The resisting force F of a supersonic plane during flight can be considered as dependent upon the length of aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of elasticity K. Express the functional relationship between these variables and resisting force.

$f(F, l, V, \mu, \rho, K) = 0$ (primary → ρ, V, l)

Ans:- $$F = l^2 V^2 \rho\ \phi\left(\frac{\mu}{V\rho l}, \frac{K}{\rho V^2}\right)$$

Prob:
The resistance R experienced by a partially submerged body depends upon the velocity v, length of the body l, viscosity of the fluid μ, density of fluid ρ and g. Obtain the dimensionless expression for R.

$$f(R, v, l, \mu, \rho, g) = 0$$

Ans:- $$R = l^2 V^2 \rho\ \phi\left[\frac{\rho V l}{\mu}, \frac{V}{\sqrt{lg}}\right]$$

Prob:- Derive on the basis of dimensional analysis suitable parameters to present the thrust developed by a propeller. Assume that the thrust T depend on the angular vel ω, speed of advance V, diameter D, dynamic viscosity μ, mass density ρ and elasticity of fluid medium which can be represented by the speed of sound c in the medium.

$$f(T, \omega, V, D, \mu, \rho, c) = 0$$ fundamental → $V D \rho$

$$T = V^2 D^2 \rho\ \phi\left[\frac{\mu}{VD\rho}, \frac{D\omega}{V}, \frac{c}{V}\right]$$

Prob: Show that Power P developed by a water turbine can be expressed as

$$P = \rho N^3 D^5 \phi\left[\frac{D}{B}, \frac{\rho D^2 N}{\mu}, \frac{ND}{\sqrt{gH}}\right]$$

D = Dia of runner   N = rpm   H = operating head
B = width of " μ = dynamic viscosity
ρ = density.

Soln: $f(P, \rho, N, D, B, \mu, H, g) = 0$ Fundamental ρ, D, N.
5 π term.

The pressure drop ΔP in a pipe of diameter D & length 'l' depends on the density 'ρ', viscosity μ, mean vel. 'V' and average height of protuberance t. Show that pressure drop can be expressed as $\Delta P = \rho V^2 f\left[\frac{l}{D}, \frac{\mu}{VD\rho}, \frac{t}{D}\right]$

The efficiency η of a fan depends on the density ρ, dynamic viscosity μ of the fluid, angular vel. 'ω', dia D of the rotor and discharge Q. Express η in terms of dimensionless parameters.

* The pi terms can only determine the pertinent dimensionless groups describing the problem but not the exact functional relationship between them.

* Any π term can be replaced by ~~raising~~ its power. or can be multiplied by a numerical constant. (then can be other π).

P.T.O

Modeling & Similitude.

Ex: A 5 cm diameter sphere is tested in water at 20°C and vely of 3.5 m/s and has a measured drag of 6 N. Make calculations for vel and drag force of a 2 m dia weather ballon moving in air at 20°C and 1 atm under similar conditions:

Given: Viscosities air 1.86×10^{-5} Pa-s water 1.01×10^{-3} Pa-s ;

Densities = air 1.20 kg/m^3 water 1000 kg/m^3

(b) A one-sixth scale model automobile is tested in a wind tunnel in the same air properties as the prototype. The prototype automobile runs on the road at a vel. of ~~50 m/s~~ 50 km/hr. For dynamically similar conditions, the drag measured on the model is 300 N. Make calculations for the drag of the prototype and the power reqd. to overcome the drag.

Soln:

(a) Re is predominant for fully immersed body

$$\therefore \left(\frac{Vl\rho}{\mu}\right)_m = \left(\frac{Vl\rho}{\mu}\right)_p$$

$$\Rightarrow V_p = V_m \cdot \frac{\rho_m}{\rho_p} \cdot \frac{l_m}{l_p} \cdot \frac{\mu_p}{\mu_m} = 3.5\times\frac{1000}{1.2}\times\frac{.05}{2}\times\frac{1.86\times10^{-5}}{1.01\times10^{-3}}$$

$$= 1.346 \text{ m/sec.}$$

Further, for dynamic similarity the parameter $R/\rho V^2 l^2$ should be same for model & prototype.

$$\therefore \left[\frac{R}{\rho V^2 l^2}\right]_p = \left[\frac{R}{\rho V^2 l^2}\right]_m$$

$$\therefore R_p = R_m \cdot \left(\frac{\rho_p}{\rho_m}\right)\times\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 6\times\frac{1.20}{1000}\times\left(\frac{1.346}{3.5}\right)^2\times\left(\frac{2}{0.05}\right)^2$$

$$= 1.704 \text{ N.}$$

(b) Speed of prototype automobile 50 km/hr.

$$= 50/3.6 = 13.89 \text{ m/s.}$$

The resistance R to the motion of a completely submerged body is given by $R = \rho V^2 l^2 \phi\left(\frac{\rho V l}{\mu}\right)$

For similarity conditions the non-dimensional terms $R/\rho V^2 L^2$ and $\rho V L/\mu$ should be same for the prototype and the model.

$\therefore$ $V_m = V_p \times \frac{L_p}{L_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p}$ [μ & ρ is same]

$\therefore$ $= 13.89 \times 6 = 83.34$ m/s.

(ii) $R_p = R_m \cdot (\rho_p/\rho_m) \times (V_p/V_m)^2 \times (L_p/L_m)^2$

$= 300 \times (1) \times (13.89/83.34)^2 \times 6^2 = 300$ N.