Material Accel^n.

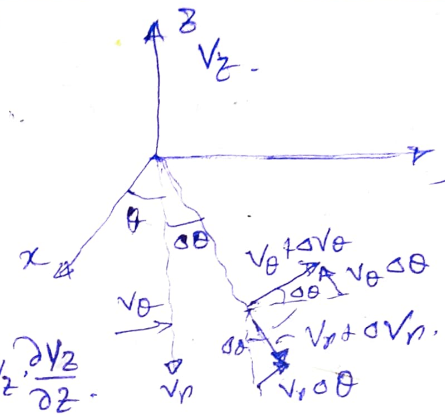
Type of flow	~~Steady~~ Temporal	Convective
Steady & uniform	0	0
steady & non-uniform	0	exists
Unsteady & uniform	exists	0
" & non-uniform	"	exists

$$a_r = \frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_\theta = \frac{DV_\theta}{Dt} + \frac{V_r V_\theta}{r} = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

$$a_z = \frac{DV_z}{Dt} = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$$

$$\vec{V}_z = V(r, \theta, z, t)$$

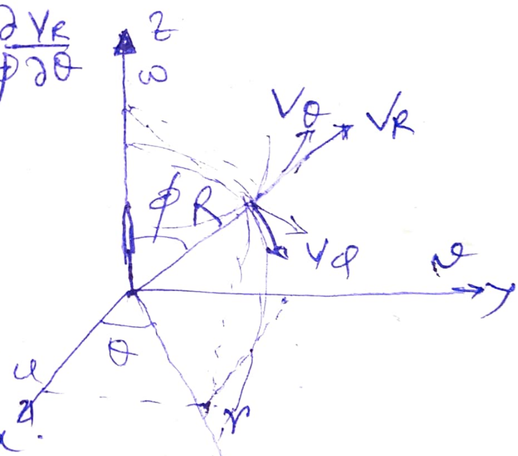


Cylindrical Polar Co-ordinate

Spherical Polar Co-ordinate.

$$a_R = \frac{\partial V_R}{\partial t} + V_R\frac{\partial V_R}{\partial R} + \frac{V_\phi}{R}\frac{\partial V_R}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\frac{\partial V_R}{\partial \theta} - \frac{V_\phi^2 + V_\theta^2}{R}$$

$$a_\phi = \frac{\partial V_\phi}{\partial t} + V_R\frac{\partial V_\phi}{\partial R} + \frac{V_\phi}{R}\frac{\partial V_\phi}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\frac{\partial V_\phi}{\partial \theta} - \frac{V_R V_\phi}{R} - \frac{V_\theta^2 \cot\phi}{R}$$



$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_R\frac{\partial V_\theta}{\partial R} + \frac{V_\phi}{R}\frac{\partial V_\theta}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta V_R}{R} + \frac{V_\theta V_\phi \cot\phi}{R}$$

Material or Substantial accⁿ = (Temporal or local accⁿ + convective accⁿ)

local accⁿ: the rate of change with time at a fixed position.

convective accⁿ: time rate of change due to change in position in the field.

Problem :- An idealized flow field $\vec{V} = 2x^3 i - 3x^2 y j$

Steady flow? $\boxed{\text{yes}}$

3D or 2D $\boxed{\text{2D}}$

Vel. & Accⁿ at P(2, 1, 3).

$u = 2x^3$, $v = -3x^2y$

at P(2,1,3) $u = 16$, $v = -12$.

$\therefore \vec{V} = 16i - 12j$ $\quad V = \sqrt{16^2 + (-12)^2} = 20$.

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y}$$
$$= 0 + 2x^3 \cdot 6x^2 + 0 = 12x^5$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y}$$
$$= 0 + 2x^3 \cdot (-6xy) + (-3x^2y) \cdot (-3x^2)$$
$$= -12x^4y + 9x^4y = -3x^4y.$$

$\therefore \bar{a} = 12x^5 i - 3x^4 y j$

at P(2,1,3) $\quad \bar{a} = 384 i - 48 j$

$\therefore a = \sqrt{384^2 + (-48)^2} = 387 \text{ m/s}^2$

Prob:- For a given velocity field $\vec{V} = 10xy i + 5x^2 j + (t^2 x + z) k$

Find the velocity and accⁿ of a fluid particle at position $\vec{r} = (i + 2j + 3k)$ when time $t = 1$.

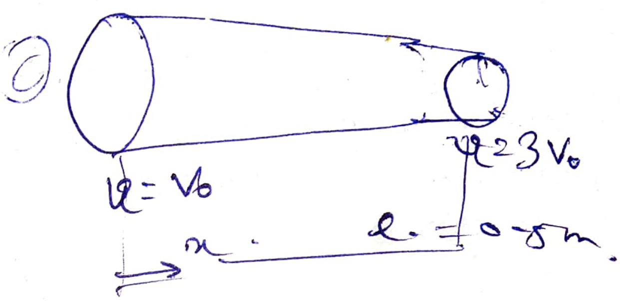
Ans:- $\boxed{\begin{aligned} V &= 21 \text{ unit} \\ a_x &= 450 \text{ unit} \\ a_y &= 200 \text{ unit} \\ a_z &= 26 \text{ units} \end{aligned}}$

Problem

1 D steady flow

$u = u(x)$

what is accn at a
the function of x



$$u(x) = V_0 + \frac{3V_0 - V_0}{l} \cdot x$$

$$= V_0\left(1 + \frac{2x}{l}\right)$$

For 1-D flow through the nozzle,

$$\vec{a} = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0 + V_0\left(1 + \frac{2x}{l}\right) \cdot V_0 \cdot \frac{2}{l}$$

$$= \frac{2}{l} \cdot V_0^2 \left(1 + \frac{2x}{l}\right)$$

at $x = 0$ $\quad a = \frac{2 \cdot 5^2}{.5} = 100\ m/s^2$ $\quad V_0 = 5\ m/s$, $l = 0.5\ m$

at $x = 0.5\ m$ $\quad a = 300\ m/s^2$

Problem For a given velocity field $\vec{V} = 10x^2y\,\hat{i} + 15xy\,\hat{j} + (25t - 3xy)\,\hat{k}$

Find accn of a fluid particle at a pt $(1, 2, -1)$ at time $t = 0.5$

(Ans:- 1531.90 units).

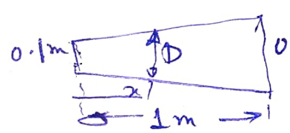
Problem

Problem] For unsteady temp field $T=(xy+z+3t)K$ and unsteady velocity field $V=(xy\,\hat{i}+z\,\hat{j}+5t\,\hat{k})$. Find the change in temp at (2, −2, 1) at time $t = 2s$. [23k/s]

Problem A two-dimensional press field $p=4x^3-2y^2$
Velocity field $\vec{V}=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$.
Determine the rate of change of pressure at pt (2, 1).
[Ans. 260 unit/sec]

Ex: (P4.22) A conical diffusing section diverges uniformly from 0.1 m dia to 0.2 m dia over length of 1 m. Find the local & convective accn at the middle of the diffuser. Consider the following two cases.
i) Flow rate is 100 l/s (constant). ii) flow rate varies uniformly from 100 l/s to 200 l/s in 5 seconds and the time of interest ~~when~~ is when $t=2s$. [Vel at any direction, ~~for~~ perpendicular to the flow direction, may be assumed to be uniform].

Soln:- Dia $D = .1+\frac{.2-.1}{1}x = .1(1+x)$



c/s area $= \frac{\pi}{4}\{0.1(1+x)\}^2$

Vel $u = \frac{Q}{A} = \frac{Q}{0.00785(1+x)^2}$; $\frac{\partial u}{\partial x} = -\frac{2Q}{0.00785(1+x)^3}$

steady flow. $a = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$ $\qquad \frac{\partial u}{\partial t} = 0$ for steady flow.

$$u\cdot\frac{\partial u}{\partial x} = \frac{Q}{0.00785(1+x)^2} \times \frac{-2Q}{0.00785(1+x)^3} = -\frac{2Q^2}{(0.00785)^2(1+x)^5}$$

at middle of diffuser
convective accn $= \frac{-2\times(0.1)^2}{0.00785\times(1+.5)^5} = -42.76\ m/s^2$

(b) 0.1 m^3/s to 0.2 m^3/s at over 5 sec.
at 2 sec $\quad Q = 0.1+\frac{.2-.1}{5} = .14\ m^3/s$.

$u\cdot\frac{\partial u}{\partial x} = -83.81\ m/s^2$

local accn $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{Q}{0.00785(1+x)^2}\right) = \frac{1}{0.00785(1+x)^2}\cdot\frac{\partial Q}{\partial t}$

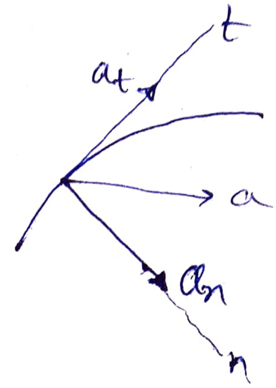
$$= \frac{1}{.00785(1+.5)^2}\times\left(\frac{.14-.1}{2}\right) = 1.132\ m/s^2$$

$a = -83.81 + 1.132 = -82.678\ m/s^2$

Motion of fluid particle along a curved path :- Normal & tangential acc^n.

Let V_s and V_n be $V_s = f_1(s,n,t)$

$V_n = f_2(s,n,t)$.



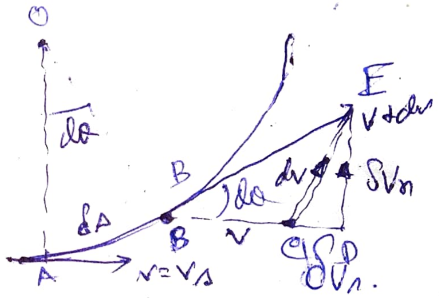
$$a_s = \frac{DV_s}{Dt} = \frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial s}\cdot\frac{ds}{dt} + \frac{\partial V_s}{\partial n}\cdot\frac{dn}{dt}$$

But. $\frac{ds}{dt} = V_s$ $\frac{dn}{dt} = V_n$

$$\therefore a_s = \frac{\partial V_s}{\partial t} + V_s.\frac{\partial V_s}{\partial s} + V_n.\frac{\partial V_s}{\partial n}$$

$$a_n = \frac{\partial V_n}{\partial t} + V_s.\frac{\partial V_n}{\partial s} + V_n.\frac{\partial V_n}{\partial n}$$

For streamline there is no flow across it and $V_n = 0$ $\left[\text{But } \frac{\partial V_n}{\partial s} \neq 0\right]$



$$\delta V_s = (V+dV)\cos d\theta - V$$

$$\delta V_n = (V+dV)\sin d\theta$$

for small $d\theta$ $\sin d\theta \simeq d\theta$

$\cos d\theta \simeq 1$

$$\delta V_s = dV \qquad \delta V_n = (V+dV)\,d\theta \simeq V d\theta$$

From triangles OAB and BDE

$$d\theta = \frac{\delta s}{r} = \frac{\delta V_n}{V} \text{ or } \frac{\delta V_n}{\delta s} = \frac{V}{r} = \frac{V_s}{r} \quad \therefore \frac{\partial V_n}{\partial s} = \frac{V_s}{r}$$

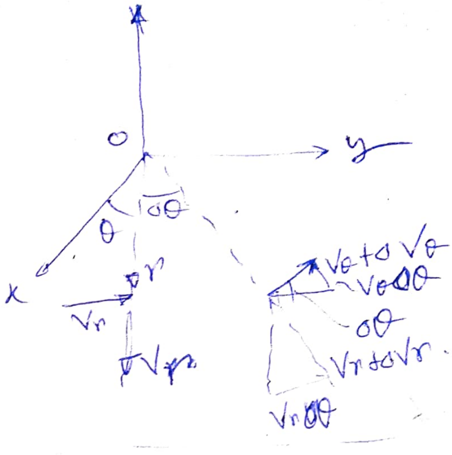
$$a_s = \frac{\partial V_s}{\partial t} + V_s.\frac{\partial V_s}{\partial s} \quad \Big/ \quad a_n = \frac{\partial V_n}{\partial t} + \frac{V_s^2}{r}$$

$\frac{\partial V_s}{\partial t}$ & $\frac{\partial V_n}{\partial t}$ are local tangential accelaration and local normal acc^n.

$V_s.\frac{\partial V_s}{\partial s}$ and $V_s.\frac{\partial V_n}{\partial s} = \frac{V_s^2}{r}$ represents convective tangential acc^n and convective normal acc^n. [$\because$ Rate of change of V_s & V_n due to change in position]

Components of acc^n in other coordinate system.

In a cylindrical polar co-ordinate system



$$a_r = \frac{DV_r}{Dt} - \frac{V_\theta^2}{r} = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\cdot\frac{\partial V_r}{\partial \theta} + V_z \cdot \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_\theta = \frac{DV_\theta}{Dt} + \frac{V_r V_\theta}{r} = \frac{\partial V_\theta}{\partial t} + V_r \cdot \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\cdot\frac{\partial V_\theta}{\partial \theta} + V_z \cdot \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

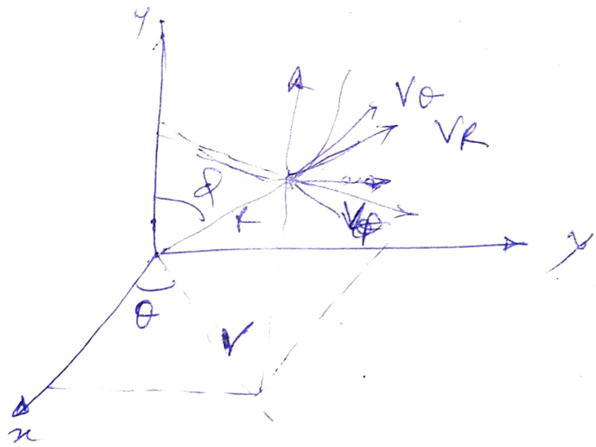
$$a_z = \frac{DV_z}{Dt} = \frac{\partial V_z}{\partial t} + V_r \cdot \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \cdot \frac{\partial V_z}{\partial z}$$

Spherical Polar Co-ordinate

$$a_R = \frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\phi}{R}\cdot\frac{\partial V_R}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\frac{\partial V_R}{\partial \theta} - \frac{V_\phi^2 + V_\theta^2}{R}$$

$$a_\phi = \frac{\partial V_\phi}{\partial t} + V_R \cdot \frac{\partial V_\phi}{\partial R} + \frac{V_\phi}{R}\cdot\frac{\partial V_\phi}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\cdot\frac{\partial V_\phi}{\partial \theta} - \frac{V_R V_\phi}{R} - \frac{V_\theta^2 \cot\phi}{R}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_R \cdot \frac{\partial V_\theta}{\partial R} + \frac{V_\phi}{R}\cdot\frac{\partial V_\theta}{\partial \phi} + \frac{V_\theta}{R\sin\phi}\cdot\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta \cdot V_R}{R} + \frac{V_\theta V_\phi \cot\phi}{R}$$



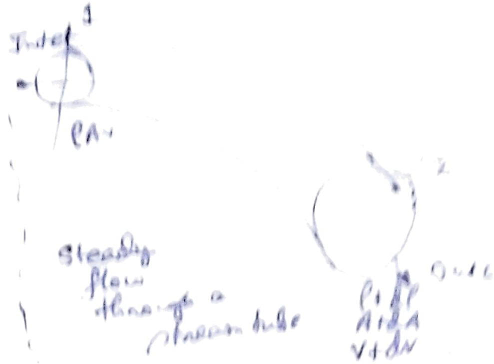
Flow rate and continuity eqn

Consider flow of an ideal fluid through a stream tube. No flow takes across streamline.

during time interval dt

mass entering $\rho A V dt$

mass outgoin $(\rho+d\rho)(A+dA)(V+dV)dt$



Steady flow through a stream tube

$dm = \rho A V dt - (A+dA)(\rho+d\rho)(V+dV)\,dt$

$\therefore \frac{dm}{dt} = -(AVd\rho + V\rho dA + A\rho dV)$

[neglecting $d\rho \cdot dA$, $dA\, dV$, $d\rho\, dV$, and $d\rho\, dA\, dV$]

for steady flow $\frac{dm}{dt} = 0$

$AVd\rho + V\rho dA + A\rho dV = 0$

$\therefore \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$ [dividing by ρAV]

or $d(\rho A V) = 0$ or $\rho A V$ = Constant

for incompressible fluid AV = const. (as ρ const)

$\therefore A_1V_1 = A_2V_2$ [continuity eqn for steady incompressible flow]

At any c/s flow in $A \times V$. $\therefore \frac{V_1}{V_2} = \frac{A_2}{A_1}$ or $V \propto \frac{1}{A}$

$\therefore$ discharge $= Q = AV$

mass flow rate $\dot{m} = \rho A V$.

[There may be several inlet and outlet]

$\sum_{inlet} \rho AV = \sum_{outlet} \rho AV$.

again unsteadily $\frac{dm}{dt} = + AVd\rho + V\rho dA + \rho A dV = 0$

$\therefore \frac{dm}{dt} + d(\rho AV) = 0$

$\frac{d}{dt}(\rho AV) +$

Problem: Water flows steadily through a pump and parabolic velocity distribution in the inlet circular pipe is prescribed by the relation

$V = 2.5\left(1 - \frac{r^2}{R^2}\right)$ m/s.

with R the radius of the inlet pipe and r is a radial distance from centre. dia = 20 cm calculate discharge and in dia = 30 cm.

$Q = \int V dA = \int 2.5\left(1-\frac{r^2}{R^2}\right) \cdot 2\pi r\, dr$

$= 5\pi \int_0^R r\left(1 - \frac{r^2}{R^2}\right) dr = \frac{5}{4}\pi R^2 = \frac{5}{4}\pi \times (0.2)^2 = .157\ m^3/s$

$$\text{Average vel} = \frac{\text{discharge}}{\text{sectional area}} = \frac{.157}{\frac{\pi}{4}(.3)^2} = 2.22 \text{ m/sec.}$$

Prob:- 2

Continuity Equation from a Closed System Approach.

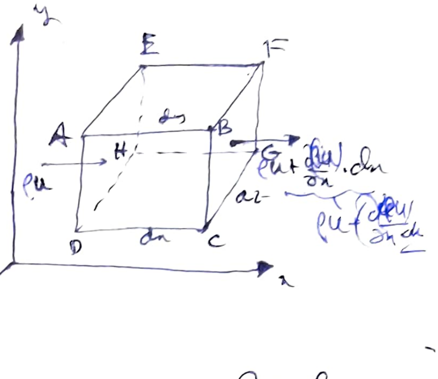
$$\frac{Dm}{Dt} = 0 \Rightarrow \frac{D}{Dt}(\rho \Delta \forall) = 0 \Rightarrow \Delta\forall \frac{D\rho}{Dt} + \rho \frac{D \Delta\forall}{Dt} = 0$$

$$\text{or} \quad \frac{D\rho}{Dt} + \rho \cdot \frac{1}{\Delta\forall} \cdot \frac{D\Delta\forall}{Dt} = 0$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \vec{V} = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0}$$

$\frac{1}{\Delta\forall} \cdot \frac{D\Delta\forall}{Dt}$ = rate of volumetric dilatation per unit volume of the elemental system. $= \nabla \cdot \vec{V}$

Differential eq^n of continuity.



consider flow of a continuous parallelopiped of dimension dx, dy, dz. The fluid is continuous both in space (no void) and time (i.e. fluid mass is neither created nor destroyed).

Element ABCD EFGH is small.

As ADHE is small it can be assumed that u prevail over the entire surface. At face BCGF the vel is $u + \frac{\partial u}{\partial x} . dx$.

Mass flow entering through surface ADHE is

$= \rho u . dy . dz . dt$ and going out through BCGF

$= \left(\rho u + \frac{\partial (\rho u)}{\partial x} dx\right) dy dz . dt$

The gain in mass $= \rho u . dy dz dt - \left[\rho u + \frac{\partial (\rho u)}{\partial x} dx\right] dy dz dt$.

$= - \frac{\partial (\rho u)}{\partial x} dx . dy dz . dt$

Like wise gain in y & z direction

$- \frac{\partial (\rho v)}{\partial y} . dx . dy . dz dt$ & $- \frac{\partial (\rho w)}{\partial z} dx . dy . dz . dt$.

Net gain $= - \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx . dy dz . dt$.

The rate of increase in mass within the parallelopiped is $\frac{\partial m}{\partial t} . dt = \frac{\partial}{\partial t}(\rho \times \text{volume}) . dt = \frac{\partial \rho}{\partial t} . dx . dy . dz dt$.

$\therefore \frac{\partial \rho}{\partial t} dx . dy dz dt = - \boxed{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)} dx dy dz dt$.

$\therefore \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$ $\left[\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{V}) = 0\right]$;

or $\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \left(u . \frac{\partial \rho}{\partial x} + v . \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}\right) = 0$

$\left[\text{or.} \frac{\partial \rho}{\partial t} + \rho . \nabla . \vec{V} + (\vec{V} \nabla) \rho = 0\right]$

For incompressible fluid $\nabla . \rho = 0$ and for steady state $\frac{\partial \rho}{\partial t} = 0$

$\therefore$ The continuity eq^n for a steady incompressible fluid is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ $\left[\nabla . \vec{V} = 0\right]$

$\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz} = 0$

$\lim_{\Delta t \to 0} \frac{dx\, dy\, dz \left[(1 + \dot{\epsilon}_{xx} \delta t)(1 + \dot{\epsilon}_{yy} \delta t)(1 + \dot{\epsilon}_{zz} \delta t) - 1\right]}{dx\, dy\, dz . \delta t} = \dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}$

The rate of volumetric dilatation per unit vol.

Problem:- In a steady two-dimensional incompressible flow, the velocity component in the x-direction is $u = 3x^2 + y^2$. Use continuity eq^n. to find the vel. v in y-direction. [$v = 0$ at $y = 0$]. Also find the ~~streamline~~ direction of streamline with respect to x-axis at point $P(1, 2)$.

For a steady 2D incompressible flow. The continuity eq^n. is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

$u = 3x^2 + y^2$ $\quad \frac{\partial u}{\partial x} = 6x$

$\therefore \quad \frac{\partial v}{\partial y} = -6x$.

$v = -6xy + c$ [$v = 0$ at $y = 0 \Rightarrow c = 0$]

$v = -6xy$

Streamline has a slope $\tan\theta = \frac{dy}{dx} = \frac{v}{u} = \frac{-6xy}{3x^2+y^2}$.

$\tan\theta_{(1,2)} = \frac{-6 \times 1 \times 2}{3 \cdot 1^2 + 2^2} = -\frac{12}{7}$ $\quad \theta = \tan^{-1} -\frac{12}{7} = -59.74°$

$\therefore$ The streamline makes an angle of $-59.74°$ at pt $(1, 2)$.

Continuity eqⁿ in polar co-ordinate.

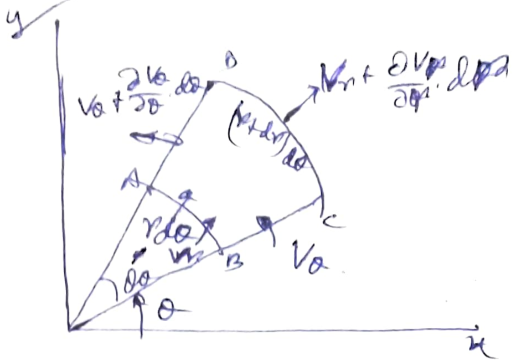
Vel V_r in radial direction and V_θ in tangential direction.

~~AB = CD~~ AD = BC = δr

$AB = r\,\delta\theta$, $CD = (r+\delta r)\,\delta\theta$

$AD = BC = \delta r$

[Thickness of element perpendicular to ~~unity~~ paper is unity].



Radial direction: [AB & CD]

Influx $\rho \cdot V_r \cdot r\,\delta\theta\,\delta t$ 　 outefflux $\rightarrow \left[\rho V_r + \frac{\partial}{\partial r}(\rho V_r)\,\delta r\right](r+\delta r)\,\delta\theta\,\delta t$

Net accumulation

$= \rho V_r \cdot r\,\delta\theta\,\delta t - \left[\rho V_r + \frac{\partial}{\partial r}(\rho V_r)\,\delta r\right](r+\delta r)\,\delta\theta\,\delta t$

$= -\left\{\left[\rho V_r\,\delta r\,\delta\theta + \frac{\partial}{\partial r}(\rho V_r)\,\delta r \cdot r\,\delta\theta\right]\delta t\right\}$ [term δr^2 is neglected]

Tangential direction:

$\left[\rho V_\theta \cdot \delta r - \left\{\rho V_\theta + \frac{\partial}{\partial \theta}(\rho V_\theta)\,\delta\theta\right\}\delta r\right]\delta t = -\frac{\partial}{\partial \theta}(\rho V_\theta)\,\delta r\,\delta\theta\,\delta t$.

Total gain

$= -\left[\rho V_r\,\delta r\,\delta\theta + \frac{\partial}{\partial r}(\rho V_r)\,\delta r \cdot r\,\delta\theta + \frac{\partial}{\partial \theta}(\rho V_\theta)\,\delta r\,\delta\theta\right]\delta t$

According to the law of mass conservation, the total gain in mass must ~~be~~ equal to the rate of change of fluid mass in the element ABCD.

$= \frac{\partial}{\partial t}\cdot(\rho \times V) \times \delta t = \frac{\partial}{\partial t}\left[\rho \cdot \frac{r\,\delta\theta + (r+\delta r)\,\delta\theta}{2} \cdot \delta r\right]\delta t$.

$= \frac{\partial}{\partial t}\,\rho\, r\,\delta\theta\,\delta r\,\delta t$

$\frac{\partial \rho}{\partial t}\, r\,\delta\theta\,\delta r\,\delta t + \left[\rho V_r + \frac{\partial}{\partial r}(\rho V_r)\, r + \frac{\partial}{\partial \theta}(\rho V_\theta)\right]\delta r\,\delta\theta\,\delta t = 0$

for steady stat $\frac{\partial \rho}{\partial t} = 0$

For steady flow.

$\left[\rho V_r + \frac{\partial}{\partial r}(\rho V_r)\, r + \frac{\partial}{\partial \theta}(\rho V_\theta)\right]\delta\theta\,\delta r = 0$

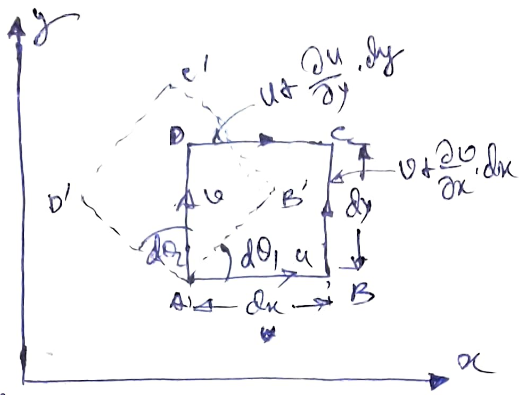
For incompressible flow

$V_r + \frac{\partial}{\partial r}(V_r)\, r + \frac{\partial}{\partial \theta}(V_\theta) = 0$.

or $\frac{\partial}{\partial r}(r \cdot V_r) + \frac{\partial}{\partial \theta}(V_\theta) = 0 \rightarrow$ Continuity eqⁿ in ~~flow~~ polar co-ordinate for 2D steady incompressible flow

Rotational Flow Rotation and Vorticity

Flow is rotational if every fluid element rotates about its axis which is perpendicular to its plane of motion.



Consider ABCD (a rectangular fluid element), on x-y plane.

vel. at A $\dot{x} = u$

at D $= u + \frac{\partial u}{\partial y} \cdot dy$ } x-direction

A $= v$

B $= v + \frac{\partial v}{\partial x} \cdot dx$ } y-direction.

These different velocities will result into angular velocities of linear element AB.

During dt AB & AD will move relative to pt A.

ω_{AB} = Angular displacement of element AB / unit time about z axis.

$$= \frac{\left(v + \frac{\partial v}{\partial x} \cdot dx - v\right) dt / dx}{dt} = \frac{\partial v}{\partial x}$$

$$\omega_{AD} = - \frac{\left[\left(u + \frac{\partial u}{\partial y} \cdot dy - u\right) dt / dy\right]}{dt} = - \frac{\partial u}{\partial y}$$

Average of angular velocities of line AB(~~dx/dx elem~~) and AD (dy) give rotation ω_z of element ABCD about z axis.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For a 3D flow $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

So $\omega = \frac{1}{2}\left[\omega_x i + \omega_y j + \omega_z k\right] = \frac{1}{2}\left(\nabla \times \vec{V}\right)$

The vector $\nabla \times \vec{V}$ is known as curl of the ~~vector~~ velocity vector.

Vorticity is taken as numerically equal to twice the value of rotation

Vorticity ξ = curl V $= (\nabla \times V)$.

$$\xi = |\nabla \times V| = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \xi_x i + \xi_y j + \xi_z k$$

$$\xi_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\xi_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad \xi_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

When this vorticity or rotation is zero at any position of the flow field, the motion there is described as <u>irrotational</u>.

<u>Prob:</u> if $u = Ayz$, $v = Azx$ & $w = Axy$ / then flow rotational ?? for incompressible fluid.

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(Ax - Ax) = 0$$

similarly $\omega_y = \omega_z = 0$ $\therefore$ the flow irrotational.

<u>Prob 2</u> 3D, incompressible steady flow.

$u = x^2 + z^2 + 5$ $\quad v = y^2 + z^2 - 3$

calculate w & check whether flow is irrotational.

<u>Soln:</u> from continuity eqn.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2y.$$

$$\Rightarrow \frac{\partial w}{\partial z} = -2(x+y).$$

$$w = -2(x+y)z + \underline{c} \Rightarrow 0 \text{ (assume)}.$$

$$\therefore w = -2(x+y)z.$$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+z^2+5 & y^2+z^2-3 & -2(x+y)z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\{-2(x+y)z\} - \frac{\partial}{\partial z}(y^2+z^2-3)\right] - j\left[\frac{\partial}{\partial x}\{-2(x+y)z\} - \frac{\partial}{\partial z}(x^2+z^2+5)\right]$$

$$\therefore -4zi + 4zj \neq 0$$

The flow is rotational.

Stream function (ψ)

Mathematical postulation such that differentiation w.r.t. y give velocity in x direction, and w.r.t. to x give velocity in y direction ($-ve$)

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

In cylindrical polar co-ordinate.

$$V_\theta = -\frac{\partial \psi}{\partial r} \quad V_r = \frac{1}{r}\cdot\frac{\partial \psi}{\partial \theta}$$

Eqn of streamline.

$$\frac{dx}{u} = \frac{dy}{v}$$

$\therefore$ $u\,dy - v\,dx = 0$

$$d\psi = \partial\psi_1 - \partial\psi_2 = 0$$

$$\partial\psi = u\,\partial y \quad u = \frac{\partial \psi}{\partial y}$$

features

$$d\psi = \frac{\partial \psi}{\partial x}\cdot dx + \frac{\partial \psi}{\partial y}\cdot dy = -v\,dx + u\,dy \rightarrow \text{for 2D motion}$$

parallel to x-y plane the streamline for an incompressible fluid is prescribe as $\frac{u}{v} = \frac{dy}{dx}$ $u\,dy - v\,dx = 0$

if $d\psi = 0$ then $\psi =$ Constant i.e. stream function is constant along streamline.

The streamlines are thus lines of constant stream function. However, varies from one streamline to another.

thus for each streamline $\psi_1 = C_1$; $\psi_2 = C_2$

* Vorticity $\xi_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$

$$= -\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational motion vorticity is zero.

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

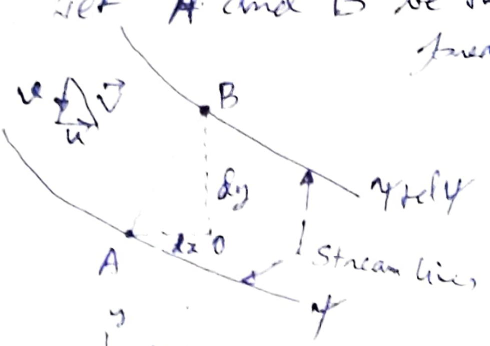
[This equation is know as Laplace equation]

* $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$

$$= \frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial x\,\partial y} = 0.$$

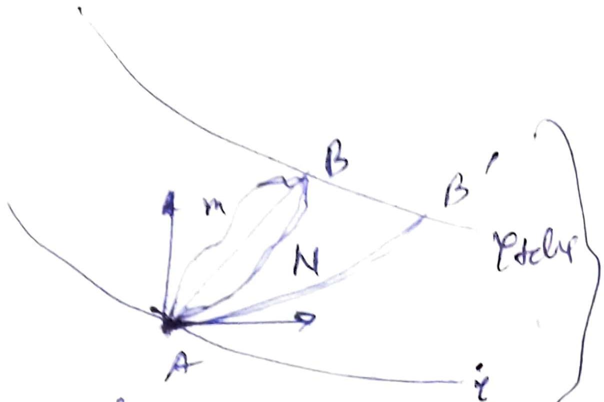
$\therefore$ Stream function satisfies the continuity equation.

* Let A and B be the two points lying of streamline ~~prescribed by the~~ the ~~stream~~ stream function ψ & $\psi + d\psi$



vel $= \vec{V}$

components u & v.



Discharge is same between two streamlines

$\vec{V}$ is perpendicular to AB.

Flow across AB = flow across AO + flow across OB

$$V \cdot ds = -v\,dx + u\,dy.$$

$\Downarrow$

as v is acting in downward direction.

$$\therefore V ds = \frac{\partial \psi}{\partial x} \cdot dx + \frac{\partial \psi}{\partial y} \cdot dy = d\psi.$$

$\therefore$ i.e $dq = d\psi$

$\Downarrow$

flow.

$\therefore$ Streamfunction can be defined as the flux or flow rate between two streamlines. The units of ψ are m^2/s. (discharge per unit thickness of flow).

* Superposition of stream function give another stream function. $\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x} = \frac{\partial}{\partial x}(\psi_1 + \psi_2) = \frac{\partial}{\partial x}(\psi).$

Potential Function (or Velocity Potential).

A function ~~that~~ (ϕ) that exists such that its derivative at any direction gives velocity in that direction.

$$u = \frac{\partial \phi}{\partial x} \quad \& \quad v = \frac{\partial \phi}{\partial y}$$

ϕ is called potential function or velocity potential. Lines of constant potential is known as equipotential lines.

$$d\phi = \frac{\partial \phi}{\partial x}.dx + \frac{\partial \phi}{\partial y}.dy. \quad = u\,dx + v\,dy$$

For equi potential lines $\quad d\phi = 0 \quad$ or $u\,dx + v\,dy = 0$

$$\therefore \frac{dy}{dx} = -\frac{u}{v}.$$

* Vorticity $\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0.$

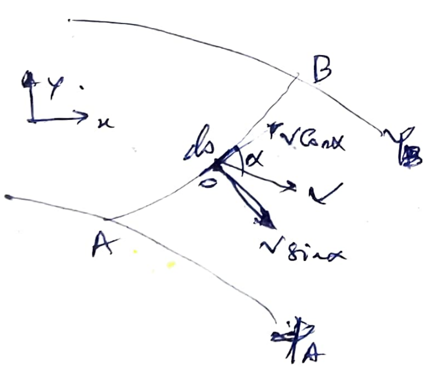
$\therefore$ ~~A potential flow is a~~

$\therefore$ Velocity potential satisfies the condition of irrotational flow. A 2D irrotational flow is known as potential flow.

* Continuity eq^n. $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\cdot\frac{\partial \phi}{\partial y} = 0$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad [\text{Laplace eq}^n]$$

$\therefore$ Velocity potential satisfies the Laplace equation.



Fluid crossing the element ds
$= V\sin\alpha\, ds.$

Volume flow rate across AOB
$= \int_{AOB} V\sin\alpha\, ds.$

if $\psi_A = 0$ (arbitrary)

then $\psi_B = \int_{AOB} V\sin\alpha\, ds.$ which is independent of path i.e. a point function

So it is possible to work with a scalar field ψ instead of velocity field $\vec{V}$.

ϕ can be defined as integral of tangential velocity component along a curve joining two points.

$$\therefore \phi_B = \int_{AOB} V\cos\alpha\, ds.$$

* As ϕ decreases along the flow some researchers prefers that $u = -\frac{\partial \phi}{\partial x}$ & $v = -\frac{\partial \phi}{\partial y}$

Conclusions:-

1. ψ applied to steady incompressible flow both rotational & irrotational.
2. ϕ exists only for irrotational flow.
3. For irrotational flow both ψ and ϕ satisfies Laplace equation, consequently they are interchangable.

Important relations: $u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$

This is known as Cauchy-Riemann equation.

Corresponding relation in cylindrical polar co-ordinates are $V_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\cdot\frac{\partial \phi}{\partial \theta}$; $V_r = \frac{1}{r}\cdot\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$

Orthogonality of Streamlines and Equipotential lines.

For a streamline $\left\{ \begin{aligned} d\psi &= \frac{\partial \psi}{\partial x}\cdot dx + \frac{\partial \psi}{\partial y}\cdot dy \\ &= -v\,dx + u\cdot dy \end{aligned} \right.$ | if $d\psi = 0$ then. $\frac{dy}{dx} = \frac{v}{u}$. . . . (i)

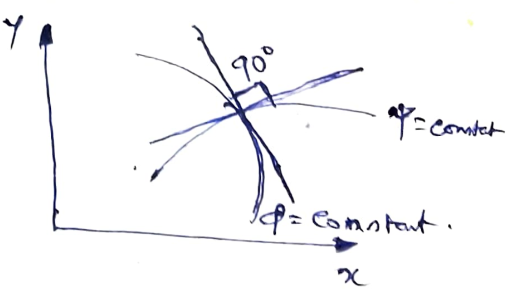
For equipotential line. $\frac{\partial y}{\partial x} = -\frac{u}{v}$. . . . (ii)

From (i) & (ii) $\frac{v}{u} \times \left(-\frac{u}{v}\right) = -1$

$\therefore$ slope of streamlines $\times$ slope of equipotential lines $= -1$

Equipotential lines are normal to the streamlines.

The orthoganality between streamlines helps to draw a flownet.



Problem: (1) A flow is described by stream function $\psi = 3\sqrt{2}\,xy$. Locate the point at which the vel. vector has a magnitude of 6 units and makes an angle of 145° with the x-axis.

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(1) $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(3\sqrt{2}\,xy) = 3\sqrt{2}\,x$. $v = -\frac{\partial \psi}{\partial x} = -3\sqrt{2}\,y$

$V = \sqrt{u^2+v^2}$; $6 = \sqrt{9(2)\,(x^2+y^2)}$

$\tan\theta = \frac{v}{u}$ $\tan 145 = \frac{-3\sqrt{2}\,y}{3\sqrt{2}\,x}$ or $-0.7 = -\frac{y}{x}$

$y = 0.7x$

$\therefore$ $6 = \sqrt{9\times 2\,[x^2+(0.7x)^2]} = 5.179x$

$x = 1.158$ and $y = 0.7x = 0.7\times 1.158 = 0.81$

Prob (2) Check whether the following functions satisfy continuity & valid potential functions.

(a) $\phi = \frac{A}{2}(x^2-y^2)$ (b) $\phi = A(\cos x + \sin y)$ (c) $\phi = A\log_e xy$

Soln: (a) For functions to satisfy both continuity & requirement for potential function. it must satisfy Laplace eqn.

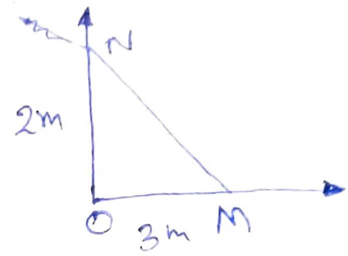
$\therefore$ $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

(a) $\phi = \frac{A}{2}(x^2-y^2)$ $\Rightarrow$ $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = A - A = 0$

(b) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \neq 0$ $\rightarrow$ ϕ is not a valid potential function.

Prob: $\psi = 4x^2y + (2+t)y^2$

Find the flow rate across the faces of triangular prism OMN having a thickness of 5 m in the z-direction at time instant $t = 2$ s.



Soln:- co-ordinates $O(0,0)$ $M(3,0)$ $N(0,2)$

$\psi_N = 0 + (2+2)\times 2^2 = 16$

$\psi_O = 0+0 = 0$ $\quad \psi_M = 0+0 = 0$

Flow rate across face $NO = 5(\psi_N - \psi_O) = 5(16-0) = 80\ m^3/s$

Flow rate across face $MO = 5(\psi_M - \psi_O) = 5(0-0) = 0$

" " " " " $MN = 5(\psi_N - \psi_M) = 5(16-0) = 80\ m^3/s$.

Prob: is $\psi = 2xy$ irrotational ? ; what is ϕ ?

Ans:- $u = \frac{\partial \psi}{\partial y} = 2x$ $\quad v = -\frac{\partial \psi}{\partial x} = -2y$

$\therefore$ velocity vector $\vec{V} = 2x\hat{i} - 2y\hat{j}$

$$(\nabla \times V) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial x}(-2y)\right] - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2x)\right] + k\left[\frac{\partial}{\partial x}(-2y) - \frac{\partial}{\partial y}(2x)\right] = 0$$

curl V is zero. Flow is irrotational and velocity potential does not exists.

$\therefore$ Laplace eqn. $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(2x) = 0+0 = 0$

(b) $d\phi = \frac{\partial \phi}{\partial x}\cdot dx + \frac{\partial \phi}{\partial y}\cdot dy = u\,dx + v\,dy$

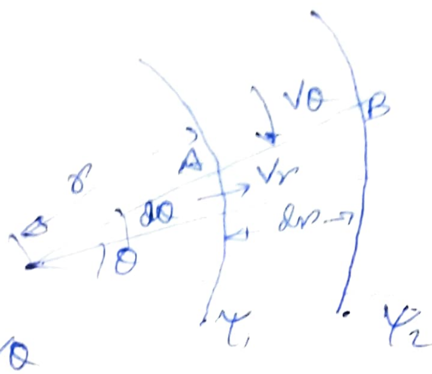
$= 2x\,dx - 2y\,dy.$

Integration gives $\phi = x^2 - y^2 + \text{constant}.$

Problem

Obtain the relationship between stream function ψ and the velocity components V_r & V_θ in cylindrical and polar co-ordinates as $V_r = \frac{1}{r} \cdot \frac{\partial \psi}{\partial \theta}$ and $V_\theta = -\frac{\partial \psi}{\partial r}$

Soln:- Consider two curved streamline ψ_1 & ψ_2
$AB = dr$, when $d\theta \to 0$



then $d\psi$ = flow between the streamlines

$= V_r(r\,d\theta) - V_\theta(dr)$ - (i)

[Sign convention :- tangential vel. V_θ is +ve in the direction of θ]

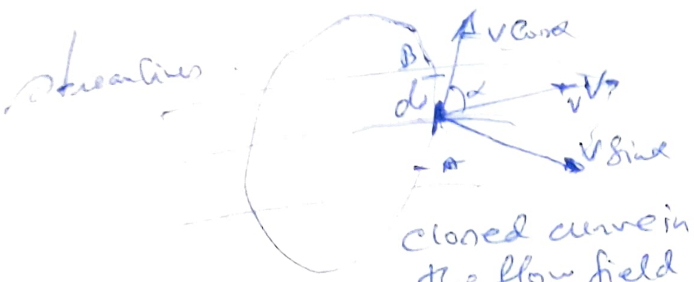
as $\psi = f(r, \theta)$

$$d\psi = \frac{\partial \psi}{\partial r} \cdot dr + \frac{\partial \psi}{\partial \theta} \cdot d\theta \quad \text{- - (ii)}$$

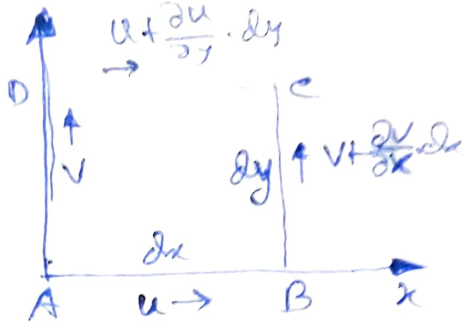
Applying (i) & (ii) $\quad V_\theta = -\frac{\partial \psi}{\partial r}$ and $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$.

$$V_r = \frac{\partial \phi}{\partial r} \qquad V_\theta = \frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta}$$

CIRCULATION



closed curve in the flow field.



Circulation around a rectangular element

Circulation Γ is the line integral of tangential velocity around a closed contour in the flow field.

Points A & B lying on the contour at a distance ds apart. Further for this element let vel. vector V inclined at an angle α with tangent to the contour. The tangential component $V\cos\alpha$; line integral for the element ds is $V\cos\alpha\, ds$.

$$\therefore \text{Circulation } \Gamma = \int V\cos\alpha \cdot ds \quad [\text{By definition}]$$

circulation +ve in anticlockwise direction.

For the rectangle

$$d\Gamma = u\,dx + \left(v+\frac{\partial v}{\partial x}dx\right)dy - \left(u+\frac{\partial u}{\partial y}dy\right)dx - v\,dy$$

$$= \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy.$$

$$\therefore \frac{d\Gamma}{dx\,dy} = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = \text{vorticity}$$

$\therefore$ vorticity = circulation / unit area.

For irrotational motion vorticity is zero. $\therefore$ Circulation is also zero.