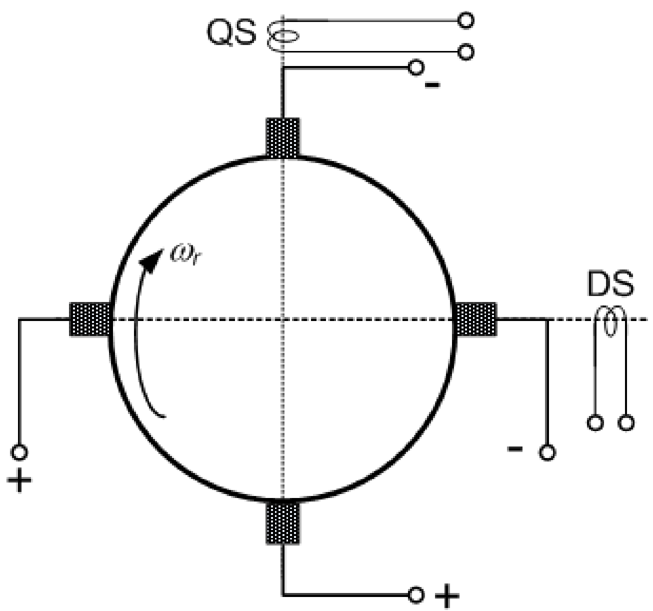
Transformation from Rotating to Stationary Axes

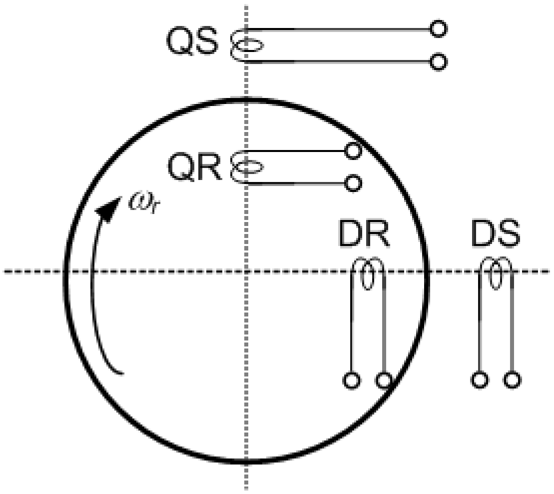
Day 9

Why Transformation?

- **Kron's Primitive (basic) 2-pole machine**
  - Two static coils in stator
  - Two pseudo-stationary coils in rotor

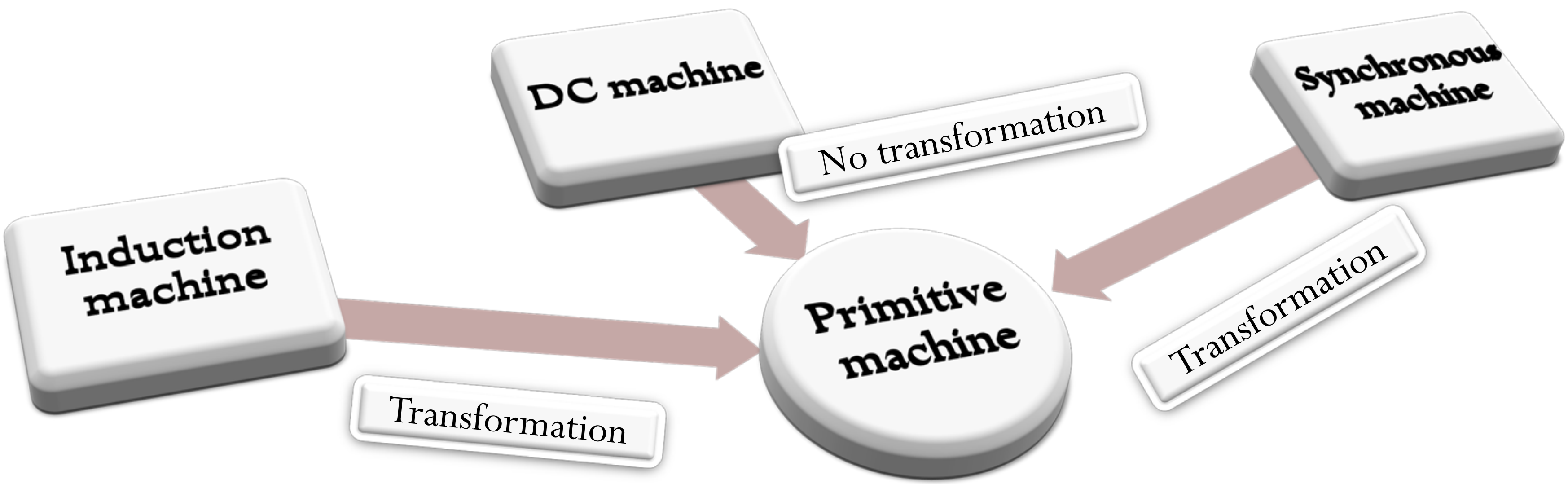
- We want to represent all rotating machines by this primitive machine structure for ease & uniformity of analysis





What is Transformation?

- The process of replacing one set of variables by another set of variables for equivalent representation electrical machines is called ***winding transformation*** or simply ***transformation***.
  - Since DC machines resemble the primitive machine structure directly, no transformation is necessary
  - Polyphase machines, however, need such transformations so that they can be fit into the primitive machine model and analyzed



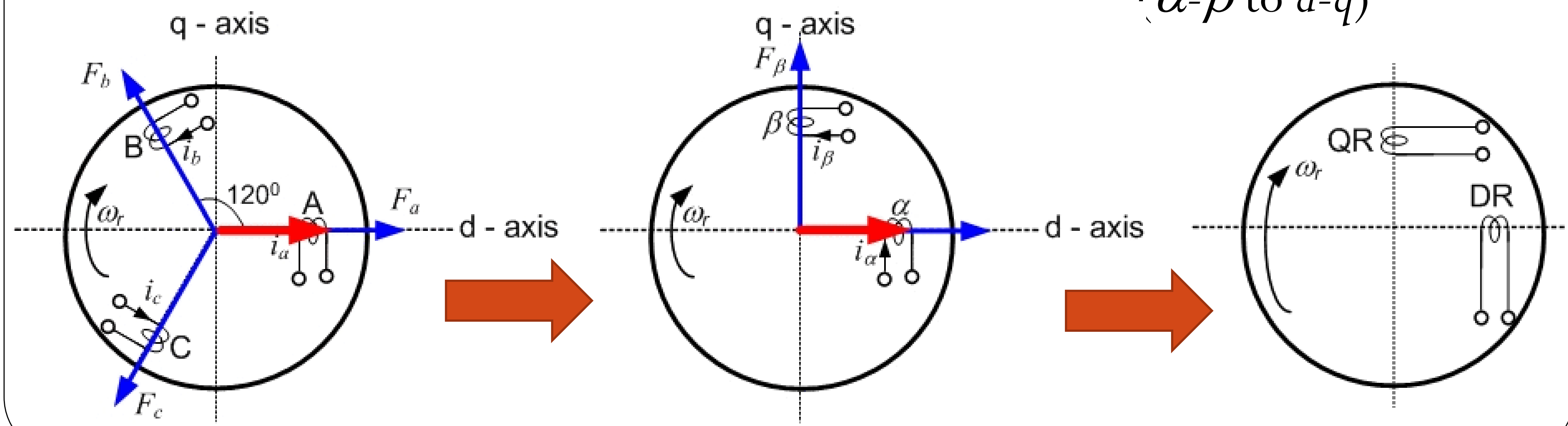
Different Transformations in machines

Transformation from a rotating 3-phase to stationary 2-phase

(a, b, c to d, q)

Transformation from rotating 3-phase to rotating 2-phase (a-b-c to α-β)

Transformation from rotating 2-phase to stationary 2-phase (α-β to d-q)



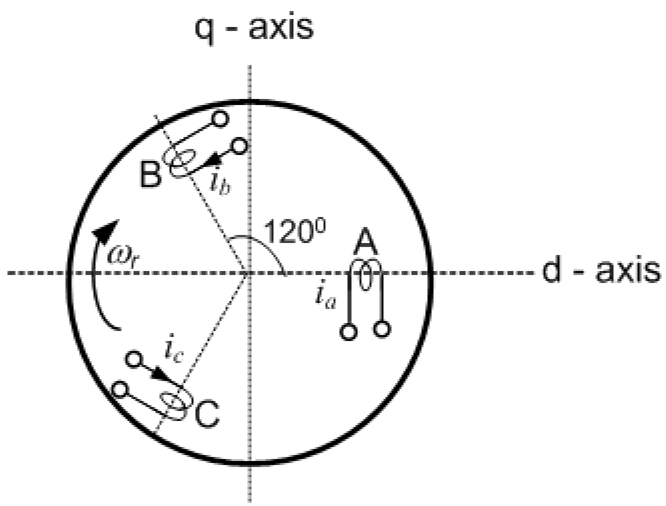
ILOs – Day9

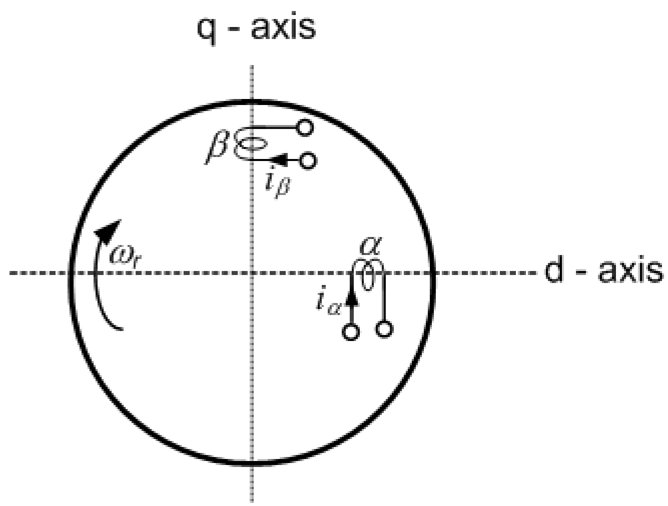
- Transform from 2-phase rotating axes to 2-phase stationary axes
  - Derive the transformation matrix (**α, β, 0 to d, q, 0**)
  - Derive the inverse transformation matrix (**d, q, 0 to α, β, 0**)

- Transform from 3-phase rotating axes to 2-phase stationary axes
  - Derive the transformation matrix (**a, b, c to d, q, 0**)
  - Derive the inverse transformation matrix (**d, q, 0 to a, b, c**)

(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - The axis of phase 'α' in 2-phase machine is assumed to coincide with the axis of phase '*A*' in 3-phase machine
  - Since the two rotors rotate at same speed and in the same direction, the two axes in question continue to coincide as the rotors rotate
  - Thus, the 2-phase axes (α,β)and 3-phase axes (*a, b, c*) are always at rest w.r.t. each other
  - Coefficients of the transformation matrix are thus constant quantities



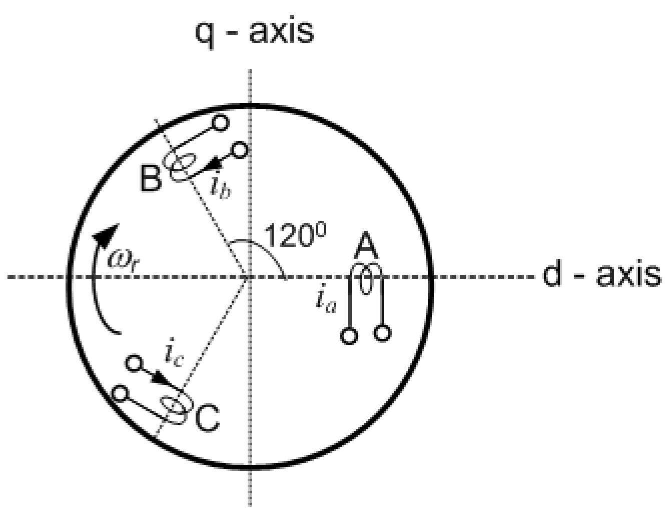


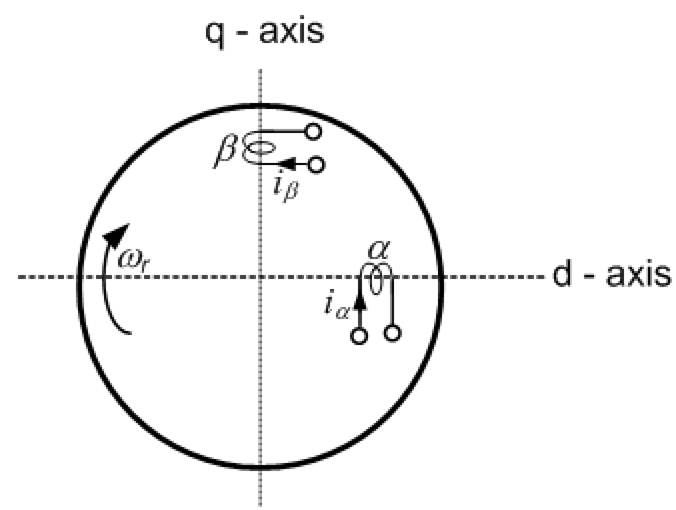
(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

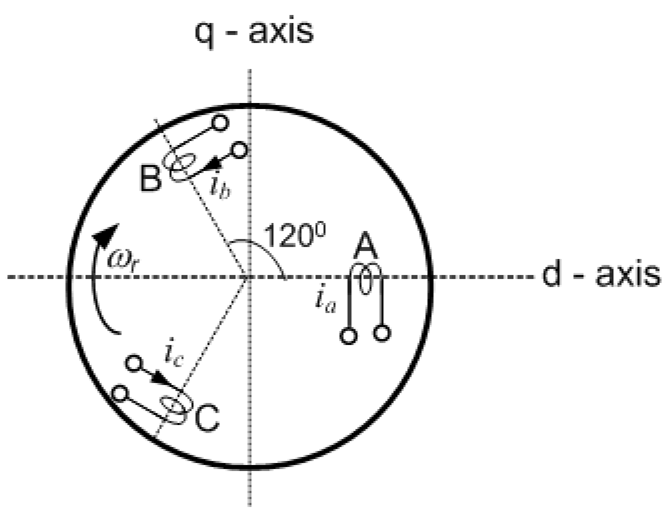
  - Thus, the 2-phase axes (α, β)and 3-phase axes (a, b, c) are always at rest w.r.t. each other
  - Coefficients of the transformation matrix are thus constant quantities

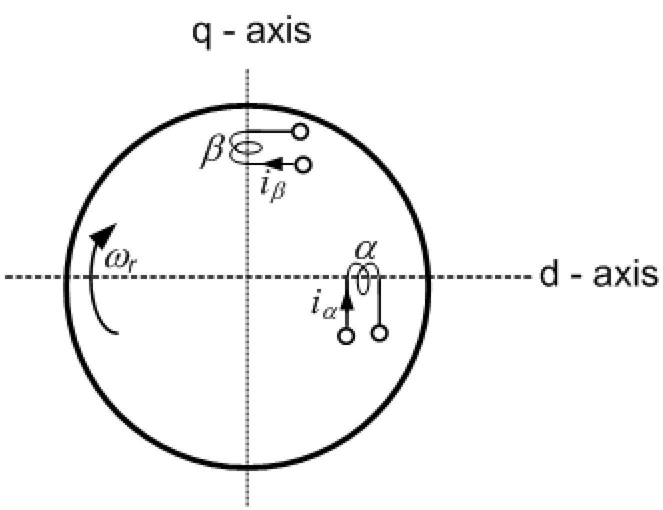




$(\alpha,\beta,0)$ to $(d, q, 0)$ transformation

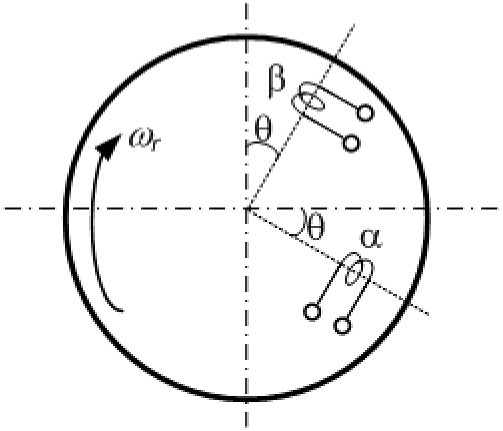
- **Rotating to stationary axes transformation**
  - The 2-phase axes (α, β) and 3-phase axes (a, b, c) are both rotating axes
  - When transformation is to be done from such a rotating axis to a stationary (fixed) axis:
    - The relative position of the rotating axis varies with respect to the stationary (fixed) axis as the rotor rotates
    - The transformation matrices in such a case will contain coefficients that are function of relative positions between the rotating **(α, β)** and fixed **(d, q)** axes



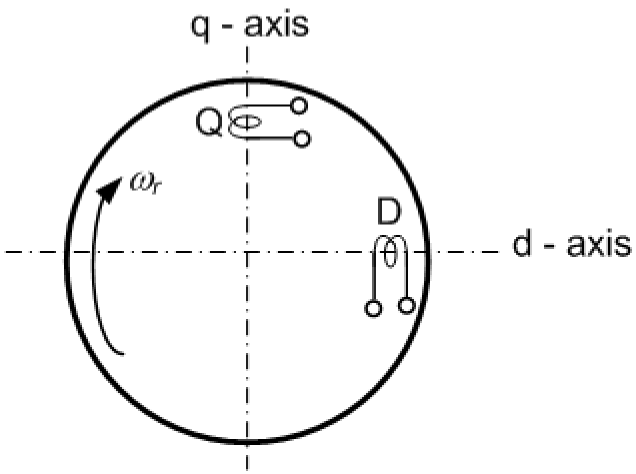


(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - The transformation process
    - The rotating coils are to be replaced by ***pseudo-stationary*** coils
    - Zero sequence quantities are not transformed, thus the required transformation is only from α-β to ***d-q*** axes
    - Revolving axes on the rotor implies that they are rotating w.r.t. the stator
    - Stationary axes on the rotor implies that they are fixed w.r.t. the stator (i.e. pseudo-stationary coil in rotor)



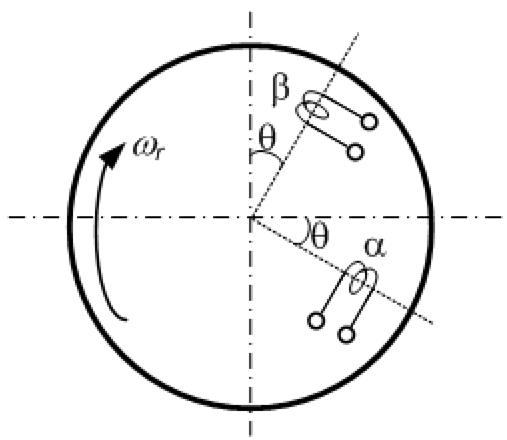
Rotating α-β axes



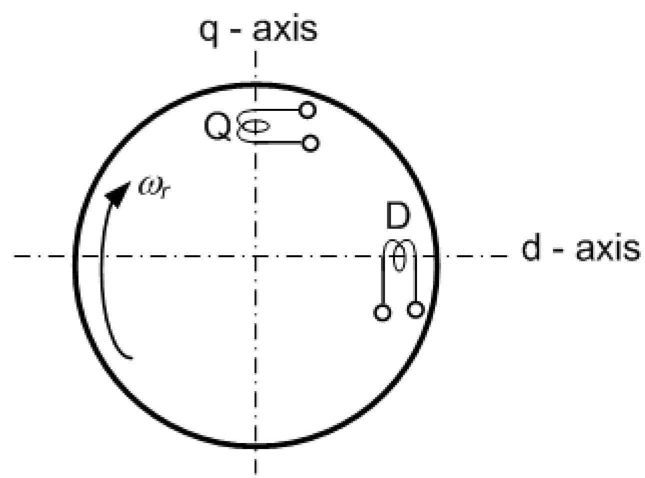
Stationary d-q axes

(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - The transformation process
    - Coils α and β of the rotating winding are shown to make an angle θ with the stationary ***d-q*** axes windings
    - The angle θ is such that at time t=0, θ=0, i.e. the rotating axes α-β coincide with the stationary axes ***d-q*** at t=0
    - At any time t, the angular displacement is $\theta=\omega_r t$, where ω_r is the angular velocity



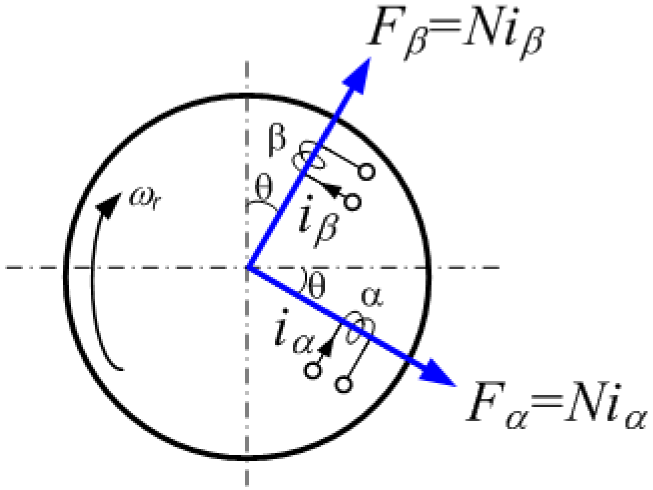
Rotating α-β axes



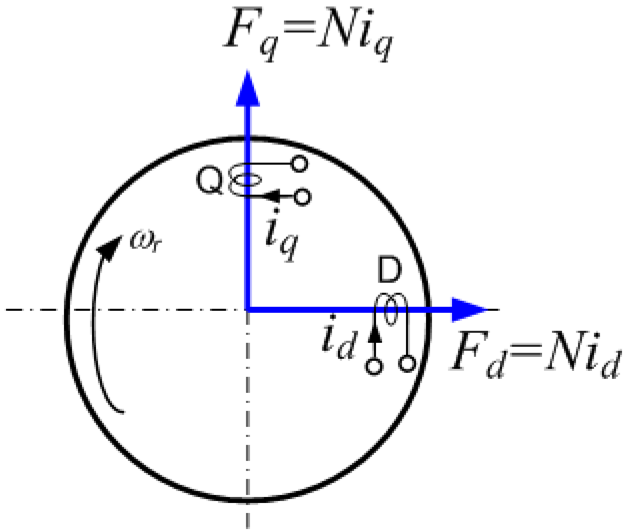
Stationary d-q axes

(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - The transformation process
    - At this instant '*t*', the MMF space phasors F_{α}, F_{β} of the ***α-β*** winding and F_{d}, F_{q} of the ***d-q*** winding are shown
    - Assume that all the windings have same number of turns "*N*'



Rotating α-β axes



Stationary *d-q* axes

(α,β,0) to (d, q, 0) transformation

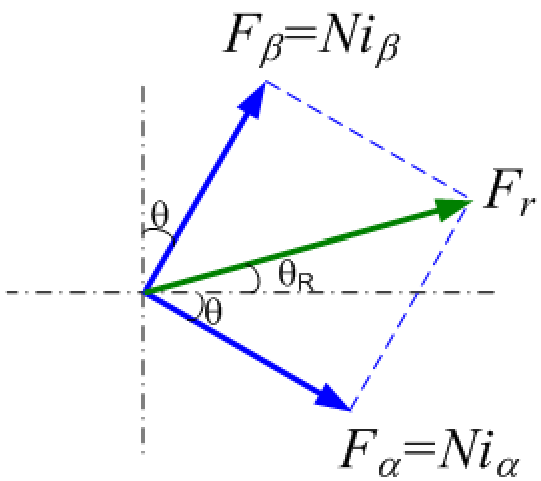
- **Rotating to stationary axes transformation**
  - The transformation process
    - The two MMFs and can be resolved along *d*- and *q*- axes:

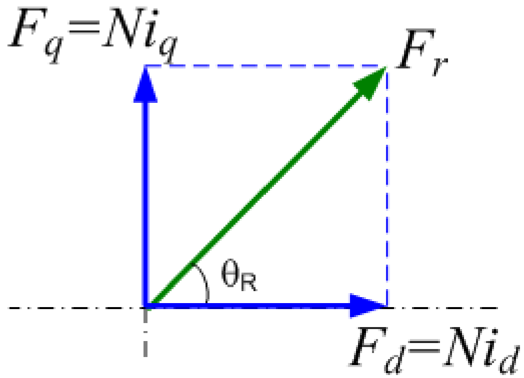
$$F_d = F_\alpha \cos\theta + F_\beta \sin\theta$$

or, $$Ni_d = Ni_\alpha \cos\theta + Ni_\beta \sin\theta$$

Thus, $$i_d = i_\alpha \cos\theta + i_\beta \sin\theta$$

Similarly, $$i_q = -i_\alpha \sin\theta + i_\beta \cos\theta$$





(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**

$$i_d = i_\alpha \cos\theta + i_\beta \sin\theta$$
$$i_q = -i_\alpha \sin\theta + i_\beta \cos\theta$$

  - In matrix form:

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{matrix} d \\ q \end{matrix} \overset{\begin{matrix}\alpha & \quad\quad \beta\end{matrix}}{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}$$

Let us define, $C = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\therefore \quad C_t = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore |C| = \cos^2\theta + \sin^2\theta = 1$$

Hence, $C^{-1} = \dfrac{C_t}{|C|} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Thus, the inverse transformation is: $\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{matrix} \alpha \\ \beta \end{matrix} \overset{\begin{matrix}d & \quad\quad q\end{matrix}}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} i_d \\ i_q \end{bmatrix}$

  - The MMF, flux and number of turns are same in both configurations
  - Thus, induced EMF (voltage) per phase will also be same
  - Hence, current and voltage will have identical transformations

(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - Let the currents in coils d-q axes are time-varying:

  (remember that they are at 90^0 in time & space)

$$\begin{array}{c} \\ \begin{bmatrix} i_d \\ i_q \end{bmatrix} \end{array} = \begin{array}{c} \\ d \\ q \end{array} \begin{array}{cc} \alpha & \beta \\ \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}$$

$$\begin{array}{c} \\ \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} \end{array} = \begin{array}{c} \\ \alpha \\ \beta \end{array} \begin{array}{cc} d & q \\ \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \begin{bmatrix} i_d \\ i_q \end{bmatrix}$$

$$i_d = I_m \sin(\omega t + \phi)$$
$$i_q = I_m \cos(\omega t + \phi)$$

Here ϕ is any arbitrary constant phase angle

The equivalent currents i_α and i_β can now be obtained using the inverse transform matrix as (note that $\omega t = \theta$):

$$\begin{aligned} i_\alpha = i_d \cos\theta - i_q \sin\theta &= I_m \sin(\omega t + \phi)\cos\theta - I_m \cos(\omega t + \phi)\sin\theta \\ &= I_m[\sin(\omega t + \phi - \theta)] = I_m[\sin(\omega t + \phi - \omega t)] \\ &= I_m \sin\phi \end{aligned}$$

$$\begin{aligned} i_\beta = i_d \sin\theta + i_q \cos\theta &= I_m \sin(\omega t + \phi)\sin\theta + I_m \cos(\omega t + \phi)\cos\theta \\ &= I_m[\cos(\omega t + \phi - \theta)] = I_m[\sin(\omega t + \phi - \omega t)] \\ &= I_m \cos\phi \end{aligned}$$

(α,β,0) to (d, q, 0) transformation

$$i_d = I_m \sin(\omega t + \phi)$$
$$i_q = I_m \cos(\omega t + \phi)$$

$$i_\alpha = I_m \sin\phi$$

$$i_\beta = I_m \cos\phi$$

- **Rotating to stationary axes transformation**
  - Currents in *d-q* axes coils are time varying with 90^0 phase shift and are also at space quadrature
    - They produce a rotating MMF (RMF)
  - Currents in α-β axes are constant (DC currents), but the coils are at space quadrature
    - The DC currents i_α, i_β can only produce a resultant stationary MMF
    - But since the coils along α-β axes physically rotate at ω_r, the resultant MMF also rotate physically along with the rotor at the same speed ω_r
  - Thus, time varying currents i_d-i_q in stationary coils *d-q* produce exactly similar RMF, rotating at same speed and in the same direction as the DC currents i_α-i_β in the rotating coils α-β produce

(α,β,0) to (d, q, 0) transformation

- **Rotating to stationary axes transformation**
  - If zero sequence currents exist:

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \begin{matrix} d \\ q \\ 0 \end{matrix} \begin{array}{c} \begin{matrix} \alpha & \beta & 0 \end{matrix} \\ \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \end{array} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix}$$

Transformation from 3-phase to 2-phase Stationary Axes

$$(a, b, c) \rightarrow (\alpha, \beta, 0) \rightarrow (d, q, 0)$$

(a, b, c) → (α,β,0) → (d, q, 0) Transformation

- **3-phase to 2-phase rotating axes transformation**
  - (a, b, c) to (α, β,0) transformation

$$\begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{matrix} \\ \alpha \\ \beta \\ 0 \end{matrix} \begin{matrix} a & b & 0 \\ \cos 0^0 & \cos 120^0 & \cos 240^0 \\ \sin 0^0 & \sin 120^0 & \sin 240^0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{matrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

- **2-phase rotating to 2-phase stationary axes transformation**
  - (α, β,0) to (d, q,0) transformation

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \begin{matrix} \\ d \\ q \\ 0 \end{matrix} \begin{matrix} \alpha & \beta & 0 \\ \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{matrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix}$$

- These two transformation matrices can be combined to obtain 3-phase to 2-phase stationary axis transformation (a, b, c) to (α, β,0) to (d, q,0)

(a, b, c) → (α,β,0) → (d, q, 0) Transformation

- **(a, b, c) to (α, β,0) to (d, q,0) axes transformation**

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \begin{array}{c} \\ d \\ q \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} \alpha & \beta & 0 \end{array} \\ \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \end{array} \sqrt{\frac{2}{3}} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} \cos 0^0 & \cos 120^0 & \cos 240^0 \\ \sin 0^0 & \sin 120^0 & \sin 240^0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

Refer appendix for 3x3 matrix multiplication

or,

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{array}{c} \\ d \\ q \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} \cos\theta & \cos\left(\theta - 120^0\right) & \cos\left(\theta - 240^0\right) \\ -\sin\theta & -\sin\left(\theta - 120^0\right) & -\sin\left(\theta - 240^0\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

Thus, the new current (i_d, i_q, i_0) in stationary axes can be expressed in terms of actual 3-phase currents (i_a, i_b, i_c)

(a, b, c) → (α,β,0) → (d, q, 0) Transformation

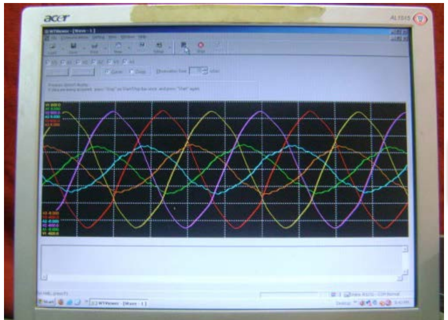
- In case the 3-phase motor is star connected without any neutral wire (as is most commonly the case in induction motors), the zero sequence current i_0 is absent, and the 3rd row will disappear from the transformation matrix:

or,

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{matrix} \\ d \\ q \end{matrix} \begin{matrix} a & b & c \\ \begin{bmatrix} \cos\theta & \cos(\theta-120^0) & \cos(\theta-240^0) \\ -\sin\theta & -\sin(\theta-120^0) & -\sin(\theta-240^0) \end{bmatrix} \end{matrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

- This gives us a transformation from ROTATING 3-phase (i_a, i_b, i_c) to STATIONARY 2-phase (i_d, i_q) transformation in stationary axes
- This technique is often used by researchers for MCSA (Motor Current Signature Analysis) for condition monitoring and fault diagnosis of induction motors

(a, b, c) → (d, q) Transformation

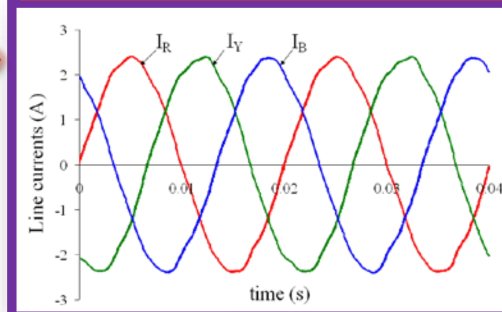


Captured 3-ph Current Signals

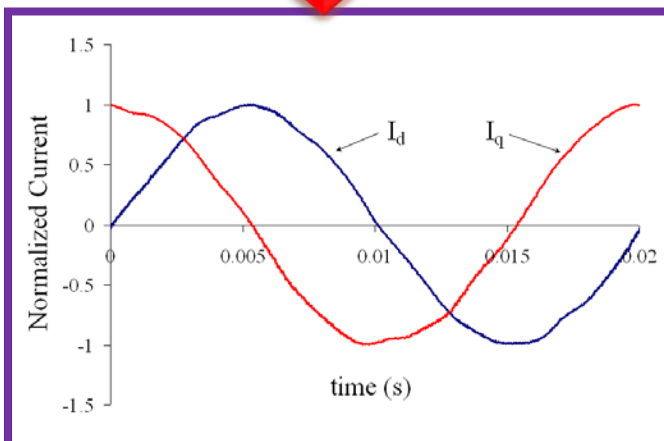
Conducted Experiments

S. Das, C. Koley, **P. Purkait**, and S. Chakravorti, “Performance of a Load-Immune Classifier for Robust Identification of Minor Faults in Induction Motor Stator Winding”, Accepted for Publication in **IEEE Transactions on Dielectrics and Electrical Insulation**, 2013.

3-phase line currents



3-ph to 2-ph transformation



Inverse Transformation

- **(d, q,0) to (a, b, c) axes transformation**
  - Expressing 3-phase currents (i_a, i_b, i_c) in terms of the equivalent stationary axes currents (i_d, i_q, i_0)

$$\begin{matrix} & & & d & q & 0 & \\ \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} & = \sqrt{\frac{2}{3}} & \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} \cos\theta & -\sin\theta & \frac{1}{\sqrt{2}} \\ \cos\left(\theta-120^0\right) & -\sin\left(\theta-120^0\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta-240^0\right) & -\sin\left(\theta-240^0\right) & \frac{1}{\sqrt{2}} \end{bmatrix} & & & \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} \end{matrix}$$

(a, b, c) → (d, q, 0) Transformation

- Similar transformation could be achieved for phase **voltages** as well
- Like i_a, i_b, i_c are related to i_d, i_q
- Similarly, v_a, v_b, v_c are related to v_d, v_q
- These relationships between the 3-phase variables (*a*, *b*, *c*) and 2-phase variables (*d*,*q*) are called **Park's Transformation**
- The 3-phase coils (*a*, *b*, *c*) and the 2-phase coils (α,β) are phase windings on the rotor that actually rotate along with the armature
  - Rotating w.r.t stator, but static w.r.t rotor
- On the other hand, the (*d*, *q*) coils are stationary w.r.t the stator
  - These are pseudo-stationary coils and hence involve the variable angular position term θ in the transformation matrix

(a, b, c) → (d, q, 0) Transformation

- **Physical concept of Park's Transformation (rotating 3-phase to stationary 2-phase)**
  - Assume that the 3-phase currents i_a, i_b, i_c are given by:

$$i_a = I_m \cos(\omega t + \alpha)$$

$$i_b = I_m \cos(\omega t + \alpha - 120^0)$$

$$i_c = I_m \cos(\omega t + \alpha - 240^0)$$

  - Here I_m is the maximum value of armature current and α is the time phase angle of i_a w.r.t the time origin at t=0

Park's transformation matrices

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{array}{c} d \\ q \end{array} \overset{\begin{array}{ccc} a & b & c \end{array}}{\begin{bmatrix} \cos\theta & \cos\left(\theta - 120^0\right) & \cos\left(\theta - 240^0\right) \\ -\sin\theta & -\sin\left(\theta - 120^0\right) & -\sin\left(\theta - 240^0\right) \end{bmatrix}} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

(a, b, c) → (d, q, 0) Transformation

- **Physical concept of Park's Transformation (rotating 3-phase to stationary 2-phase)**
  - Using Park's transformation:

$$i_a = I_m \cos(\omega t + \alpha)$$
$$i_b = I_m \cos(\omega t + \alpha - 120^0)$$
$$i_c = I_m \cos(\omega t + \alpha - 240^0)$$

$$i_d = \sqrt{\frac{2}{3}}\left[i_a \cos\theta + i_b \cos\left(\theta - 120^0\right) + i_c \cos\left(\theta - 240^0\right)\right]$$

$$\cos A \cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$$

Substituting $\theta = \omega t$:

$$i_d = \sqrt{\frac{2}{3}} I_m \left[\cos\omega t \cos\left(\omega t + \alpha\right) + \cos\left(\omega t - 120^0\right)\cos\left(\omega t + \alpha - 120^0\right) + \cos\left(\omega t - 240^0\right)\cos\left(\omega t + \alpha - 240^0\right)\right]$$

$$= \sqrt{\frac{2}{3}} I_m \frac{1}{2}\left[\cos\left(2\omega t + \alpha\right) + \cos\alpha + \cos\left(2\omega t + \alpha - 240^0\right) + \cos\alpha + \cos\left(2\omega t + \alpha - 480^0\right) + \cos\alpha\right]$$

$$= \sqrt{\frac{2}{3}} I_m \frac{1}{2}\left[\cos\left(2\omega t + \alpha\right) + \cos\left(2\omega t + \alpha - 240^0\right) + \cos\left(2\omega t + \alpha - 120^0\right) + 3\cos\alpha\right]$$

Park's transformation matrices

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{array}{c} \\ d \\ q \end{array} \begin{array}{ccc} a & b & c \\ \cos\theta & \cos\left(\theta - 120^0\right) & \cos\left(\theta - 240^0\right) \\ -\sin\theta & -\sin\left(\theta - 120^0\right) & -\sin\left(\theta - 240^0\right) \end{array} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

(a, b, c) → (d, q, 0) Transformation

- **Physical concept of Park's Transformation (rotating 3-phase to stationary 2-phase)**

$$i_a = I_m \cos(\omega t + \alpha)$$
$$i_b = I_m \cos(\omega t + \alpha - 120^0)$$
$$i_c = I_m \cos(\omega t + \alpha - 240^0)$$

$$i_d = \sqrt{\frac{2}{3}} I_m \frac{1}{2}\left[\cos(2\omega t + \alpha) + \cos(2\omega t + \alpha - 240^0) + \cos(2\omega t + \alpha - 120^0) + 3\cos\alpha\right]$$

For a balanced 3-phase system:

$$\cos(2\omega t + \alpha) + \cos(2\omega t + \alpha - 240^0) + \cos(2\omega t + \alpha - 120^0) = 0$$

Thus: $i_d = \sqrt{\frac{2}{3}} I_m \frac{1}{2}[3\cos\alpha] = \sqrt{\frac{2}{3}} I_m \frac{3}{2}\cos\alpha = \sqrt{\frac{3}{2}} I_m \cos\alpha$

Similarly, it can be shown that: $i_q = \sqrt{\frac{3}{2}} I_m \sin\alpha$

- The two currents i_d and i_q are thus not time-varying, i.e. they are DC currents
- But still they produce RMF due to physical rotation of the rotor
- This RMF has the same speed and direction as that produced by the three time varying currents i_a, i_b, and i_c

Stator winding Transformation

- In addition to rotor, the stator may also have 1-phase, 2-phase, or 3-phase winding
  - If the stator has a 1-phase winding, it is taken along *d*-axis
  - If the stator has 2-phase windings, they are taken along *d*-*q* axes
  - If the stator has 3-phase windings, transformations need to be done to convert from 3-phase axes (a, b, c) to (α, β) or (d, q) axes
- Since the stator coil axes are always stationary w.r.t. the stator and its winding, coefficients of the transformation matrix are always constant