ANALYSIS OF PLATES USING TRIANGULAR ISOPARAMETRIC FINITE ELEMENTS

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REGISTRATION NO. : 128864 of 2014-2015

EXAM ROLL NO. : M4CIV1605

Under the Guidance of

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A Thesis Paper to be submitted

In Partial Fulfillment of the Requirements for the Degree of

Master of Engineering in Civil Engineering

(Specialization: Structural Engineering)

At the

Department of Civil Engineering

Faculty of Engineering and Technology

Jadavpur University

Kolkata – 700032

2014 – 2016

DEPARTMENT OF CIVIL ENGINEERING

FACULTY OF ENGINEERING AND TECHNOLOGY

JADAVPUR UNIVERSITY

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CERTIFICATE OF RECOMMENDATION

This is to certify that the thesis entitled, "**Analysis of Plates Using Isoparametric Triangular Finite Elements**", that is being submitted by Sourav Chanda (Roll No. 001410402009) to Jadavpur University for Partial fulfillment of the requirements for awarding the degree of Master of Civil Engineering (Structural Engineering) is a record of bona fide research work carried out by him under my direct supervision and guidance.

The work contained in the thesis has not been submitted in part or full to any other university or institution or professional body for the award of any degree or diploma.

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ACKNOWLEDGEMENT

I am extremely grateful and deeply indebted to **Professor Arup Guha Niyogi,** Department of Civil Engineering, Jadavpur University, Kolkata, for his excellent guidance and whole hearted involvement during my thesis work without whose invaluable suggestions, meticulous efforts, versatility and untiring guidance, this project would not have been feasible. I am also indebted to him for his encouragement and moral support throughout this programme.

It is my privilege and pleasure to express my deep sense of gratitude and sincere thanks to him for his timely help, valuable support and suggestions which have helped to orient my Thesis paper.

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ABSTRACT

In this thesis, Isoparametric Finite Elements have been used to find out static deflection of Kirchhoff's plates. Each triangular element has three nodes at the three corners and each of these elements has three degrees of freedom per node. The problem considered in this thesis work is that the plate is simply supported on the four edges and it is subjected to uniformly distributed load throughout the plate. Various mesh sizes are developed such as 4*4, 8*8, 16*16 and 32*32. The convergence study for various mesh sizes is made and conclusion is done accordingly. MATLAB Software is used to develop the program of the problem mentioned above.

The results obtained from the program are compared with analytical results obtained from the solution of governing equation of thin plates.

Again, deflection for various edge conditions are then obtained from MATLAB program and validated with the analytical results.

Later, MATLAB code has been deployed to obtain mass matrices for triangular finite elements. Finally, eigenvalues and eigenvectors have been extracted to derive natural frequencies and the mode shapes of a plate, respectively. The mode shapes are plotted accordingly.

SYMBOLS

Symbols	Description	Unit
h	Thickness of the plate	m
E	Modulus of Elasticity	N/m^2
μ	Poisson's Ratio	
θ_x	Angle of orientation about x - axis	Degree
θ_y	Angle of orientation about y - axis	Degree
a	Length of the plate	m
b	Width of the plate	m
A	Area of the element	m^2
V	Volume of the element	m^3
f_b	Force vector	
K	Element stiffness matrix	N/m
B	Strain – displacement matrix	
ζ	Natural coordinate along x – axis	
η	Natural coordinate along y – axis	
ψ	Shape functions	
w	Downward deflection along z – axis	m
D	Constitutive relationship matrix	
q	Intensity of uniformly distributed load applied on the plate	N/m^2
{b}	Body force vector	
M_x	Bending moments along x – axis	N-m
M_y	Bending moments along y – axis	N-m
M_{xy}	Twisting moment	N-m
Q_x	Vertical shear forces acting per unit length parallel to the y axes	N
Q_y	Vertical shear forces acting per unit length parallel to the x axes	N

τ_{xz}	Shear stress acting on xz plane	N/m^2
τ_{yz}	Shear stress acting on yz plane	N/m^2
T	Kinetic energy	N-m
V	Potential energy	N-m
W	Work done	N-m
m	Mass of a system	kg
k	Stiffness of a system	N/m
c	Viscous damping	N-sec/m
M	Mass matrix	
K	Stiffness matrix	
C	Viscous damping matrix	
λ	Eigen value	
ω	Natural frequency	Rad/sec or Hertz

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CHAPTER – 1

INTRODUCTION

1.1. Introduction to Finite Element Method

The conventional analytical approaches to solution of various structural problems are based on determining the necessary equations governing the behavior of the structure by taking into account equilibrium and compatibility within the structure [1] [2] [3]. The solution to these equations exists only for special cases of loading and boundary conditions. Due to the complex nature of the shape of the structure, loading pattern, irregularities in geometry or material, the analytical solution normal1y becomes difficult and even impossible to solve such problems for displacements, stress or strains within the structure, the need for some other technique, such as suitable numerical methods for tackling the more complex structures with arbitrary shapes, loading and boundary conditions, is then essential. Several approximate numerical methods have evolved over the years. One of the common methods is the Finite Difference scheme in which an approximation to the governing equations is used. The solution is formed by writing difference equations for a grid points. The solution is improved as more points are used. With this technique, some fairly difficult problems can be treated, but for example, for problems of irregular geometries or unusual specification of boundary conditions, the solution becomes more complex and difficult to obtain. On the other hand, the Finite Element Method (FEM), can take care of all these complex problems, and hence has become more widespread in finding solutions to complex structural and non-structural problems.

The Finite Element Method (FEM), or Finite Element Analysis (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in everyday life as well as in engineering. FEM is a powerful method for the analysis of continuous structures including complex geometrical configurations, material properties, or loading. These structures are idealized as consisting of one, two or three-dimensional elements connected at the nodal points, common edges, or surfaces.

1.2. History of Finite Element Method

The modern development of the finite element method in the field of structural engineering dates back to 1941 and 1943, when its key features were published by Courant [1], Hrenikoff [4] and McHenry [5]. The work of Courant is particularly significant because of its concern with problems governed by equations applicable to structural mechanics and other situations. He proposed setting up the solution of stresses in the variational form. Then he introduced piecewise interpolation functions (shape functions) over triangular sub-regions making the whole region to obtain the approximate numerical solution.

In 1947, Levy [6] developed the flexibility method (force method) and in1953 he suggested that the use of the displacement method could be a good alternative for the analyzing statically redundant aircraft wings [7]. This method became popular only later after the invention of the high-speed computers.

In 1954, Argyris and Kelsey [8] gave a very general formulation of the stiffness matrix method based on the fundamental energy principles of elasticity. This illustrated the importance of the energy principles and their role in the development of the finite element method.

In 1956, Turner, Clough, Martin and Topp [9] presented the first treatment of the two dimensional elements. They derived the stiffness matrices for triangular and rectangular elements based on assumed displacements and they outlined the procedure commonly known as the Direct Stiffness Method for assembling the total stiffness matrix of the structure. This is regarded as one of the key contributions in the discovery of the finite element method.

The technology of finite elements has advanced through a number of in distinct phases in the period since the mid 1950's. The formulation of the triangular and rectangular elements for plane stress has motivated the researchers to continue and establish element relationships for solids, plates in bending and thin shells.

In 1960's, linear strain triangular element was developed [10]. This element has 6 nodes with 2 degrees of freedom per node. The derivation of the stiffness matrix for this element was difficult. The isoparametric formulation was then developed [11] in which both the element geometry and displacements are defined by the same interpolation functions. This formulation was then applied to two and three dimensional stress analysis where higher order triangular and rectangular plane elements were developed. Also, brick elements were developed for three dimensional stress analysis. Elements created can be non-rectangular and have curved sides.

By the early 1970's, this method was further developed for use in the aerospace and nuclear industries where the safety of the structures is critical. Since the rapid decline in the cost of computers, FEM has been developed to an incredible precision. Currently, there exist commercial finite element packages that are capable of solving the most sophisticated problems for static as well as dynamic loading, in a wide range of structural as well as non-structural applications.

Before reviewing the available finite element solutions for two dimensional structures, a brief introduction to the finite element method is presented showing a description of the procedure for obtaining the stiffness matrix of the general (triangular or rectangular) plane element.

1.3. Procedure of Finite Element Method [12]

A. Discretization of the domain: The continuum is divided into a number of finite elements by imaginary lines or surfaces. The interconnected elements may have different sizes and shapes. The choice of the simple element or higher order element, straight or curved, its shape, refinement is to be decided before the mathematical formulation starts.

B. Identification of variables: It is assumed that the elements are connected at their intersecting points referred to as nodal points. At each node, unknown displacements are to be prescribed. They are dependent on the problem at hand.

C. Choice of approximating functions: Once the variables and local coordinate system have been chosen, the next step is the choice of the displacement function. In fact, it is the displacement function that is the starting point of the mathematical analysis. The function represents the variation of the displacement within the element. The function can be approximated in a number of ways. The displacement function may be approximated in the form of a linear function or higher – order functions. A convenient way to express it is by using polynomial expressions.

D. Formation of the element stiffness matrix: After the continuum is discretised with desired element shapes, the element stiffness matrix is formulated. This can be done in a number of ways. Basically it is a minimization procedure whatever may be the approach adopted. For certain elements, the form involves great deal sophistication. With the exception of a few simple elements, the element stiffness matrix for majority of elements is not available in explicit form. As such, they require numerical integration for their evaluation.

   The geometry of the element is defined with reference to the global frame. In many problems such as those of rectangular plates, the global and local axis systems are coincident and for them no further calculation is needed at the element level beyond computation of element stiffness matrix in local coordinates. Coordinate transformation must be done for all elements where it is needed.

E. Formation of the overall stiffness matrix: After the element stiffness matrices in global coordinates are formed, they are assembled to form of the overall stiffness matrix. The assembly is done through the nodes which are common to adjacent elements. At the nodes, the continuity of the displacement function and possibly their derivatives are established. The overall stiffness matrix is symmetric and banded.

F. Incorporation of boundary conditions: The boundary restraint conditions are to be imposed in the stiffness matrix. There are various techniques available to satisfy the boundary conditions. In some of these approaches, the size of the stiffness matrix may be reduced or condensed in its final form. To ease the computer programming aspect and to elegantly incorporate the boundary conditions, the size of the overall stiffness matrix is kept the same.

G. Formation of the element loading matrix: The loading forms an essential parameter in any structural engineering problem. The loading inside an element is transferred at the nodal points and consistent element loading matrix is formed. Sometimes, based on the typicality of the problem, the loading matrix may be simplified.

H. Formation of the overall loading matrix: Like the overall stiffness matrix, the element loading matrices are assembled to form the overall loading matrix. This matrix has one column per loading case and it is either a column vector or a rectangular matrix depending on the number of loading conditions.

I. Solution of simultaneous equations: All the equations required for the solution of the problem are developed. In the displacement method, the unknowns are the nodal displacements. Rarely is the stiffness matrix stored as a solid matrix. Advantages are taken of the symmetric nature of the problem and banded properties. The Gauss elimination and Cholsky's factorization are the most commonly used procedures for the solution of simultaneous equations. These methods are well suited to a small or moderate number of equations. For large sized problems, frontal technique is one of the methods of obtaining solutions. For systems of large order, Gauss-Seidel or Jacobi iterations are more suited.

1.4. Literature Review

Kirchhoff [13] and Love [14] developed a theory for thin plates. The theory was developed in 1888 by Love using assumptions proposed by **Kirchhoff [13]**. It is assumed that a mid-surface plane can be used to represent the three dimensional plate in two dimensional form, i.e. no shear deformation occurs to the plate.

Mindlin [15]–Reissner [16] plate theory is applied for analysis of thick plates, where the shear deformations are considered, rotations and lateral deflections are decoupled. It does not require the cross – sections to be perpendicular to the axial forces after deformation.

Vanam et al. [17] has done static analysis of an isotropic rectangular plate using finite element analysis (FEA). The aim of study was static bending analysis of an isotropic rectangular plate with various boundary conditions and various types of load applications. In this study finite element analysis has been carried out for an isotropic rectangular plate by considering the master element as a four noded quadrilateral element. Numerical analysis has been carried out by developing programming in mathematical software MATLAB and the results obtained from MATLAB are giving good agreement with the results obtained by classical method - exact solutions. Later, for the same structure, analysis has been carried out using finite element analysis software ANSYS.

P. S. Gujar, K. B. Ladhane [18] studied static bending analysis of an isotropic circular plate using analytical method i.e. Classical Plate Theory and Finite Element software ANSYS. Circular plate analysis is done in cylindrical coordinate system by using Classical Plate Theory. The axisymmetric bending of circular plate is considered in this study. Both simply supported and clamped boundary conditions subjected to uniformly distributed load and center concentrated load have been considered. In this work, the effect of varying thickness of the plate on its deflection and bending stress is studied. Modeling and analysis of circular plate is done in ANSYS. Once deflection is obtained by using CPT, bending moments and bending stresses are easily calculated by usual relations. Analytical results of CPT are validated with ANSYS results.

Neffati M. Werfalli, Abobaker A. Karoud [19] has done a study of free vibration of thin isotropic rectangular plates with various edge conditions. This study involves the obtaining of natural frequencies by solving the mathematical model that governs the vibration behavior of the plate using a Galerkin-based finite element method. Cubic quadrilateral serendipity sub parametric elements with twelve degrees of freedom are used in this analysis. Even though the order of polynomial used is the lowest possible, the effectiveness of the method for calculating the natural frequencies accurately is demonstrated by comparing the solution obtained against the existing analytical results.

Ebirim, Stanley I., Ezeh, J.C, Ibearugbulem, Owus M. [20] presents a theoretical formulation based on Ibearugbulem's shape function and application of Ritz method. In this study, the free vibration of simply supported plate with one free edge was analyzed. The Ibearugbulem's shape function derived was substituted into the potential energy functional, which was minimized to obtain the fundamental natural frequency. Aspect ratios from 0.1 to 2.0 with 0.1 increments were considered. The values of fundamental natural frequencies of the first mode were determined for different aspect ratio. Comparison was made for values of non-dimensional parameter of fundamental natural frequencies obtained in this study with those of previous research works. It was seen that there is no significant difference between values obtained in this study with those of previous studies.

T. Sakiyama and M. Huang [21] proposed an approximate method for analyzing the free vibration of rectangular plates with variable thickness. The approximate method is based on the Green function of a rectangular plate. The Green function of a rectangular plate with arbitrary variable thickness is obtained as a discrete form solution for deflection of the plate with a concentrated load. The discrete form solution is obtained at each discrete point equally distributed on the plate. It is shown that the numerical solution for the Green function has the good convergence and accuracy. By applying the Green function, the free vibration problem of plate is translated into the eigenvalue problem of matrix. The convergence and accuracy of the numerical solutions for the natural frequency parameter calculated by the proposed method are investigated, and the frequency parameters and their modes of free vibration are shown for some rectangular plates.

Sai Sudha Ramesha, C.M. Wang, J.N. Reddy and K.K. Ang [22] were concerned with the development of a higher-order triangular plate element based on the first-order shear deformation plate theory. The present study brings out the shortcomings of conventional lower-order finite element interpolation when applied for the bending analysis of plates with free edges. The stress resultants, especially the transverse shear forces and twisting moments, obtained using the lower-order displacement finite elements fail to satisfy the natural (or force) boundary conditions accurately. On the other hand, the plate element with higher-order interpolation of the field variables enables the accurate prediction of stress resultants for plates. The superior performance of the higher order plate bending element is demonstrated through bending analyses of plates of various shapes and free edges and examining the distribution of the stress resultants and the satisfaction of the natural boundary conditions.

A graph-theoretical method is presented by **A. Kaveh, M.S. Massoudi** [23] for the formation of sparse, banded and highly accurate null basis matrices for finite element models with triangular and rectangular bending elements. These bases are generated much faster than those obtained by the algebraic methods.

1.5. Objective and Scope of work

To model difficult geometries requirement of triangular finite element is immense. So, in the present work, it is intended to develop a finite element procedure to analyze isotropic Kirchhoff's plates using isoparametric triangular elements. Bending problems and free vibration are targeted.

A finite element code is proposed to develop in MATLAB to study the bending and free vibration of the rectangular plate structure. The model has to be validated using analytical deflection formulae and published results. Later the frequencies and mode shapes are to be plotted to obtain a visual feel.

1.6. Organization of Thesis

The present thesis contains six chapters. Chapter 1 contains the motivation of the present work with a brief description of history of finite element method, literature review and the procedure for finite element formulation. At the end of this chapter, the scope of present work is outlined.

Chapter 2 deals with Kirchhoff's plate theory, analytical and finite element formulation of three-noded triangular plate elements with three degrees of freedom at each node. This section consists of derivation of shape functions for the triangular element, development of element stiffness matrix and force vector.

Chapter 3 consists of free vibration of plate. Degrees of freedom, matrix formulation for multi-degrees of freedom system and determination of eigenvalues and eigenvectors are described in this chapter.

In Chapter 4, the algorithm of the programming to compute the deflection and frequencies and mode shape of the square or rectangular plate divided into triangular element is shown in Flow chart form.

In Chapter 5, the results obtained from the MATLAB program for various mesh sizes are checked for convergence and then validated with the analytical results. On the next phase of this chapter, results for various edge conditions are computed and then verified with the analytical values.

Finally the conclusion and future scope of present work are enlisted in Chapter 6. The list of references which are extremely valuable for the present work is cited at the end.

CHAPTER – 2

PLATE THEORY AND FINITE ELEMENT FORMULATIONS

2.1. Kirchhoff – Love theory for thin plates [13]

The Kirchhoff–Love theory is an extension of Euler–Bernoulli beam theory to thin plates. The theory was developed in 1888 by Love using assumptions proposed by Kirchhoff. It is assumed that a mid-surface plane can be used to represent the three dimensional plate in two dimensional form.

The following kinematic assumptions that are made in this theory:

- Straight planes normal to the mid surface remain straight after deformation.
- The thickness of the plate does not change during a deformation.

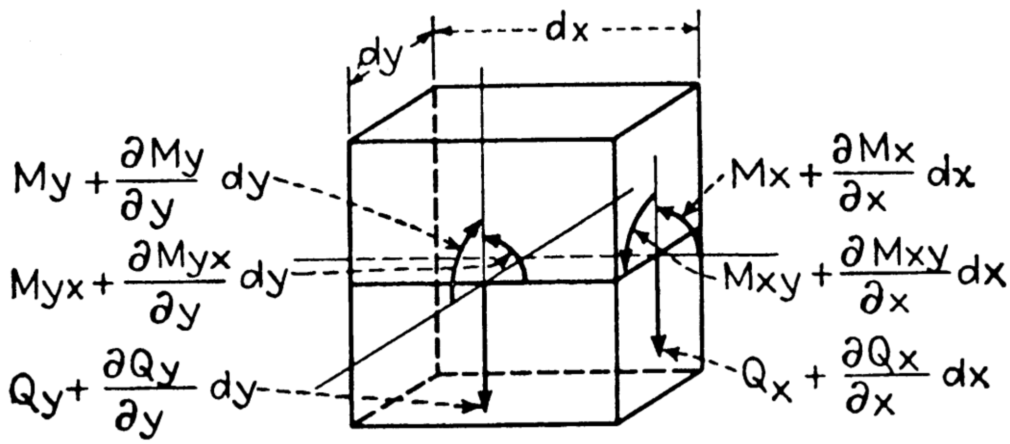


Figure2.1: Plate with loads and moments acting on it

Let us consider the coordinate axes x and y in the middle plane of the plate and the z axis perpendicular to that plane. Let us further consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes as shown in figure 2.1. In addition to the bending moments M_x and M_y, the twisting moments M_{xy} and vertical shear forces are considered on the sides of the element. The magnitude of these shear forces per unit length parallel to the y and x axes we denote by Q_x and Q_y, respectively, so that

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \qquad (2.1.1)$$

One must also consider the load distributed over the upper surface of the plate. The intensity of this load we denote by q, so that the load acting on the element is qdxdy.

Projecting all the forces acting on the element onto the z axis one obtains the following equation of equilibrium:

$$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dydx + q\, dxdy = 0$$

(2.1.2)

From which,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

(2.1.3)

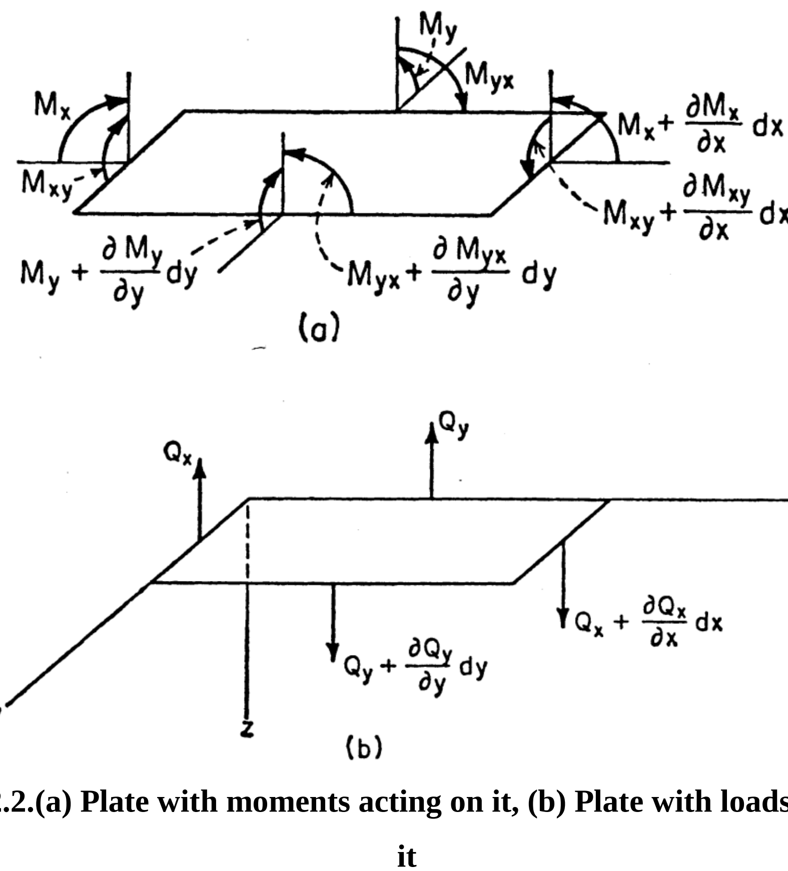


Figure 2.2.(a) Plate with moments acting on it, (b) Plate with loads acting on it

Taking moments of all the forces acting on the element with respect to the x axis, the equation of equilibrium becomes,

$$\frac{\partial M_{xy}}{\partial x}dxdy - \frac{\partial M_{yx}}{\partial y}dydx + Q_y dxdy = 0$$

(2.1.4)

The moment of the load q and the moment due to change in the force Q_y are neglected in this equation, since they are small quantities of a higher order than those retained. After simplification the equation becomes,

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{yx}}{\partial y} + Q_y = 0$$

(2.1.5)

In the same manner, by taking moments with respect to the y axis,

$$\frac{\partial M_{yx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x = 0$$

(2.1.6)

Since there are no forces in the x and y directions and no moments with respect to the z axis, the three equations (2.1.3), (2.1.5) and (2.1.6) completely define the equilibrium of the element. Eliminating the shearing forces Q_x and Q_y from these equations by determining them from equations (2.1.5) and (2.1.6) and substituting into equation (2.1.3), one obtains

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q$$

(2.1.7)

Observing that $M_{yx} = - M_{xy}$, by virtue of $\tau_{xy} = \tau_{yx}$, we finally represent the equation equilibrium (2.1.7) in the following form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q$$

(2.1.8)

Now,

$$M_x = -D(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2})$$

$$M_y = -D(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2})$$

$$M_{xy} = -M_{yx} = D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

(2.1.9)

Substituting these expressions in equation (g), we obtain

$$\boldsymbol{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}}$$

(2.1.10)

This is the governing differential equation of laterally loaded thin plates.

2.2. Development of Shape functions of a Three-noded isotropic triangular element

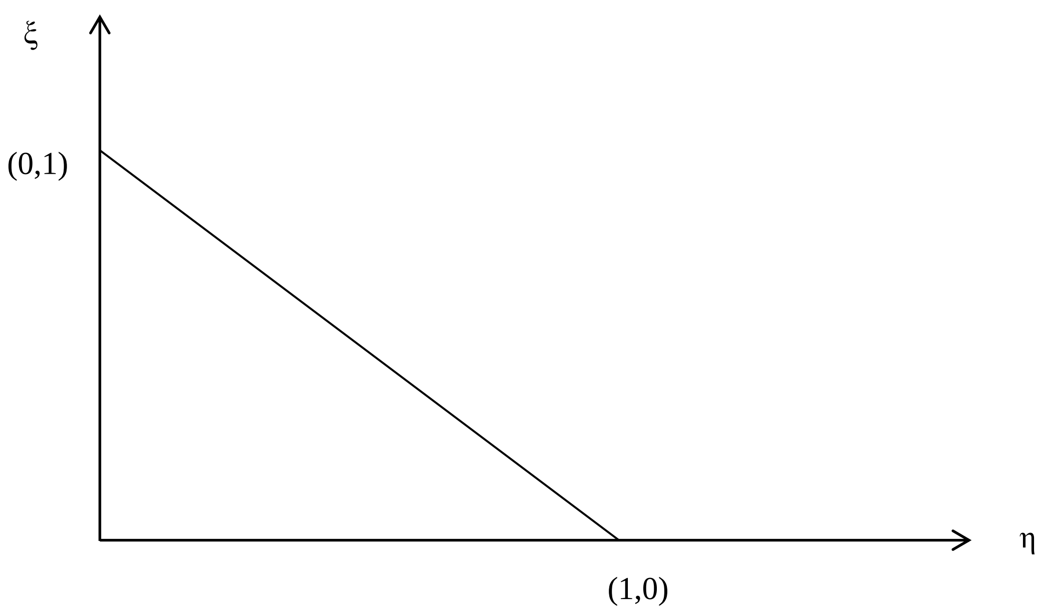


Figure 2.3: Three-node triangular element in Natural Coordinate System

Let, $\psi_1 = a_1+b_1\zeta+c_1\eta$

At node 1, $\zeta=0$, $\eta=0$, $\psi_1=1$

So, $a_1 = 1$

At node 2, $\zeta=1$, $\eta=0$, $\psi_1=0$

So, $b_1 = -1$

At node 3, $\zeta=0$, $\eta=1$, $\psi_1=0$

So, $c_1 = -1$

$$\Psi_1 = 1 - \zeta - \eta \qquad (2.2.1)$$

Let, $\psi_2 = a_2+b_2\zeta+c_2\eta$

At node 1, $\zeta=0$, $\eta=0$, $\psi_2=0$

So, $a_2 = 0$

At node 2, $\zeta=1$, $\eta=0$, $\psi_2=1$

So, $b_1 = 1$

At node 3, $\zeta=0$, $\eta=1$, $\psi_2=0$

So, $c_1 = 0$

$$\Psi_2 = \zeta \qquad (2.2.2)$$

Let, $\psi_3 = a_3+b_3\zeta+c_3\eta$

At node 1, $\zeta=0$, $\eta=0$, $\psi_3=0$

So, $a_3 = 0$

At node 2, $\zeta=1$, $\eta=0$, $\psi_3=0$

So, $b_3 = 0$

At node 3, $\zeta=0$, $\eta=1$, $\psi_3=1$

So, $c_3 = 1$

$$\Psi_3 = \eta \tag{2.2.3}$$

2.3. Finite element formulation of a three-noded triangular isotropic plate element [24]

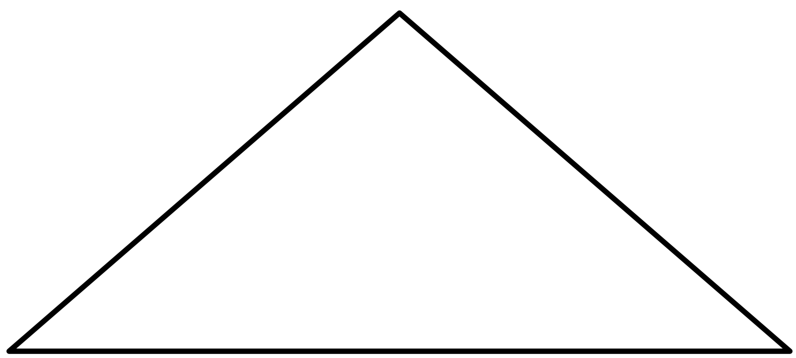


Figure 2.3: Triangle with three corner nodes and their displacements

$$w = \psi_1 q_1 + \psi_2 q_4 + \psi_3 q_7 \tag{2.3.1}$$

$$\theta_x = \psi_1 q_2 + \psi_2 q_5 + \psi_3 q_8 \tag{2.3.2}$$

$$\theta_y = \psi_1 q_3 + \psi_2 q_6 + \psi_3 q_9 \tag{2.3.3}$$

θ_x is the rotation about x – axis, θ_y is the rotation about y – axis and w is the downward deflection.

q_1, q_4 and q_7 are the downward deflection of node 1, 2 and 3 respectively.

q_2, q_5 and q_8 are the rotation about x – axis of node 1, 2 and 3 respectively.

q_3, q_6 and q_9 are the rotation about y – axis of node 1, 2 and 3 respectively.

A three noded triangular plate element will have 9 degrees of freedom totally. Hence the displacement function for w should consist of 9 terms in the polynomial form. The polynomial function is assumed as:

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8y^3 + a_9(x^2y + xy^2)$$

$$\theta_x = \frac{\partial w}{\partial x} = a_2 + 2a_4x + a_5y + 3a_7x^2 + a_8y^3 + a_9(2xy + y^2) \qquad (2.3.4)$$

$$\theta_y = \frac{\partial w}{\partial y} = a_3 + a_5x + 2a_6y + 3a_8y^2 + a_9(x^2 + 2xy)$$

Applying equation (2.3.4) at the three nodes with $(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$ as the degrees of freedom at each node,

$$w_1 = a_1 + a_2x_1 + a_3y_1 + a_4x_1^2 + a_5x_1y_1 + a_6y_1^2 + a_7x_1^3 + a_8y_1^3 + a_9(x_1^2y + x_1y_1^2)$$

$$\left(\frac{\partial w}{\partial x}\right)_1 = a_2 + 2a_4x_1 + a_5y_1 + 3a_7x_1^2 + a_8y_1^3 + a_9(2x_1y_1 + y_1^2)$$

$$\left(\frac{\partial w}{\partial y}\right)_1 = a_3 + a_5x_1 + 2a_6y_1 + 3a_8y_1^2 + a_9(x_1^2 + 2x_1y_1)$$

Similarly,

$$\left(\frac{\partial w}{\partial y}\right)_3 = a_3 + a_5x_3 + 2a_6y_3 + 3a_8y_3^2 + a_9(x_3^2 + 2x_3y_3)$$

$$\mathbf{u} = \begin{Bmatrix} w_1 \\ \frac{\partial w}{\partial x_1} \\ \frac{\partial w}{\partial y_1} \\ w_2 \\ \frac{\partial w}{\partial x_2} \\ \frac{\partial w}{\partial y_2} \\ w_3 \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial w}{\partial y_3} \end{Bmatrix} = \mathbf{P} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix} = \mathbf{P}\,\mathbf{a} \qquad (2.3.5)$$

a = P^{-1} u

where, P is

$$P = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & x_1^3 & y_1^3 & (x_1^2y_1 + x_1y_1^2) \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 & 3x_1^2 & 0 & (2x_1y_1 + y_1^2) \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 & 0 & 3y_1^2 & (x_1^2 + 2x_1y_1) \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & x_2^3 & y_2^3 & (x_2^2y_2 + x_2y_2^2) \\ 0 & 1 & 0 & 2x_2 & y_2 & 0 & 3x_2^2 & 0 & (2x_2y_2 + y_2^2) \\ 0 & 0 & 1 & 0 & x_2 & 2y_2 & 0 & 3y_2^2 & (x_2^2 + 2x_2y_2) \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & x_3^3 & y_3^3 & (x_3^2y_3 + x_3y_3^2) \\ 0 & 1 & 0 & 2x_3 & y_3 & 0 & 3x_3^2 & 0 & (2x_3y_3 + y_3^2) \\ 0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & 3y_3^2 & (x_3^2 + 2x_3y_3) \end{bmatrix} \quad (2.3.6)$$

Using equation (2.3.5) in equation (2.3.6), the strain – displacement relationship matrix **B** is obtained.

$$Q = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & 0 & -2y \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -6y & -2x \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4(x+y) \end{bmatrix} \quad (2.3.7)$$

B = Q P^{-1}

Constitutive Relationship matrix is given by,

For Plane stress problem,

$$D = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad (2.3.8)$$

For Plane strain problem,

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & (1-2\mu) \end{bmatrix} \quad (2.3.9)$$

Hence, the stiffness matrix can be obtained as,

$$K = |J| \sum_1^{NGP} B^T . D . B . W \quad (2.3.11)$$

Where,

|J| = determinant of Jacobian matrix

NGP = no. of Gaussian sampling points

W = weight coefficients

Table 2.1:- Gauss quadrature: Sampling points and weights [Appendix – A]

Number of points, n	Triangular Coordinates	Weights
1	1/3, 1/3, 1/3	1
2	½, ½, 0	1/3
	0, ½, ½	1/3
	½, 0, ½	1/3
3	1/3, 1/3, 1/3	-27/48
	0.6, 0.2, 0.2	25/48
	0.2, 0.6, 0.2	25/48
	0.2, 0.2, 0.6	25/48
4	1/3, 1/3, 1/3	-9/32
	3/5, 1/5, 1/5	25/96

2.4. Development of Force vector

Body force vector is given by,

$f_b = \int_V^0 \psi^T \{b\} dV$ (2.4.1)

where, ψ = Shape function matrix

b = uniformly distributed load per unit volume

V = volume of the body

For three noded triangular plate element with three degrees of freedom, w = uniformly distributed load per unit area

h = thickness of the plate element

A = area of the plate element

Hence,

$$f_b = \begin{Bmatrix} h\int \psi_1 w dA \\ 0 \\ 0 \\ h\int \psi_2 w dA \\ 0 \\ 0 \\ h\int \psi_3 w dA \\ 0 \\ 0 \end{Bmatrix} \qquad (2.4.2)$$

As there is no force acting in direction of q_2,q_3, q_5, q_6, q_8, q_9, the corresponding rows of body forces of the force vector are zero.

Now, from the definition of shape functions of triangle,

$\int \psi_1 dA$ = A/3

Similarly, $\int \psi_2 dA$ = $\int \psi_3 dA$ = A/3 (2.4.3)

Substituting the values of equation (2.4.3) into equation (2.4.2), one can represent the force vector as,

$$f_b = \begin{Bmatrix} \frac{hwA}{3} \\ 0 \\ 0 \\ \frac{hwA}{3} \\ 0 \\ 0 \\ \frac{hwA}{3} \\ 0 \\ 0 \end{Bmatrix} \qquad (2.4.4)$$

CHAPTER – 3

FREE VIBRATION OF PLATE

3.1. Degrees of Freedom and Generalized Coordinates

The number of degrees of freedom used in the analysis of a mechanical system is the number of kinematically independent coordinates necessary to completely describe the motion of every particle in the system. Any such set of coordinates is called a set of generalized coordinates.

3.2. Types of Vibration

If a system, after an initial disturbance is left to vibrate on its own, the resulting vibration is known as free vibration. The frequency of free vibration is known as natural frequency of vibration.

If a system is subjected to an external repeating type of force, the resulting vibration is known as forced vibration. The frequency of forced vibration is known as forced frequency of vibration.

If forced frequency of vibration of the system matches with the natural frequency of vibration of the system, the amplitude of vibration is very high and this condition is referred as resonance.

If no energy is dissipated in friction or other resistance during vibration, the resulting vibration is known as undamped vibration.

If energy is dissipated due to resistance during vibration, it is known as damped vibration.

3.3. Equivalent Systems Analysis

All linear 1-degrees-of-freedom systems with viscous damping can be modeled by the simple mass-spring-dashpot system as shown in Figure 3.1.

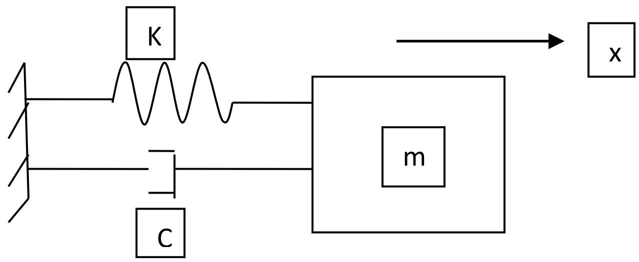


Figure 3.1. Simple mass-spring-dashpot system

Let x be the chosen generalized coordinate. The kinetic energy of a linear system can be written in the form,

$$T = \frac{1}{2} m\dot{x}^2$$

(3.3.1)

The potential energy of a linear system can be written in the form,

$$V = \frac{1}{2} kx^2$$

(3.3.2)

The work done by the viscous damping force in a linear system between two arbitrary locations x_1 and x_2 can be written as,

$$W = -\int_{x_1}^{x_2} c\,\dot{x}\,dx$$

(3.3.3)

Comparing with the kinetic energy, potential energy and work done by the actual system with equations (3.3.1), (3.3.2) and (3.3.3), mass, stiffness and damping of a system can be obtained.

3.4. Derivation of Differential Equations

All linear 1-degrees-of-freedom systems can be modeled using the system shown in Figure 3.1. The principle of conservation of energy or Lagrange's equations is used to derive differential equations. The system of Figure 3.1 is used as a model when generalized coordinates is a linear displacement coordinate. Its governing differential equation is,

$$m\ddot{x} + c\dot{x} + kx = 0$$

(3.4.1)

3.5. Matrix Formulation of Differential Equations for Linear Multi-Degree-of-Freedom system

For linear multi-degree-of-freedom system, the potential and kinetic energies have quadratic forms:

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\, x_i x_j$$

(3.5.1)

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\, \dot{x}_i \dot{x}_j$$

(3.5.2)

If viscous damping, independent of the generalized coordinates, are the only non-conservative forces, the virtual work can be expressed as,

$$\delta W = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, \dot{x}_i\, \delta x_j$$

(3.5.3)

Application of lagrange's equation to the lagrangian developed using equations (3.5.1) and (3.5.2) and the virtual work of equation (3.5.3) leads to,

$$\boldsymbol{M\ddot{x} + C\dot{x} + Kx = 0} \tag{3.5.4}$$

Where M is the n*n mass matrix whose elements are m_{ij}, K is the n*n stiffness matrix whose element are k_{ij}, C is the n*n viscous damping matrix whose elements are c_{ij}, x is the n*1 displacement vector whose elements are x_i. The matrices are symmetric. For example, $m_{ij} = m_{ji}$.

For undamped vibration,

$$\boldsymbol{M\ddot{x} + Kx = 0} \tag{3.5.5}$$

3.6. Eigen Value Problem

For undamped vibration system, to complete the solution of equation (3.5.5), we must determine the displacements x_i (i = 1,2, …..n). To this end, equation (3.5.5) can be rewritten as,

$$(K - \lambda M)x_i = 0 \tag{3.5.6}$$

This constitute a set of n homogeneous algebraic equations in the unknowns x_i, with $\lambda = \omega^2$ playing the role of a parameter. The problem of determining the values of ω^2 for which nontrivial solutions x_i (i = 1,2, …..n) of equation (3.5.6) exist is known as the characteristic value or eigenvalue problem.

It is convenient to write equation (3.5.6) in the matrix form as.

$$Kx = \omega^2 Mx$$

(3.5.7)

Equation (3.5.7) represents the eigenvalue problem associated with matrices M and K and it possesses nontrivial solutions if and only if the determinant of the coefficients vanishes. This can be expressed in the form,

$$det|K - \omega^2 M| = 0$$

(3.5.8)

Equation (3.5.8) is called the characteristic equation. The characteristic polynomial is of degree n in ω^2, and possesses in general n distinct roots, referred to as characteristic values or eigenvalues. The n roots are denoted by ω_1^2, ω_2^2, …….., ω_n^2 and the square roots of these quantities are the system natural frequencies ω_r (r = 1,2,……., n). the natural frequencies can be arranged in increasing order of magnitude. The lowest frequency ω_1 is referred to as the fundamental frequency, and for many practical problems it is the most important one.

Associated with every one of the frequencies ω_r, there is a certain nontrivial vector x_r, whose components x_{ir} are real numbers, where x_r is a solution of the eigenvalue problem, and hence it satisfies,

$$Kx_r = \omega_r{}^2 M x_r$$

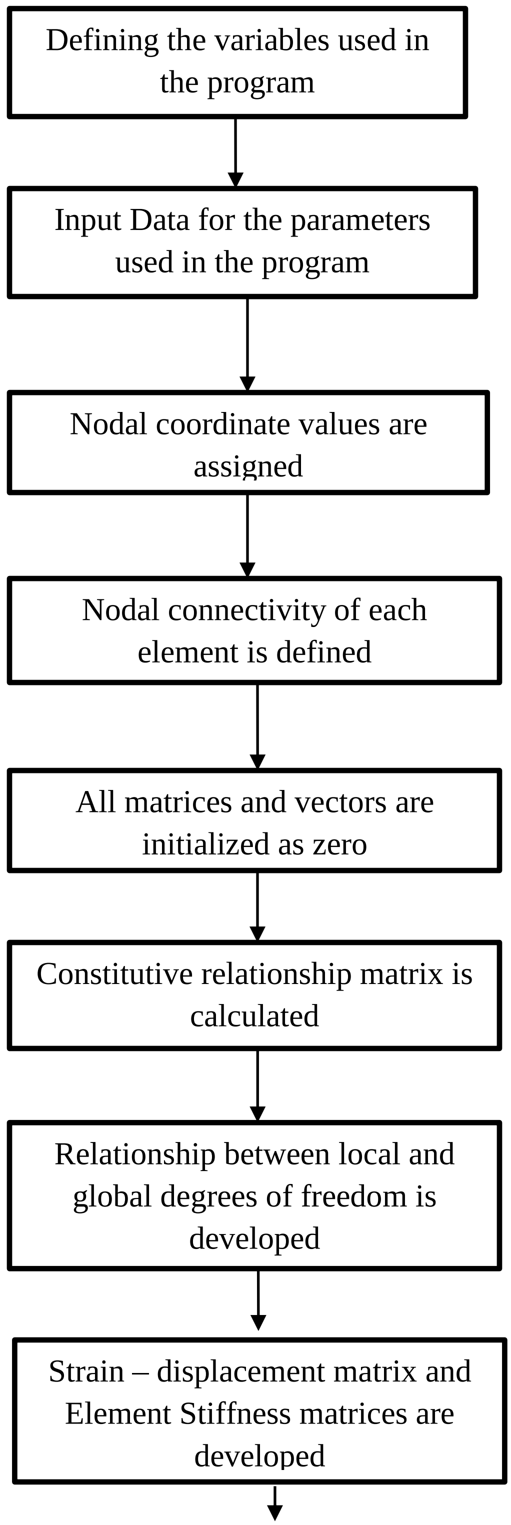
r = 1,2,…….,n (3.5.9)

The vectors x_r (r = 1,2,…….,n) are known as characteristic vectors or eigenvectors. The eigenvectors are also referred to as modal vectors and represent physically the so-called natural modes.

CHAPTER – 4

THE ALGORITHM

4.1. Algorithm to Programming



All the element stiffness matrices assembled to form the assembled stiffness matrix

Force vector for each element are computed

Force vectors are assembled to form assembled force vector

Boundary conditions are applied

The rows and columns of assembled stiffness matrix corresponding to locked degrees of freedom are deleted

The matrix equation is solved to calculate the displacements

Deflection curves are plotted for the corresponding deflection values

Element mass matrices are calculated

All the element mass matrices assembled to form the assembled mass matrix

The rows and columns of assembled mass matrix corresponding to locked degrees of freedom are deleted

The eigenvalues and eigenvectors corresponding to the assembled stiffness matrix and assembled mass matrices are computed

The frequencies and mode shapes of the system are computed

CHAPTER – 5

RESULTS AND DISCUSSION

5.1. Introduction

In the present chapter the developed finite element code is first validated with the analytical results available for maximum deflection of plates under bending. The plate with various mesh size are developed and the result is validated in each cases.

5.2. Convergence Study

A convergence study is done on an isotropic square plate with all its edges simply supported with varying mesh size. A uniformly distributed load of intensity ‘q’ is applied throughout the plate. The schematic diagram of the plate is given in Fig. 4.1.

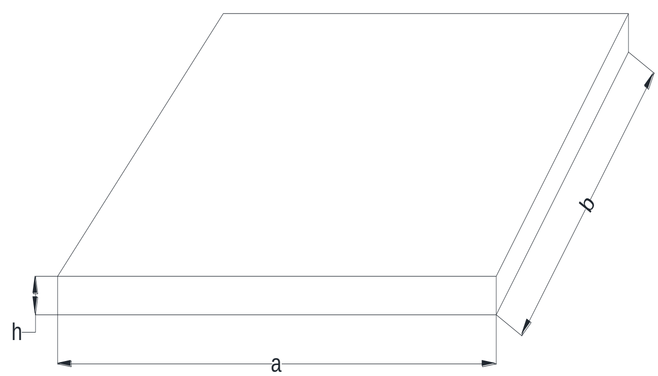


Figure 5.1. Isotropic Plate

The geometric and material properties for the plate are shown below:

Isotropic Material Properties:

Material: Steel

Young’s modulus: $2*10^5$MPa

Poisson’s ratio: 0.3

Geometric Properties:

Length of plate, a = 1m

Width of plate, b = 1m

Thickness of plate, h = 0.01m

Intensity of loading, q = 0.001MPa

The maximum deflection of the plate simply supported at all the four edges is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{4qa^4}{\pi^6 D} \tag{5.2.1}$$

where, $D = \frac{Eh^3}{12(1-\mu^2)}$ = Modulus of Rigidity of plate

The analytical maximum deflection is obtained as, w_{max} = 0.227mm.

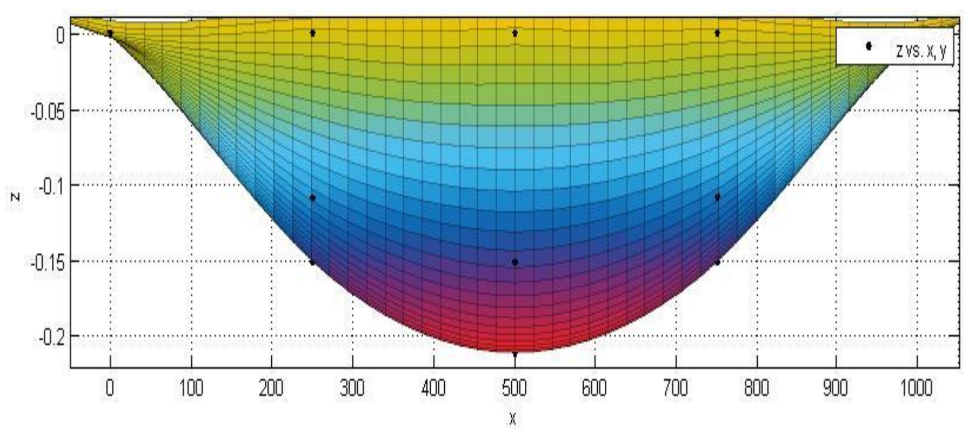


Figure 5.2. Deflection of plate simply supported at four edges

Table 5.1:- Maximum deflection of isotropic plate simply supported at four edges with varying mesh size

Mesh Size	Maximum Deflection (mm)	Percentage Error from Analytical Value (%)	Convergence percentage error (%)
4*4	0.2101059	7.442	-
8*8	0.218959	3.542	4.214
16*16	0.2211086	2.595	0.982
32*32	0.2216328	2.364	0.236

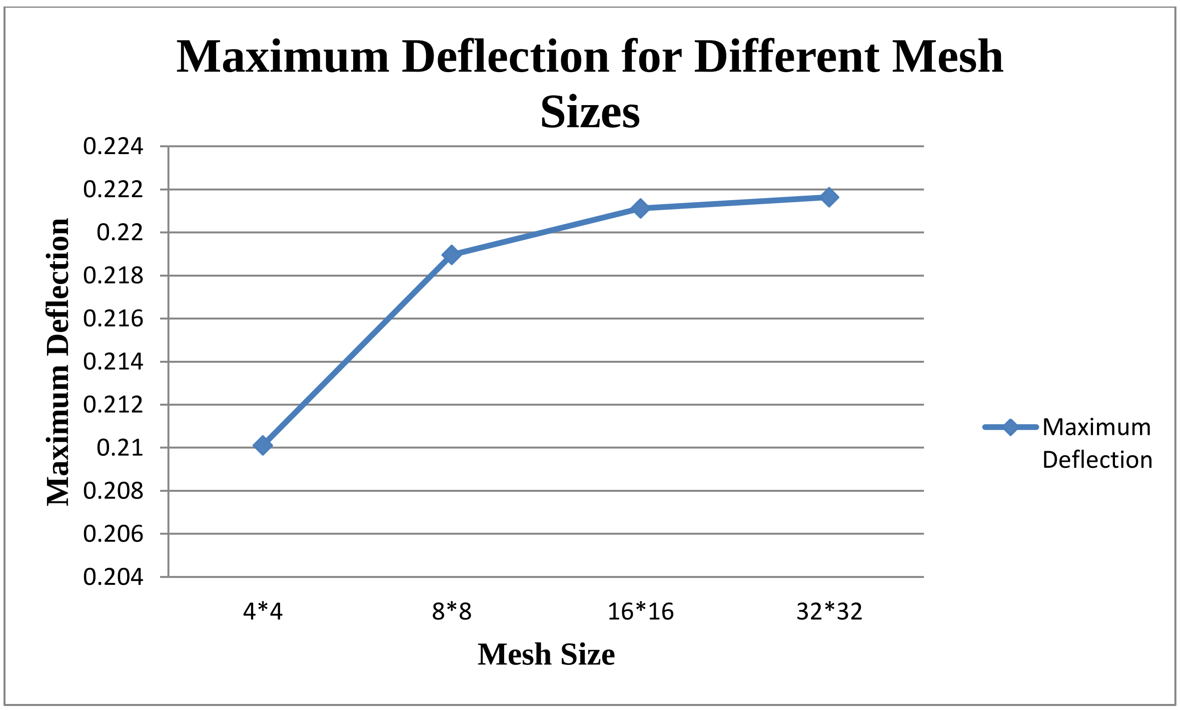


Figure 5.3. Convergence graph for maximum deflection of plate simply supported at four edges for varying mesh

It is seen from Table 5.1 and the convergence graph in figure 5.3 that as the number of elements increases, the deflection values converges very well and convergence percentage are reducing. And the result has almost converged for mesh size of 16*16 and 32*32. Now, the convergence of results for various edge conditions for different mesh sizes is discussed.

5.3. Case Studies of Plates with various edge conditions:

5.3.1. Square Plate with all the edges clamped

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.00126qa^4}{D} \tag{5.3.1.1}$$

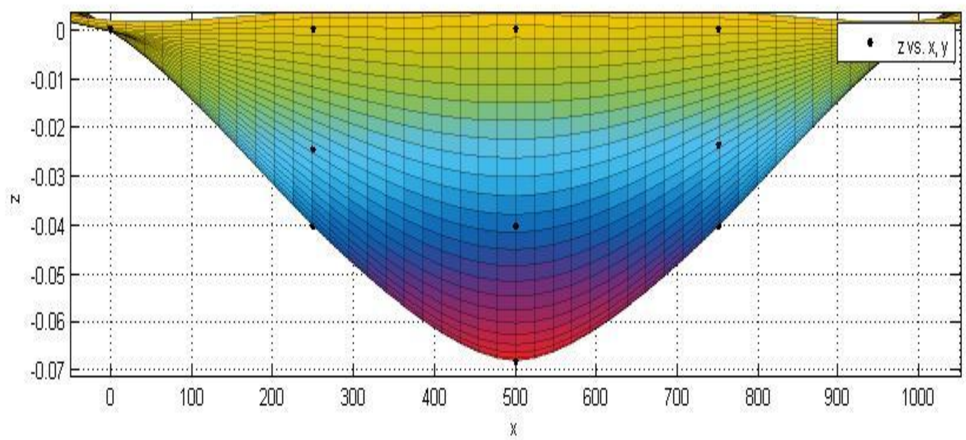


Figure 5.4. Deflection of plate with all edges clamped

Table 5.2:- Maximum deflection of plate with all edges clamped

Mesh Size	Maximum Deflection (mm)	Theoretical Deflection (mm)
4*4	0.0679	0.0688
8*8	0.0689	
16*16	0.069627	
32*32	0.069623	

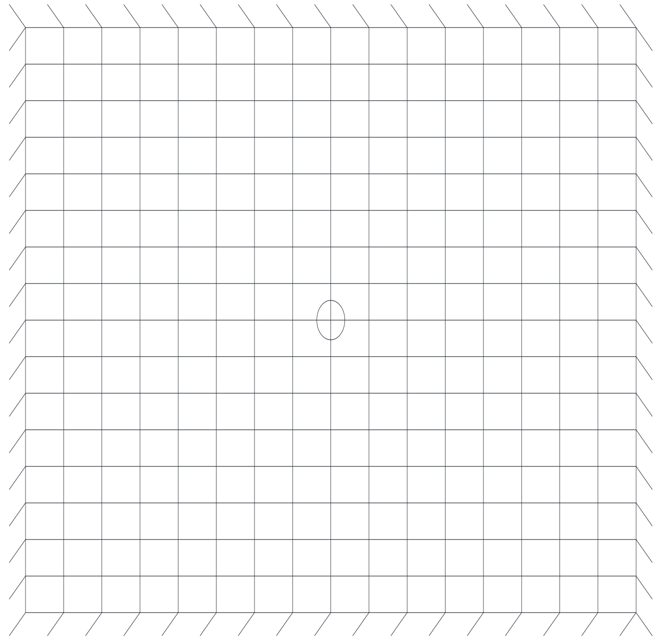


Figure 5.5.Plate with all the edges clamped

The maximum deflection occurs at the center of the square plate.

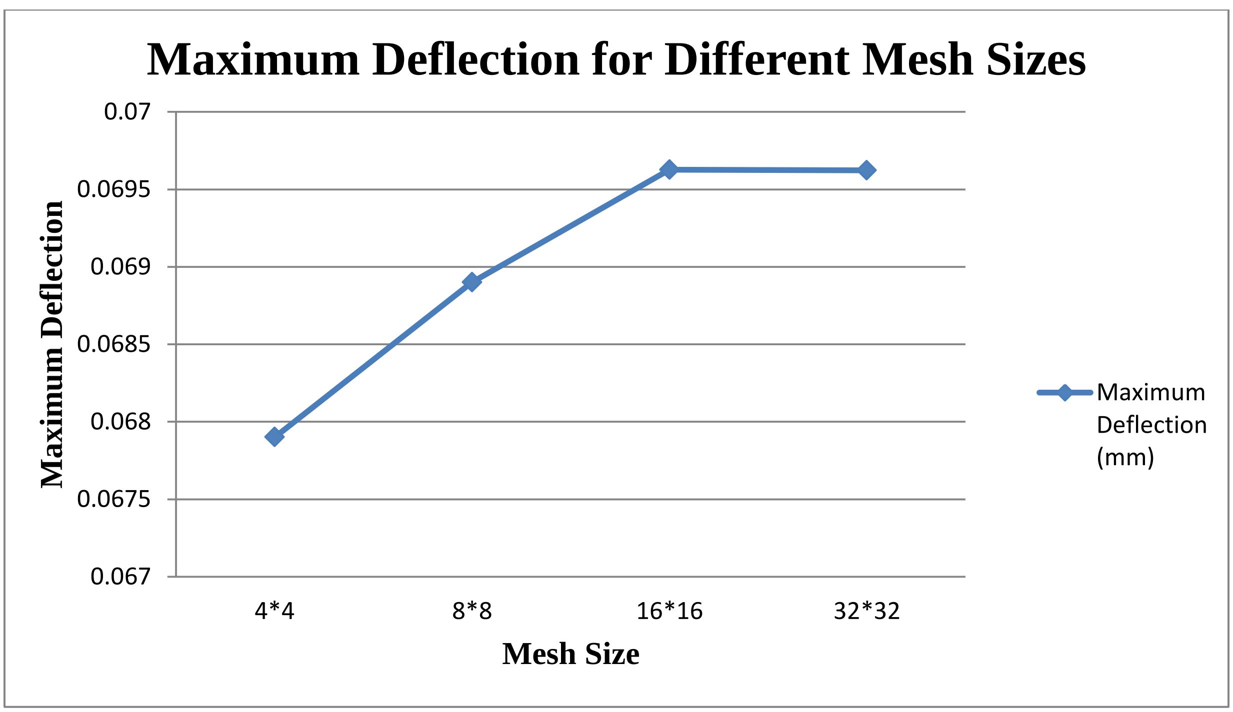


Figure 5.6. Convergence graph for maximum deflection of plate with all edges clamped for varying mesh

It is seen from the above table 5.2 and the convergence graph figure 5.6 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.3.2. Square Plate with two opposite edges simply supported and the other two edges clamped

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.00192qa^4}{D} \tag{5.3.2.1}$$

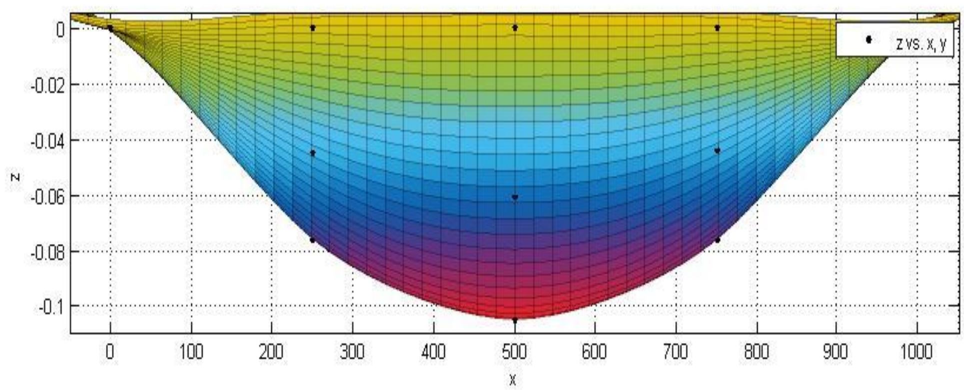


Figure 5.7. Deflection of plate with two opposite edges simply supported and the other two edges clamped

Table 5.3:- Maximum deflection for plate with two opposite edges simply supported and the other two edges clamped

Mesh Size	Maximum Deflection (mm)	Theoretical Deflection (mm)
4*4	0.1048883	0.104832
8*8	0.1058999	
16*16	0.1065441	
32*32	0.1082271	

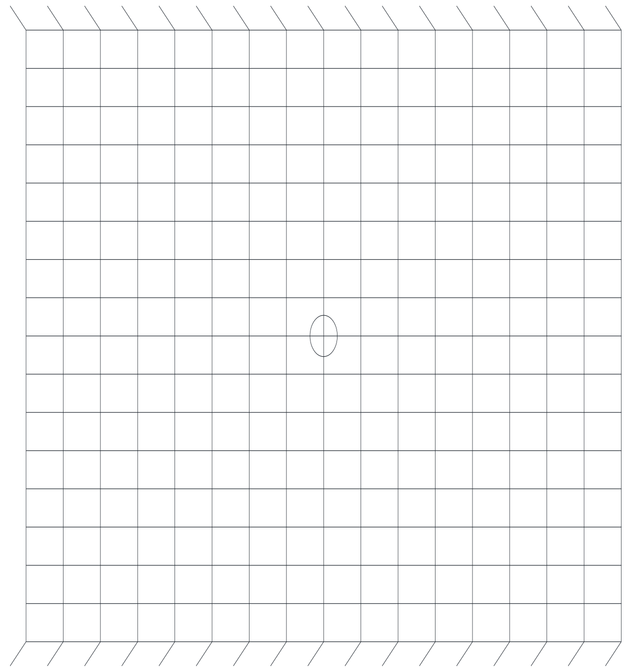


Figure 5.8.Plate with two opposite edges simply supported and the other two edges clamped

The maximum deflection occurs at the center of the square plate.

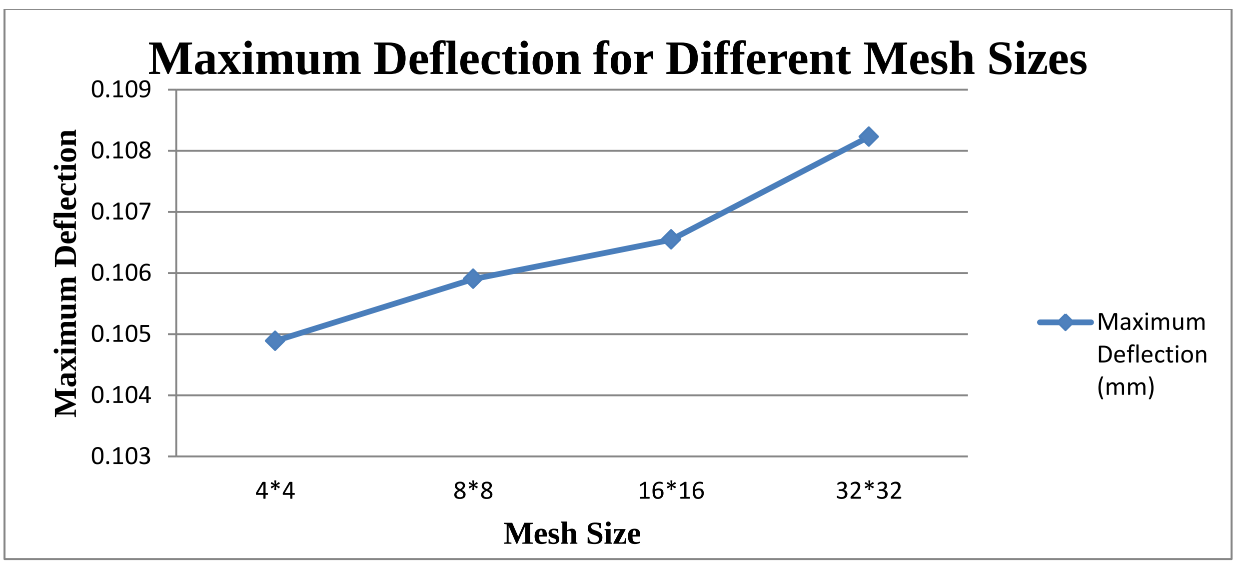


Figure 5.9. Convergence graph for maximum deflection of plate with two opposite edges simply supported and the other two edges clamped for varying mesh

It is seen from the above table 5.3 and the convergence graph figure 5.9 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.3.3. Square Plate with three edges simply supported and one edge built in

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.0028qa^4}{D} \tag{5.3.3.1}$$

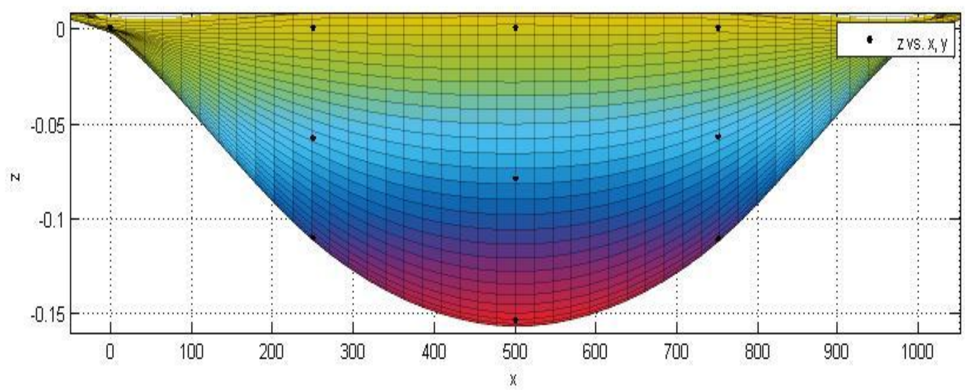


Figure 5.10. Deflection of plate with three edges simply supported and one edge built in

Table 5.4:- Maximum deflection for plate with three edges simply supported and one edge built in

Mesh Size	**Maximum Deflection (mm)**	**Theoretical Deflection (mm)**
4*4	0.1527119	0.15288
8*8	0.1538067	
16*16	0.1548917	
32*32	0.1549821	

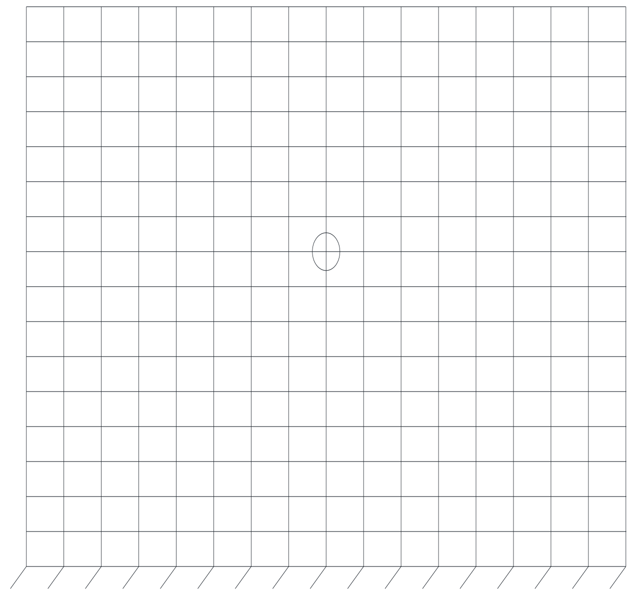


Figure 5.11.Plate with three edges simply supported and one edge built in

The maximum deflection occurs at the node no. 162, i.e. the node just above the central node of 16*16 division mesh.

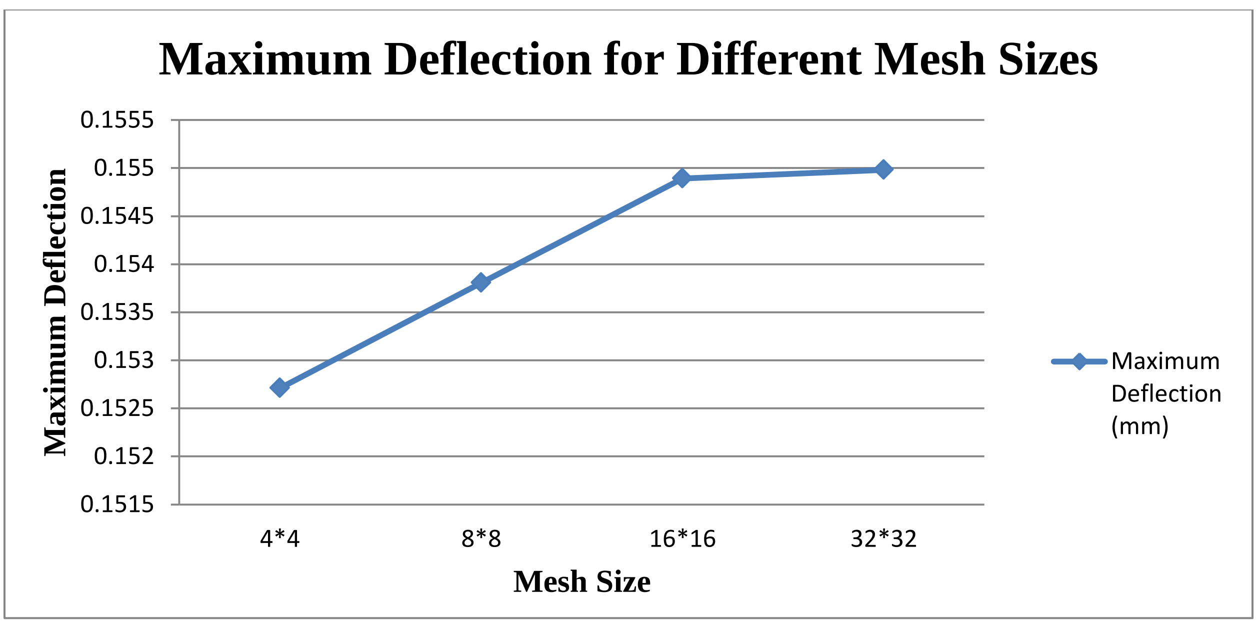


Figure 5.12. Convergence graph for maximum deflection of plate with three edges simply supported and one edge built in for varying mesh

It is seen from the above table 5.4 and the convergence graph figure 5.12 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.3.4. Square Plate with one edge simply supported and other three edges built in

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.00157qa^4}{D} \tag{5.3.4.1}$$

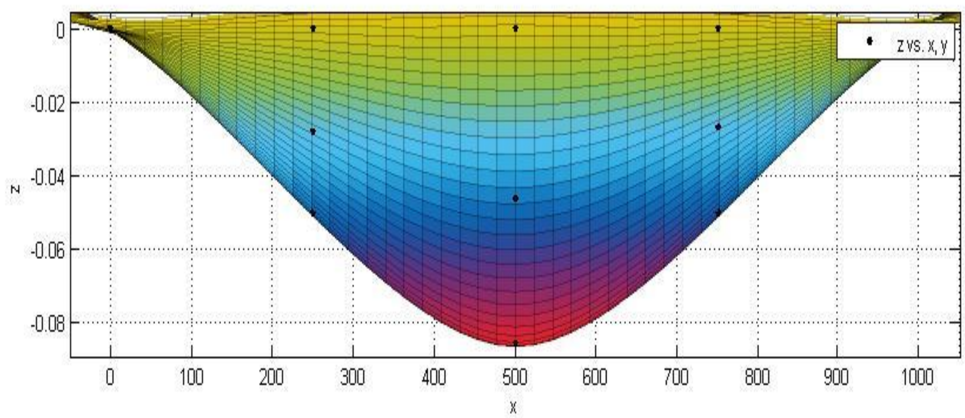


Figure 5.13. Deflection of plate with one edge simply supported and other three edges built in

Table 5.5:- Maximum deflection for plate with one edge simply supported and other three edges built in

Mesh Size	Maximum Deflection (mm)	Theoretical Deflection (mm)
4*4	0.08463584	0.08572
8*8	0.08584063	
16*16	0.085993	
32*32	0.08609803	

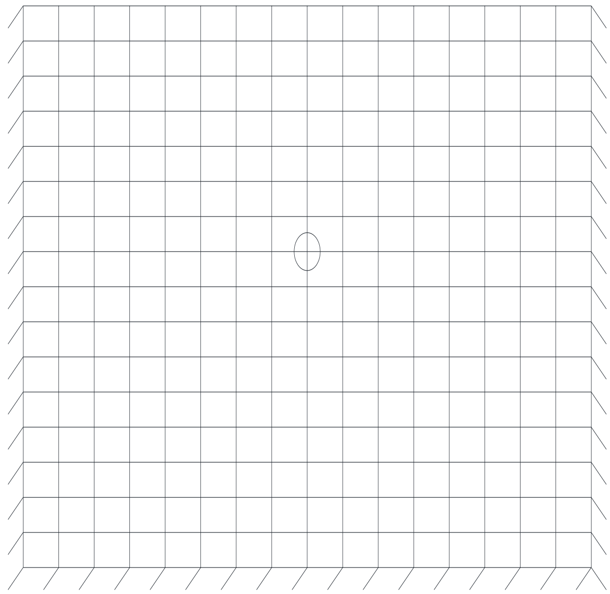


Figure 5.14.Plate with one edge simply supported and other three edges built in

The maximum deflection occurs at the node no. 162, i.e. the node just above the central node of 16*16 division mesh.

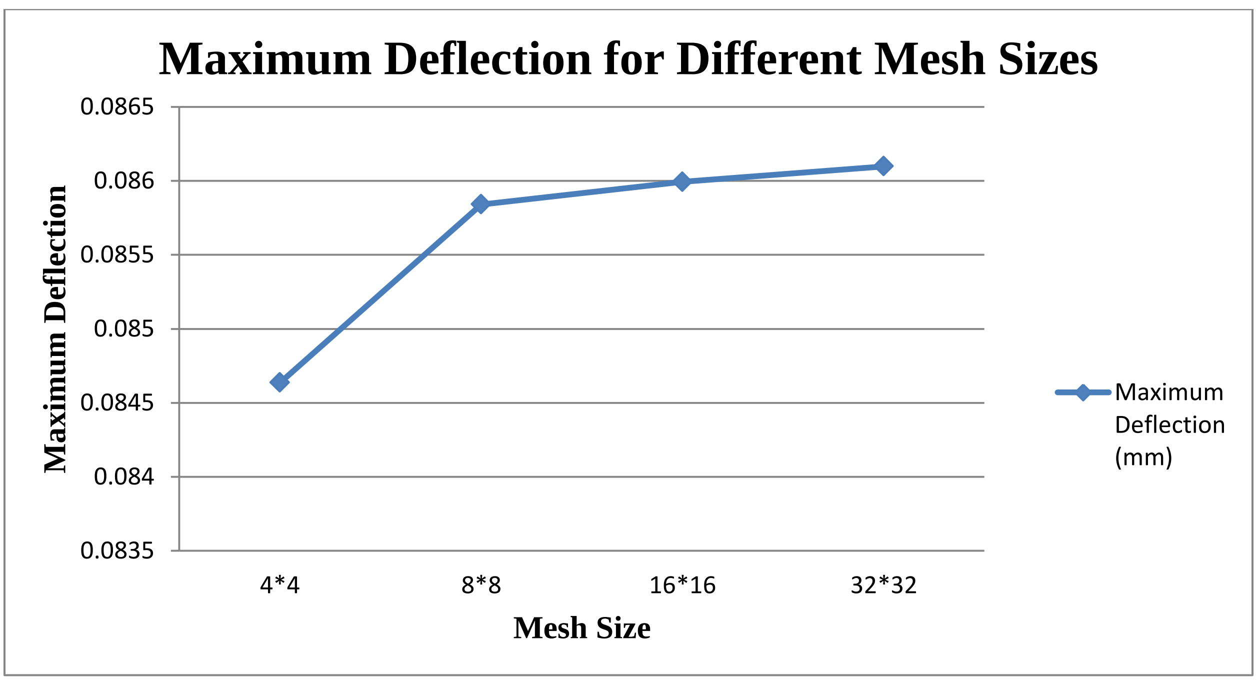


Figure 5.15. Convergence graph for maximum deflection of plate with one edge simply supported and three edges built in for varying mesh

It is seen from the above table 5.5 and the convergence graph figure 5.15 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.3.5. Square Plate with two opposite edges simply supported, third edge free and fourth edge built in

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.0113qa^4}{D} \tag{5.3.5.1}$$

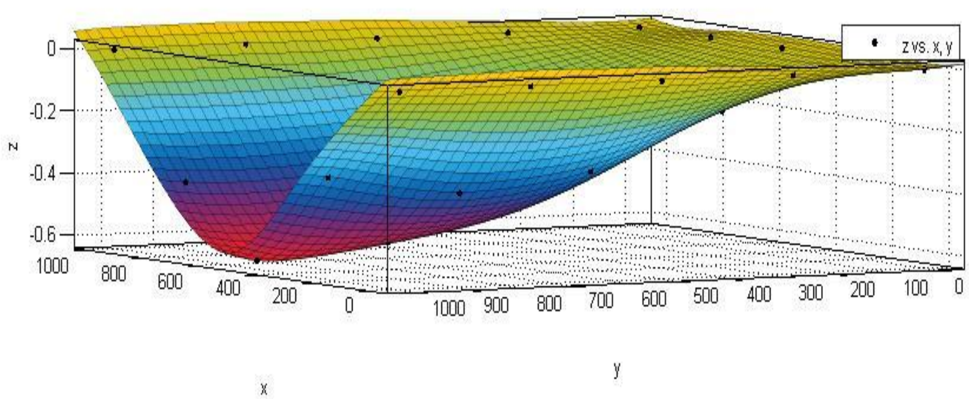


Figure 5.16. Deflection of plate with two opposite edges simply supported, third edge free and fourth edge built in

Table 4.6:- Maximum deflection for plate with two opposite edges simply supported, third edge free and fourth edge built in

Mesh Size	Maximum Deflection (mm)	Theoretical Deflection (mm)
4*4	0.59355492	0.617
8*8	0.60768718	
16*16	0.61475331	
32*32	0.62181944	

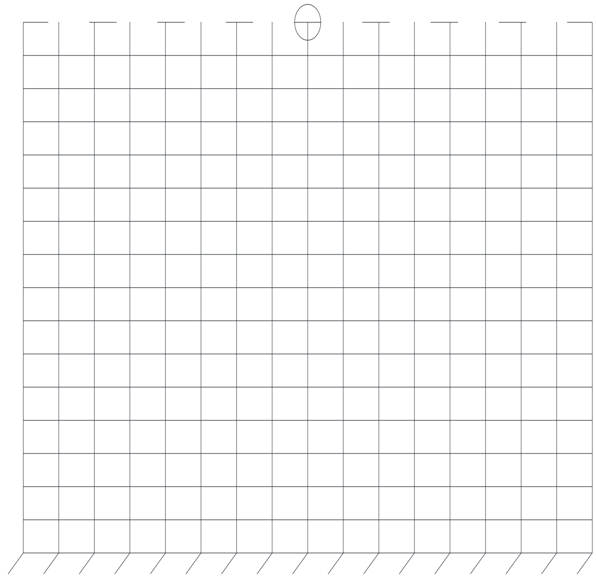


Figure 5.17.Plate with two opposite edges simply supported, third edge free and fourth edge built in

The maximum deflection occurs at the node no. 281, i.e. at the central node of the free edge of 16*16 division mesh.

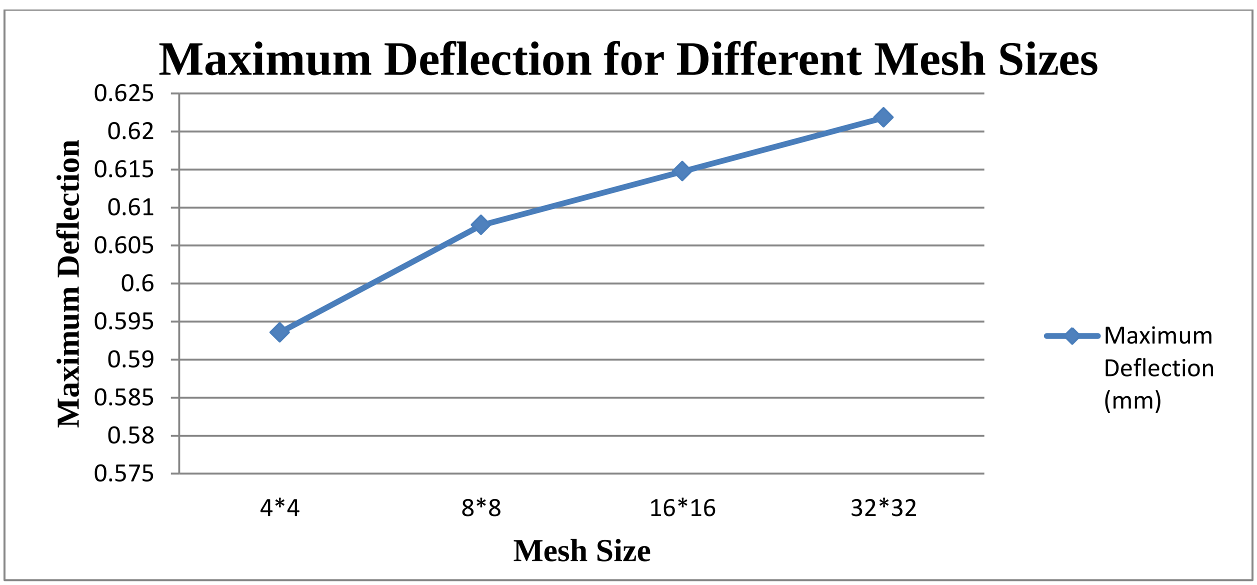


Figure 5.18. Convergence graph for maximum deflection of plate with two edges simply supported, third edge free and fourth edge built in for varying mesh

It is seen from the above table 5.6 and the convergence graph figure 5.18 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.3.6. Square Plate with three edges built in and fourth edge free

The maximum deflection of the plate is calculated analytically by the formula [13] given below:

$$w_{max} = \frac{0.01286qa^4}{D} \tag{5.3.6.1}$$

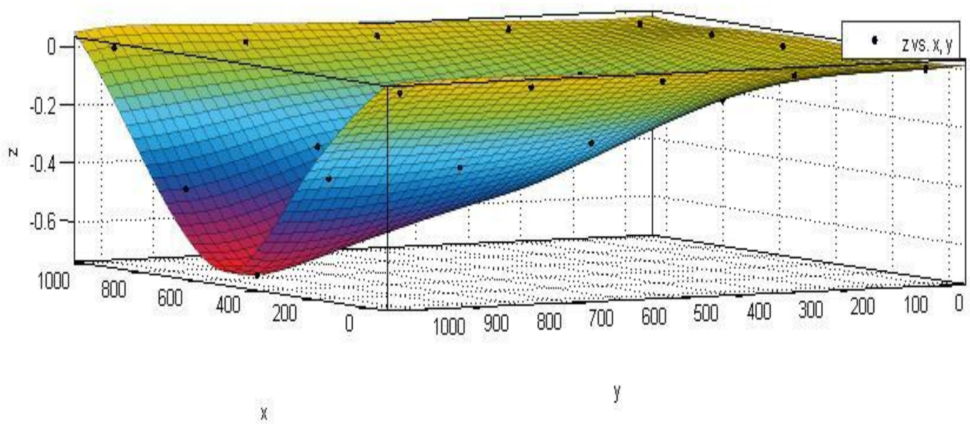


Figure 5.19. Deflection of plate with three edges built in and fourth edge free

Table 5.7:- Maximum deflection for plate with three edges built in and fourth edge free

Mesh Size	Maximum Deflection (mm)	Theoretical Deflection (mm)
4*4	0.6862605	0.702156
8*8	0.69998571	
16*16	0.702848315	
32*32	0.70327962	

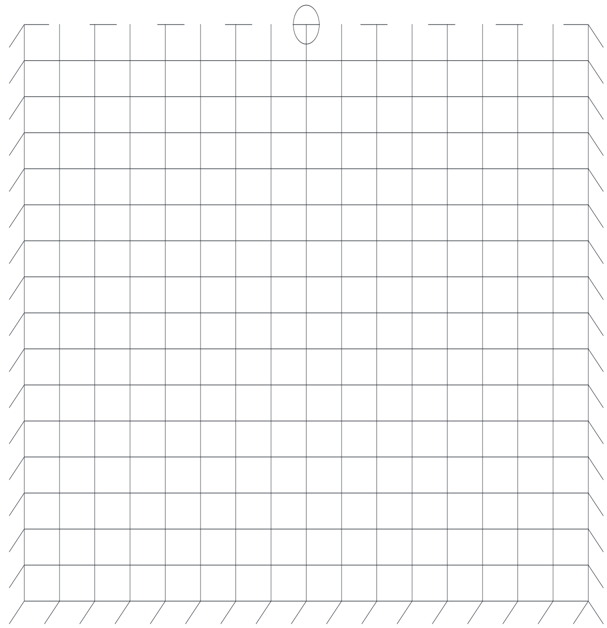


Figure 5.20.Plate with three edges built in and fourth edge free

The maximum deflection occurs at the node no. 281, i.e. at the central node of the free edge of 16*16 division mesh.

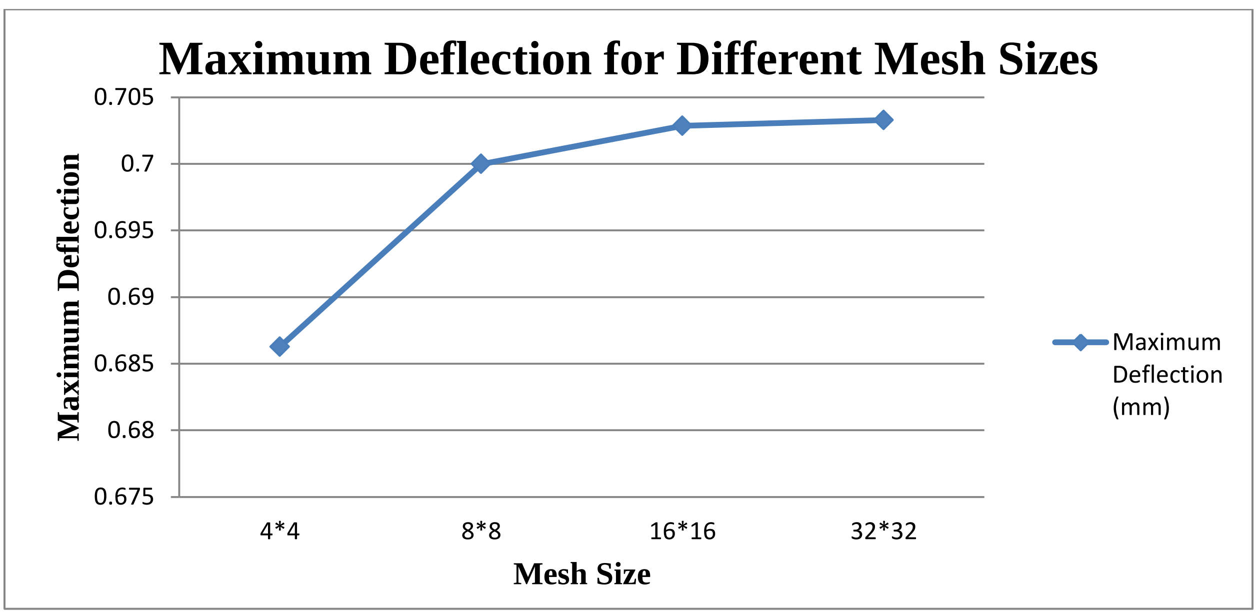


Figure 5.21. Convergence graph for maximum deflection of plate with three edges built in and fourth edge free for varying mesh

It is seen from the above table 5.7 and the convergence graph figure 5.21 that as the number of elements is increased, the deflection values has almost converged for mesh size of 16*16 and 32*32.

5.4. Mode shapes for plate for various edge conditions and varying mesh sizes

5.4.1. Mode shapes for plate simply supported at four edges

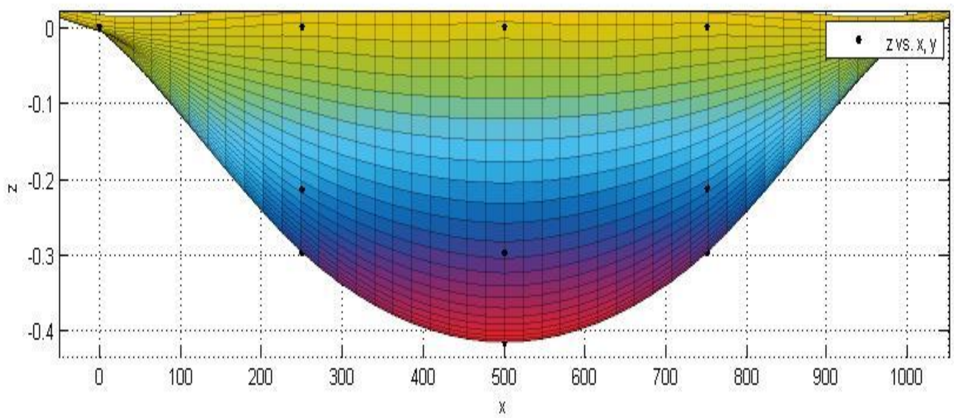


Figure 5.22. Mode shape 1 for simply supported plate

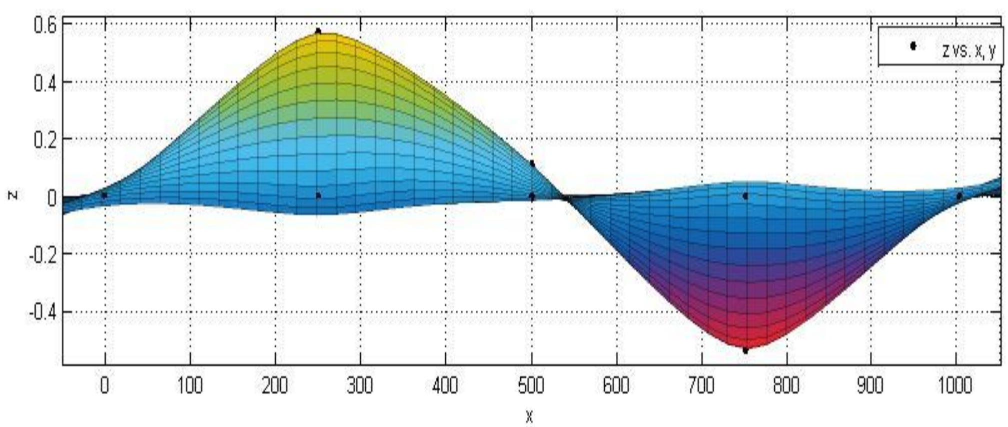


Figure 5.23. Mode shape 2 for simply supported plate

5.4.2. Mode shapes for plate clamped at four edges

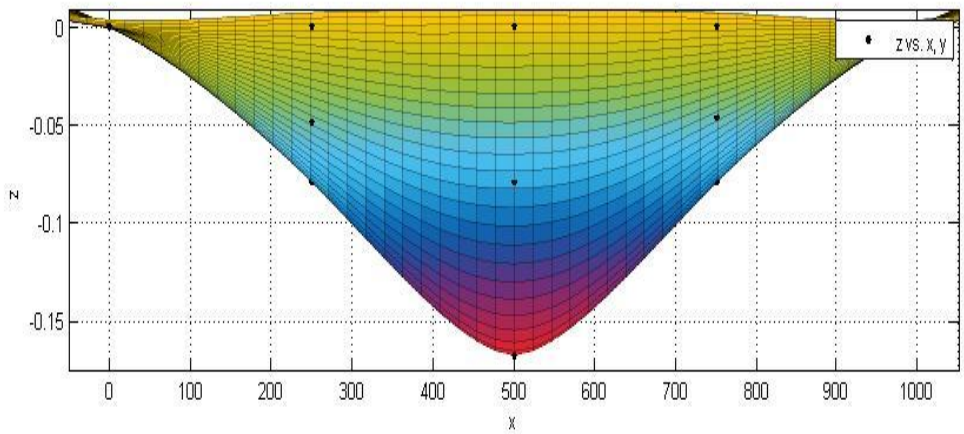


Figure 5.24. Mode shape 1 for fixed plate

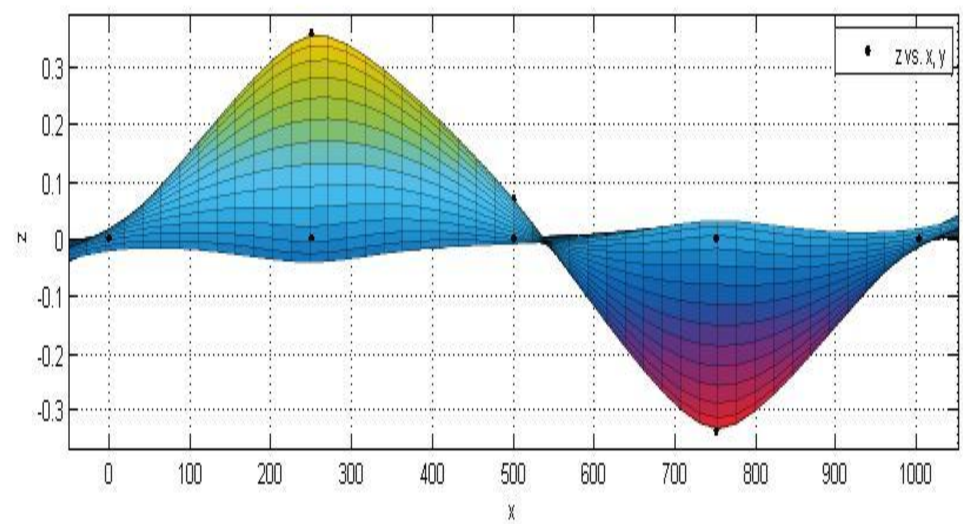


Figure 5.25. Mode shape 2 for fixed plate

CHAPTER – 6
CONCLUSIONS AND SCOPE FOR FUTURE WORK

6.1. Conclusions

The present work is concerned with the comparison of the results of deflection of a square or rectangular plates divided into triangular elements developed from program with the analytical results. Natural frequencies and mode shapes for specific boundary conditions and mesh sizes are also calculated. MATLAB software is used for developing the finite element code for the problem described above.

After doing a thorough study of existing literatures, it has been found out that ample works has been done on free vibration of plate, shear deformation of plate, higher order triangular element under bending, but limited work has been done on isoparametric formulation of triangular elements. Hence, this topic has been chosen for analysis and to obtain results.

First of all, a square or rectangular plate divided into triangular elements subjected to uniformly distributed load throughout the plate with simply supported at all the four edges is analysed for bending and the deflection is found out for different mesh sizes such as 4*4, 8*8, 16*16 and 32*32. As the mesh sizes are increased, the deflection gets more converged.

The stiffness matrix formulation is validated from "Finite Element Analysis and Programming: An Introduction" by S.S. Shivaswamy [24]. The stiffness matrix of an example problem given in the book is validated with the MATLAB program.

Deflection plots are made for various edge conditions and convergence graph for various mesh size with various edge conditions are drawn.

Again, the mass matrices are formulated for simply supported and fixed boundary conditions only corresponding frequencies and mode shapes are calculated. First two mode shapes are plotted for each case.

Eigenvalue and eigenvector formulation is validated from an example problem from "Fundamentals of Vibrations" by Leonard Meirovitch [25].

Since variables are linearly distributed along the edges, higher number of elements is necessary to obtain convergence of results.

6.2. Scope for future work

The work report in this thesis is a limited part of a vast area of research of isoparametric triangular plate element under bending. It can be said that there are various problems in the mentioned area that needs to be addressed as there has been very few works done on this area. Some of the important aspects pertaining to the present work that need attention are listed as follows:

1. A study on isoparametric triangular element with six nodes, nine nodes can be developed and further finite element formulation can be done and convergence can be checked for various mesh sizes and various edge conditions.
2. Folded plate can be analyzed from the development of triangular plate element.
3. Besides bending analysis of the plate element, shear deformation analysis could be included to conduct static analysis, as well as, free and forced vibration analyses.

Appendix – A

Formulation of integrals over a triangular area[26]

The finite element method for two – dimensional problems with triangular elements requires the numerical integration of shape functions on a triangle. Since an affine transformation makes it possible to transform any triangle into the two – dimensional standard triangle T with coordinates (0, 0), (0, 1), (1, 0) in Cartesian frame, we have to consider just the numerical integration T. The integral of an arbitrary function, f, over the surface of a triangle T is given by,

$$I = \iint f(x,y)dxdy = \int_0^1 dx \int_0^{1-x} f(x,y)dy = \int_0^1 dy \int_0^{1-y} f(x,y)dx \tag{A}$$

It is now required to find the value of the integral by a quadrature formula:

$$I = \sum_{m=1}^{N} c_m \; f(x_m , y_m) \tag{B}$$

Where c_m are the weights associated with specific points (x_m, y_m) and N is the number of pivotal points related to the required precision.

The integral I of equation (A) can be transformed into an integral over the surface of the square: $\{(u,v) \mid 0 \le u, v \le 1\}$ by the substitution:

$$x = u, y = (1-u)v \tag{C}$$

Then the determinant of the Jacobian and the differential area are:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (1)(1-u) - 0(-v) = 1-u$$

And

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,dudv = (1-u)dudv$$

(D)

Then, on using equations (C) and (D) in equation (A), we have

$$\int_0^1\int_0^{1-x} f(x,y)dydx = \int_0^1\int_0^1 f\,(u,(1-u)v)(1-u)dudv$$

(E)

The integral I of equation (E) can be transformed further into an integral over the standard 2 – square: {(ξ, η) | -1 ≤ ξ, η ≤ 1} by the substitution

$$u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2}$$

(F)

Then clearly the determinant of the Jacobian and the differential area are:

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - (0)(0) = \frac{1}{4}$$

$$du\,dv = \frac{\partial(u,v)}{\partial(\xi,\eta)}\,d\xi d\eta = \frac{1}{4}d\xi d\eta$$

(G)

Now, on using equations (F) and (G) in equation (E), we have:

$$I = \int_0^1\int_0^{1-x} f(x,y)dydx = \int_0^1\int_0^1 f\,(u,(1-u)v)(1-u)dudv$$

$$= \int_{-1}^1\int_{-1}^1 f\left(\frac{1+\xi}{2},\frac{(1-\xi)(1+\eta)}{4}\right)\left(\frac{1-\xi}{8}\right)d\xi d\eta$$

(H)

Equation (H) represents an integral over the surface of a standard 2 – square: $\{(\xi, \eta) \mid -1 \leq \xi, \eta \leq 1\}$.

From equation (H), we can write:

$$I = \int_{-1}^{1}\int_{-1}^{1} f\big(x(\xi,\eta), y(\xi,\eta)\big)\left(\frac{1-\xi}{8}\right) d\xi d\eta$$

$$I = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1-\xi_i}{8}\right) w_i w_j f\left(x\left(\xi_i,\eta_j\right), y\left(\xi_i,\eta_j\right)\right)$$

(I)

Where ξ_i, η_j are Gaussian points in the ξ, η directions respectively, and w_i and w_j are the corresponding weights.

We can rewrite equation (I) as:

$$I = \sum_{k=1}^{N=n*n} c_k \; f(x_k, y_k)$$

(J)

where c_k, x_k and y_k can be obtained from the relations:

$$c_k = \frac{(1-\xi_i)}{8} w_i w_j \,, x_k = \frac{(1+\xi_i)}{2} \,, y_k = \frac{(1-\xi_i)(1+\eta_j)}{4}$$

(k = 1, 2, ………., n), (i, j = 1, 2, 3, ………, n) (K)

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