ANALYSIS OF FREE VIBRATION AND BUCKLING OF LAMINATED COMPOSITE PLATE

THESIS

Submitted by

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Under the Guidance of

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CERTIFICATE OF RECOMMENDATION

This is to certify that SUBHANKAR PRAMANIK (Exam Roll No M4CIV 1613, Registration no 107752 of 2009-2010 has carried out the thesis work entitled "**ANALYSIS OF FREE VIBRATION AND BUCKLING OF LAMINATED COMPOSITE PLATE"** under my direct supervision & guidance .He carried out this work independently .I hereby recommend that the thesis be accepted in partial fulfillment of the requirements for awarding the degree of "**MASTER OF ENGINEERING IN CIVIL ENGINEERING (STRUCTURAL ENGINEERING**)".

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This foregoing thesis is hereby approved as credible study on an Engineering Subject carried out and presented in a manner satisfactory to warrant its acceptance as a prerequisite to the degree for which it has been submitted.

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DECLARATION

I, Subhankar Pramanik a student of Master of engineering in civil engineering dept. (structural engineering), Jadavpur university, Faculty of engineering & technology, hereby declare that the work being presented in the thesis work entitled, **"Analysis of Free Vibration and Buckling of Laminated Composite Plate"** is authentic record of work that has been carried out at the Department of civil engineering**,** Jadavpur university, under Assistant Professor (Dr.) Sreyashi Das(Pal) Department of civil engineering**,** Jadavpur university.

The work contained in the thesis has not yet been submitted in part or full to any other university or institution or professional body for award of any degree or diploma or any fellowship.

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CHAPTER 1 INTRODUCTION

1.1 Composite material

In general, a material that is formed by combining different materials on a macroscopic scale is known as composite material. A composite material usually derives it's properties from it's constituents and sometimes it's properties may be quite different from those of the constituents. Which are plywood and reinforced concrete are some of the composite materials, which are being used for a long time. A composite material may be classified as fibrous, laminated and particulate. In fibrous composite material, the fibres are embedded in a matrix. The load is mainly carried by fibres. The matrix binds the fibres, distributes the load along the fibres and prevents the fibres from direct exposure to the environment. The fibres and the matrix may be of the same material or different materials. The fibres used in this material are characterised by their near crystal size diameter. In laminated composite materials, layers of differing properties are bonded together to act as an integral part. In particulate composite materials, particle of different materials are held in a matrix.

Nowadays, fibre reinforced plastics are being increasingly used in aerospace applications due to their high specific strength, high specific stiffness and low density. In addition, they have good corrosive resistance. A designer can easily tailor these materials for different applications. Epoxy polyester, vinyl-ester, phenol are commonly employed as a matrix. High performance thermoplastics are also being utilised on a large scale. It is generally considered that the reduction in weight up to 25% can be achieved by using fibre reinforced plastics in place of conventional materials of air craft. Glass-epoxy composite materials were the first to be used in aircraft structures in mid-forties. Due to low specific stiffness glass-epoxy, compared to conventional aircraft materials, it was not used in major applications. In around 1960 graphite and boron fibres were developed. Since, graphite-epoxy and boron-epoxy composite materials are superior to conventional metals used in aircrafts, in terms of both strength and stiffness; they were used in aircraft applications to a significant level. Graphite-epoxy which is much cheaper than boron-epoxy became very popular and has been used in aircraft structures. Kevlar-epoxy composite are also widely used in aircraft structures, but to smaller extents compared to graphite-epoxy composite materials.

In fibre reinforced plastic composites, first thin lamina is prepared from fibres and matrix (sometimes a lamina may also be made from woven fabric). Laminae with differing fibre orientations are bonded together to form an integral structural components, which is known as laminate. A lamina is considered to be homogeneous as macroscopic level. It has three plane of symmetry, and hence termed as orthotropic. It’s stress-strain behaviour is usually treated as linear elastic the laminates may be symmetric or antisymmetric or unsymmetric. They are also referred to as cross-ply or angled ply depending on fibre orientations of lamina. If the fibre orientations in a laminate are zero degree or ninety degree it is called cross ply and or any other fibre orientations, it is known as angled ply.

1.2 Finite element method

With the development of computers, engineer in this century turned to matrix algebra for solving structural engineering problems. This has led to development of matrix analysis for skeletal structures and then the more generalised finite element analysis of continuum structures based on Variational Principles. In the finite element method, the continuum structure is discretised into elements of finite size, called as finite elements. The continuum structure is considered as an assemblage of elements connected at finite number of joints, known as nodes. This follows the concept of discretisation used in the finite difference method, which was popular prior to the advent of finite element method for solving structural engineering problems for which closed form solution is not obtainable. This method can be implemented in two ways viz, the displacements as unknowns called as the finite element displacement method, and the internal forces as unknowns called as finite element force method. The finite element displacement method is more commonly used. In the finite element displacement method, the element displacement field is expressed as simple functions, known as shape functions, and the displacements at the nodes of an element the strains and stress within the element are also expressed in terms of the element nodal displacements. Then the Variational Principles, total potential energy or virtual work, is applied for the element to derive the equilibrium equations in matrix form. The equilibrium equations for the entire continuum structures are obtained by proper assembling of element matrix so that continuity of displacement at the nodes, where the elements are connected is ensured.

The primary of the finite element method is that it can be conveniently applied to study the buckling and free vibrations effects on the structural response of laminated composite plates. Reasonably accurate solutions may be obtained by the finite element method for free vibration and buckling or arbitrary laminated composite plates for various boundary conditions. Since, for fibre reinforced laminated plates, the traverse shear deformation cannot be ignored it can be easily incorporated in the finite element analysis by using the finite element method in the laminated plates.

1.3 Objectives of the investigation

Among the most important structural functions that a system can provide are stiffness, strength, fracture toughness, ductility, fatigue strength, energy absorption, damping, and thermal stability. With conventional structural materials, it has been difficult to achieve simultaneous improvement in multiple structural functions, but the increasing use of composite materials has been driven in part by the potential for such improvements. The use of nano reinforcements in polymer composites has produced unprecedented improvements in mechanical properties of the composites material. In recent years, the use of composite laminates is increasing in various engineering fields where weight saving is crucial, mainly because of their high strength-to-weight, stiffness-to weight ratios, good energy, and sound absorption, and often also low production cost. The laminates have a large number of parameters involved with the production and fabrication processes. Complete control of these parameters is not economically feasible and also not practically possible. Hence variations in the system properties, such as material properties, geometric properties, etc., are inherent in nature. As the composites are used in various fields such as naval structures, aircraft structures etc., which are subjected to different dynamic loading, in order to obtain an effective design of the composite plates, a study of their free vibration and buckling due to different loading is important. Composite materials have inherent uncertainties due to the large number of design variables that a re-involve in fabrication and the lack of total control over the manufacturing and processing techniques. Hence, deterministic analysis is insufficient to provide complete information about the structural response. Consequently, there has been considerable interest over the last few decades in developing stochastic formulations for predicting the actual static and dynamic behaviour of composite plates with random system parameters. The modelling and determination of the foundation properties are also a matter of concern in the practice.

The mathematical model is complicated since the material is of orthotropic nature. First order transverse shear deformation is accounted along with rotary inertia of the material. A shear correction factor of 5/6 is taken in the analysis. Eight-nodded iso-parametric plate finite elements have been implemented in the present computations.

In this paper the free vibration and buckling behaviour of laminated composite rectangular plates are being provided through parametric studies conducted by incorporating variation in different aspect ratios, different fibre orientations, different number of layers, various boundary conditions and their overall thicknesses. A Finite Element program in MATLAB is developed. Detail interpretations of the results are provided.

The classical plate theory (CPT) [1] and first order shear deformation theory (FSDT) [2] are commonly used theory for the analysis of laminated composite plates. However, CPT predicts good results for thin plates only, because the transverse shear deformation is omitted in CPT. FSDT does not satisfy shear stress free conditions at top and bottom surfaces of plates. The shear correction factor is needed to appropriately take into account the strain energy of shear deformation. Its value depends on the material coefficients, geometry, stacking scheme, boundary conditions and loading conditions, which cannot be easily determined for practical problems. Further, FSDT is not capable of properly constraining all the displacements at the clamped supports of beams and plates. Higher order shear deformation theories are therefore developed to overcome these limitations of classical laminated plate theory for the better representation of the bending, buckling and vibration of the laminated composite. Several review articles on laminated composite and sandwich plates have been reported in the literature by various researchers, such as Reddy [3], Kapania and Raciti [4], Noor et al. [5], Bert [6], K. S. Sai Ram and P. K. Sinha[7],A. GuhaNiyogi [8], Mallikarjuna and Kant [9]. Several books are also available on vibration of plates such as, Leissa [10], Reddy [11], Liew et al. [12], Yang [13], Bathe [115] and Chakraverty[14]. Various methods for the analysis of plates are available in the literature. This article reviews the application of these methods for the free vibration and buckling analysis of laminated composite plates. The research reported from year 2000 to 2013 is reviewed with some classical references.

2.1. Navier'sMethod

Navier's solution technique is used only for simply supported boundary conditions. Many higher order shear deformation theories have been reported in the literature for the free vibration analysis of simply supported plates using Navier's method. Reddy [15] has developed a well known third order shear deformation theory which is further used by many researchers for their research. Recently, Aghababaei and Reddy [16] reformulated the third-order shear deformation plate theory of Reddy [15] using the nonlocal linear elasticity theory and applied for the bending and vibration ofplates. Ray [17] has developed Zeroth order shear deformation theory and applied for the free vibration analysis of laminated composite plates. Shimpi et al. [18], Zenkour [19], Ghugal and Sayyad [20], Neves et al. [21] developed some trigonometric shear deformation theories for the free vibration analysis of isotropic, orthotropic, laminated composite, sandwich and functionally graded plates. Recently, several new hyperbolic shear deformation theories are developed by Neves et al. [22] and Zenkour [23]. Karama et al. [24] have developed an exponential shear deformation theory for the free vibration and buckling analysis of laminated and composite plates which is further used by Thai and Choi [25] and Xiang et al. [26] developed nth-order shear deformation theory for thefree vibration analysis of isotropic, laminated composite and sandwich plates. Forced vibration analysis of antisymmetric laminated rectangular plates with distributed patch mass using higher order shear deformation theory of Reddy has been carried out by Alibeigloo and Kari [27].K. S. Sai Ram [7],Chen et al. [28] presented buckling and vibration of initially stressed composite plates with temperature dependent material in thermal environments.

2.2. Levy's Method

Xiang andWei [29] employed Levy's solution technique for the free vibration and buckling analysis of multi-span rectangular plates. Thaiand Kim [30] also employed Levy type solution technique for free vibration analysis of orthotropic plates based on two variable plate theory. The eigen function system of the Hamiltonian operator appearing in the free vibration of rectangular Kirchhoff plates with two opposite edges simply supported is studied by Bai and Chen [30]. Aydogdu and Ece [31] presented the buckling and free vibrationanalysis of rectangular isotropic plates with non-ideal boundary conditions based on classical plate theory using Levy type solution.

2.3 Rayleigh–Ritz Method

Narita [32] has developed a modified Ritz method to calculate natural frequencies of anisotropic rectangular plates with classical boundary conditions. Free vibration response of isotropic skew plates was studied by using conventional Rayleigh–Ritz method. Bert [33], Analas and Goker [34],Wang et al[35], presented a free vibration analysis of skew sandwich plates with laminated faces using Ritz method. Adam [36] employed Rayleigh–Ritz method to carry out vibration analysis of orthotropic plates. Zhou et al. [37] presented three-dimensional vibration analysis of thick rectangular plates using Ritz method. The study on vibration analysis of cross-ply laminated square plates subjected to different sets of boundary conditions has been carried out by Aydogdu and Timarci [38] using Ritz Method. Gupta et al. [39] studied vibration of polar orthotropic circular plate resting onWinkler foundation using Ritz method. Nallim and Grossi [40] employed Rayleigh–Ritz method for the vibration analysis of symmetrically laminated elliptical and circular plates. Lal and Kumar [41] used Rayleigh–Ritz method for the free vibrations of non-homogeneous orthotropic rectangular plates with bilinear thickness via classical plate theory. Carrera et al. [42] presented free-vibration analysis of anisotropic simply supported plates using Rayleigh–Ritz Method based on layer-wise, equivalent single layer and zig-zag models. The effect of non-homogeneity of the material of plate structures on the vibration frequencies was presented by **Chakraverty et al. [43]** using Rayleigh Ritz method. Watkins and Barton [45] studied the free vibration analysis of laminated and sandwich plates on elastic foundation using Rayleigh–Ritz method. Iurlaro et al. [45] applied Rayleigh–Ritz approach for the free vibration analysis of laminated composite and sandwich plates using refined zigzag theory.

2.4. Differential Quadrature Method

Shu [46],Malekzadeh et al.[47]   has presented detail information of differential quadrature method with its engineering applications. They also studied vibration analysis of symmetrically laminated plates based on FSDT using the moving least squares differential quadrature method. The inter laminar stresses and deflections of a laminated rectangular plate under thermal vibration are presented by using the generalized differential quadrature method.

2.5 GalerkinMethod

Zhang and Sainsbury [48] applied the Galerkin element method to the vibration of rectangular damped sandwich plates. Gorman [49] reported free vibration analysis of completely free rectangular plates by the superposition Galerkin method. Chenet al. [50] studied free vibration analysis of thin plates of complicated shapes using Galerkin's method. Muthurajan et al. [51] and Chien and Chen [52] employed Galerkin's method for the nonlinear vibration analysis of laminated composite rectangular plates. Givli et al. [53] presented free vibrations analysis of delaminated sandwich panels using a modified Galerkin approach. Gupta and Kumar [54] studied the vibration of non-homogenous rectangular plate of linearly varying thickness using Galerkin Method. Liu et al. [55] also employed Galerkin method for the free vibration analyses of sandwich panels with square-honeycomb cores. Qianet al. [56] studied free and forced vibration of thick rectangular plates by using higher order shear and normal deformable theory and meshless local Petroc–Galerkin method.

2.6 Discrete Singular Convolution (DSC) Method

Wei et al. [57] studied vibration of plates by discrete singular convolution method. Zhao et al. [58] examined discrete singular convolution for the prediction of high frequency vibration of plates. Ng et al. [59] Civalek [60].Gurses et al. [61] , Wang and Xu [62] ,Zhu and Wang [63] carried out free vibration analysis of thin isotropic and anisotropic rectangular plates by the discrete singular convolution method.

2.7. Exact Solutions

Exact natural frequencies of thick multilayered laminated composite plates were presented by Srinivas and Rao [64] and Noor [65]. Batra and Aimmanee [66] pointed out and presented the in-plane distortional modes of vibration missing from the solution of Srinivas and Rao [64]. Leissa and Kang [67] and Kang and Shim [68] presented exact solutions for free vibration analysis of rectangular plate subjected to linearly varying in-plane stresses. Zhang et al. [69] have reported three-dimensional theory of elasticity for free vibration analysis of composite laminates via layer wise differential quadrature modelling. The first-known exact solutions for vibration of stepped rectangular Mindlin plates with two opposite edges simply supported and the remaining two edges being either free, simply supported or clamped was presented by Wei [70]. He also proposed a novel Bessel function method to obtain the exact solutions for the free vibration analysis of rectangular thin plates with different boundary conditions. Demasi [71] presented three-dimensional closed-form solutions and exact thin plate theories for isotropic plates. Saeidifar and Ohadi [72] developed exact solution for investigating vibration of non-uniform plate with time-dependent boundary conditions. Xing and Liu [73], Liu and Xing [74] developed new exact solutions for the free vibration analysis of isotropic and orthotropic plates Lim et al. [77] developed simplistic elasticity approach for exact free vibration solutions of rectangular Kirchhoff plates. Messina [78] presented the influence of different sets of edge–boundary conditions on the dynamics of freely vibrating isotropic and cross-ply multilayer laminated rectangular plates using three-dimensional theory of elasticity.

2.8.Finite Element Method

K. S. Sai Ram and P. K. Sinha [7] presented hygrothermal effects on Free Vibration and buckling of Laminated Composite Plates.Kant and Swaminathan [79], Swaminathan and Patil[138,139], Rao and Desai [80] have carried out free vibration analysis of laminated composite and sandwich plates based on higher order shear and normal deformation theory using finite element method.. Free vibration analysis of composite plates was carried out by Singh et al. [81] using higher order shear deformation theory with random material properties. Rikards et al.[82] analyzed vibration and buckling of plates using triangular elements. Setoodeh and Karami [83] presented the free vibration and buckling analysis of composite laminates with elastically restrained edges using finite element analysis. Hull and Buchanan [84] presented vibration analysis of square orthotropic stepped plates based on finite element method. Forced vibration analysis of rectangular plates using finite element method has been carried out by Ahmadian and Zangeneh [85]. An accurate, three-dimensional, higher order, mixed finite element modelling for the free vibration analysis ofmulti-layered laminated composite plates was presented by Desai et al. [86]. A triangular element based on Reissner–Mindlin plate theory is developed by Sheikh et al. [87] for the free vibration and buckling analysis of plates. A non-conforming C^1 finite element triangular model was presented by Chakrabarti and Sheikh [88] for the free vibration analysis of laminated plates. **S. Pal, A. GuhaNiyogi** [89] presented folded plate formulation in a stiffened laminated composite and sandwich folded plate vibration using finite element approach. Lal et al. [90] reported nonlinear free vibration of laminated composite plates on elastic foundation with random system properties. Nonlinear free vibration analysis of simply supported piezo-laminated plates has been carried out by Tanveer and Singh [91].Vibration analysis of composite laminated plates with variable fiber spacing using finite element method was studied by Kuoand Shiau [92]. Brischetto and Carrera [93] used Carrera Unified Formulation to study free vibration response of simply supported multi layered orthotropic composite plates. Shariyat[94] has developed a generalized global–local high-order theory for the vibration analyses of sandwich plates subjected to thermo-mechanical loads. Singh and Lal [95] studied stochastic non-linear free vibration analysis of laminated composite plates on elastic plates. **S. K. Singh and A. Chakrabarti**[96] also studied the stochastic free vibration analysis of laminated composite platessubjected to a thermal loading with general boundary conditions using finite element method. Carrera et al. [97] presented refined finite element model for the dynamic analysis of multi layered plates. Van et al. [98] presented vibration analysis of laminated composite plate/shell structures via a smoothed quadrilateral flat shell element with in-plane rotations. Srinivasa et al. [99] presented free flexural vibration on laminated composite skew plates using finite element analysis whereas Manna [100] studied free vibration of tapered isotropic rectangular plates. The free vibration behaviour of sandwich functionally graded plates is investigated using finite element method by Natarajan and Manickam [101]. Eftekhariand Jafari [102] used mixed finite element and differentialquadrature formulation for free vibration of rectangular and skew Mindlin plates with general boundary conditions. Elmalich and Rabinovitch [103] investigated the dynamic behaviour of soft-core sandwich plates using finite element method. A higher order displacement based formulation to investigate the plane strain edge vibrations or end modes in composite laminated sandwich plates has been developed by Chitnis et al. [104]. Cetkovic and Vuksanovic [105] studied vibrations of isotropic, orthotropic and laminated composite plates with various boundary conditions using finite element method. Chalak et al. [106]

presented finite element model for the free vibration analysis of laminated composite and sandwich plates. Singh and Chakrabarti [107] alsostudied static, vibration and buckling behaviour of laminated composite and sandwich skew plates under thermo-mechanical loading using finite element model based on refined higher order zigzag theory. Li et al. [108] carried out finite element analysis for the free vibration of composite sandwich plates. Ribeiro [109]used a Hierarchical finite element for geometrically non-linear vibration of thick plates. Kucukrendeci and Kucuk [110] applied finite element method for the vibration analysis of laminated composite plates on elastic foundation. **Thai et al.** [111] presented a finite element formulation for static, free vibration and buckling analyses of laminated composite plates using a new higher order shear deformation theory. This paper is used to verify the result of the programme.

3.1 Governing Equations of Lamina

A fibre reinforced composite lamina is an orthotropic material, since the presence of fibre say along direction 1, imparts more strength and stiffness in this direction compared to direction normal to it. As a lamina is basically a very thin plate, plane stress assumptions can be applied to this case. Considering unidirectional lamina as shown in Figure 3.1, the stress-strain relation in the principal material directions 1 and 2, arranged as per right hand cork screw rule, in presence of temperature and moisture can be given as

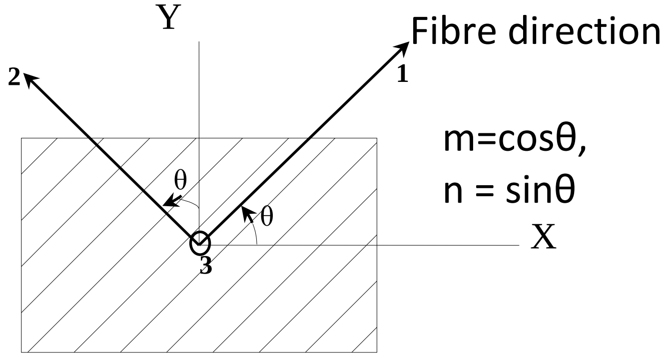


Figure3.1 Principal directions and local plate axes of a lamina.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 - e_1 \\ \varepsilon_2 - e_2 \\ \varepsilon_4 \text{ or } \gamma_{23} \\ \varepsilon_5 \text{ or } \gamma_{31} \\ \varepsilon_6 \text{or } \gamma_{12} \end{Bmatrix} \tag{2.1}$$

Where, $c_{11}= \frac{E_1}{1-\nu_{12}\nu_{21}}$, $c_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$,

$c_{12}= \nu_{12}\, c_{22} = \nu_{21}\, c_{11}$,

$c_{44}= G_{23}$, $c_{55}= G_{31}$, $c_{66}= G_{12}$. (2.2)

In general though the transverse shear stresses σ_4 (or τ_{23}), σ_5 (or τ_{31}) are not used in classical plane stress analysis, here these stresses are accounted to adopt Mindlin's plate theory instead of the conventional Kirchhoff's theory.

The above stress strain relation based on fibre direction is often termed as on-axis relation. On the contrary the off – axis relation are obtained by applying suitable transformation on the on-axis relations. Hence the stress strain relations of the lamina, with respect to the X, Y and Z axes (Figure 3.1) are expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} c'_{11} & c'_{12} & c'_{16} & & \\ c'_{12} & c'_{22} & c'_{26} & & \\ c'_{16} & c'_{26} & c'_{66} & & \\ & & & c'_{44} & c'_{45} \\ & & & c'_{45} & c'_{55} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - e_x \\ \varepsilon_y - e_y \\ \varepsilon_{xy} - e_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} \tag{2.3}$$

Where,

$C'_{11} = m^4C_{11} + 2m^2n^2 (C_{12}+2C_{66}) + n^4C_{22}$

$C'_{22} = n^4C_{11} + 2m^2n^2 (C_{12}+2C_{66}) + m^4C_{22}$

$C'_{12} = m^2n^2 (C_{11}+ C_{12} - 4C_{66}) + (m^4 + n^4) C_{12}$

$C'_{16} = m^3n (C_{11}- C_{12} - 2C_{66}) - mn^3 (C_{22} - C_{12} - 2C_{66})$ (2.4)

$C'_{26} = mn^3 (C_{11} - C_{12} - 2C_{66}) - m^3n (C_{22}- C_{12} - 2C_{66})$

$C'_{66} = m^2n^2 (C_{11}+ C_{22} - 2C_{12}) + (m^2 - n^2) C_{66}$

$C'_{44} = m^2C_{44} + n^2C_{55}$

$C'_{45} = mn (C_{55} - C_{44})$

$C'_{55} = m^2C_{55} + n^2C_{44}$ and

$e_x = m^2e_1 + n^2 e_2$

$e_y = n^2e_1 + m^2 e_2$

$e_{xy} = 2mn(e_1 - e_2)$ (2.5)

3.2 Governing Equations for a Laminate

Consider a laminated composite plate of thickness *t* consisting of unidirectional lamina bonded together to act as an integral part. The bonds are infinitesimally thin and are not shear deformable. Hence, the displacements are continuous through the thickness of the laminate.

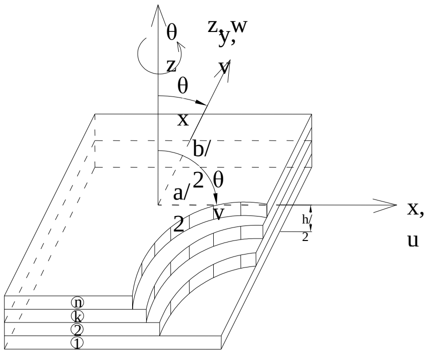


Figure 3. 2 The laminated composite Plate

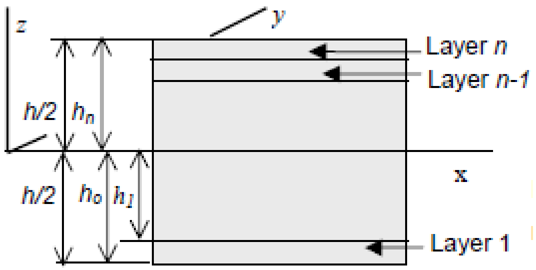


Figure 3.3 Composite plate nomenclatures

The following assumptions are made according to the Young-Norris-Stavsky theory [13], which is the generalization of the Mindlin theory, to accommodate laminated composite plates:

1. The material behaviour is linear and elastic.
2. The thickness, *t* of the laminate is small compared to the other two dimensions.
3. Displacements *u*, *v*, and *w* are small compared to the laminate thickness, *h*.
4. Normal to the mid-plane before deformation remains straight but not necessarily normal to the mid-plane after deformation. Hence Young-Norris-Stavsky theory is termed as first order shear deformation theory.
5. Stresses normal to the mid-plane are neglected.

The deformed geometry of the laminated composite plate is shown in Figure 3. 4.

The in-plane displacements *u* and *v* of any point at any distance *z* from the mid-plane are given by:

$$u(x,y,z) = u_0(x,y) + z\theta_y \,,$$
$$v(x,y,z) = v_0(x,y) - z\theta_x \qquad (2.6)$$

The shear rotation of the plate can be expressed as

$$\varphi_x = \theta_y + w_{,x} \,,$$
$$\varphi_y = -\theta_x + w_{,y} \qquad (2.7)$$

With the displacements, defined in Eq. (2.6), the linear in-plane strains of laminate at a distance *z* from the mid-surface can be found out as,

$$\varepsilon_x = u_{,x} = u_{0,x} + z\,\theta_{y,x} = \varepsilon^0_x + zK_x$$

$\varepsilon_y = v_{,y} = v_{0,y} - z\,\theta_{x,y} = \varepsilon^0{}_y + zK_y$

$$\gamma_{xy} = u_{,y} + v_{,x} = u_{0,y} + v_{0,x} + z(\theta_{y,y} - \theta_{x,x}) = \varepsilon^0{}_{xy} + zK_{xy} \qquad (2.8)$$

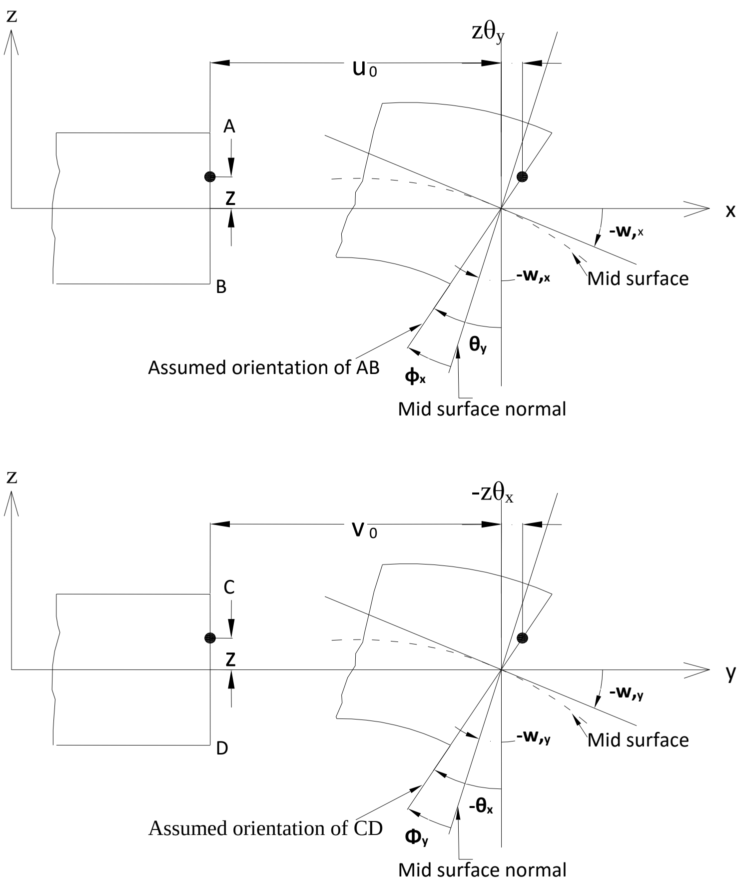


Figure 3.4. Detail deformation of the laminated composite plate

Since, the transverse shear deformation is assumed to be same across the thickness of the laminate, these are given by,

$$\gamma_{xz} = \varphi_x,\ \ \gamma_{yz} = \varphi_y,\ \ \varepsilon_z = 0 \qquad (2.9)$$

Eq.(2.8) can be expressed as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{bmatrix} + z \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix}, \qquad (2.10)$$

Where

$$\begin{bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{bmatrix} = \begin{bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{bmatrix}$$

are the mid-surface in plane strains, and

$$\begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix} = \begin{bmatrix} \theta_{y,x} \\ -\theta_{x,y} \\ \theta_{y,y} - \theta_{x,x} \end{bmatrix}$$

are the mid-surface curvature.

Inserting the above relationship in Eq. (2.3) , the following relation is obtained.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[c'_{ij}\right]_k \begin{Bmatrix} \varepsilon_x^0 + zK_x \\ \varepsilon_y^0 + zK_y \\ \varepsilon_{xy}^0 + zK_{xy} \end{Bmatrix} - \left[c'_{ij}\right]_k \{e\}_k \qquad \text{(i, j = 1, 2, 6)}$$

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \alpha \left[c'_{ij}\right]_k \begin{Bmatrix} \varphi_x \\ \varphi_y \end{Bmatrix} \qquad \text{(i, j = 4, 5)} \qquad (2.11)$$

where, $\{e\}_k = \{e_x \quad e_y \quad e_{xy}\}^T$

And α is a shear correction factor, taken as 5/6 [8], to take account for the non uniform distribution of the transverse shear strain across the thickness of the laminate.

These expressions have to be integrated over the entire plate thickness to account for the laminate property. As a result the stress terms are replaced by the stress-resultant terms, while the conventional strain terms are replaced by the mid-plane strain terms. Hence, assuming that the laminate comprises of *n* laminate, the in-plane forces,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} \, [\varepsilon]_{x,y} \, \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} \, [e]_k \, \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} \left([\varepsilon^0]_{x,y} + [zK]_{x,y}\right) \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} \, [e]_k \, \mathrm{dz} \qquad (2.12)$$

Moments,

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k zdz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k z\mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z \, c'^{k}_{ij} \, [\varepsilon]_{x,y} \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z \, c'^{k}_{ij} \, [e]_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} \left([z\varepsilon^0]_{x,y} + [z^2\kappa]_{x,y}\right) dz - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z \, c'^{k}_{ij} \, [e]_k \mathrm{dz} \qquad (2.13)$$

Shear forces,

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \alpha \left[c'^{k}_{ij}\right]_k \begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}_k \mathrm{dz} \qquad (2.14)$$

From equations [2.12~ 2.14] the internal force and moment resultants can be expressed as,

$\{F\} = [D]\ \{\varepsilon\} - \{F^N\}$ (2.15)

where, $\{F\} = \{N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ Q_x\ Q_y\}^T$

$\{\varepsilon\} = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_{xy}\ K_x\ K_y\ K_{xy}\ \varphi_x\ \varphi_y\}^T = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_{xy}\ K_x\ K_y\ K_{xy}\ \varepsilon_{xz}\ \varepsilon_{yz}\}^T$

$\{F^N\}$ = In plane force and Moments = $\{N^N_x\ N^N_y\ N^N_{xy}\ M^N_x\ M^N_y\ M^N_{xy}\ 0\ 0\}^T$

$$[D]=\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix}$$

Here,

$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij}\ [1, z, z^2]_k dz$ (i, j = 1, 2 and 6)

and $(A_{ij}) = \alpha \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} c'^{k}_{ij} dz$ (i, j = 4, 5) (2.16)

Since the deflection w does not vary with z, the non-linear portion of the overall strains in Eq(2.8) in a laminated plate can be expressed as

$\varepsilon_{xnl} = \frac{1}{2}(u^2_x + v^2_x + w^2_x)$

$\varepsilon_{xnl} = \frac{1}{2}(u^2_y + v^2_y + w^2_y)$

$\varepsilon_{xynl} = (u_x\ u_y + v_x\ v_y + w_x\ w_y)$

$\varepsilon_{xznl} = (u_x\ u_z + v_x\ v_z)$

$\varepsilon_{yznl} = (u_y\ u_z + v_y\ v_z)$ (2.17)

From the Strain-displacement relation in Eq(2.10)

$\varepsilon_{xnl} = \frac{1}{2}[u_0{}^2_x + v_0{}^2_y + w^2_x + 2z(u_{0x}\theta_{yx} - v_{0x}\theta_{xx}) + z^2\ (\theta^2_{yx} + \theta^2_{xx})]$

$\varepsilon_{ynl} = \frac{1}{2}[u_0{}^2_y + v_0{}^2_y + w^2_y + 2z(u_{0y}\theta_{yy} - v_{0y}\theta_{xy}) + z^2\ (\theta^2_{xy} + \theta^2_{yy})]$

$\varepsilon_{xynl} = [u_{0x}\ u_{0y} + v_{0x}\ v_{0y} + w_x\ w_y + z(u_{0y}\theta_{yx} + u_{0x}\theta_{yy}) - z(v_{0y}\theta_{xx} + v_{0x}\theta_{xy})$
$+ z^2\ (\theta_{yx}\ \theta_{yy} + \theta_{xx}\ \theta_{xy})]$

$\varepsilon_{xznl} = [u_{0x}\ \theta_y - v_{0x}\ \theta_x + z(\theta_y\theta_{yx} + \theta_x\theta_{xx})]$

$\varepsilon_{yznl} = [u_{0y}\ \theta_y - v_{0y}\ \theta_x + z(\theta_y\theta_{yy} + \theta_x\theta_{xy})]$ (2.18)

3.3 Governing Differential Equations from Energy Principles

The potential energy of deformation of a laminated plate is given

$U_a = \iiint_v \{\varepsilon_{nl}^a\}^T \{\sigma^a\}\, dV$ (2.20)

where, $\{\varepsilon_{nl}^a\}^T = \{\ \varepsilon_{xnl}\ \ \varepsilon_{ynl}\ \ \varepsilon_{xynl}\ \}^T$ (2.21)

$\{\sigma^a\} = \{\sigma^a_x\ \sigma^a_y\ \ \tau^a_{xy}\ \}^T$ (2.22)

in which $\sigma^a_x, \sigma^a_y, \tau^a_{xy}$, are the in plane stresses produced by applied in-plane loads.

The potential energy of inertia force and moment is expressed as

$V_i = -\iint_A \{u\}^T \{X\} dA.$ (2.23)

where, $\{X\} = \{pu_0\omega^2_n, pv_0\omega^2_n, pw\omega^2_n, I\theta_x\omega^2_n, I\theta_y\omega^2_n,\}^T$

$p = \sum_{k=1}^{n}(z_k - z_{k-1})\,\rho_k$, $I = \frac{1}{3}\sum_{k=1}^{n}(z_k^3 - z_{k-1}^3)\rho_k$

The total potential energy is

$\prod = U + V_i$ (2.24)

According to the principle of total potential energy the first variation of $\prod$ in eq.(2.24) is stationary for equilibrium of the laminated plate. Thus by equating $\delta\prod$ to zero in eq (2.24) the respective equilibrium conditions are obtained.

3.4 Boundary Conditions:

In the present investigation two types of boundary conditions are employed. They are simply-supported and clamped. The restrained displacements are described below for various cases.

Anti-symmetric cross-ply laminates, simply-supported

X=-a/2, a/2 : $\dot{v}$=0, w=0, $\theta_x = 0$

Y=-b/2, b/2 : $\dot{u}$=0, w=0, $\theta_y = 0$

Anti-symmetric angle-ply laminates, simply-supported

X= -a/2, a/2 : $\dot{u}$=0,w=0, $\theta_y = 0$

Y= -b/2, b/2 : $\dot{v}$=0, w=0, θ_y= 0

Anti-symmetric cross-ply and angle-ply laminates, clamped

X=-a/2, a/2; y=-b/2, b/2 : $\dot{u}$=0,$\dot{v}$=0,w=0 ,$\theta_x = 0, \theta_y = 0$

FINITE ELEMENT FORMULATIONS

3.5 INTRODUCTION

The finite element formulation and the solution details are presented in this chapter. The formulation is based on the governing equations derived in the presenting chapter. The element stiffness, geometric stiffness and mass matrices as well as the load vectors are derived using the principle of total potential energy. An eight nodded isoperimetric element is employed, both the geometry and displacement field of which are expressed in terms of the same shape functions. The present element in local natural coordinate system can be mapped to an arbitrary shape in the Cartesian coordinate system.

3.6 Quadratic Isoperimetric Element

A flat Mindlin eight noded plate element with six degrees of freedom (D.O.F) at each node, i.e., u_0, v_0, w, θ_x, θ_y , θ_z is used in the analysis. The co-ordinates and the elastic parameters inside the element can be interpolated using shape function (interpolation function) N_i as given in Figure 3.5.

$$x = \sum_{i=1}^{8} N_i(\xi,\eta)x_i \qquad y = \sum_{i=1}^{8} N_i(\xi,\eta)y_i \tag{2.26}$$

where x_i and y_i are the global co-ordinates at a node i.

$$u_0 = \sum_{i=1}^{8} N_i(\xi,\eta)u_{o_i} \quad v_0 = \sum_{i=1}^{8} N_i(\xi,\eta)v_{o_i} \quad w = \sum_{i=1}^{8} N_i(\xi,\eta)w_i$$

$$\theta_x = \sum_{i=1}^{8} N_i(\xi,\eta)\theta_{x_i} \quad \theta_y = \sum_{i=1}^{8} N_i(\xi,\eta)\theta_{y_i} \tag{2.27}$$

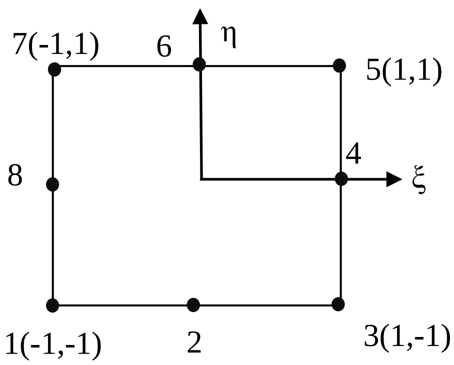


Figure 3.5 8 Noded element

in which u_{oi}, u_{oi}, v_{oi}, w_i, θ_{xi}, θ_{yi} are the displacement at a node i.

The shape functions N_i in eqs.(2.26) and (2.27) are defined as

$N_i = (1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)/4$; i = 1, 3, 5, 7

$N_i = (1-\xi^2)(1+\eta\eta_i)/2$; i = 2,6

$N_i = (1-\eta^2)\ (1+\xi\xi_i)\ /2\ ;\ i = 4{,}8$ (2.28)

Where ξ and η are the local natural co-ordinates of the element and ξ_i and η_i are the value of them at node i.

The strains at the mid-plane of the plate are given by,

$$\begin{bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \\ K_x \\ K_y \\ K_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \sum_{j=1}^{8} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} \\ 0 & 0 & 0 & -N_{i,y} & 0 \\ 0 & 0 & 0 & -N_{i,x} & N_{i,y} \\ 0 & 0 & N_{i,x} & 0 & N_i \\ 0 & 0 & N_{i,y} & -N_i & 0 \end{bmatrix} \begin{bmatrix} u_{0j} \\ v_{0j} \\ w_{0j} \\ \theta_{xj} \\ \theta_{yj} \end{bmatrix} \qquad (2.29)$$

Or, $\{\varepsilon\} = [B]\{d\}$, where, $[B]$ is the linear strain- displacement matrix.

3.7 Elastic Stiffness matrix for the element

The potential energy of deformation for the element from eq. (2.30), is

$$U = \frac{1}{2}\iint_{Ae}\{\varepsilon\}^T\,[D]\,\{\varepsilon\}dA \qquad (2.30)$$

$$\{\varepsilon\} = [B]\{\delta_e\} = [[B_1]\ [B_2]\ \ldots\ldots\ldots[B_8]]\ \{\delta_e\} \qquad (2.31)$$

where $\{\varepsilon\} = \{\ \varepsilon^0{}_x\ \varepsilon^0{}_y\ \gamma^0{}_{xy}\ K_x\ K_y\ K_{xy}\ \gamma_{xz}\ \gamma_{yz}\}^T$

$$\{\delta_e\} = \{u_{0j},\ v_{0j},\ w_{0j},\ \theta_{xj},\ \theta_{yj}\}^T\ ,\ j=1,8, \qquad (2.32)$$

then

$$U = \frac{1}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\{\delta e\}^T\,[B]^T[D][B]\,\{\delta e\}\,dx\,dy$$

$$= \frac{1}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\{\delta e\}^T\,[K_e]\{\delta_e\}$$

in which $[K_e] = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}[B]^T[D][B]dx\,dy$ = element stiffness matrix. (2.33)

Since dx dy = |J| dξ dη , where |J| is the determinant of the Jacobin matrix, the element stiffness matrix can be expressed in local natural coordinates of the element.

From eq. (2.33).

$$[K_e] = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\,|J|d\xi\,d\eta \qquad (2.34)$$

Here, $[J] = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}$ (2.35)

A 3x3 integration of Gauss Quadrature is used in the evaluation of bending stiffness where as a 2x2 reduced integration is employed for shear stiffness terms. The purpose of reduced integration is to reduce the shear stiffness of the element [2.29].

3.8 Element Geometric Stiffness Matrix

The non-linear strains due to applied in plane load

The non-linear strains $\varepsilon_{xnl}, \varepsilon_{ynl}, \varepsilon_{xynl}$, given by equations(2.18) are expanses

$[\varepsilon_{nl}^{0}]=\{\varepsilon_{xnl}, \varepsilon_{ynl}, \gamma_{xynl}\}^{T}=1/2[U]\{f\},$ (2.35)

Where $\{f\}=\{\bar{u}_{x}, \bar{u}_{y}, \bar{v}_{x}, \bar{v}_{y}, w_{x,} w_{y}, \theta_{xx}, \theta_{xy}, \theta_{yx}, \theta_{yy}\}^{T}$

and [U] is obvious from equations (2.18) and (2.35)

If $\{f\}= [H][\delta_{e}]=[[H_{1}]\ldots\ldots\ldots\ldots[H_{2}]]\{\delta_{e}\}$ (2.36)

In Plane Loading Conditions:

a. Uniaxial Loading

b. Biaxial Loading

c. All in Plane Loading

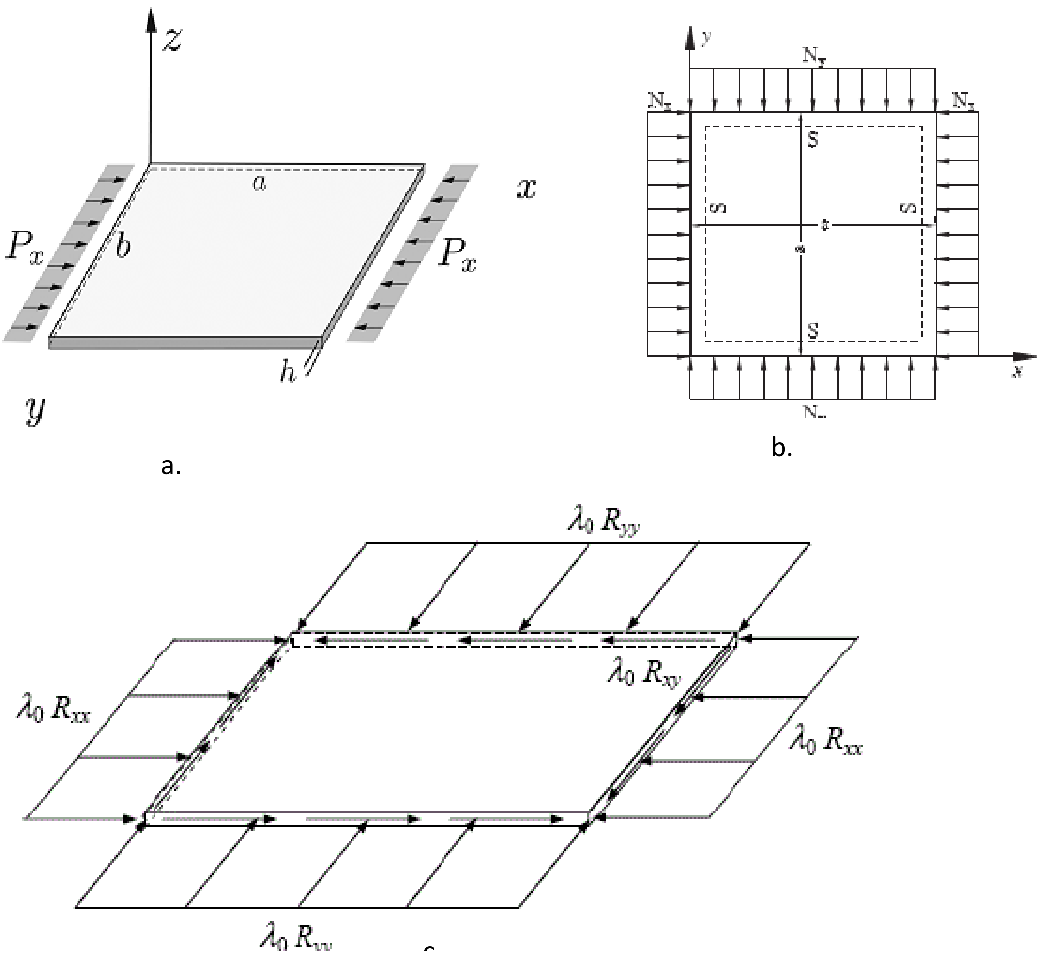


Figure.3.6 Different loading conditions

a. Uniaxial loading b. Biaxial loading

c. All in plane loading

Then from equation (2.20) the potential energy of in-plane stress, produced by applied in-plane load, for the element can be expressed as

Uae=1/2$\iiint_{ve}\{\delta_e\}^T[H]^T[U]^T\{\sigma^a\}$ dV (2.37)

[H_i] in equation (2.36) is given by

$$[H_i]=\begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ N_{i,y} & 0 & 0 & 0 & 0 \\ 0 & N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 \\ 0 & 0 & N_{i,x} & 0 & 0 \\ 0 & 0 & N_{i,y} & 0 & 0 \\ 0 & 0 & 0 & N_{i,x} & 0 \\ 0 & 0 & 0 & N_{i,y} & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} \\ 0 & 0 & 0 & 0 & N_{i,y} \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (i=1\text{ to }8) \qquad (2.38)$$

Since $[U]^T\{\sigma^a\}=[\sigma^a][H]\{\delta_e\}$, (2.39)

U_{ae}=1/2$\iiint_{ve}\{\sigma\}^T[H]^T[\sigma]^a[H]\delta_e\}$ dV

=1/2 $\{\{\delta_e\}^T[K_{Ge}^a]\{\delta_e\}$,

In which

$[K_{Ge}^a]=\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2}[H]^T[\sigma^a][H]dx\,dy\,dz$ (2.40)

Is the element geometric stiffness matrix due to in- plane stresses produced by applied in-plane load.

By performing analytical integration in z direction, $[K_{Ge}^a]$ in the element local co-ordinates can be written as

$[K_{Ge}^a]=\int_{-1}^{1}\int_{-1}^{1}[H]^T[S^a][H][J]d\zeta d\eta$. (2.41)

The matrix [S^a] is given

The 3x3 Gauss quadrature is employed to evaluate $[K_{Ge}^a]$.

Since the stress distribution is not uniform in a plate with a cut-out when subjected to in-plane loads, the in-plane stress as resultants N_x^a, N_Y^a and N_{XY}^a at each Gauss point are obtained separately. The geometric stiffness matrix $[K_{Ge}^a]$ is foamed for these stress resultants.

$$\{\sigma^a\}=\begin{bmatrix} \sigma_x^a & & & & & & & & & \\ \tau_{xy}^a & \sigma_y^a & & & & & & & & \\ 0 & 0 & \sigma_x^a & & & & symmetic & & & \\ 0 & 0 & \tau_{xy}^a & \sigma_y^a & & & & & & \\ 0 & 0 & 0 & 0 & \sigma_x^a & & & & & \\ 0 & 0 & 0 & 0 & \tau_{xy}^a & \sigma_y^a & & & & \\ 0 & 0 & -z\sigma_x^a & -z\tau_{xy}^a & 0 & 0 & z^2\sigma_x^a & & & \\ 0 & 0 & -z\tau_{xy}^a & -z\sigma_y^a & 0 & 0 & z^2\tau_{xy}^a & z^2\sigma_y^a & & \\ z\sigma_x^a & z\tau_{xy}^a & 0 & 0 & 0 & 0 & 0 & 0 & z^2\sigma_x^a & \\ z\tau_{xy}^a & z\sigma_y^a & 0 & 0 & 0 & 0 & 0 & 0 & z^2\tau_{xy}^a & z^2\sigma_y^a \end{bmatrix} \quad (2.41)$$

$$[S^a]=\begin{bmatrix} N_x^a & & & & & & & & & \\ N_{xy}^a & N_y^a & & & & & & & & \\ 0 & 0 & N_x^a & & & & & & & \\ 0 & 0 & N_{xy}^a & N_y^a & & & symmetric & & & \\ 0 & 0 & 0 & 0 & N_x^a & & & & & \\ 0 & 0 & 0 & 0 & N_{xy}^a & N_y^a & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_x^a t^2}{12} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_{xy}^a t^2}{12} & \frac{N_y^a t^2}{12} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_x^a t^2}{12} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_{xy}^a t^2}{12} & \frac{N_y^a t^2}{12} \end{bmatrix} \quad (2.42)$$

3.9 Elemental mass matrix

The potential energy of inertia forces for the element is obtained from eq. (2.23),

$$V_i = -\iint_{Ae}\{u\}^T\{X\}dA. \quad (2.43)$$

where, $\{X\} = \omega_n^2\,[P]\,\{u_0\ v_0\ \ w\ \theta_x\ \ \theta_y\}^T$

Here, $[P] = \begin{bmatrix} p & & & & \\ 0 & p & & & \\ 0 & 0 & p & & \\ 0 & 0 & 0 & I & \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$, ($p$ and I is already given in Eq. (2.23)) (2.44)

Therefore,

$$V_{ie} = -\omega_n^2\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\{\delta_e\}^T[N]^T[P]\,[N]\,\{\delta_e\}dx\,dy$$

$$= -\omega_n^2 \{\delta_e\}^T [M_e] \{\delta_e\} \quad (2.45)$$

where $[M_e] = \int_{-1}^{1}\int_{-1}^{1}[N]^T[P]\,[N]\,|J|\,d\xi\, d\eta$ = element mass matrix

[N] = [$[N_1]$ $[N_2]$ ....$[N_8]$]

$$\text{and } [N_i] = \begin{bmatrix} N_i & & & & \\ 0 & N_i & & & \\ 0 & 0 & N_i & & \\ 0 & 0 & 0 & N_i & \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (\text{ i = 1 to 8}) \quad \ldots\ldots\ldots\ldots \quad (2.46)$$

A 2x2 integration of Gauss Quadrature is used in the evaluation of mass matrix for an element.

3.10 Solution Process

The minimization of ∏ in the equation (2.24) leads to the following equilibrium conditions

for the free vibrations of the laminated plates ([K] – $\omega_n^2[M_e]$) {δ} = 0 ...... (2.47)

From this equation frequencies and different mode shape are determined.

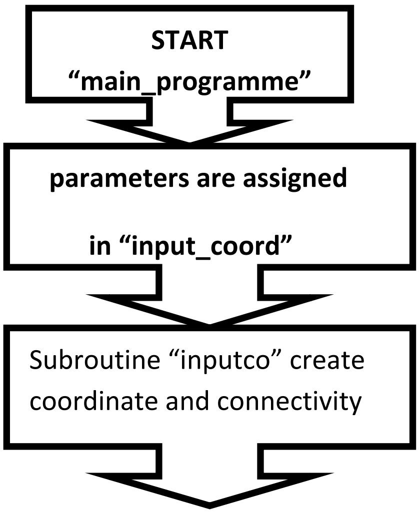
for the buckling of the laminated plates in in-plane loading problem this equation are found.

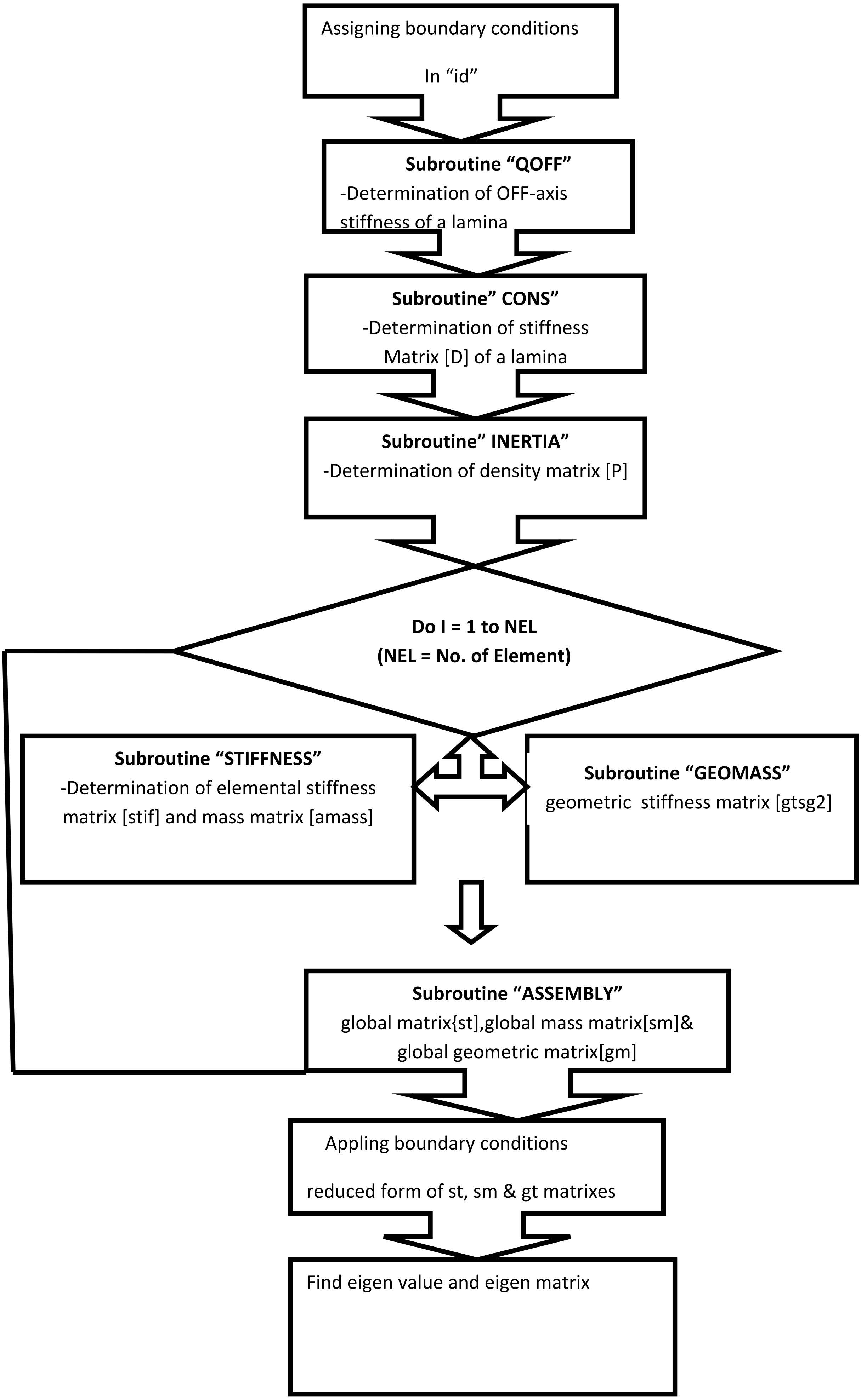
([K] – $\mu[K_{Ge}^{a}]$) {δ} = 0 ..... (2.48)

From which the critical loads are determined.

Both the problem are solved by developing a Matlab programme (appendix). Here Eigen values are calculated by using Matlab function.

3.11 Flow Chart of the Computer Programme




Assigning boundary conditions
In “id”
Subroutine “QOFF”
-Determination of OFF-axis stiffness of a lamina
Subroutine” CONS”
-Determination of stiffness Matrix [D] of a lamina
Subroutine” INERTIA”
-Determination of density matrix [P]
Do I = 1 to NEL
(NEL = No. of Element)
Subroutine “STIFFNESS”
-Determination of elemental stiffness matrix [stif] and mass matrix [amass]
Subroutine “GEOMASS”
geometric stiffness matrix [gtsg2]
Subroutine “ASSEMBLY”
global matrix{st],global mass matrix[sm]&
global geometric matrix[gm]
Appling boundary conditions
reduced form of st, sm & gt matrixes
Find eigen value and eigen matrix

CHAPTER4. NUMERICAL RESULT

The finite element formulation has been used to compare the present results with the published papers. It has also been used to generate new results to study the effects of various parameters on the dynamic and buckling behaviour of the composite plate structure. Generally 3 x 3 integration is applied for bending stiffness and 2 x 2 for shear stiffness and mass matrices to avoid shear locking phenomenon. Here 3x3 integration is used for mass matrix for dynamic study and geometric stiffness for buckling study.

I. Numerical studies 1 for free vibration of composite laminated plate

II. Numerical studies 2 for buckling behaviour of composite laminated plate

4.1 Numerical Studies 1 for Free Vibration of Composite Laminated Plate

4.1.1 Study on Mesh Convergence

A mesh convergence study is shown in the following examples.

Angle ply (30^0/-30^0/30^0) laminate is analyzed for first five natural frequencies with different mesh size, as shown in Table 4.1.1. The material properties are as given in as

Table 4.1.1: Material properties used in mesh convergence study

Material 1	E_1(GPa)	E_2(GPa)	G_{12} = G_{13} (GPa)	G_{23}(GPa)	ν_{12}	ρ(KG/m^3)
Glass Epoxy	60.7	24.8	12.0	12.0	0.23	1300

The geometry of the structure is shown in Figure [4.1.1]. The plate is clamped on one side. A 0.01m thick 1m x 1m square cantilever plate is analyzed.

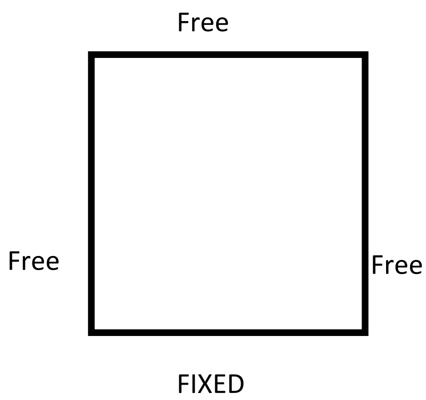


Fig.4.1.1 Boundary condition

Table 4.1.2. Natural Frequency (Hz) for square plate for different meshes

Meshing	4x4	6x6	% Error	8x8	%Error	10x10	% Error
Mode1	9.836	9.835	0.013	9.834	0.008	9.833	0.005
Mode2	21.685	21.651	0.157	21.634	0.077	21.627	0.032
Mode3	56.868	56.782	0.153	56.774	0.013	56.771	0.007
Mode4	66.482	66.233	0.376	66.190	0.065	66.178	0.019
Mode5	83.572	83.117	0.548	83.006	0.134	82.969	0.044

It can be seen that by modelling the composite laminated plate using 8x8 or 10x10 elements provides quite accurate result , taking minimum computation time and computer memory. Hence in the further studies of the composite laminated plate 4x4, 6X6 elements are considered. Sometimes 10X10 meshes is used for more accurate results.

4.1.2 Validation for free vibration

Validation has been performed for composite laminated plate for free vibration analysis to observe the accuracy and reliability of the present code in modelling of composite laminated plate of simply supported boundary conditions. For validation following material properties has been used.

Table 4.1.3: Material properties used in validation study 1

Material2	E_1(GPa)	E_2(GPa)	G_{12} = G_{13} (GPa)	G_{23}(GPa)	ν_{12}	ρ(KG/m^3)
Graphite epoxy	280	7	4.2	3.5	0.25	1300

The dimension of the composite laminated plate are a x b x h = 1m x 1m x .010m. The lamination of plate is $(45^o/-45^o/45^o)$.Here 8 noded with 6 degrees of freedom serendipity element is considered. A 10x10 mesh is taken for the study.

Table 4.1.4: Comparison of non dimensional frequencies for validation study

Boundary Conditions	A\H Ratio	Present(λ)	Paper(λ)[111]	Percent Difference
SSSS	10	15.1575	15.12	0.248
	20	17.68	17.6438	0.205
	50	18.702	18.6586	-0.046
	100	18.891	18.8205	0.3745

[Non dimensional frequency $\lambda = \omega_n a^2 (\rho/E_2\, t^2)^{1/2}$].

The program has been validated with ANSYS also.

Table 4.1.5 Material properties used in validation study with ANSYS

Material3	E_1(GPa)	E_2(GPa)	G_{12}=G_{13} (GPa)	G_{23}(GPa)	ν_{12}, ν_{21}	ρ(KG/m^3)
Carbon epoxy	130	9.5	6	3.	0.23 0.017	1600

The dimension of the composite laminated plate is a x b x h = 1m x 1m x .010m. The lamination of plate is ($0^0/90^0/90^0/0^0$). Here CFFF boundary condition has been used. A 10x10 mesh is taken for the study. There is no difference with ANSYS result.

Table 4.1.6 Comparison of frequencies (Hz) with ANSYS

Mode No	Present	ANSYS	%Difference
1	6.339	6.339	0.000
2	13.569	13.564	0.035
3	39.875	39.870	0.012
4	52.452	52.418	0.064
5	90.816	90.732	0.093

4.1.3. Case Study

To analyze different case the previous element properties has been used and here some new element properties also have been used.

Table 4.1.7: Material properties used in different case study

Material Name	E_1(GPa)	E_2(GPa)	G_{12} = G_{13} (GPa)	G_{23}(GPa)	ν_{12}	ρ(KG/m^3)
Material 4	**9.5**	**9.5**	**3.6**	**3.6**	**0.3**	**1600**
Material 5	**E_1/E_2**	**9.5**	**5.7**	**4.75**	**0.25**	**1600**

Case Study-1: Study of dynamic behaviour of laminated composite plate for different boundary condition.

Case Study-2: Study of dynamic behaviour of laminated composite plate by varying ply orientation at different layer.

Case Study-3: Study of dynamic behaviour of laminated composite plate by varying cross ply orientation at different layer.

Case Study-4: Study of dynamic behaviour of laminated composite plate by varying angle ply orientation at different layer.

Case stusy-5: Study of dynamic behaviour of laminated composite plate for different aspect ratio.

Case stusy-6: Study of dynamic behaviour of laminated composite plate structure for different orthotropy ratio (E1/E2).

4.1.3.1. Case Study-1: Study of dynamic behaviour of laminated composite plate structure for different boundary condition.

In this problem the change in dynamic behaviour of laminated composite plate for different boundary conditions and a/h ratio is discussed. In the following analysis 0/90/90/0 ply composite square plate is used. Material 5 [Table 4.1.7] is taken for the analysis with E1/E2=40.

Table 4.1.8.Frequency (Hz) for Different Boundary Conditions

Boundary Condition		CFFF	CCFF	SSSS	CFCF	CCCF	SCSC	CCCC
a/h=100	mode 1	9.546	26.383	73.214	60.897	66.771	151.645	167.220
	mode 2	15.524	67.797	134.106	64.394	166.682	202.466	258.802
	mode 3	60.012	147.219	267.629	169.898	176.477	354.273	427.164
	mode 4	68.851	170.451	280.161	172.262	256.853	413.744	429.473
	mode 5	150.878	178.197	317.196	177.034	368.821	469.892	510.651
a/h=10	mode 1	93.026	232.051	537.278	232.051	587.830	771.951	878.084
	Non dimensional frequency $\lambda = \omega_n a^2 (\rho/E_2 h^2)^{1/2}$							
a/h=10	mode 1	2.399	5.984	16.628	15.832	15.158	19.905	22.642
a/h=100	mode 1	3.687	6.812	24.334	23.449	18.891	39.103	43.317

C=Clamped, F=Free, S=Simply Supported

The boundary condition is defined by CSCF.

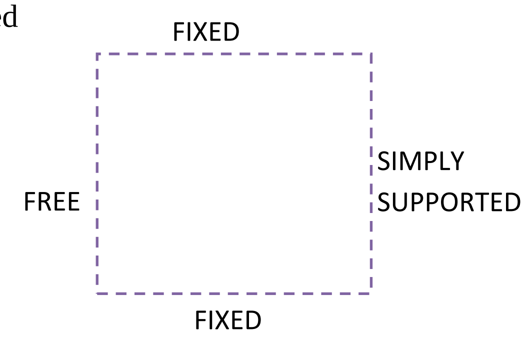


Fig.4.1.2

In the table 4.1.8 frequency for two a/h ratios are shown to clarify the result, after that non dimensional frequency is shown. From table 4.1.8, it is observed that CFFF boundary conditions gives very low frequency and CCCC gives very high frequency. Hence with clamped boundary conditions, stiffness increases. Frequency verses different mode for different boundary conditions are shown in the Fig 4.1.3.1. Variations with non dimensional frequency are shown in Fig 4.1.3.2.For thinner (a/h=100) plate non dimensional frequency is higher than thicker (a/h=10) plate, but in actual frequency is higher at thicker plate because of higher stiffness. For further study non dimensional frequency is used to simplify the non dimensional parameter for different cases.

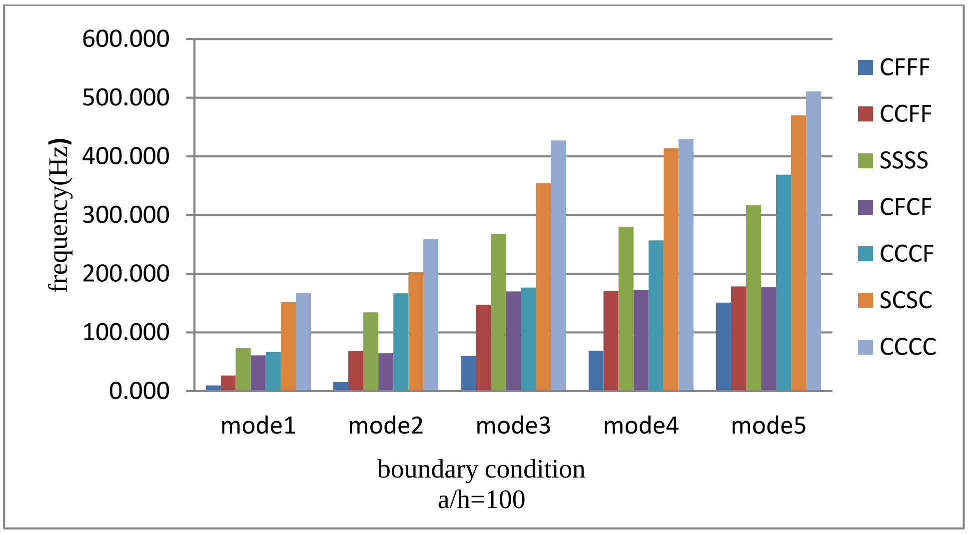


Fig 4.1.3.1 Frequency vs boundary condition

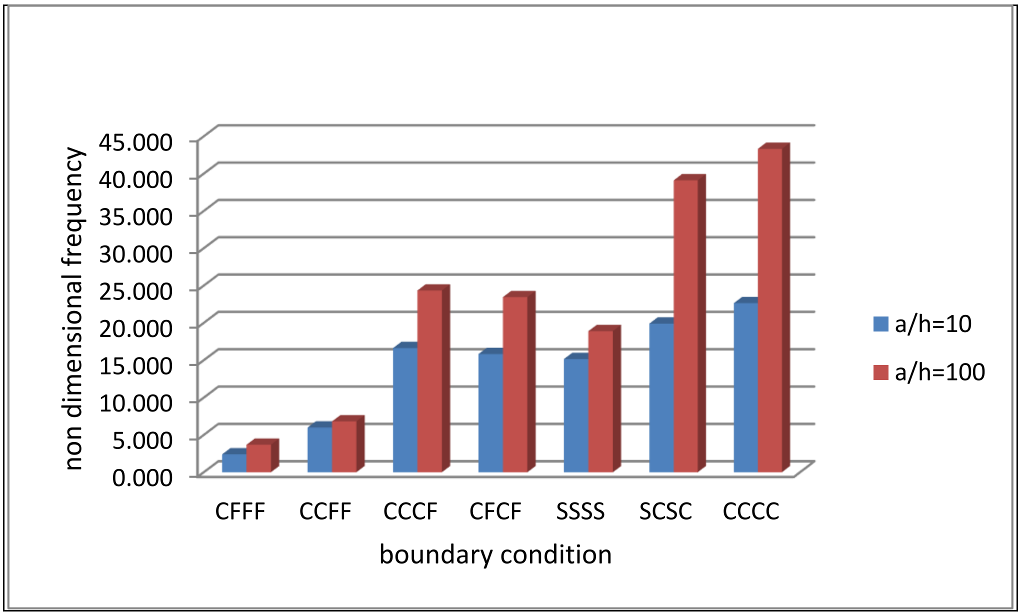


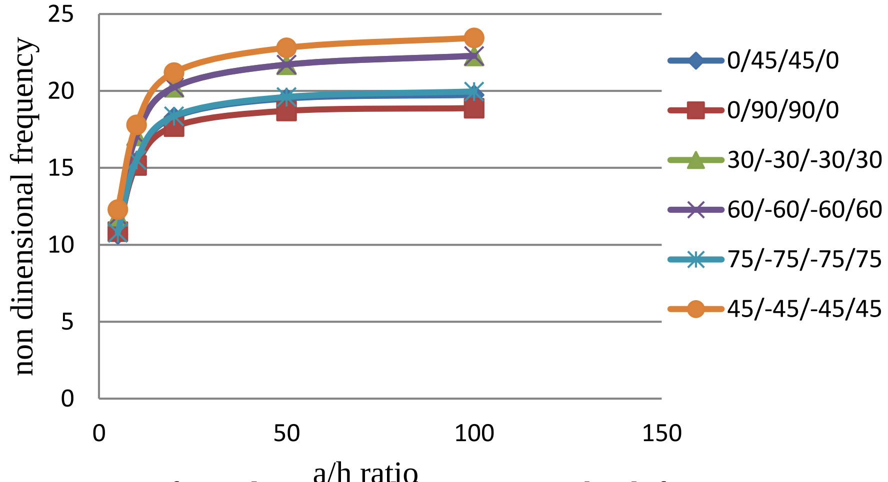
Fig. 4.1.3.2 Non dimensional Frequency for Different Boundary Condition

4.1.3.2. Case Study-2: Study of dynamic behaviour of laminated composite plate structure for variation in a/h ratio and ply orientation at different layer:

In this case study, the variation of non dimensional frequency for simply supported plate is shown (Table 4.1.9 and Fig. 4.1.4), for the plate with the material constants as in the previous case, for a/h ratio varying from 5 to 100 and different layup sequence. From the result, it is observed that non dimensional frequency increases with increase in a/h ratio. Though the rate of increasing is decreases with increase in a/h value. For a particular a/h ratio, stiffness increases for angle ply orientation and stiffness is less for cross ply plates.

Table 4.1.9 Variation of non dimensional frequency with a/h for different ply orientation

a/h	5	10	20	50	100
0/45/45/0	**10.683**	15.413	18.301	19.5107	19.747
0/90/90/0	10.862	**15.157**	**17.68**	**18.702**	**18.878**
30/-30/-30/30	11.866	17.0809	20.24	21.71	22.281
60/-60/-60/60	11.866	17.081	20.2416	21.711	22.2819
75/-75/-75/75	10.791	15.516	18.365	19.607	19.964
45/-45/-45/45	**12.295**	**17.804**	**21.198**	**22.806**	**23.446**

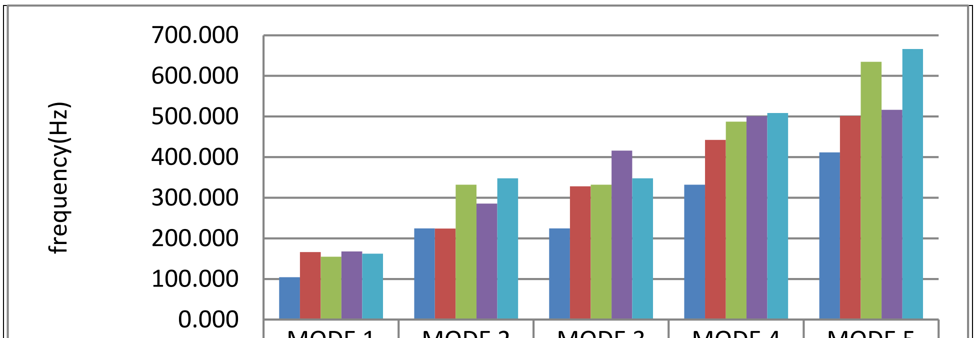


variation of non dimensional frequency with a/h for different ply orientation
Fig.4.1.4

4.1.3.3. Case Study-3: Study of dynamic behaviour of laminated composite plate structure by varying cross ply orientation at different layer.

In this study total thickness of the layer is kept constant and numbers of layers are increased. Here CCCC boundary conditions and square composite plate of material 5 [Table 4.1.7] is used. Dimension of the plate is 1m x1m x.01m and E1/E2=40.

Table 4.1.10 Frequencies (Hz) for different cross ply orientations



	MODE 1	MODE 2	MODE 3	MODE 4	MODE 5
0/90	104.293	224.409	224.409	331.971	411.469
0/90/0	166.414	224.307	327.997	442.385	501.599
0/90/0/90	154.932	332.235	332.235	487.099	634.627
0/90/0/90/0	167.676	285.486	416.140	500.833	516.490
0/90/0/90/0/90	162.455	347.762	347.762	508.363	665.992

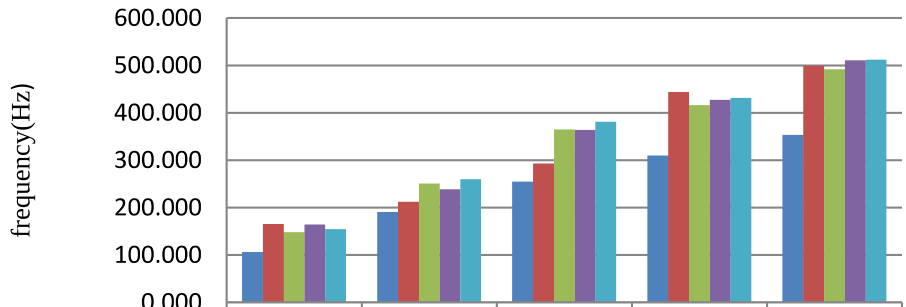
Fig.4.1. 5 Frequencies (Hz) for different cross ply orientations

Here frequency in 0/90/0 is more than 0/90/0/90 and frequency for 0/90/0/90/0 is more than 0/90/0/90/0/90. In general it is found that for same thickness stiffness is increase with the number of layer and stiffness is decreases due to decrease in layer number. It is observed that for successive odd number of layer frequency is more than the even number of layers though the number of layer higher.

4.1.3.4. Case Study-4: Study of dynamic behaviour of laminated composite plate structure for different angle ply orientation with varying layers.

Here same condition and material are used to study the behaviour of angle ply composite plate.

Table 4.1.11 Frequencies (Hz) for different angle ply orientations



	MODE 1	MODE 2	MODE 3	MODE 4	MODE 5
0/45	106.467	190.614	255.065	310.065	353.437
0/45/0	165.587	212.411	293.165	443.901	499.510
0/45/0/45	148.318	251.020	364.981	416.213	492.062
0/45/0/45/0	164.552	238.571	363.907	427.311	510.792
0/45/0/45/0/45	154.671	259.908	381.210	431.613	512.069

Fig.4.1.6 Frequencies (Hz) for different angle ply orientations

In fibre orientation 0/45/0 frequency is more than 0/45 and it is the highest frequency among all cases for first mode, after 0/45/0 higher frequency than 0/45/0/45/0 and 0/45/0/45/0/45 respectively. It is found that for same thickness 0/45/0 has highest stiffness and it is not increasing though the number of layer increases. But stiffness is decreased due to increase in layer number. In general frequency increases with increasing no of layer for higher mode but mode1 gives higher value for odd no of layers. The odd layers are symmetric in nature whereas the even layers are antisymmetric. As a result, there stiffness increases with the increasing number of layers.

4.1.3.5. Case stusy-5: Study of dynamic behaviour of laminated composite plate structure for different aspect ratio:

Here deviation of non-dimensional frequency with respect to a/h has been shown for ply orientation 0/90/90/0. In this analysis simply supported square plate has been used with material properties 5 [Table 4.1.7] with E_1/E_2 ratio 40. Effect of length to thickness ratio on non dimensional frequency parameter [$\lambda = \omega_n a^2 (\rho/E_2 h^2)^{1/2}$] has shown in the table 4.1.13.

Table 4.1.12. variation of non dimensional frequency with varying a/h

a/h ratio	5	10	20	50	100
mode1(Hz)	842.533	587.830	342.837	145.059	73.214
mode2(Hz)	1498.760	1064.257	621.293	263.396	134.106
mode3(Hz)	1659.023	1442.585	1055.289	514.924	267.629
mode4(Hz)	2075.681	1707.026	1197.369	538.893	280.161
mode5(Hz)	2396.536	1896.570	1211.001	582.921	317.196
a/h	Non dimensional frequency				
a/b=1	10.8626	15.1575	17.6805	18.7021	18.8787
a/b=2	19.1773	27.1885	31.6969	33.4909	33.7951
a/b=3	29.7682	46.664	58.9745	64.8248	65.8449

From the table, it is seen that with increase in a/b ratio, stiffness of the structure increases. It is true for any a/h ratio.

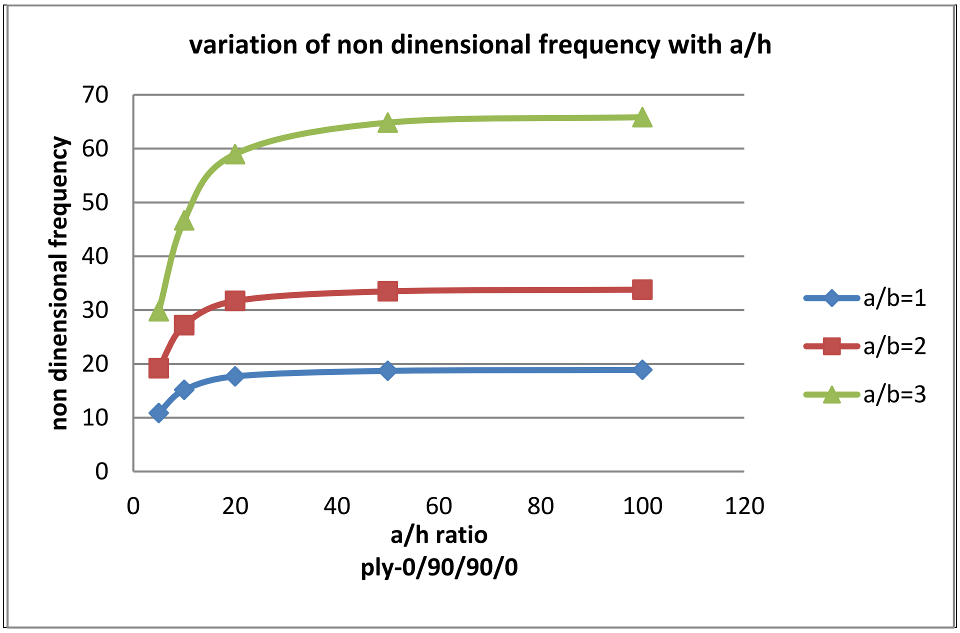


Fig.4.1.7

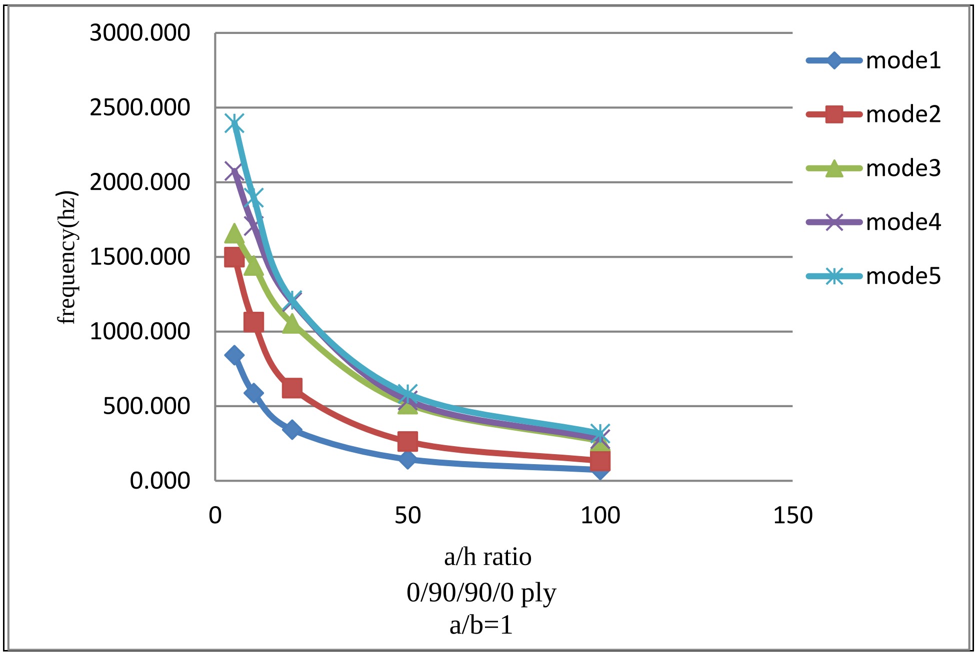


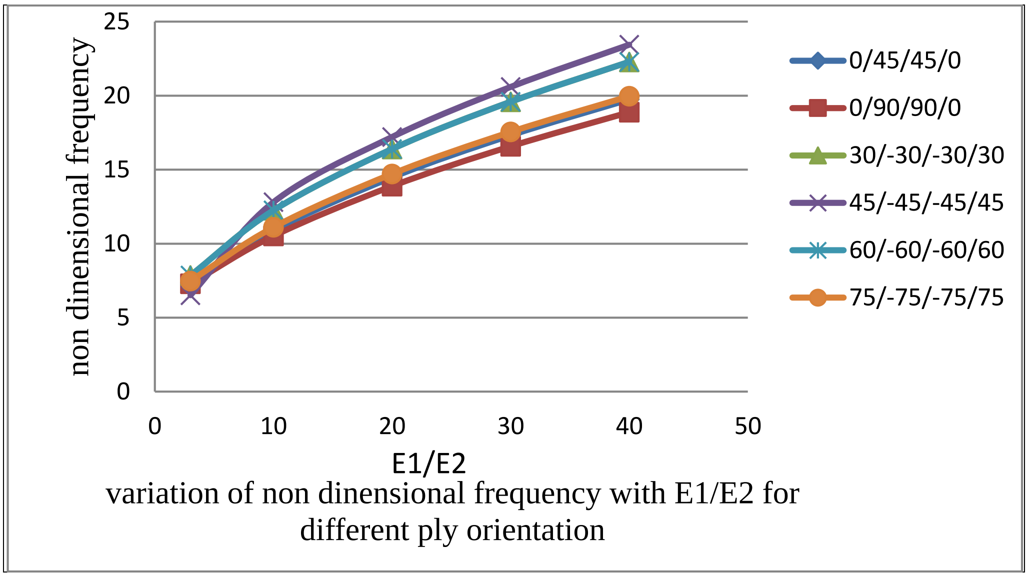
Fig.4.1.8

4.1.3.6. Case stusy-6: Study of dynamic behaviour of laminated composite plate structure for different orthotropy ratio (E1/E2):

Variation of non dimensional frequency with modular ratios (E1/E2) for different ply orientation has been shown for square simply supported laminated composite plate with different modular ratios for a/h ratio 10. Here materials 5[Table 4.1.7] have been used to find non dimensional frequency. From the table and corresponding figure, it is clear, that frequencies are increasing with E1/E2 ratio irrespective of different lay-up sequence. From the table it is observed that for thicker plate, 45/-45/-45/45 gives least stiffness and 60/-60/-60/60 gives the highest stiffness, whereas for thinner plates, 45/-45/-45/45 gives maximum stiffness and 0/90/90/0 gives the least stiffness.

Table 4.1.13. Dimensional Frequency with Modular Ratios

E1/E2	3	10	20	30	40
0/45/45/0	7.393	10.899	14.481	17.32	19.747
0/90/90/0	7.286	**10.52**	**13.886**	**16.575**	**18.878**
30/-30/-30/30	**7.851**	12.253	16.393	19.585	22.281
45/-45/-45/45	**6.517**	**12.804**	**17.21**	**20.592**	**23.446**
60/-60/-60/60	7.851	12.253	16.393	19.585	22.281
75/-75/-75/75	7.477	11.106	14.714	17.551	19.964

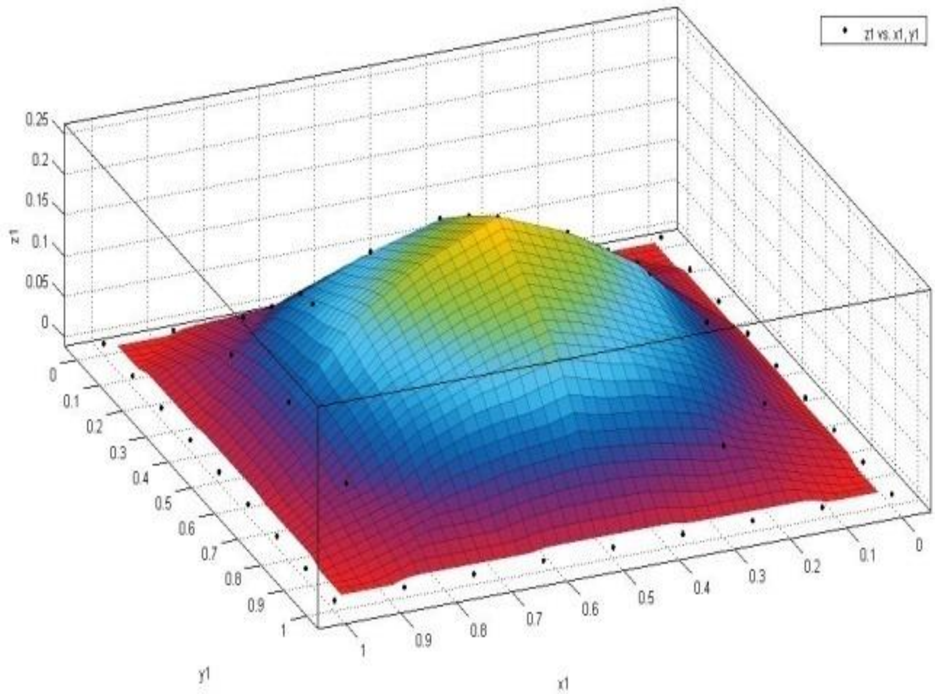


variation of non dinensional frequency with E1/E2 for different ply orientation

Fig.4.1.9

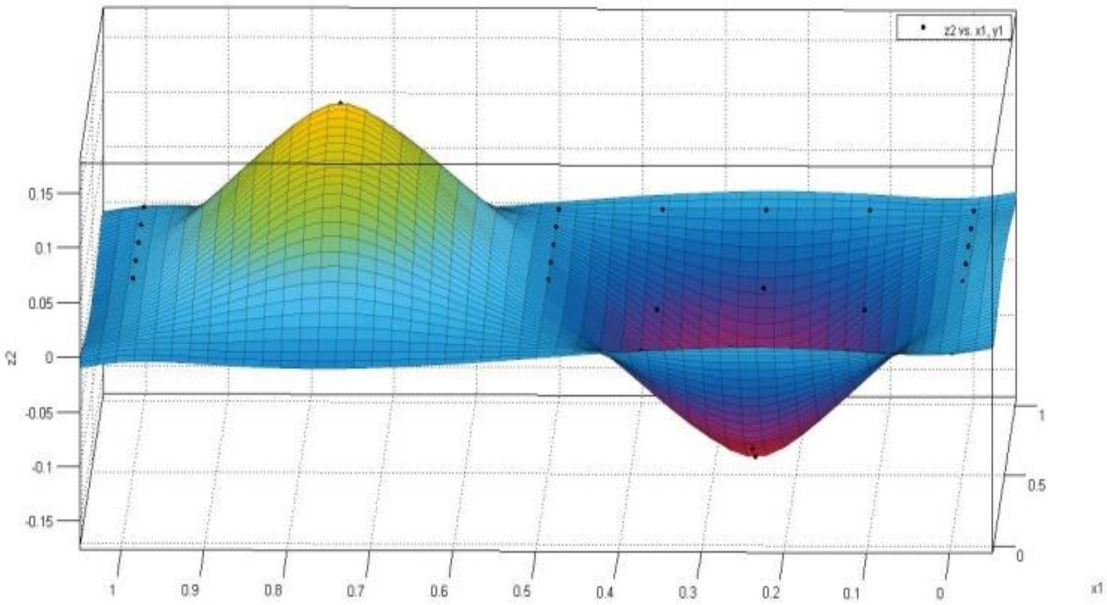
4.1.4. Mode Shape for Different Boundary Condition

Different mode shape are shown in the following figure for all clamped (CCCC), clamped free (CFCF) and cantilever plate (CFFF) for a square plate with 0/90/90/0 lamina. Here the same material has been used.



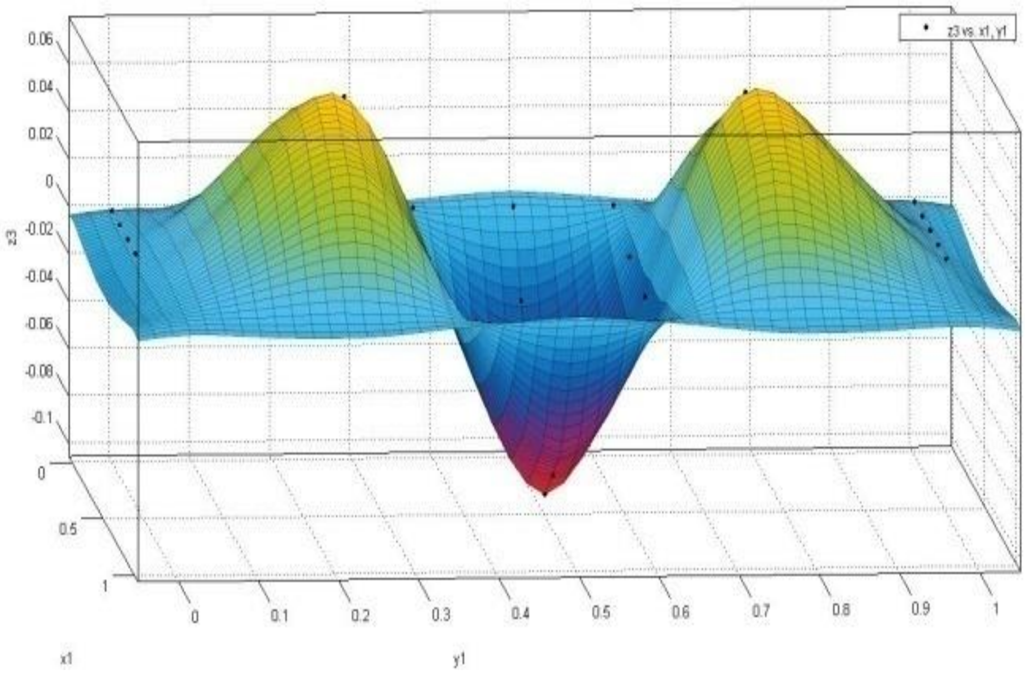
Mode Shape 1-CCCC

Fig.4.1.10



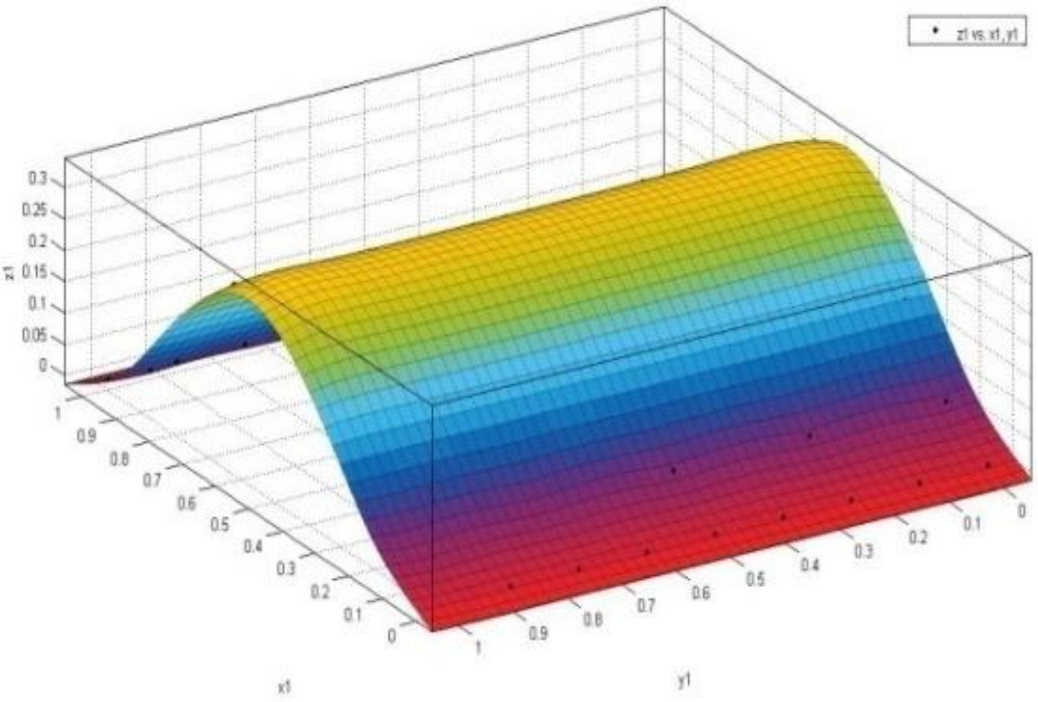
Mode Shape 2-CCCC

Fig.4.1.11



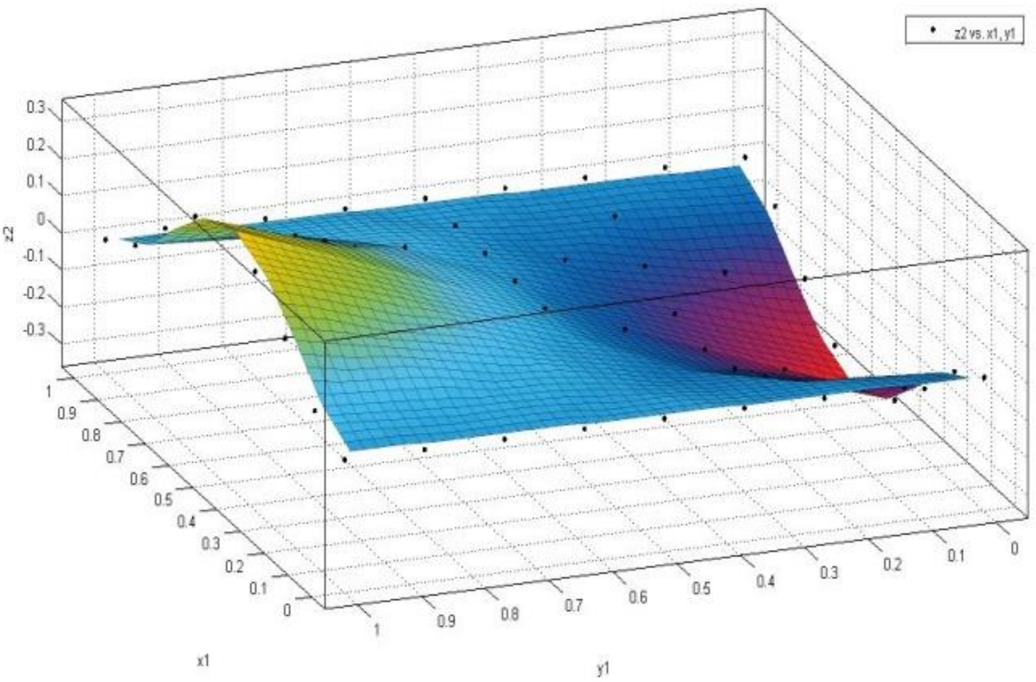
Mode Shape 3-CCCC

Fig.4.1.12



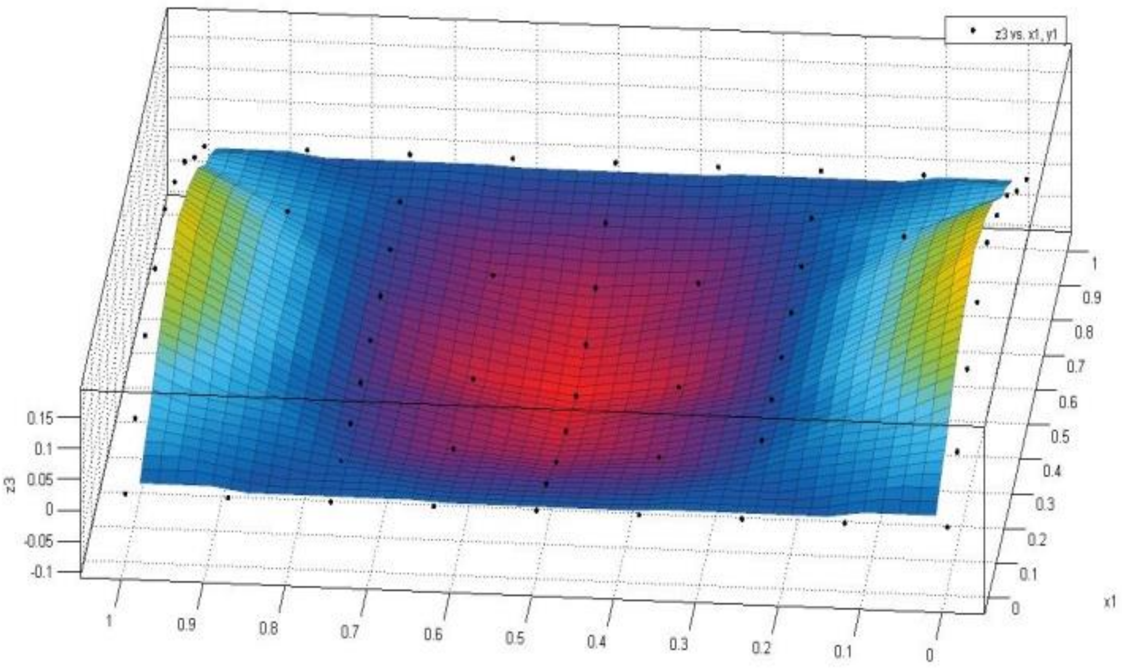
Mode Shape 1- CFCF

Fig.4.1.13



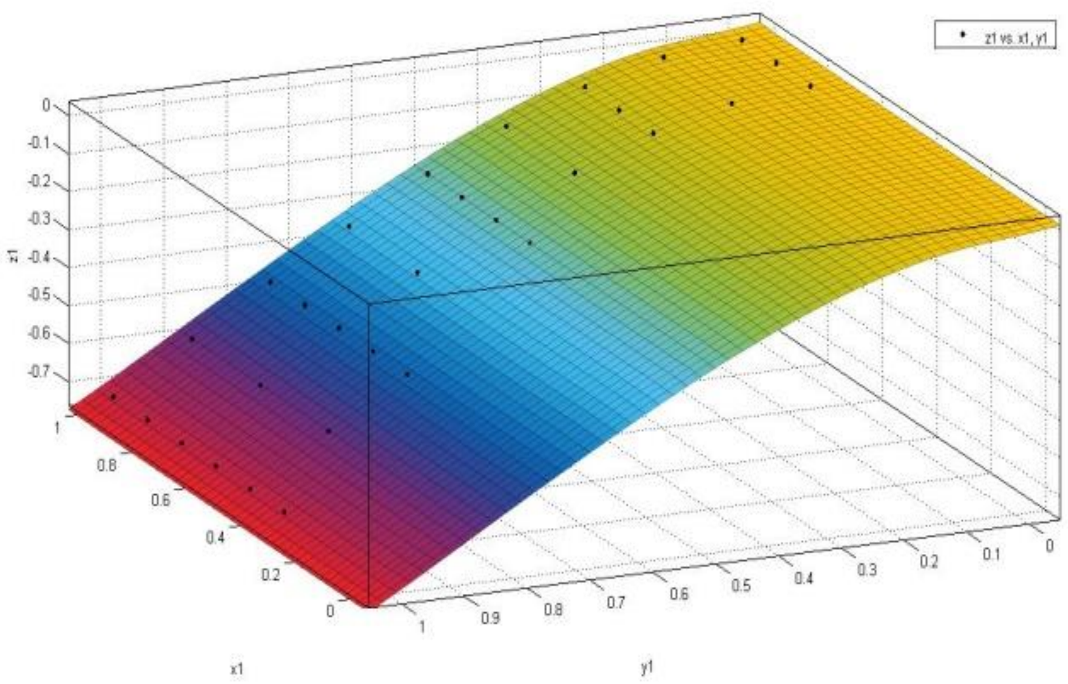
Mode Shape 2- CFCF

Fig.4.1.14



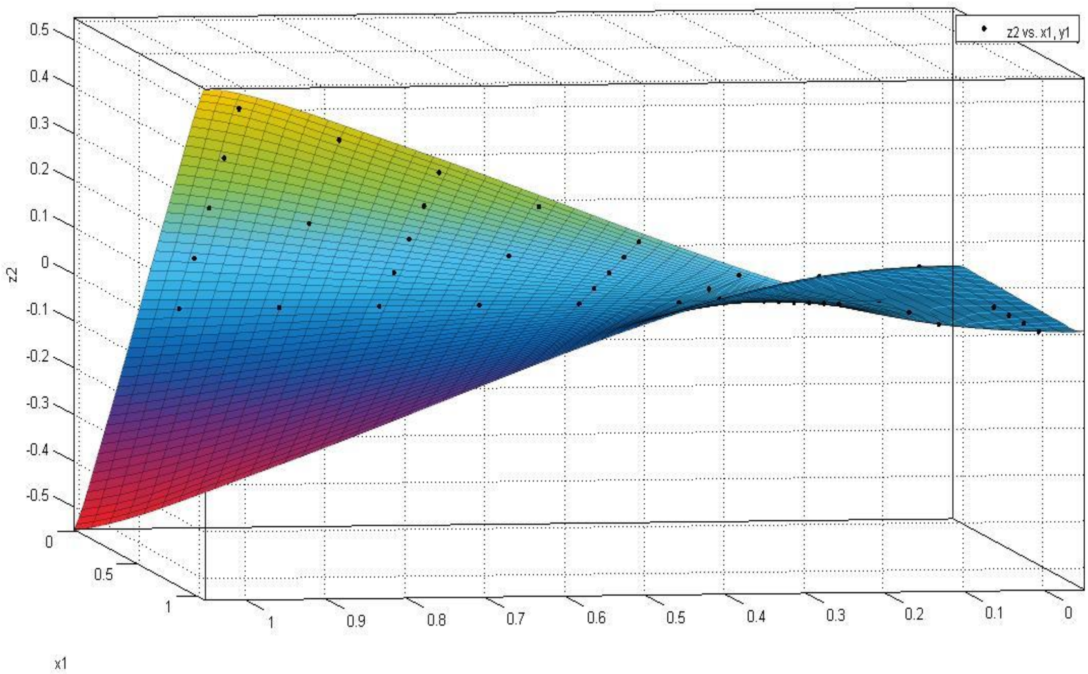
Mode Shape 3- CFCF

Fig.4.1.15



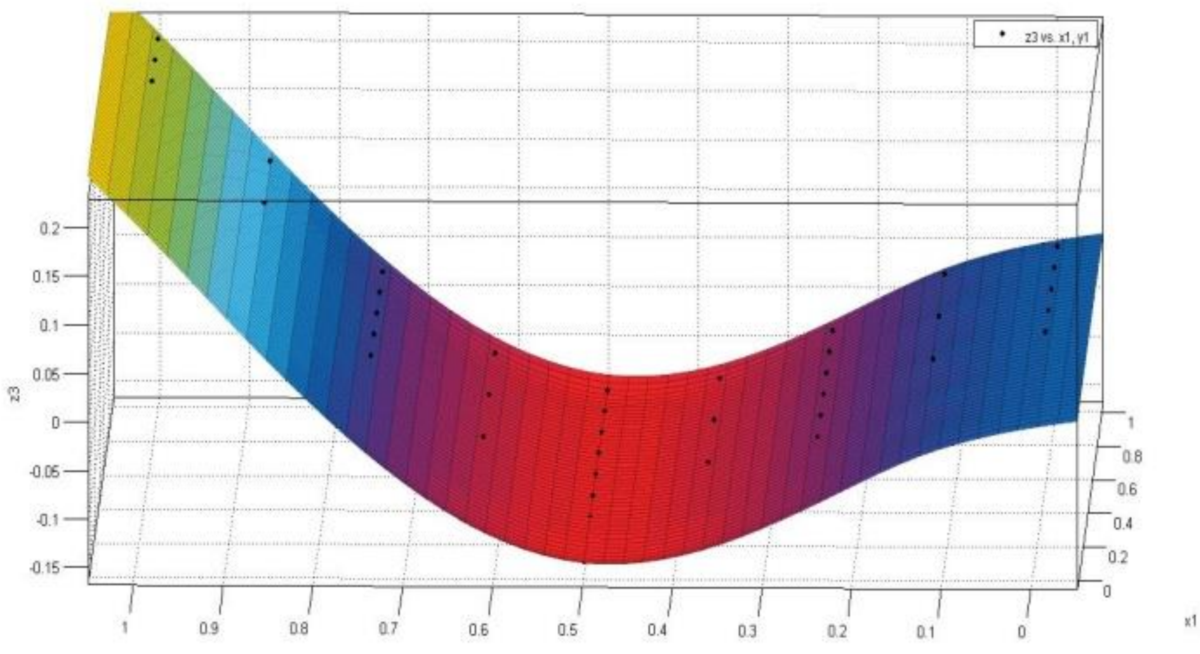
Mode Shape 1- CFFF

Fig.4.1.16



Mode Shape 2- CFFF

Fig.4.1.17



Mode Shape 3- CFFF

Fig. 4.1.18

4.2. NUMERICAL STUDIES 2 FOR BUCKLING OF COMPOSITE PLATE

Validation is performed for composite laminated plate formulation for buckling analysis.

4.2. 1.Validation

4.2.1.1 For isotropic element

To observe the accuracy and reliability of the present code in modelling of composite laminated plate for buckling analysis of simply supported condition of plates has used. It is analyzed numerically and compared with the paper of Chakrabarti and S.K. Singh [96].The dimension of the composite laminated plate are a x b = 1m x 1m. The lamination of plate is 8 noded with 6 degrees of freedom element is considered. A 10x10 mesh is taken for the study. Material4 [Table 4.1.7] is used for isotropic element and material 5 [Table 4.1.7] is used in other all cases. In this example simply supported square isotropic plate is subjected to uni-axial loading. The analysis is carried out for different thickness ratio (*a*/*h* = 100, 10). The critical buckling load is obtained by using present FE model and has been compared with the results of Chakrabarti and S.K.Shing [96] based on refined higher order shear deformation theory (RFSDT). Normalized Critical buckling loads ($\lambda_{cr}= \lambda\ a^2/\pi\ ^2D$ where $D=E_2h^3/12(1-\upsilon_{21}\upsilon_{12})$ are shown in table 4.2.2 for square isotropic plate. It is observed that the result matches well with paper [96].

Table.4.2.2 Comparison of non dimensional buckling load for isotropic plate

Description	a/h	present	Paper[96]
Uniaxial loading	100	4.033	4
	10	3.714	3.782

4.2.1.2 For orthotropic plate element

In this example simply supported square laminated composite plate subjected to uni-axial loading is considered. The critical buckling load obtained by using present FE model obtained are presented with the results of Chakrabarti and S.K. Singh [96] based on higher order zigzag theory (HZT). It is observed that the present results are little lesser than [96] for very thick element.

Table.4.2.3 Validations of Non dimensional Buckling Load

Descriptions	ply	a/h	E1/E2	Present	Paper[17]	%Error
Uniaxial Loading	0/90/90/0	10	10	9.567725	9.76	1.970031
			20	15.02848	15.064	0.23579
			30	19.49515	19.46	-0.1806
			40	23.24245	23.13	-0.48618
	0/90/0	10	10	9.473153	9.629	1.618514
			20	14.66063	14.64	-0.14094
			30	18.76243	18.61	-0.81909
			40	22.10016	21.8527	-1.13242

$\lambda_{cr}=\lambda\frac{a}{E_2h^3}$ nondimensional critical buckling load for simply supported condition.

4.2.2 Case study:

Case study 1. Effect of different boundary conditions on buckling load.

Case Study2. Normalized critical buckling loads ($\lambda\ cr$) for various aspect ratios.

Case Study3. Variation with Different Modular Ratio.

Case Study 4. Variation of critical buckling load with different loading.

4.2.2.1 Case study 1.Effect of different boundary condition on buckling load

In this case 0/90/90/0 cross ply with material 5 (Table 4.1.7) is used to calculate critical buckling load of a square plate. The non dimensional critical buckling load has been calculated from the formulae $\lambda_{cr}=\lambda\frac{a}{E_2h^3}$. Here two case has been shown for a/b ratio (10,100). From the result it has been shown that buckling load increases with restrain conditions because of increasing stiffness. Here CFFF has lowest and CCCC has highest buckling load.

Table4.2.4. Non dimensional Buckling Load for Deferent Boundary Condition

a/h	CFFF	CCFF	SSSS	CFCF	CCCF	SCSC	CCCC
10	1.667	7.681	25.122	17.515	19.657	37.166	40.771
100	2.003	9.010	36.102	36.879	42.452	82.937	147.952

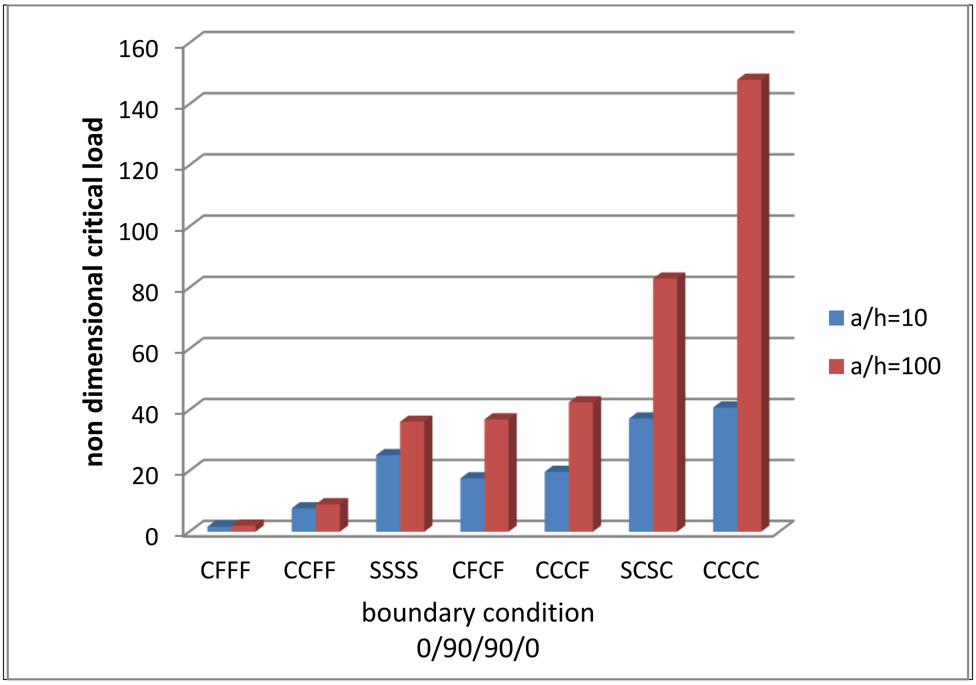


Fig.4.2.1

4.2.2.2 CASE STUDY2: Normalized Critical Buckling Loads with Various Aspect Ratios.

Normalized critical buckling loads (λ_{cr}) have been calculated with various aspect ratios (a/b) for simply supported laminated composite plate with cross ply [0/90/90/0] .Analysis has been done with material 5 [Table 4.1.7] with E_1/E_2 equal to 40.

Table 4.2.5. Variations of non dimensional buckling load for different a/h ratio.

Uniaxial loading	E_1/E_2 =40				
a/h	5	10	20	50	100
a/b=1	11.41	23.24	31.62	35.38	36.06
a/b=2	11.56	45.74	93.50	113.47	115.54
a/b=3	11.72	46.63	150.34	219.54	236.84
Biaxial loading					
a/b=1	5.97	11.62	15.81	17.69	18.03
a/b=2	7.44	14.96	20.33	22.69	23.11
a/b=3	6.13	22.03	35.18	42.51	43.86

From the above data it has been shown that critical bucking load is increasing with a/h ratio for constant aspect ratio and also critical bucking load is increasing with increasing a/b ratio for constant a/h ratio. It has been shown that uniaxial buckling load (N_{xcr}) is more than the bi-axial buckling load (N_{xcr},N_{xcr}).

If the loading N_x and N_y both applied it is equivalent to load N_{xy}.

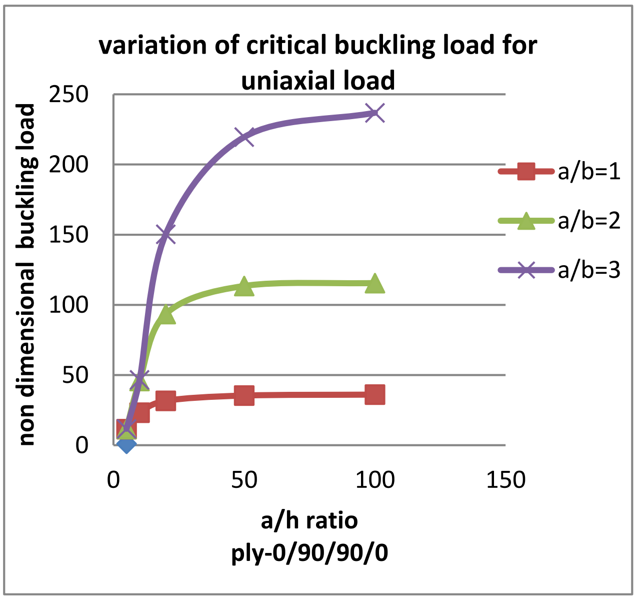


Fig.4.2.2

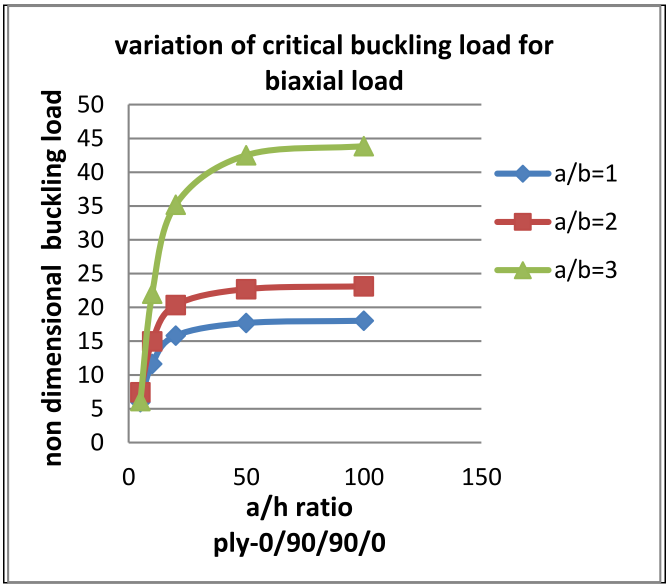


Fig.4.2.3

4.2.2.3 CASE STUDY3: Variation with Different Modular Ratio

Here analysis has been done for cross ply composite one having four and other having three layers. Simply supported square plate is used to analyse the plate with a/h = 10. Uniaxial and biaxial both loading conditions are used for different E_1/E_2 ratios. Here it has been shown that non dimensional buckling load is increasing with increasing E_1/E_2 ratios.

Table 4.2.6. Non dimensional critical buckling load for different loading

E1/E2	0/90/90/0 Uniaxial loading	0/90/90/0 Biaxial loading	0/90/0 Uniaxial loading	0/90/0 Biaxial loading
3	4.928005	2.464003	4.924774	2.462387
10	9.567725	4.783856	9.473153	4.736569
20	15.02848	7.514215	14.66063	7.312692
30	19.49515	9.747526	18.76243	8.736476
40	23.24245	11.62116	22.10016	9.992563

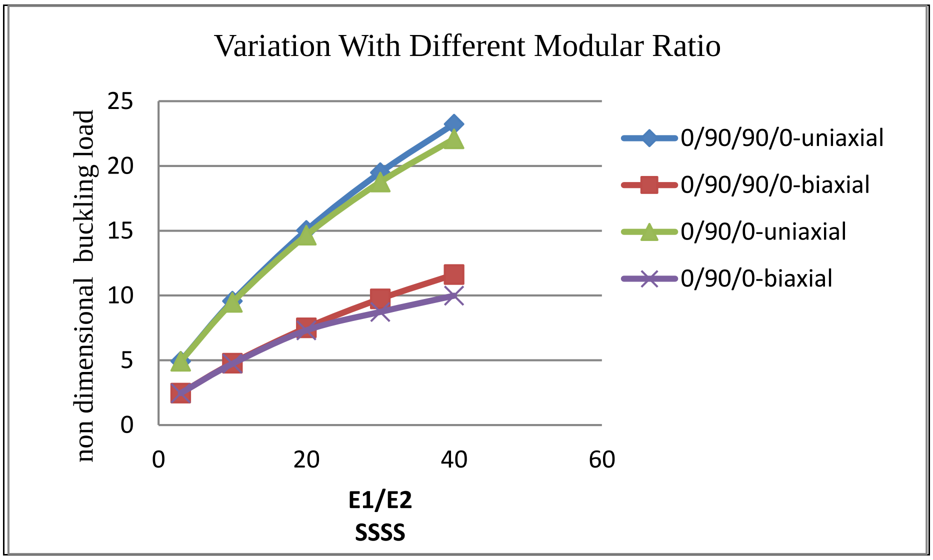


Fig. 4.2.4

From the graph, it is observed that 4 layer composite plates gives relatively higher critical buckling load than 3 layers. Another case study is introduced for different lay-up sequence. Analyses have been done for different ply orientation with constant thickness and no of layers. A comparison has been done for constant a/h ratio 100 with simply supported boundary conditions of a square plate. From the table it is noticed that ply 45/-45/-45/45 has higher buckling load than others ply orientations though for ply 30/-30/-30/30 has highest buckling load for lower E1/E2(=3) ratio for uni axial loading.

Table 4.2.7 Non dimensional buckling load for different ply orientations for uniaxial loading (N_x)

E1/E2	10	20	30	40
0/45/45/0	12.012	21.196	30.317	39.389
0/90/90/0	11.201	19.509	27.793	36.055
30/-30/-30/30	15.034	26.659	37.827	48.757
45/-45/-45/45	**16.377**	**29.241**	**41.556**	**53.6**
60/-60/-60/60	14.999	25.974	35.366	44.4028
75/-75/-75/75	**10.464**	**14.859**	**19.095**	**23.213**

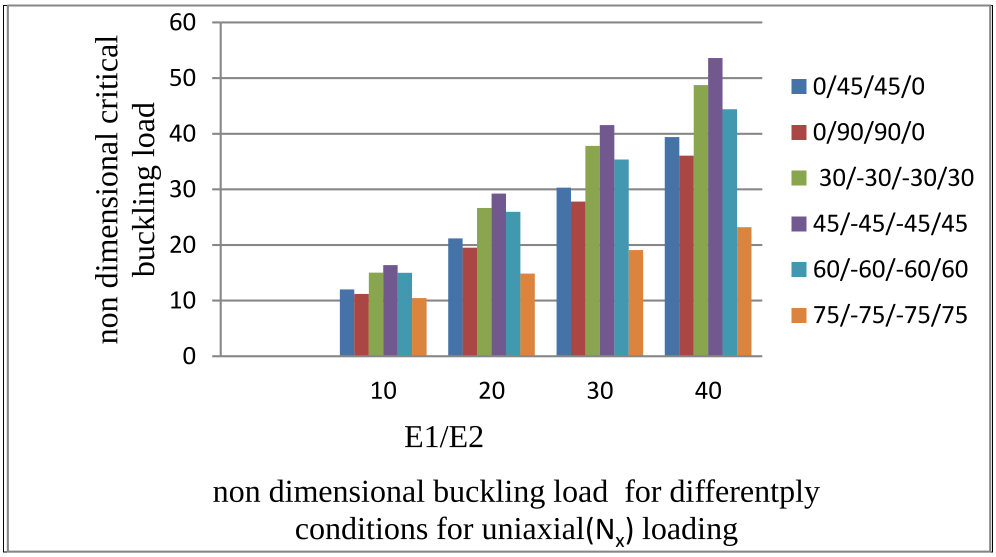
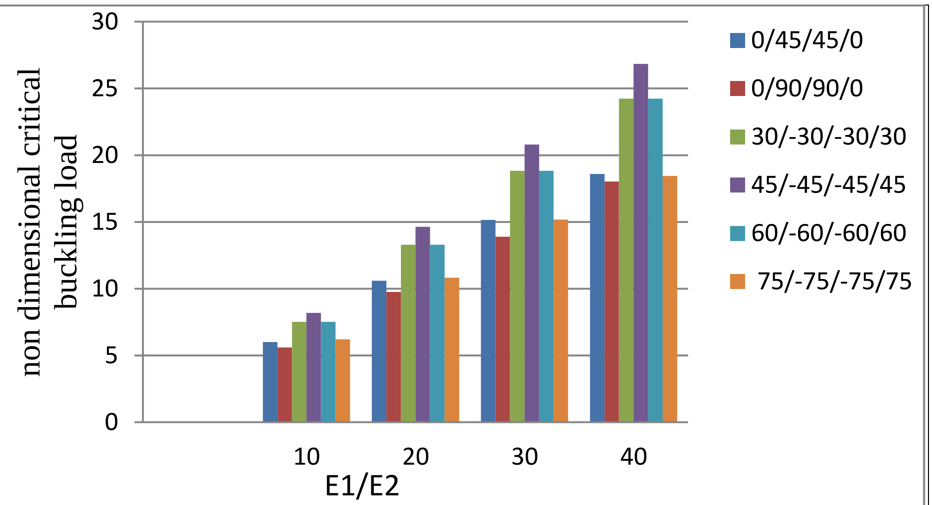


Fig. 4.2.5

In biaxial loading same behaviour has been observed keeping other conditions same. Same analysis has been done for all in plane loading along x,y and xy directions. Here it has been shown that ply- 45/-45/-45/45 has highest buckling load and ply-0/90/90/0 has lowest buckling loads.

Table 4.2.8 Non dimensional buckling load for different ply orientations for biaxial loading(N_x,N_y)

E1/E2	10	20	30	40
0/45/45/0	6.006	10.597	15.157	18.59
0/90/90/0	5.6	9.754	13.896	18.027
30/-30/-30/30	7.512	13.297	18.832	24.238
45/-45/-45/45	8.19	14.632	20.802	26.837
60/-60/-60/60	7.512	13.297	18.832	24.238
75/-75/-75/75	6.208	10.826	15.178	18.445

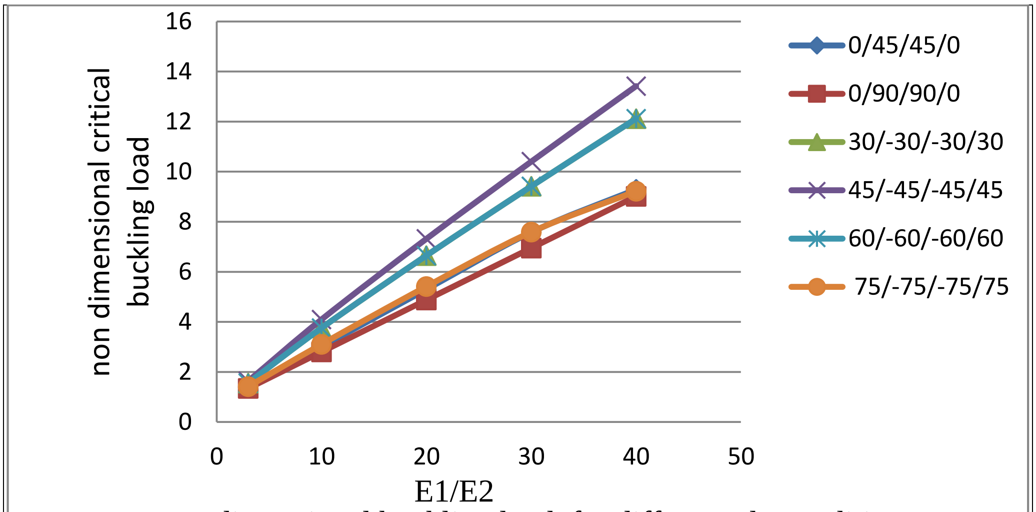


non dimensional buckling load for different ply orientations for biaxial(N_x,N_y) loading

Fig.4. 2.6

Table 4.2.9 Non dimensional buckling load for different ply orientations for all in plane loading(N_x,N_y,and N_{xy})

E1/E2	3	10	20	30	40
0/45/45/0	1.382	3.003	5.2988	7.578	9.2951
0/90/90/0	1.342	2.8	4.877	6.948	9.014
30/-30/-30/30	1.556	3.756	6.648	9.416	12.119
45/-45/-45/45	1.6292	4.095	7.316	10.401	13.418
60/-60/-60/60	1.556	3.756	6.6485	9.416	12.119
75/-75/-75/75	1.413	3.104	5.413	7.589	9.222



non dimensional buckling load for different ply conditions for N_x,N_y,and N_{xy} loading

Fig.4.2.7

4.2.2.4 Case Study 4: Variation of Critical Buckling Load with a/h Ratio for Different Ply Orientation

Here it has been examined how the buckling load are dependent with loading condition for different ply orientation. This calculation has been carried out for simply supported square plate for material 5 (E_1/E_2 =40). It is shown that non dimensional buckling load is increasing with a/h ratio. Actually stiffness is decreases with increasing a/h ratio due to thinner section. But in non dimensional form it is increases with a/h ratio. Ply orientation 30/-30/-30/30 gives highest buckling loads and (75/-75/-75/75) gives the lowest buckling loads for uniaxial loading. The result is similar for bi-axial loading, too. For all in plane loading conditions, (30/-30/-30/30) gives the highest buckling load whereas (75/-75/-75/75) gives the lowest buckling load except for the case a/h=100. Ply orientation (30/-30/-30/30) gives higher stiffness because internal angle between two layer is maximum (120°) and for (75/-75/-75/75) it is lowest (30°).

Table 4.2.10. Non dimensional buckling load [$\lambda_{cr}=\lambda\frac{a}{E_2 h^3}$] for different ply orientations for uniaxial (N_x) loading

a/h	5	10	20	50	100
45/-45/-45/45	10.578	27.571	43.102	50.161	53.6
0/45/45/0	11.386	23.882	33.763	38.421	39.389
30/-30/-30/30	**11.796**	**28.437**	**39.751**	**45.859**	**48.757**
60/-60/-60/60	8.155	19.767	31.214	39.445	44.403
75/-75/-75/75	**6.079**	**12.407**	**17.658**	**21.11**	**23.213**
0/90/90/0	11.408	23.242	31.624	35.384	36.055

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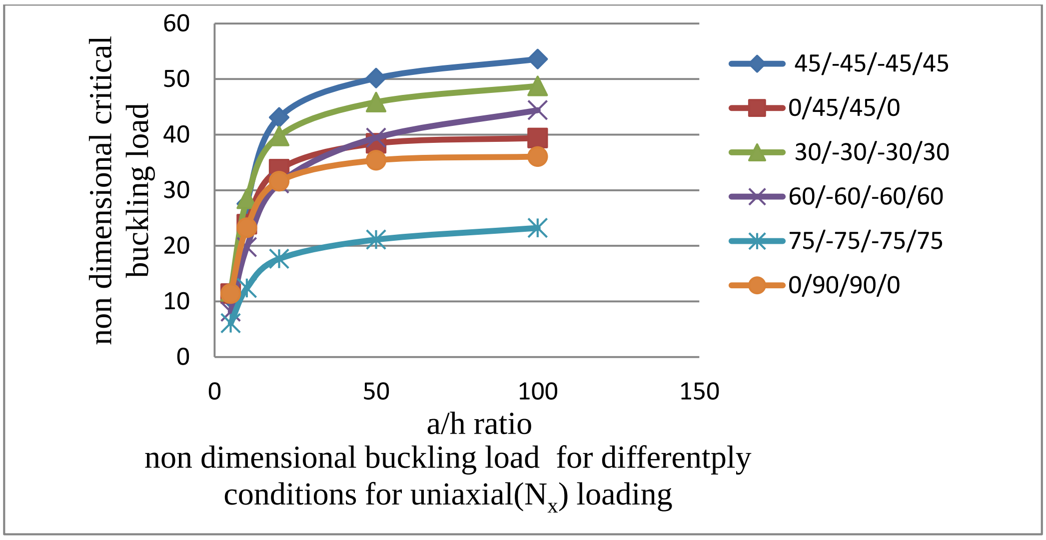


Fig. 4.2.8

Table 4.2.11 Non dimensional buckling load for various a/h ratio for different ply orientations in biaxial (N_x,N_y) loading

a/h	5	10	20	50	100
0/45/45/0	5.356	10.628	15.0139	17.4919	18.59
0/90/90/0	5.968	11.621	15.811	17.692	18.027
30/-30/-30/30	**11.866**	**14.124**	**19.71**	**22.753**	**24.238**
60/-60/-60/60	**6.73**	**14.124**	**19.714**	**22.753**	**24.238**
75/-75/-75/75	**4.942**	**9.866**	**14.008**	**16.752**	**18.445**
45/-45/-45/45	7.421	15.355	21.644	25.129	26.837

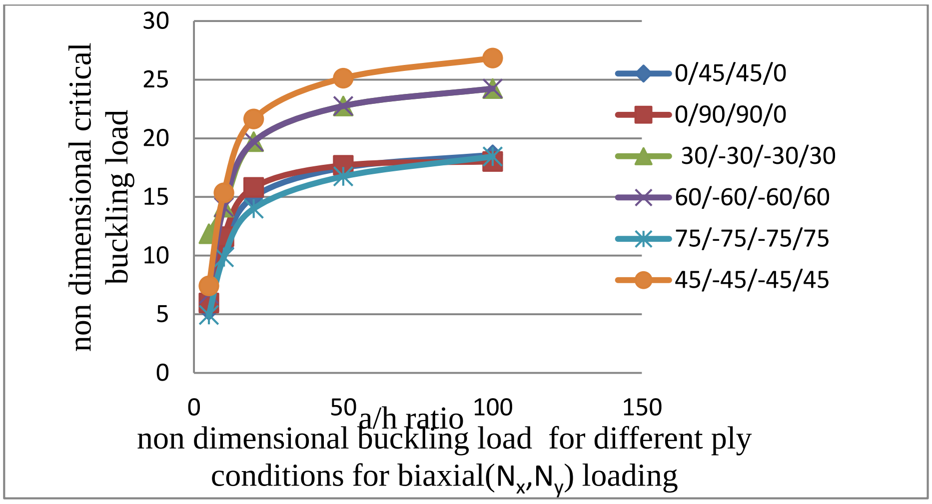


Fig. 4.2.9

Here N_x, N_y, and N_{xy} loading is applied to find the critical buckling loads. Here simply supported plate also has been used.

Table 4.2.12. Non dimensional buckling load for various a/h ratio for different ply orientations in all plane loading

a/h	5	10	20	50	100
0/45/45/0	2.678	5.314	7.506	8.7459	9.2951
0/90/90/0	2.984	5.81	7.905	8.846	**9.013**
45/-45/-45/45	**3.711**	**7.677**	**10.822**	**12.564**	**13.418**
30/-30/-30/30	3.365	7.062	9.857	11.376	12.119
60/-60/-60/60	3.365	7.062	9.857	11.376	12.119
75/-75/-75/75	**2.471**	**4.933**	**7.004**	**8.376**	**9.222**

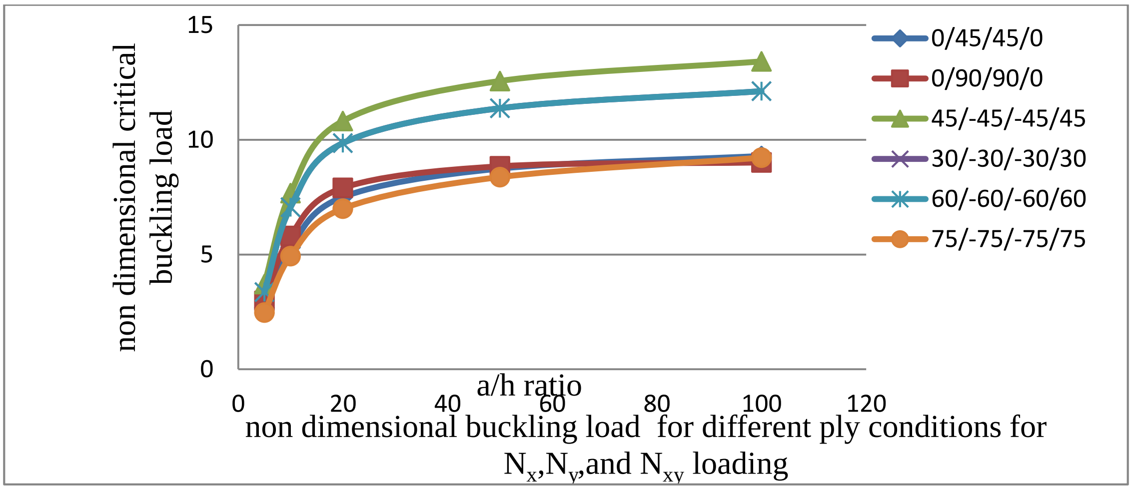
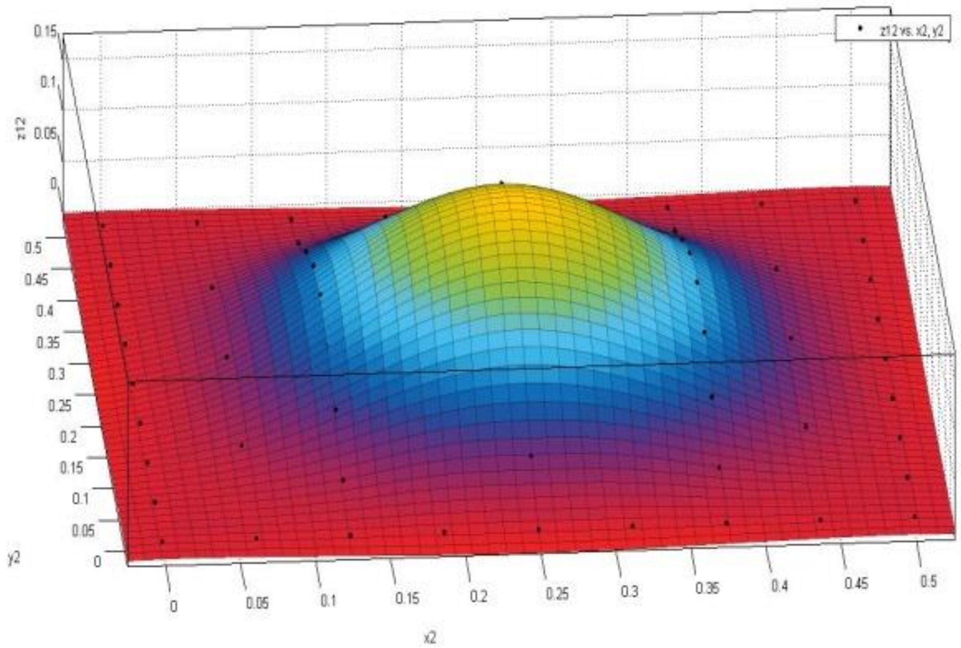


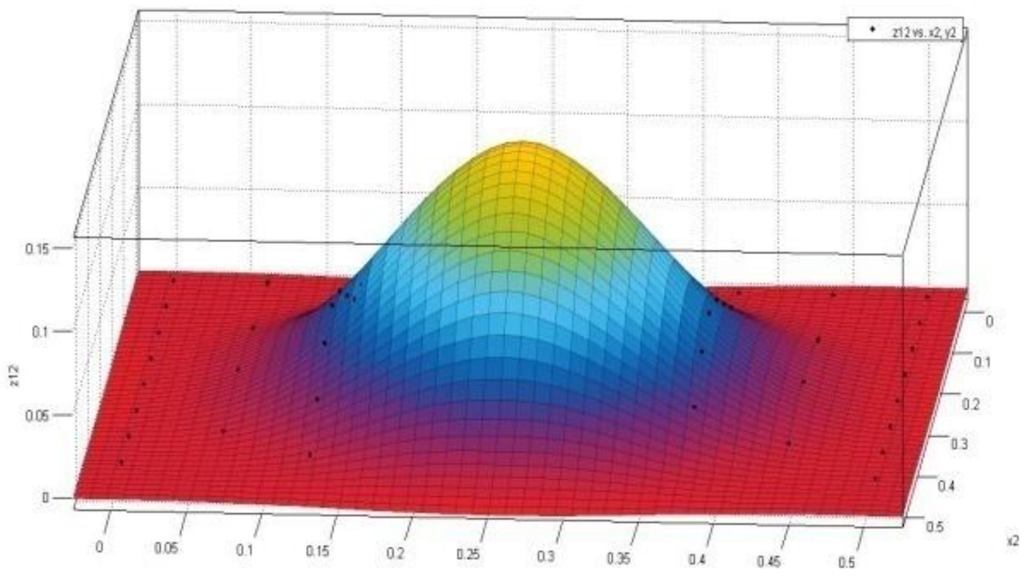
Fig. 4.2.10

4.2.3 Buckled Shape for Buckling



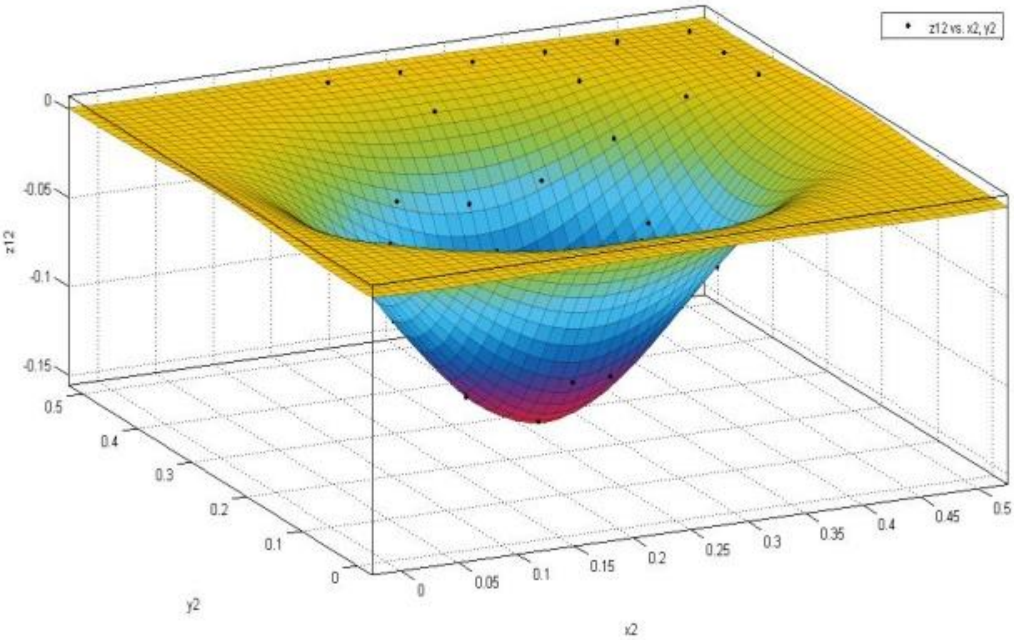
CCCC uniaxial loading

Fig. 4.2.11



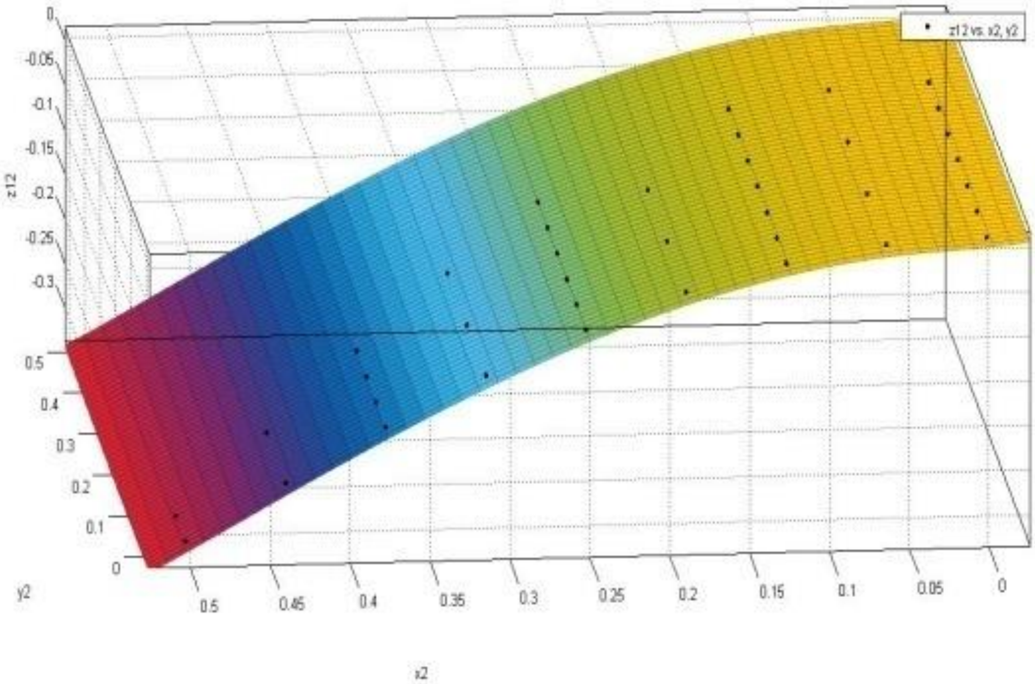
CCCC biaxial loading

Fig. 4.2.12



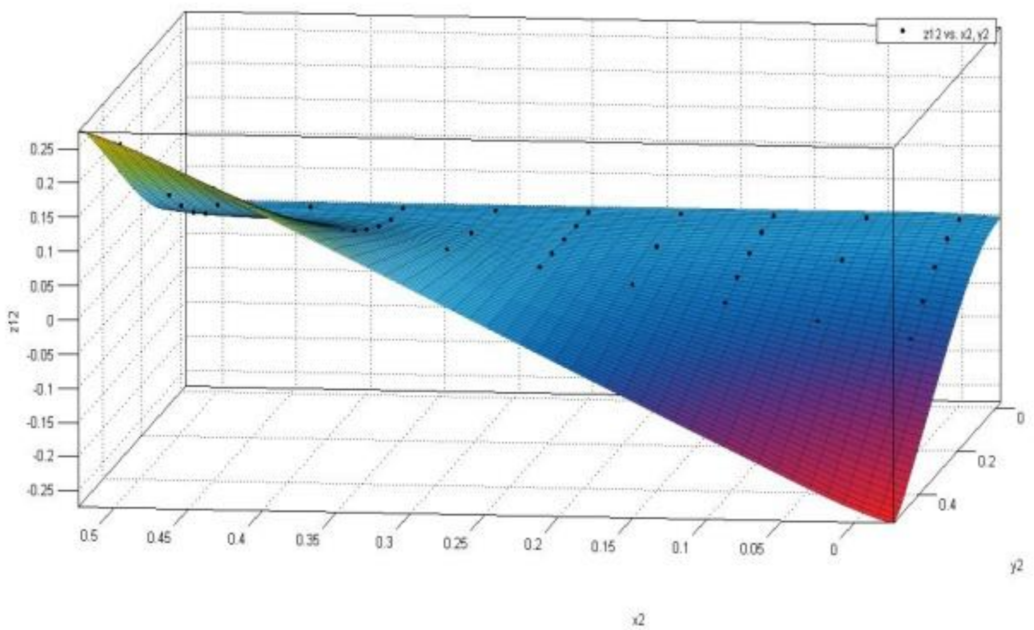
CCCC all inplane loading

Fig. 4.2.13



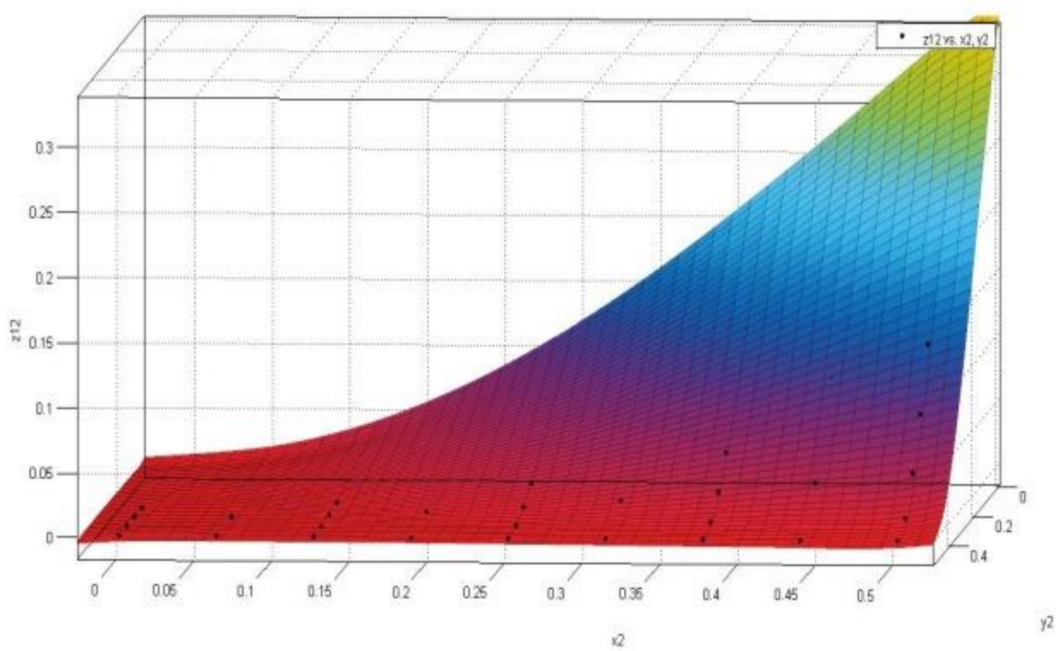
CFFF-unixial loading

Fig. 4.2.14



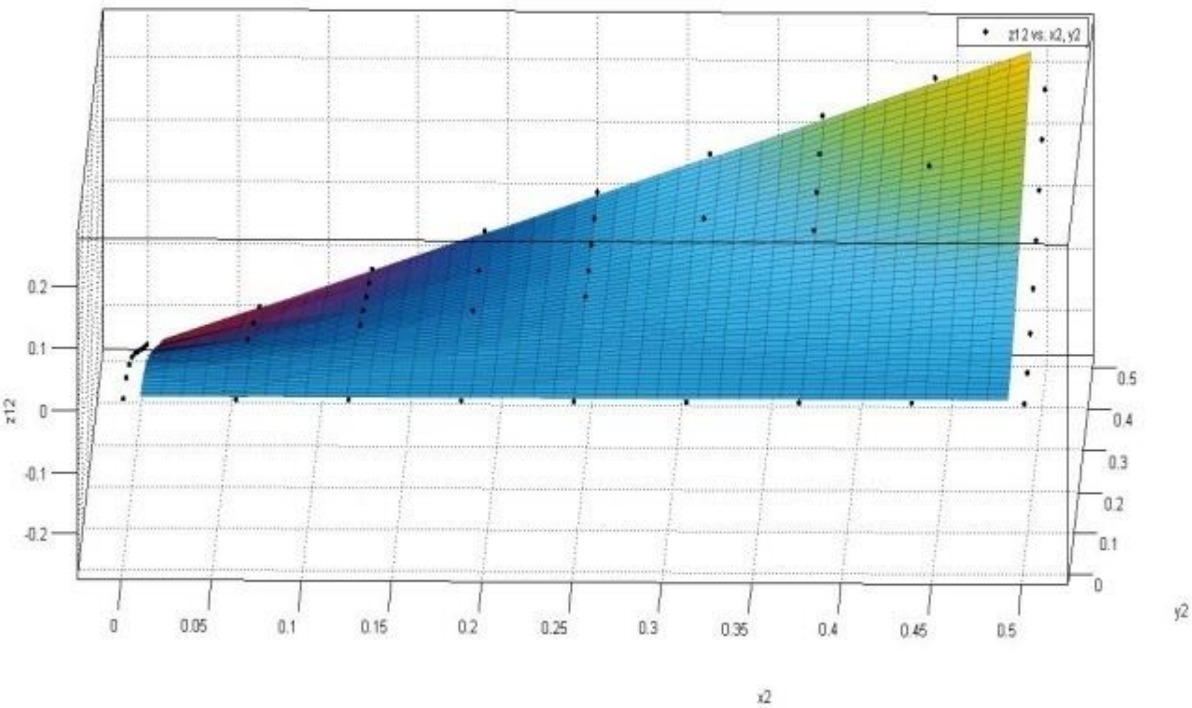
CFFF biaxial loading

Fig. 4.2.15



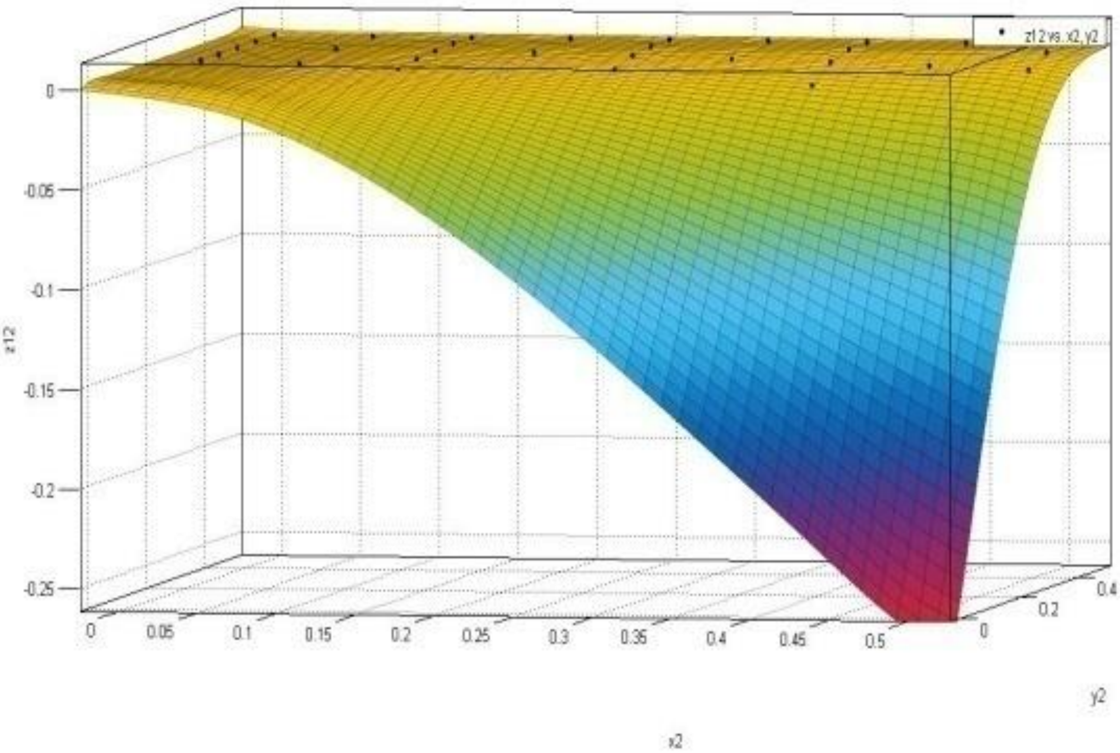
CCFF-uniaxial loading

Fig. 4.2.16



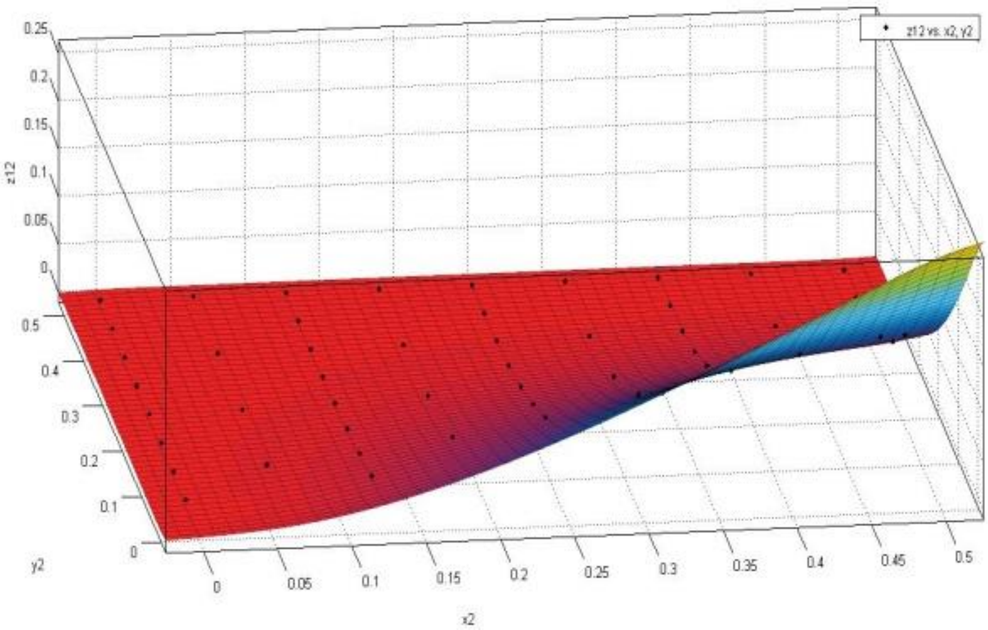
CFFF-all inplane loading

Fig. 4.2.17



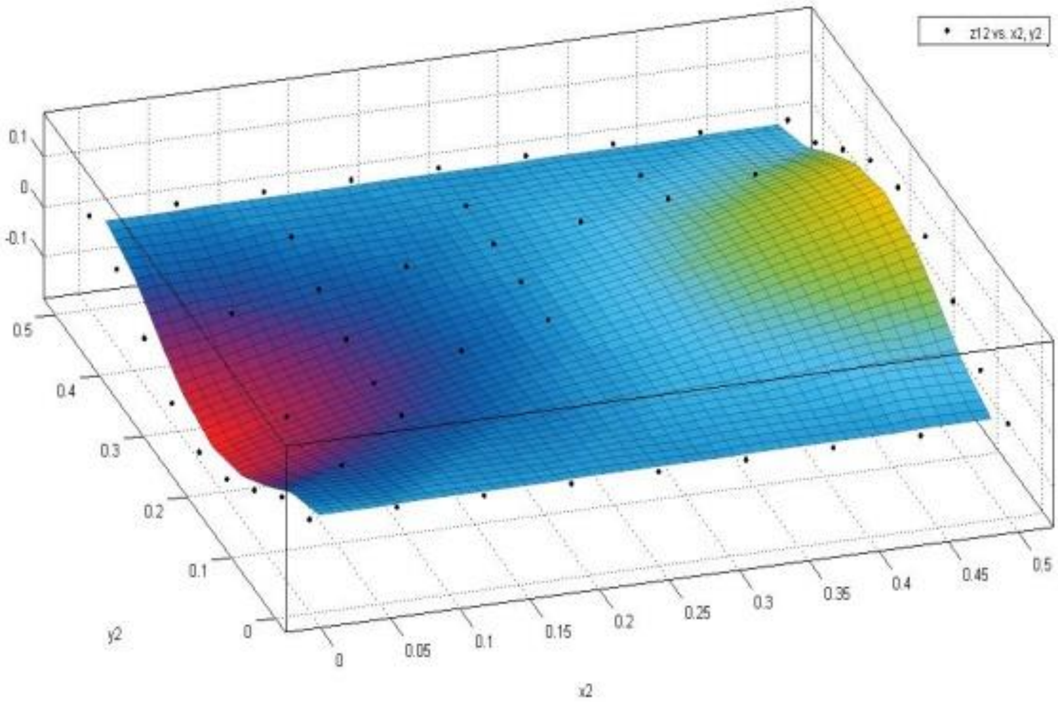
CCFF-biaxial loading

Fig. 4.2.18



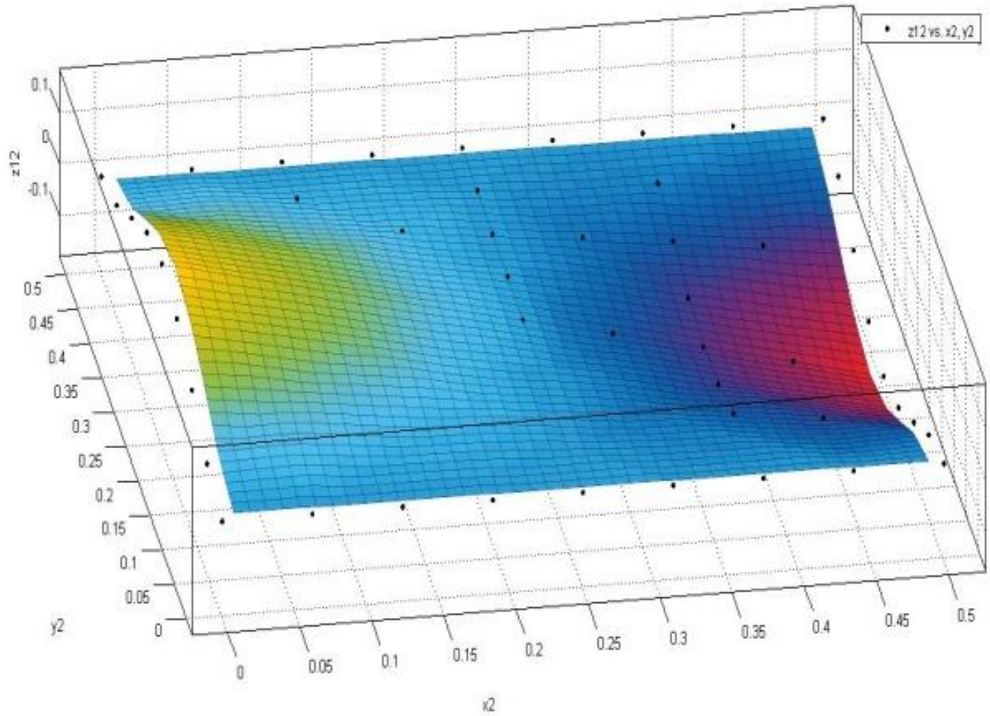
CCFF-all inplane loading

Fig. 4.2.19



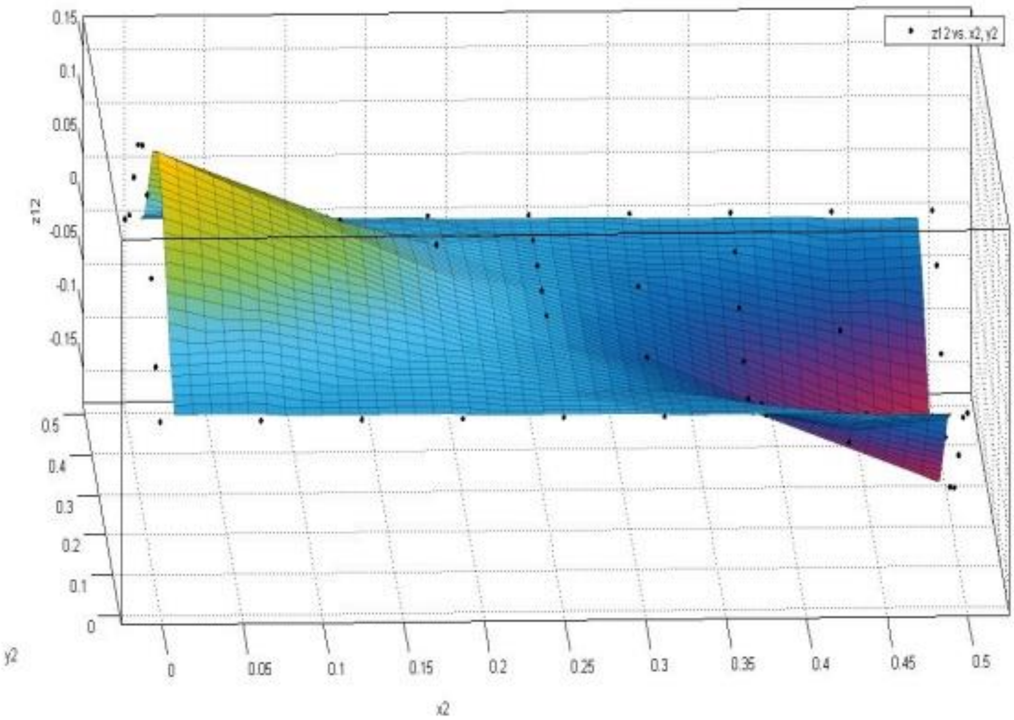
CFCF uniaxial loading

Fig. 4.2.20



CFCF-biaxial loading

Fig. 4.2.21



CFCF-all inplane loading
Fig. 4.2.22

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