

# **MULTI-ROBOT COORDINATION BY MACHINE LEARNING AND EVOLUTIONARY ALGORITHMS**

**Thesis submitted  
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## CERTIFICATE FROM THE SUPERVISOR

*This is to certify that the thesis entitled “**Multi-Robot Coordination by Machine Learning and Evolutionary Algorithms**” submitted by **Shri Arup Kumar Sadhu**, who got his name registered on **30/12/2013** for the award of Ph.D. (Engineering) degree of Jadavpur University is absolutely based upon his own work under the supervision of **Prof. Amit Konar** and that neither his thesis nor any part of the thesis has been submitted for any Degree or any other academic award anywhere before.*

Signature of the Supervisor  
and date with Official seal



**The Thesis is dedicaTed To**  
*Reviewers of my papers*

## Preface

Coordination is a fundamental trait in lower level organisms as they used their collective effort to serve their goals. Hundreds of interesting examples of coordination are available in nature. For example, ants individually cannot carry a small food item, but they collectively carry quite a voluminous food to their nest. The tracing of the trajectory of motion of an ant following the pheromone deposited by its predecessor also is attractive. The queen bee in her nest directs the labor bees to specific directions by her dance patterns and gestures to collect food resources. These natural phenomena often remind us the scope of coordination among agents to utilize their collective intelligence and activities to serve complex goals.

Coordination and planning are closely related terminologies from the domain of multi-robot system. Planning refers to the collection of feasible steps required to reach a predefined goal from a given position. However, coordination indicates the skillful interaction among the agents to generate a feasible planning step. Therefore, coordination is an important issue in the field of multi-robot coordination to address complex real-world problems. Coordination usually is of three different types: cooperation, competition and mixed. As evident from their names, cooperation refers to improving the performance of the agents to serve complex goals, which otherwise seems to be very hard for an individual agent because of the restricted availability of hardware/software resources of the agents or deadline/energy limits of the tasks. Unlike cooperation, competition refers to serving conflicting goals by two (team of) agents. For example, in robot soccer, the two teams compete to win the game. Here, each team plans both offensively and defensively to score goals and thus act competitively. Mixed coordination indicates a mixture of cooperation and competition. In the example of a soccer game, inter-team competition and intra-team cooperation is the mixed coordination. Most of the common usage of coordination in robotics lies in cooperation of agents to serve a common goal. The thesis deals with the cooperation of robots/robotic agents to efficiently complete a complex task.

In recent times, researchers are taking keen interest to employ machine learning in multi-robot cooperation. The primary advantage of machine learning is to generate the action plans in sequence from the available sensory readings of the robots. In case of a single robot, learning the action plans from the sensory readings is straight-forward. However, in the context of multi-robot, the positional changes of the other robots act as additional inputs for the learner robot, and thus learning is relatively difficult. Several machine learning and evolutionary algorithms have been adopted over the last two decades to handle the situations. The simplest of all is the supervised learning technique that requires an exhaustive list of sensory instances and the action plan by the robots. Usually, a human experimenter provides these data from his long acquaintance with such problems or by direct measurement of the sensory instances and

decisions. The training instances being too large, sometimes has a negative influence to the engineer, and he/she feels it uncomfortable not to miss a single instance that carries valuable mapping from sensory instance to action plan by the robots.

Because of the difficulty of generating training instances and excessive computational overhead to learn those instances, coupled with the need for handling dynamic situations, researchers felt the importance of reinforcement learning (RL). In RL, we need not provide any training instance, but employ a critic who provides a feedback to the learning algorithm about the possible reward/penalty of the actions by the agent. The agent/s on receiving the approximate measure of penalty/reward understands which particular sensory-motor instances they need to learn for future planning applications. The dynamic nature of environment thus can easily be learned by RL. In the multi-agent scenario, RL needs to take care of learning in joint state/action space of the agents. Here, each agent learns the sensory-motor instances in the joint state/action space with an ultimate motive to learn the best actions for itself to optimize its rewards.

The superiority of evolutionary algorithms (EA) in optimizing diverse objective functions is subjected to the No Free Lunch Theorem (NFLT). According to NFLT, the expected effectiveness of any two traditional EAs across all possible optimization problems is identical. A self-evident implication of NFLT is that the elevated performance of one EA, say A, over the other, say B, for one class of optimization problems is counterbalanced by their respective performances over another class. It is therefore practically difficult to devise a universal EA that would solve all the problems. This apparently paves the way for hybridization of EAs with other optimization strategies, machine learning techniques, and heuristics.

In evolutionary computation paradigm, hybridization refers to the process of integrating the attractive features of two or more EAs synergistically to develop a new hybrid EA. The hybrid EA is expected to outperform its ancestors with respect to both accuracy and complexity over application-specific or general benchmark problems. The fusion of EAs through hybridization hence can be regarded as the key to overcome their individual limitations.

Hence, apart from the RL, hybridization of the evolutionary algorithms (EA) is also an effective approach to serve the purpose of multi-robot coordination in a complex environment. The primary objective of an EA in the context of multi-robot coordination is concerned with the minimization of the time consumed by the robots (i.e., the length of the path to be traversed by the robots) for complete traversal of the planned trajectory. In other words, robots plan their local trajectory, so that robots shifted from given positions to the next positions (sub-goals) in a time-optimal sense avoiding collision with the obstacles or the boundary of the world-map. The optimization algorithm is executed in each local planning

step to move a small distance. Hence, cumulatively robots move to the desired goal position using the sequence of local planning. There are traces of literature on hybridization of the EAs.

Several algorithms for multi-agent learning are available in the literature, each with one specific flavor to optimize certain learning intents of the agents. Of these algorithms, quite a few interesting works on the MAQL have been reported in the literature. Among the state-of-the-art MAQL algorithms, the following need special mentions. Claus and Boutilier, aimed at solving the coordination problem using two types of reinforcement learners. The first one, called independent learner (IL), takes care of the learning behavior of individual agents by ignoring the presence of other agents. The second one, called joint action learner (JAL), considers all agents including the self to learn at joint action-space. Unlike JAL, in Team Q-learning proposed by Littman, an agent updates its Q-value at a joint state-action pair without utilizing associated agents' reward; rather the value function of the agent at the next joint state is evaluated by obtaining the maximum Q-value among the joint actions at the next joint state. Vile proposed Asymmetric-Q learning (AQL) algorithm, where the leader agents are capable of maintaining all the agents Q-tables. However, the follower agents are not allowed to maintain all the agents' Q-tables and hence, they just maximize their own rewards. In AQL, agents always achieve the pure strategy Nash equilibrium (NE), although there does exist mixed strategy NE. Hu and Wellman extended the Littman's Minimax Q-learning to the general-sum stochastic game (where the summation of all agents' payoff is neither zero nor constant) by taking into account of other agents' dynamics using NE. They also offered a proof of convergence of their algorithm. In case of multiple NE occurrences, one is selected optimally. Littman proposed Friend-or-Foe Q-learning (FQL) algorithm for general-sum games. In this algorithm, the learner is instructed to treat each other agent either as a friend in Friend Q-learning or as a foe in Foe Q-learning. Friend-or-Foe Q-learning provides a stronger convergence guarantee in comparison to that of the existing NE based learning rule. Greenwald and Hall proposed correlated Q-learning (CQL) employing correlated equilibrium (CE) to generalize both Nash Q-learning (NQL) and FQL. The bottlenecks of the above MAQL algorithms are update policy selection for adaptation of the Q-tables in joint state-action space and the curse of dimensionality with an increase in the number of learning agents. Several attempts have been made to handle the curse of dimensionality in MAQL. Jelle and Nikos proposed Sparse Cooperative Q-learning, where a sparse representation of the joint state-action space of the agents is done by identifying the need for coordination among the agents at a joint state. Here, agents undertake coordination by their actions only in a few joint states. Hence, each agent maintains two Q-tables: one is the individual-action Q-table for un-coordinated joint states and another one is the joint action Q-table to represent the

coordinated joint states. In case of uncoordinated states, a global Q-value is evaluated by adding the individual Q-values. Zinkevich offers a neural network based approach for generalized representation of the state-space for multi-agent coordination. By such generalization, agents (here robots) can avoid collision with an obstacle or other robots by collecting minimum information from the sensors. Reinaldo et al. proposed a novel algorithm to heuristically accelerate the TMAQL algorithms.

In the literature of MAQL agents either converge to NE or CE. The equilibrium-based MAQL algorithms are most popular for their inherent ability to determine optimal strategy (equilibrium) at a given joint state. Hu et al. identified the phenomenon of similar equilibria in different joint states and introduced the concept of equilibrium transfer to accelerate the state-of-the-art equilibrium-based MAQL (NQL and CQL). In equilibrium transfer, agents recycle the previously computed equilibria having very small transfer-loss. Recently Zhang et al. attempted to reduce the dimension of the Q-tables in NQL. The reduction is done by allowing the agents to store the Q-values in joint state-individual action space, instead of joint state-action space.

In the state-of-the-art MAQL (NQL and CQL), balancing exploration/exploitation during the learning phase is an important issue. Traditional approaches used to balance exploration/exploitation in MAQL are summarized here. The greedy exploration, although has wide publicity, needs to tune the value of which is time-costly. In the Boltzmann strategy, the action selection probability is controlled by tuning a control parameter (temperature) and by utilizing the Q-values due to all actions at a given state. Here, the setting of temperature to infinity (zero) implies pure exploration (exploitation). Unfortunately, the Boltzmann strategy antagonistically affects the speed of learning. Evolution of the Boltzmann strategy towards better performance is observed in a series of literature. However, the above selection mechanisms are not suitable for selecting a joint action preferred for the team (all the agents) because of the dissimilar joint Q-values offered by the agents at a common joint state-action pair. There are traces of literature concerning joint action selection at a joint state during learning. However, with the best of our knowledge, there is no work in the literature, which considers the work, presented in this thesis.

The thesis includes six (6) chapters. Chapter 1 provides an introduction to the multi-robot coordination algorithms for complex real-world problems, including transportation of a box/stick, formation control for defense applications and soccer playing by multiple robots utilizing the principles of reinforcement learning, the theory of games, dynamic programming, and/or evolutionary algorithm. Naturally, this chapter provides a thorough survey of the existing literature of reinforcement learning with a brief overview of the

evolutionary optimization to examine the role of the algorithms in the context of multi-agent coordination. Chapter 1 includes multi-robot coordination employing evolutionary optimization, and especially reinforcement learning for cooperative, competitive, and their composition for application to static and dynamic games. The latter part of the chapter deals with an overview of the metrics used to compare the performance of the algorithms while coordinating. Fundamental metrics for performance analysis are defined to study the learning and planning algorithms.

Chapter 2 offers learning-based planning algorithms, by extending the traditional multi-agent Q-learning algorithms (Nash Q-Learning and Correlated Q-Learning) for multi-robot coordination and planning. This extension is achieved by employing two interesting properties. The first property deals with the exploration of the team-goal (simultaneous success of all the robots) and the other property is related to the selection of joint action at a given joint state. The exploration of team-goal is realized by allowing the agents, capable of reaching their goals, to wait at their individual goal states, until remaining agents explore their individual goals synchronously or asynchronously. Selection of joint action, which is a crucial problem in traditional multi-agent Q-learning, is performed here by taking the intersection of individual preferred joint actions of all the agents. In case the resulting intersection is a null set, the individual actions are selected randomly or otherwise following classical techniques. The superiority of the proposed learning and learning-based planning algorithms are validated over contestant algorithms in terms of the speed of convergence and run-time complexity respectively.

In chapter 3, it is shown that robots may select the suboptimal equilibrium in presence of multiple types of equilibria (here Nash equilibrium or correlated equilibrium). In the above perspective, robots need to adapt to such a strategy, which can select the optimal equilibrium in each step of the learning and the planning. To address the bottleneck of the optimal equilibrium selection among multiple types, chapter 3 presents a novel consensus Q-learning for multi-robot coordination, by extending the equilibrium-based multi-agent Q-learning algorithms. It is also shown that a consensus (joint action) jointly satisfies the conditions of the coordination type pure strategy Nash equilibrium and the pure strategy correlated equilibrium. The superiority of the proposed consensus Q-learning algorithm over traditional reference algorithms in terms of the average reward collection are shown in the experimental section. In addition, the proposed consensus-based planning algorithm is also verified considering the multi-robot stick-carrying problem as the testbed.

Unlike correlated Q-learning, Chapter 4 proposes an attractive approach to adapt composite rewards of all the agents in one Q-table in joint state-action space during learning, and subsequently, these rewards are employed to compute correlated equilibrium in the planning

phase. Two separate models of multi-agent Q-learning have been proposed. If the success of only one agent is enough to make the team successful, then model-I is employed. However, if an agent's success is contingent upon other agents and simultaneous success of the agents is required then model-II is employed. It is also shown that the correlated equilibrium obtained by the proposed algorithms and by the traditional correlated Q-learning are identical. In order to restrict the exploration within the feasible joint states, constraint versions of the said algorithms are also proposed. Complexity analysis and experiments have been undertaken to validate the performance of the proposed algorithms in multi-robot planning on both simulated and real platforms.

Chapter 5 hybridizes the Firefly Algorithm and the Imperialist Competitive Algorithm. The above explained hybridization results in the Imperialist Competitive Firefly Algorithm, which is employed to determine the time-optimal trajectory of a stick, being carried by two robots, from a given starting position to a predefined goal position amidst static obstacles in a robot world-map. The motion dynamics of fireflies of the Firefly Algorithm is embedded into the socio-political evolution-based meta-heuristic Imperialist Competitive Algorithm. Also, the trade-off between the exploration and exploitation is balanced by modifying the random walk strategy based on the position of the candidate solutions in the search space. The superiority of the proposed Imperialist Competitive Firefly Algorithm is studied considering run-time and accuracy as the performance metrics. Finally, the proposed algorithm has been verified in a real-time multi-robot stick-carrying problem.

Chapter 6 concludes the thesis based on the analysis made, experimental and simulation results obtained from the earlier chapters. The chapter also examines the prospects of the thesis in view of the future research trends.

In summary, the thesis aimed at developing multi-robot coordination algorithms with a minimum computational burden and less storage requirement as compared to the traditional algorithms. The novelty, originality, and applicability of the thesis are illustrated below.

Chapter 1 introduces fundamentals of the multi-robot coordination. Chapter 2 offers two useful properties, which have been developed to speed-up the convergence of TMAQL algorithms in view of the team-goal exploration, where team-goal exploration refers to the simultaneous exploration of individual goals. The first property accelerates exploration of the team-goal. Here, each agent accumulates high (immediate) reward for team-goal state-transition, thereby improving the entries in the Q-table for state-transitions leading to the team-goal. The Q-table thus obtained offers the team the additional benefit to identify the joint action leading to a transition to the team-goal during the planning, where TMAQL-based planning stops inadvertently. The second property directs an alternative approach to speed-up the convergence of TMAQL by identifying the preferred joint action for the team. Finding

preferred joint action for the team is crucial when robots are acting synchronously in a tight cooperative system. The superiority of the proposed algorithms in Chapter 2 is verified both theoretically as well as experimentally in terms of the convergence speed and the run-time complexity.

Chapter 3 proposes the novel consensus Q-learning (CoQL), which addresses the equilibrium selection problem. In case multiple equilibria exist at a joint state by adapting the Q-functions at a consensus. Analytically it is shown that a consensus at a joint state is a coordination type pure strategy NE as well as a pure strategy CE. Experimentally, it is shown that the average rewards earned by the robots are more when adapting at consensus, than by either NE or CE.

Chapter 4 introduces a new dimension in the literature of the traditional CQL. In traditional CQL, CE is evaluated both in learning and planning phases. In Chapter 4, CE is computed partly in the learning and the rest in the planning phases, thereby requiring CE computation once only. It is shown in an analysis, that the CE obtained by the proposed techniques is same as that obtained by the traditional CQL algorithms. In addition, the computational cost to evaluate CE by the proposed techniques is much smaller than that obtained by traditional CQL algorithms for the following reasons. Computation of CE in the traditional CQL requires consulting  $m$  Q-tables in joint state-action space for  $m$  robots, whereas in the present context, we use a single Q-table in the joint state-action space for evaluation of CE. Complexity analysis (both time-and space-complexity) undertaken here confirms the last point. Two schemes are proposed: one for a loosely-and the other one for a tightly-coupled multi-robot system. Also, the problem-specific constraints are taken care of in Chapter 4 to avoid unwanted exploration of the infeasible state-space during the learning phase, thereby saving additional run-time complexity during the planning phase. Experiments are undertaken to validate the proposed concepts in simulated and practical multi-agent robotic platform (here Khepera-environment).

Chapter 5 offers the evolutionary optimization approach to address the multi-robot stick-carrying problem using the proposed Imperialist Competitive Firefly Algorithm (ICFA). ICFA is the synergistic fusion of the motion dynamics of a firefly in the Firefly Algorithm (FA) and the local exploration capabilities of the Imperialist Competitive Algorithm. In ICA, an evolving colony is not guided by the experience of more powerful colonies within the same empire. However, in ICFA each colony attempts to contribute to the improvement of its governing empire by improving its socio-political attributes following the motion dynamics of a firefly in the FA. To improve the performance of the above mentioned hybrid algorithm further, the step-size for random movement of each firefly is modulated according to its relative position in the search space. An inferior solution is driven by the explorative force



while a qualitative solution should be confined to its local neighborhood in the search space. The chapter also recommends a novel approach of evaluating the threshold value for uniting empires without imposing any serious computational overhead on the traditional ICA. Simulation and experimental results confirm the superiority of the proposed ICFA over the state-of-art techniques. Chapter 6 concludes the thesis with interesting future research directions.

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# Table of Contents

Preface .....	vi
Acknowledgement .....	xiv
Table of Contents .....	xv
List of Figures .....	xxi
List of Tables .....	xxiv
<b>CHAPTER 1 INTRODUCTION: MULTI-ROBOT COORDINATION BY MACHINE LEARNING AND EVOLUTIONARY ALGORITHMS .....</b>	<b>1</b>
1.1 INTRODUCTION .....	2
1.2 SINGLE AGENT PLANNING .....	3
1.2.1 Terminologies used in single agent planning .....	4
1.2.2 Single agent search-based planning algorithms .....	9
1.2.2.1 Dijkstra's algorithm .....	10
1.2.2.2 A* (A-star) Algorithm .....	12
1.2.2.3 D* (D-star) Algorithm .....	14
1.2.2.4 Planning by STRIPS-like language .....	16
1.2.3 Single agent reinforcement learning .....	16
1.2.3.1 Multi-Armed Bandit Problem .....	17
1.2.3.2 Dynamic programming and Bellman equation .....	19
1.2.3.3 Correlation between reinforcement learning and Dynamic programming .....	20
1.2.3.4 Single agent Q-learning .....	20
1.2.3.5 Single agent planning using Q-learning .....	23
1.3 MULTI-AGENT PLANNING AND COORDINATION .....	24
1.3.1 Terminologies related to multi-agent coordination .....	24
1.3.2 Classification of multi-agent system .....	25
1.3.3 Game theory for multi-agent coordination .....	27
1.3.3.1 Nash equilibrium (NE) .....	30
1.3.3.1.1 Pure strategy NE (PSNE) .....	31
1.3.3.1.2 Mixed strategy NE (MSNE) .....	33
1.3.3.2 Correlated equilibrium (CE) .....	36
1.3.3.3 Static game examples .....	37
1.3.4 Correlation among RL, DP, and GT .....	39
1.3.5 Classification of MARL .....	39
1.3.5.1 Cooperative multi-agent reinforcement learning .....	41
1.3.5.1.1 Static .....	41
1.3.5.1.1.1 Independent Learner (IL) and Joint Action Learner (JAL) .....	41

1.3.5.1.1.2 Frequency maximum Q-value (FMQ) heuristic .....	44
1.3.5.1.2 Dynamic .....	46
1.3.5.1.2.1 Team-Q .....	46
1.3.5.1.2.2 Distributed –Q .....	47
1.3.5.1.2.3 Optimal Adaptive Learning .....	50
1.3.5.1.2.4 Sparse cooperative Q-learning (SCQL) .....	52
1.3.5.1.2.5 Sequential Q-learning (SQL) .....	53
1.3.5.1.2.6 Frequency of the maximum reward Q-learning (FMRO) .....	53
1.3.5.2 Competitive multi-agent reinforcement learning .....	55
1.3.5.2.1 Minimax-Q Learning .....	55
1.3.5.2.2 Heuristically–accelerated multi-agent reinforcement learning .....	56
1.3.5.3 Mixed multi-agent reinforcement learning .....	57
1.3.5.3.1 Static .....	57
1.3.5.3.1.1 Belief-based Learning rule .....	57
1.3.5.3.1.1.1 Fictitious play .....	57
1.3.5.3.1.1.2 Meta strategy .....	58
1.3.5.3.1.1.3 Adapt When Everybody is Stationary, Otherwise Move to Equilibrium (AWESOME) .....	60
1.3.5.3.1.1.4 Hyper-Q .....	62
1.3.5.3.1.2 Direct policy search based .....	63
1.3.5.3.1.2.1 Fixed learning rate .....	63
1.3.5.3.1.2.1.1 Infinitesimal Gradient Ascent (IGA) .....	63
1.3.5.3.1.2.1.2 Generalized Infinitesimal Gradient Ascent (GIGA) .....	65
1.3.5.3.1.2.2 Variable learning rate .....	66
1.3.5.3.1.2.2.1 Win or Learn Fast-IGA (WoLF-IGA) .....	66
1.3.5.3.1.2.2.2 GIGA-Win or Learn Fast (GIGA-WoLF) .....	66
1.3.5.3.2 Dynamic .....	67
1.3.5.3.2.1 Equilibrium dependent .....	67
1.3.5.3.2.1.1 Nash-Q Learning .....	67
1.3.5.3.2.1.2 Correlated-Q Learning (CQL) .....	68
1.3.5.3.2.1.3 Asymmetric-Q Learning (AQL) .....	68
1.3.5.3.2.1.4 Friend-or-Foe Q-learning .....	70
1.3.5.3.2.1.5 Negotiation-based Q-learning .....	71
1.3.5.3.2.1.6 MAQL with equilibrium transfer .....	74
1.3.5.3.2.2 Equilibrium independent .....	76
1.3.5.3.2.2.1 Variable learning rate .....	76

1.3.5.3.2.2.1.1 Win or Learn Fast Policy hill-climbing (WoLF-PHC) .....	76
1.3.5.3.2.2.1.2 Policy Dynamic based Win or Learn Fast (PD-WoLF).....	78
1.3.5.3.2.2.2 Fixed learning rate .....	78
1.3.5.3.2.2.2.1 Non-Stationary Converging Policies (NSCP) .....	78
1.3.5.3.2.2.2.2 Extended Optimal Response Learning (EXORL).....	79
1.3.6 Coordination and planning by MAQL.....	80
1.3.7 Performance analysis of MAQL and MAQL-based coordination.....	81
1.4 COORDINATION BY OPTIMIZATION ALGORITHM.....	83
1.4.1 Particle Swarm Optimization (PSO) Algorithm.....	84
1.4.2 Firefly Algorithm (FA).....	87
1.4.2.1 Initialization .....	87
1.4.2.2 Attraction to Brighter Fireflies.....	87
1.4.2.3 Movement of Fireflies .....	88
1.4.3 Imperialist Competitive Algorithm (ICA) .....	89
1.4.3.1 Initialization .....	89
1.4.3.2 Selection of Imperialists and Colonies.....	89
1.4.3.3 Formation of Empires .....	89
1.4.3.4 Assimilation of Colonies .....	90
1.4.3.5 Revolution .....	91
1.4.3.6 Imperialistic Competition .....	91
1.4.3.6.1 Total Empire Power Evaluation.....	91
1.4.3.6.2 Reassignment of Colonies and Removal of Empire .....	92
1.4.3.6.3 Union of Empires .....	92
1.4.4 Differential evolutionary (DE) algorithm.....	93
1.4.4.1 Initialization .....	93
1.4.4.2 Mutation.....	93
1.4.4.3 Recombination .....	93
1.4.4.4 Selection .....	93
1.4.5 Offline optimization .....	94
1.4.6 Performance analysis of optimization algorithms .....	94
1.4.6.1 Friedman test .....	94
1.4.6.2 Iman–Davenport test.....	95
1.5 SCOPE OF THE THESIS.....	95
1.6 SUMMARY .....	98
References .....	98

<b>CHAPTER 2 IMPROVING SPEED OF CONVERGENCE OF MULTI-AGENT Q-LEARNING FOR COOPERATIVE TASK-PLANNING BY A ROBOT-TEAM .....</b>	<b>107</b>
2.1 INTRODUCTION .....	108
2.2 LITERATURE REVIEW .....	112
2.3 PRELIMINARIES .....	114
2.3.1 <i>Single agent Q-learning</i> .....	114
2.3.2 <i>Multi-agent Q-learning</i> .....	115
2.4 PROPOSED MULTI-AGENT Q-LEARNING .....	118
2.4.1 <i>Two useful properties</i> .....	119
2.5 PROPOSED FCMQL ALGORITHMS AND THEIR CONVERGENCE ANALYSIS .....	120
2.5.1 <i>Proposed FCMQL algorithms</i> .....	120
2.5.2 <i>Convergence analysis of the proposed FCMQL algorithms</i> .....	121
2.6 FCMQL-BASED COOPERATIVE MULTI-AGENT PLANNING .....	122
2.7 EXPERIMENTS AND RESULTS .....	123
2.8 CONCLUSIONS .....	130
2.9 SUMMARY .....	131
2.10 APPENDIX 2.1 .....	131
2.11 APPENDIX 2.2 .....	135
References .....	152
<b>CHAPTER 3 MULTI-ROBOT COOPERATIVE PLANNING BY CONSENSUS Q-LEARNING.....</b>	<b>157</b>
3.1 INTRODUCTION .....	158
3.2 PRELIMINARIES .....	159
3.2.1 <i>Single agent Q-learning</i> .....	159
3.2.2 <i>Equilibrium-based multi-agent Q-learning</i> .....	160
3.3 CONSENSUS .....	161
3.4 PROPOSED CONSENSUS Q-LEARNING AND PLANNING .....	162
3.4.1 <i>Consensus Q-learning</i> .....	162
3.4.2 <i>Consensus-based multi-robot planning</i> .....	164
3.5 EXPERIMENTS AND RESULTS .....	165
3.5.1 <i>Experimental setup</i> .....	165
3.5.2 <i>Experiments for CoQL</i> .....	165
3.5.3 <i>Experiments for consensus-based planning</i> .....	166
3.6 CONCLUSIONS .....	168
3.7 SUMMARY .....	168
References .....	168

<b>CHAPTER 4 AN EFFICIENT COMPUTING OF CORRELATED EQUILIBRIUM FOR COOPERATIVE Q-LEARNING BASED MULTI-ROBOT PLANNING.....</b>	<b>171</b>
4.1 INTRODUCTION .....	172
4.2 SINGLE-AGENT Q-LEARNING AND EQUILIBRIUM BASED MAQL .....	175
4.2.1 <i>Single Agent Q learning</i> .....	175
4.2.2 <i>Equilibrium based MAQL</i> .....	175
4.3 PROPOSED COOPERATIVE MULTI-AGENT Q-LEARNING AND PLANNING .....	176
4.3.1 <i>Proposed schemes with their applicability</i> .....	176
4.3.2 <i>Immediate rewards in Scheme-I and -II</i> .....	177
4.3.3 <i>Scheme-I induced MAQL</i> .....	178
4.3.4 <i>Scheme-II induced MAQL</i> .....	180
4.3.5 <i>Algorithms for scheme-I and II</i> .....	182
4.3.6 <i>Constraint <math>\Omega</math>QL-I/<math>\Omega</math>QL-II(C<math>\Omega</math>QL-I/C<math>\Omega</math>QL-II)</i> .....	183
4.3.7 <i>Convergence</i> .....	183
4.3.8 <i>Multi-agent planning</i> .....	185
4.4 COMPLEXITY ANALYSIS .....	186
4.4.1 <i>Complexity of Correlated Q-Learning</i> .....	187
4.4.1.1 <i>Space Complexity</i> .....	187
4.4.1.2 <i>Time Complexity</i> .....	187
4.4.2 <i>Complexity of the proposed algorithms</i> .....	188
4.4.2.1 <i>Space Complexity</i> .....	188
4.4.2.2 <i>Time Complexity</i> .....	188
4.4.3 <i>Complexity comparison</i> .....	189
4.4.3.1 <i>Space complexity</i> .....	190
4.4.3.2 <i>Time complexity</i> .....	190
4.5 SIMULATION AND EXPERIMENTAL RESULTS.....	191
4.5.1 <i>Experimental platform</i> .....	191
4.5.1.1 <i>Simulation</i> .....	191
4.5.1.2 <i>Hardware</i> .....	192
4.5.2 <i>Experimental approach</i> .....	192
4.5.2.1 <i>Learning phase</i> .....	193
4.5.2.2 <i>Planning phase</i> .....	193
4.5.3 <i>Experimental results</i> .....	194
4.6 CONCLUSION .....	201
4.7 SUMMARY .....	202
4.8 APPENDIX .....	203

References .....	209
<b>CHAPTER 5 A MODIFIED IMPERIALIST COMPETITIVE ALGORITHM FOR MULTI-ROBOT STICK-CARRYING APPLICATION .....</b>	<b>213</b>
5.1 INTRODUCTION .....	214
5.2 PROBLEM FORMULATION FOR MULTI-ROBOT STICK-CARRYING .....	219
5.3 PROPOSED HYBRID ALGORITHM .....	222
5.3.1 <i>An Overview of Imperialist Competitive Algorithm (ICA)</i> .....	222
5.3.1.1 Initialization .....	222
5.3.1.2 Selection of Imperialists and Colonies .....	223
5.3.1.3 Formation of Empires .....	223
5.3.1.4 Assimilation of Colonies .....	223
5.3.1.5 Revolution .....	224
5.3.1.6 Imperialistic Competition .....	224
5.3.1.6.1 Total Empire Power Evaluation .....	225
5.3.1.6.2 Reassignment of Colonies and Removal of Empire .....	225
5.3.1.6.3 Union of Empires .....	226
5.4 AN OVERVIEW OF FIREFLY ALGORITHM (FA) .....	226
5.4.1 <i>Initialization</i> .....	226
5.4.2 <i>Attraction to Brighter Fireflies</i> .....	226
5.4.3 <i>Movement of Fireflies</i> .....	227
5.5 PROPOSED IMPERIALIST COMPETITIVE FIREFLY ALGORITHM .....	227
5.5.1 <i>Assimilation of Colonies</i> .....	229
5.5.1.1 Attraction to Powerful Colonies .....	230
5.5.1.2 Modification of Empire Behavior .....	230
5.5.1.3 Union of Empires .....	230
5.6 SIMULATION RESULTS .....	232
5.6.1 <i>Comparative Framework</i> .....	232
5.6.2 <i>Parameter Settings</i> .....	232
5.6.3 <i>Analysis on Explorative Power of ICFA</i> .....	232
5.6.4 <i>Comparison of Quality of the Final Solution</i> .....	233
5.6.5 <i>Performance Analysis</i> .....	233
5.7 COMPUTER SIMULATION AND EXPERIMENT .....	240
5.7.1 <i>Average total path deviation (ATPD)</i> .....	240
5.7.2 <i>Average Uncovered Target Distance (AUTD)</i> .....	241
5.7.3 <i>Experimental Setup in Simulation Environment</i> .....	241
5.7.4 <i>Experimental Results in Simulation Environment</i> .....	242



5.7.5 Experimental Setup with Khepera Robots.....	244
5.7.6 Experimental Results with Khepera Robots .....	244
5.8 CONCLUSION .....	245
5.9 SUMMARY .....	247
5.10 APPENDIX 5.1 .....	248
References .....	249
<b>CHAPTER 6 CONCLUSIONS AND FUTURE DIRECTIONS .....</b>	<b>255</b>
6.1 CONCLUSIONS .....	256
6.2 FUTURE DIRECTIONS.....	257

## List of Figures

Fig. 1.1 Single agent system.....	4
Fig. 1.2 Three discrete states in an environment .....	4
Fig. 1.3 Robot executing action Right (R) at state s1 and moves to the next state s2.....	5
Fig. 1.4 Deterministic state-transition .....	6
Fig. 1.5 Stochastic state-transition.....	6
Fig. 1.6 Two dimensional 5×5 grid environment .....	7
Fig. 1.7 Refinement approach in robotics .....	8
Fig. 1.8 Hierarchical tree.....	9
Fig. 1.9 Hierarchical model .....	9
Fig. 1.10 Two dimensional 3×3 grid environment .....	11
Fig. 1.11 Corresponding graph of Fig. 1.10 .....	11
Fig. 1.12 Two dimensional 3×3 grid environment with an obstacle .....	14
Fig. 1.13 Structure of reinforcement learning .....	17
Fig. 1.14 variation of average reward with the number of trial for different $\epsilon$ in 10-Armed Bandit Problem.....	19
Fig. 1.15 Correlation between the RL and DP .....	20
Fig. 1.16 Single agent Q-learning.....	21
Fig. 1.17 Possible next state in stochastic situation.....	22
Fig. 1.18 Single agent planning.....	24
Fig. 1.19 Multi-agent system with $m$ agents.....	24
Fig. 1.20 Robots executing joint action $\langle R, L \rangle$ at joint state $\langle 1, 8 \rangle$ and move to the next state $\langle 4, 5 \rangle$ .....	25
Fig. 1.21 Classification of multi-robot systems.....	26
Fig. 1.22 Hands gestures in rock-paper-scissor game .....	29
Fig. 1.23 rock-paper-scissor game.....	29

Fig. 1.24 Reward mapping from joint Q-table to reward matrix.....	31
Fig. 1.25 Pure strategy Nash equilibrium evaluation .....	32
Fig. 1.26 Evaluation of mixed strategy Nash equilibrium.....	34
Fig. 1.27 reward matrix for tennis game .....	35
Fig. 1.28 Reward matrix of in a common reward two-agent static game.....	35
Fig. 1.29 Pure strategy Egalitarian equilibrium, which is one variant of CE.....	36
Fig. 1.30 Game of Chicken.....	37
Fig. 1.31 Reward matrix in Game of Chicken.....	37
Fig. 1.32 Constant-sum-game.....	38
Fig. 1.33 matching pennies.....	38
Fig. 1.34 Reward matrix in Prisoner’s Dilemma Game .....	39
Fig. 1.35 Correlation among the MARL, DP, and GT .....	39
Fig. 1.36 Classification of multi-agent reinforcement learning.....	40
Fig. 1.37 The climbing game reward matrix .....	44
Fig. 1.38 The penalty game reward matrix.....	45
Fig. 1.39 The penalty game reward matrix.....	45
Fig. 1.40 Individual Q-values obtained in the climbing game reward matrix by Distributed Q-learning.....	49
Fig. 1.41 The penalty game reward matrix.....	49
Fig. 1.42 Individual Q-values obtained in the penalty game reward matrix by Distributed Q-learning.....	49
Fig. 1.43 Reward matrix of a three player coordination game .....	50
Fig. 1.44 reward matrix in a two-player two-agent game .....	64
Fig. 1.45 Nonstrict EDNP in Normal-form game .....	73
Fig. 1.46 multistep negotiation process between agent A and B.....	74
Fig. 1.47 Multi-robot coordination for the well-known stick-carrying problem.....	81
Fig. 1.48 Multi-robot local planning by swarm/evolutionary algorithm.....	84
Fig. 1.49 Surface plot of (1.97) .....	86
Fig. 1.50 Surface plot of (1.98) .....	86
Fig. 1.51 Steps of Differential evolutionary (DE) algorithm .....	93
Fig. 2.1 Block diagram of reinforcement leaning (RL).....	114
Fig. 2.2 Experimental workspace for two agents during the learning phase.....	123
Fig. 2.3 Convergence plot of NQLP12 and reference algorithms for two agents .....	126
Fig. 2.4 Average of average reward (AAR) plot of NQLP12 and reference algorithms for two agents.....	126
Fig. 2.5 Joint action selection strategy in EQLP12 and reference algorithms for two agents.....	127

Fig. 2.6 Cooperative path planning to carry a triangle by three robots in deterministic situation .....	127
Fig. 2.7 Cooperative path planning to carry a stick by two Khepera-II mobile robots using NQIMP algorithm (bigger dimension in Fig. A2. 2.15 [64]) .....	129
Fig. 2.8 Cooperative path planning to carry a stick by two Khepera-II mobile robots using Algorithm 3 .....	129
Fig. 3.1 Equilibrium selection in multi-agent system.....	163
Fig. 3.2 AAR versus learning epoch for two agent system .....	166
Fig. 3.3 AAR versus learning epoch for three agent system .....	166
Fig. 3.4 planning path offered by the consensus-based multi-agent planning algorithm .....	167
Fig. 3.5 planning path offered by the Nash Q- learning based planning algorithm.....	167
Fig. 4.1 Corner cell, boundary cell and other cell .....	189
Fig. 4.2 Feasible joint states for two agent systems in stick-carrying problem.....	189
Fig. 5.1 Diagram illustrating the calculation of $d$ .....	221
Fig. 5.2 Evolution of the expected population variance .....	233
Fig. 5.3 Relative performance in mean best objective function versus function evaluation for f05 with Max_FEs=500000.....	236
Fig. 5.4 Relative performance in mean best objective function versus function evaluation for f17 with Max_FEs=300000.....	236
Fig. 5.5 Relative performance in mean best objective function versus function evaluation for f07 with Max_FEs=100000.....	236
Fig. 5.6 Relative performance in mean best objective function versus function evaluation for f20 with Max_FEs=500000.....	237
Fig. 5.7 Relative performance in accuracy versus function evaluation for ICFA over other competitive algorithms for f25 with Max_FEs= $5 \times 10^6$ .....	238
Fig. 5.8 Variation of FEs required for convergence to predefined threshold accuracy ( $1.00e-08$ ) with increase in search space dimensionality for f04 and f11.....	239
Fig. 5.9 Graphical representation of Bonferroni-Dunn's procedure considering ICFA as control method.....	240
Fig. 5.10 Initial (a) and final configuration of the world-map after execution of the (b) ICFA- (c) ICA-DE- (d) ICAAI- (e) FA- and (f) ICA-based simulations with 5 obstacles requiring 23, 29, 32 and 34 steps respectively .....	242
Fig. 5.11 Average total path traversed versus number of obstacles .....	243
Fig. 5.12 Average total path deviation versus number of obstacles .....	243
Fig. 5.13 Average uncovered target distance vs. number of steps with number of obstacles= 5 (constant).....	243

Fig. 5.14 Final configuration of the world-map after experiment using Khepera-II mobile robots.....	245
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## List of Tables

Table 1.1 Trace of Dijkstra’s algorithm for Fig. 1.11 .....	12
Table 1.2 Trace of A* algorithm from Fig. 1.10.....	13
Table 1.3 Trace of D* algorithm from Fig. 1.12 .....	15
Table 1.4 Expected reward of R1 and R2 at MSNE.....	33
Table 2.1 List of acronyms.....	108
Table 2.2 Details of 10×10 grid maps .....	124
Table 2.3 Run-time complexity of Algorithm 2.3 over reference algorithms in deterministic situation.....	128
Table 2.4 Run-time complexity of Algorithm 2.3 over reference algorithms in stochastic situation.....	128
Table 2.5 Time taken by Khepera-II mobile robots to reach a team-goal with different speeds in Algorithm 2.3 .....	130
Table 3.1 List of acronyms.....	158
Table 3.2 Planning performance.....	167
Table 4.3 Average run-time complexity of different planning algorithms (second).....	201
Table 5.1 COMPARATIVE ANALYSIS OF PERFORMANCE OF THE PROPOSED ICFA WITH OTHER ALGORITHMS BASED ON SOLUTION QUALITY FOR f01 TO f25 .....	234
Table 5.2 COMPARATIVE ANALYSIS OF PERFORMANCE OF THE PROPOSED ICFA WITH OTHER ALGORITHMS BASED ON CONVERGENCE TIME IN SECONDS FOR f01 TO f25 .....	235
Table 5.3 AVERAGE RANKINGS OBTAINED THROUGH FRIEDMAN’S TEST.....	240
Table 5.4 COMPARISON OF NUMBER OF STEPS, AVERAGE PATH TRAVERSED AND AVERAGE TOTAL PATH DEVIATION BY THE KHEPERA ROBOTS .....	244

# **Chapter 1**

## **Introduction: Multi-Robot Coordination by Machine Learning and Evolutionary Algorithms**

This chapter provides an introduction to the multi-robot coordination by reinforcement learning and evolutionary algorithms. A robot (agent) is an intelligent programmable manipulator capable of performing complex tasks and decision-making like the human beings. Mobility is part and parcel of modern robots. Mobile robots employ sensing-action cycles to sense the world around them with an aim to plan their journey to the desired destination. Coordination is an important issue in modern robotics. In recent times, researchers are taking keen interest to synthesize multi-agent-coordination in complex real-world problems, including transportation of a box/stick, formation control for defense applications and soccer playing by multiple robots by utilizing the principles of reinforcement learning, theory of games, dynamic programming, and/or evolutionary optimization algorithms. This chapter provides a thorough survey of the exiting literature of reinforcement learning with a brief overview of evolutionary optimization to examine the role of the algorithms in the context of multi-agent coordination. The study includes the classification of multi-agent coordination based on different criterion, such as, the level of cooperation, knowledge sharing, communication, and the like. The chapter also includes multi-robot coordination employing evolutionary optimization, and specially reinforcement learning for cooperative, competitive, and their composition for application to static and dynamic games. The later part of the chapter deals with an overview of the metrics used to compare the performance of the algorithms in coordination. Two fundamental metrics of performance analysis are defined, where the first one is required to study the learning performance, while the other to measure the performance of the planning algorithm. Conclusions are listed at the end of the chapter with possible explorations for the future real-time applications.

## 1.1 Introduction

A robot is an intelligent and programmable manipulator, targeted at developing the functionality similar to those of a living creature [1]. It can serve complex and/or repetitive tasks efficiently. Based on the ability of locomotion, robots are categorized into two basic types: fixed base robots and mobile robots. Depending upon the type of locomotion, mobile robots are categorized into three types: wheeled/legged robots, winged/flying robots, underwater robots, where for the last one, locomotion is controlled by water thrust. In this chapter, we would deal with wheeled robots only.

Agency is a commonly used jargon in modern robotics [1]. An agent is a piece of program/hardware that helps a robot to serve a directed goal. Like humans, when complexity of the problem grows, collective intelligence of the agents is required to achieve the target. The thesis is on collective/group behavior of agents, who can sense and act rationally. On occasions, agents can share the sensory information or its decision with its teammates directly through a communication network or by displaying its gestural/postural patterns, carrying a specific signature, to communicate a message to its team members.

Communication is a vital issue to generate plans by the agents. However, communication is time-costly and thus is often disregarded for real world robotic applications. In the present thesis, we attempted to learn the agent behavioral patterns by a process of learning, and thus avoid communication overhead in real-time planning [1].

There exists quite a vast literature on planning algorithms [2]-[30]. One of the early robot planning algorithms is due to Nilsson in connection with his research on reasoning based planning undertaken in Stanford AI research laboratory, which later was adopted in STRIPs [31]-[33]. In late 1980s to early 1990's several planning algorithms, including A\* [32], [33] Voronoi diagrams [34], Quad tree, and potential field [35] were evolved. These algorithms presume static world. At the beginning of 1990's Michalewicz in one of his renowned papers introduced genetic operators to undertake dynamic planning with local adaptation in trial solutions by specialized mutation operators. The period: 1990-2000 has seen significant changes in the planning algorithm with the introduction of supervised/unsupervised neural learning in planning algorithms [33]. The neural algorithms worked in both static and dynamic environments. Typically, in dynamic environments they predict the direction and speed of motion to determine possible avoidance of collisions. However, they had limited learned experience, and thus were unable to handle planning in presence of random motions of dynamic obstacles/persons in the environment. Almost at the beginning of the first quarter of 1990's, Sutton proposed reinforcement learning (RL) algorithm [36], which can help the robot learn its environment through semi-supervised learning. We would deal with multi-agent RL (MARL) in this chapter.

Planning and coordination are two closely used terms in multi-agent robotics [31]. While planning is concerned with determining the sequence of steps to achieve a goal, coordination refers to skilful interaction among the agents to serve their individual short-run/long-run purposes. Apparently, coordination among the agents is required to implement the steps of planning. In centralized planning, the agents need not require coordination, as the central manager takes care of all the agents' states as if its own state and generate a planning cycle by taking care of all the agents' states and goals jointly. Unfortunately, centralized planning is very slow and single point failure may occur. Thus centralized planning is not amenable for real-time applications, when the number of agents is excessive. In distributed planning, each agent generates one step of planning by coordinating with other active agents.

Coordination is broadly divided into two types: cooperation and competition [37]. As the names indicate, cooperation requires agents to work hand-in-hand to purposefully serve the common objective of the team. Competition, on the other hand, leads to the success of one team against the failure of its opponent. For instance, in robot soccer, teammates work harmoniously in a cooperative manner, while each team of agents competes for winning at the cost of defeat of the other team.

Researchers are taking keen interest to model agent cooperation/competition by various models/tools. A few of these that need special mention include RL, theory of games (GT) [38]-[44], [84] dynamic programming (DP) [45], [46], evolutionary optimization (EO) [47]-[55], and many others [16]-[30], [56]-[58]. In RL, agents learn the most profitable joint action at each joint state through a feedback from the environment, and use them for subsequent planning applications [36]. GT requires for strategic analysis in multi-agent domain. In GT, agents evaluate the equilibrium, representative of the most-profitable joint action for the team in a joint state, and execute the joint action for joint state-transition in a loop until the joint goal is explored [38], [41]-[43], [59]. In DP [45], a complex problem is divided into finite overlapping sub problems. Each sub problem is solved by a DP algorithm and the solution is stored in a database. In the subsequent iterations, if a sub problem already addressed reappears, then that sub problem is not readdressed, but its solution is exploited from the database. In EO algorithm [60]-[70] the constraint to satisfy the cooperation is checked on the members of the trial solutions before the solutions are entertained for the next generation. Recently, researchers aimed at developing multi-agent RL (MARL) fusing RL, DP, and GT [71], [72]. In this thesis, we would explore new algorithms of MARL and novel EO.

## **1.2 Single agent planning**

In single agent planning [5], an agent searches for the sequence of actions, for which it reaches its predefined goal state from a given state optimally in terms of predefined performance metric. The section describes the single agent planning terminologies and

algorithms. Here single agent planning algorithm includes search-based and learning-based planning algorithms.

### 1.2.1 Terminologies used in single agent planning

**Definition 1.1:** An *agent* [1] is a mathematical entity that acts on its environment and senses the changes in the environment due to its action. The agent is realized by hardware/software means. A hardwired agent has an actuator (motors/levers) and a sensor to serve the purpose of actuation and sensing respectively.

A learning agent learns its right action at a given location/grid, called state, from its sensory-action doublets. A planning agent identifies its best action at its current state to obtain maximum reward for its action in the given environment.

In a single agent system, the environment includes a single agent. Naturally, the learning/planning steps/moves of the agent is undisturbed by the environment. Fig. 1.1 offers architecture of a single agent system.

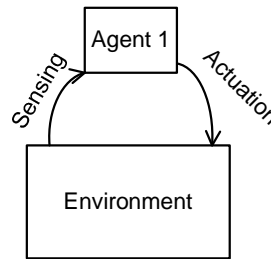


Fig. 1.1 Single agent system

**Definition 1.2:** The *state* of an agent represents a situation of the agent, concerning the position and/or orientation of the agent in the environment at an instant.

A state-space is a collective set of states of an agent. The definition of the state-space is required *a priori*, to address a planning problem. Such description of the state-space is problem specific. The state-space may be discrete or continuous. We in this thesis, however, deal with discrete state-space. Fig. 1.2 illustrates three discrete states (s1, s2, and s3) of an environment.

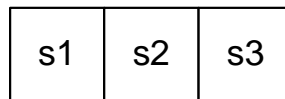


Fig. 1.2 Three discrete states in an environment

**Definition 1.3:** The *action selection* by an agent is done randomly or using specific strategies, such as  $\epsilon$ -greedy strategy [36] or the Boltzmann strategy [73]. Random action selection



sometimes is inefficient, when the same action is selected repeatedly during the learning phase.

The  $\epsilon$ -greedy strategy [36] allows an agent to select random actions from a pool with a probability  $= \epsilon$ . For example, if  $\epsilon = 0.2$ , then the agent would select 20 actions randomly and 80 greedy actions out of 100 trials from a pool of actions.

Unlike the above, the Boltzmann strategy [73] employs a probability distribution based on the reward function value obtained for individual actions. Usually, an exponential distribution is used to determine the probability of an action in a pool of actions. The larger is the individual reward, the higher is the action selection probability. One control parameter temperature is used to tune the action selection probabilities.

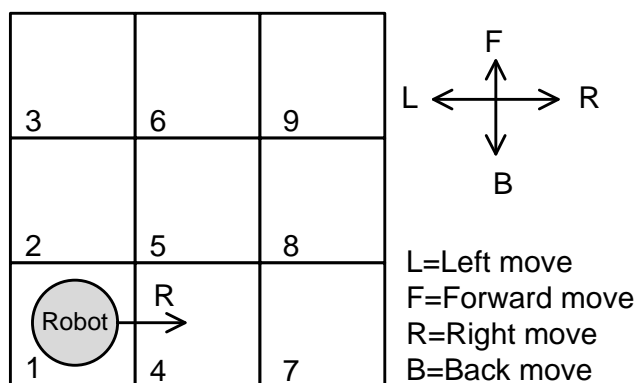


Fig. 1.3 Robot executing action Right (R) at state  $s_1$  and moves to the next state  $s_2$

In Fig. 1.3, we consider one agent capable of state-transitions using only four actions: Left-move (L), Forward-move (F), Right-move (R) and Back-move (B).

**Definition 1.4:** A *state-transition* [36] function at state  $s \in \{s\}$  due to action  $a \in \{a\}$  is a mapping from  $(s, a)$  to  $s' \in \{s\}$ , where  $s'$  be a next state, i.e.

$$s' \leftarrow \delta(s, a). \tag{1.1}$$

In deterministic system, for each pair of  $(s, a)$  we have a fixed  $s'$ . In non-deterministic (or stochastic) situation, for each pair of  $(s, a)$  we may have different  $s'$ . Traditionally, non-determinism is handled in an easier way by assigning a probability mass for each state-transition  $\delta(s, a)$ , such that sum of the state-transition probabilities is equal to one.

Non-determinism creeps into the system by various ways. For instance, in robot planning application, the condition of floor, such as its “slippery condition” is a guiding factor to determine the transition probabilities.

Suppose, in Fig. 1.4, a robot executes an action  $a$  at state  $s$  and moves to the next state  $s'$ , receiving an immediate reward  $r(s,a)$  as a feedback from the environment. Suppose the floor on which the robot moves on is slippery. In that case, from a state  $s$  because of an action  $a$  the robot can have more than one state transition, each with a state-transition probability of  $P(s' | (s,a))$ ,  $s' \in [s_1, s_2, s_3]$ , where,

$$\sum_{\forall s'} P(s' | (s,a)) = 1 \quad (1.2)$$

as shown in Fig. 1.5 . For each state-transition agent receives an individual immediate reward  $r(s,a)$  with its corresponding state-transition probability.

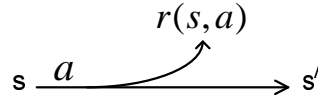


Fig. 1.4 Deterministic state-transition

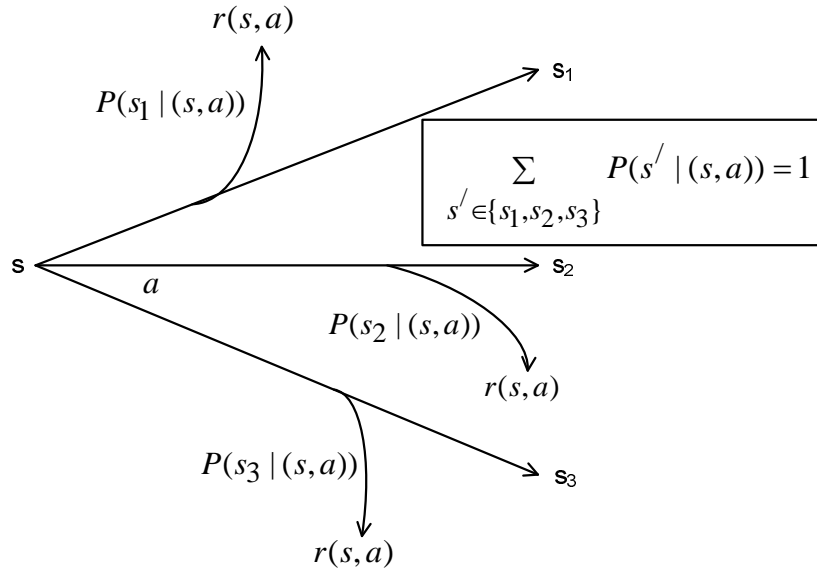


Fig. 1.5 Stochastic state-transition

**Definition 1.5:** A *policy* [36]  $\pi$  is a decision making mapping function, representing the probability assignment to a set of actions  $\{a\}$  at a given state  $s \in \{s\}$  such that,

$$\sum_{\forall a} \pi(s,a) = 1, \text{ i.e.,} \quad (1.3)$$

$$\pi : s \times \{a\} \rightarrow [0,1], \quad (1.3)$$

$$\text{subject to } \sum_{\forall a} \pi(s,a) = 1. \quad (1.4)$$

holds for each state  $s$ .

In Fig. 1.3, at state  $s_1$  there is a set of finite possible actions: L, F, R and B. Now, random selection of an action from this finite set infinite times results in a policy,  $\pi(s_1, a) = 0.25, a \in \{L, F, R, B\}$ .

In a planning problem, an agent starts by executing its individual action from a predefined state (starting state) with an aim to reach its individual predefined absorbing state (goal state), optimally in terms of time, path length, energy, and the like. Feasibility and optimality are two desired criterions need to be satisfied while addressing the planning problem [31].

**Definition 1.6:** *Feasibility* refers to the locomotion of an agent to a feasible next state because of an action form the current state.

**Definition 1.7:** *Optimality* indicates the performance optimization of the planning algorithm in each step, by minimizing the system resource utilization.

**Definition 1.8:** The sequence of actions lead to the predefined goal state from a given starting state maintaining the feasibility and optimality jointly in each step is well-known as *plan*.

To understand the concept of planning, Example 1.1 is given to realize the movement of a single agent (here robot) in a two dimensional discrete environment.

**Example 1.1:** Suppose a robot moves in a two dimensional  $5 \times 5$  grid environment as shown in

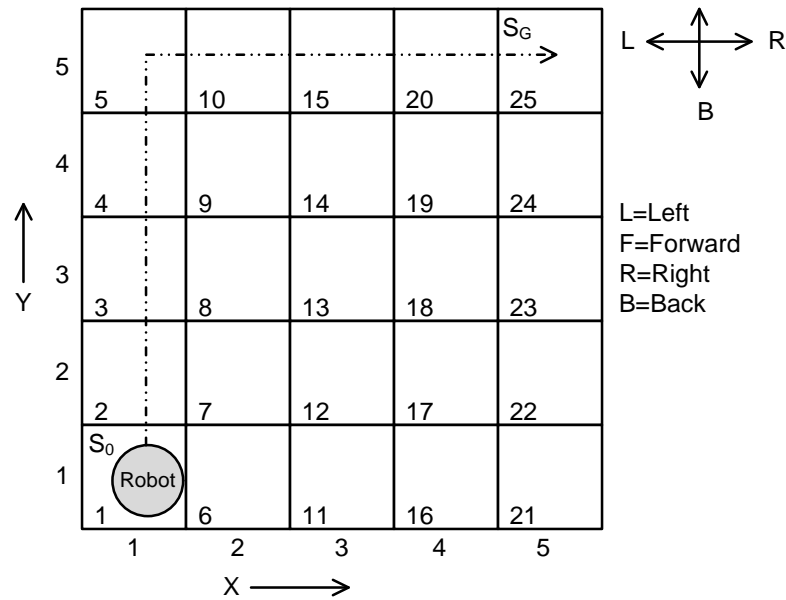


Fig. 1.6 Two dimensional  $5 \times 5$  grid environment

Fig. 1.6. There are 25 states and each state is represented by an integer or the Cartesian coordinate  $(x, y)$ , where  $x \in [1, 5]$  and  $y \in [1, 5]$ . An agent can execute one among the four possible actions  $a \in \{L, F, R, B\}$  at a state  $s \in [1, 25]$ . After executing an action  $a$  at a state  $s$

a state-transition takes place and the robot moves from  $s$  to the next state  $s' \in [1,25]$  by (1.1). The collection of state transitions for which the robot moves from its current state '1' to the goal state '25' is called a feasible path. Among the feasible paths the optimal one is chosen. One optimal path (here in terms of number of state transitions) is shown in Fig 1.6 by dotted lines. The example can be made more interesting by adding obstacles in the optimal path.

After finalizing a plan (sequence of actions) by an agent, the agent follows the plan either by execution, refinement, or hierarchical approach.

**Execution:** In the execution phase, planner's plan is executed in a simulator or by a robot connected to the real environment. There are two types of robots for execution. In the first type, the robot is programmable and acts as an autonomous agent. This approach has the provision of updating the plans after finite time interval. However, most planning algorithms are designed to tackle new situations during the planning phase and hence, the above type of execution is not preferred. The second one is the special-purpose robot designed to solve a specific task given to it.

**Refinement:** Refinement is the evolution of the planning algorithms towards the better performance as shown in Fig. 1.7. In Fig. 1.7, agents first compute a collision-free path in presence of obstacles after that agents optimize (smoothen) the path. Finally, a trajectory is planned following the path and a feedback controller is added for that.

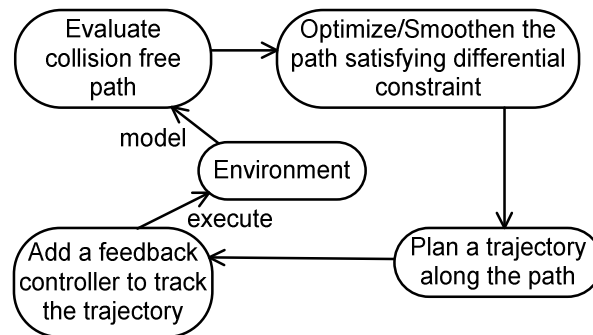


Fig. 1.7 Refinement approach in robotics

**Hierarchical:** In hierarchical model, each plan is considered as an action under a larger plan. The same plan may also be defined as a subroutine under the larger plan. In Fig. 1.8, the master plan is known as the *root node*. Remaining subsequent plans act as an action for the master plan or plan. There may be infinite number of plans under a master plan or plan. In Fig. 1.8,  $n$ ,  $m$ , and  $p$  are the real positive integer number. In Fig. 1.9 (hierarchical model), agent 1 interacts with environment 1 and agent 2 with environment 2. Again in Fig. 1.9,

environment 2 includes agent 1 and environment 1. So, Agent 2 interacts with the environment 2 as well as 1.

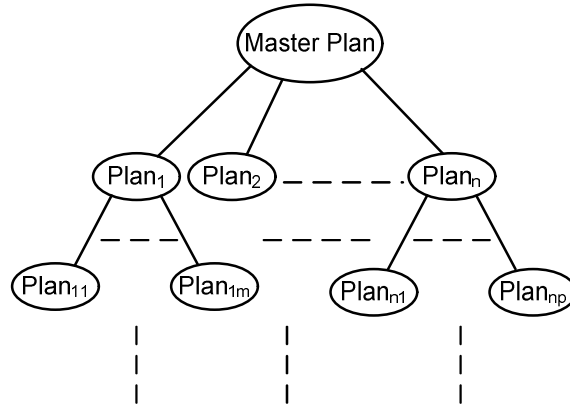


Fig. 1.8 Hierarchical tree

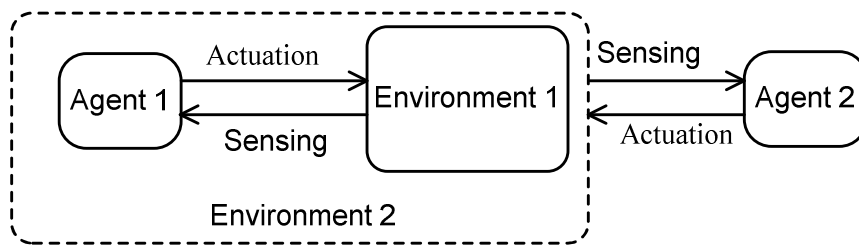


Fig. 1.9 Hierarchical model

The search-based planning algorithms are employed to evaluate low cost planning paths in terms of path length, time, energy and the like, for single-robot planning. The search-based planning algorithms are popular mainly because of their simplicity. The search-based algorithm comprises of the following two parts:

- 1) In the first part, the realization of the goal following a number of feasible plans is done by employing a search algorithm.
- 2) The second part is related to the optimal planning, which employs principle of optimality to reduce the computational effort in the planning algorithms.

The search-based planning algorithms avoid the geometric models or differential equations. The search based algorithms also avoids uncertainty and hence it avoids complications due to probability calculation.

### 1.2.2 Single agent search-based planning algorithms

By search-based planning algorithms a plan (or sequence of feasible actions) is searched by one of the following methods: forward search, backward search, and bidirectional search [31].

Forward search algorithm deals with the three variant of states. First one is the state which has not been visited yet or the unexplored one is known as *unvisited* state. If all possible state-transitions are explored in a given state, then the state is referred to as a *dead* state. The state which has been visited but still there exist a few unexplored next state is defined as *alive* state. Breadth first [31], Depth first [31], Dijkstra's [74], Best first search [31], Iterative deepening [31], A-star ( $A^*$ ) [33], D-star ( $D^*$ ) [6] are the examples of forward search algorithms. The above forward search algorithms are extendable to the backward search algorithm, by solving the same planning problem by traversing from the goal state to the starting state. The bidirectional search is the combination of forward and backward search. In every search-based planning algorithm, a tree is maintained. For the forward (backward) search initial (goal) state is the root node of the tree. The advantage of bidirectional search is the radical reduction in the exploration required. In this chapter, only the Dijkstra's,  $A^*$  and  $D^*$  and STRIPS like algorithms are discussed as given below.

### 1.2.2.1 Dijkstra's algorithm

Dijkstra's algorithm was proposed by computer scientist Edsger W. Dijkstra's in [74]. Dijkstra's algorithm is employed to find out the shortest path between two nodes in a graph. In case of robotics, each state is represented by a node of the graph. The starting state is denoted by the source node and instead of finding the shortest path from the source node to all other nodes, shortest path is obtained from the source node to a specific goal node (goal state of the robot).

#### Algorithm 1.1 Dijkstra's algorithm

**Input:** Mark all the unvisited nodes and the current node is set as the source node;  
Generate a search graph  $G$ , including the starting node  $x$ . Mark node  $x$  as an open;  
**Output:** The optimal path;  
**Begin**  
Initialize: Set a distance value to all the nodes in the graph. Set zero for the source node (here state 1) and  $\infty$  for the remaining nodes;  
**Repeat**  
1. From the current node, explore all the unvisited neighbors and evaluate their distances from the initial node. (For example, let the current node,  $x$  has a distance of 3 unit from the source node, and an edge connecting  $x$  with another node  $y$  has distance of 2. Now, the distance to  $y$  through  $x$  from the source node becomes  $3+2=5$ . Compare the currently evaluated distance with the previously recorded distance ( $\infty$  at the beginning). If the currently evaluated distance is less than previously recorded distance, then update the database by the currently evaluated distance, otherwise do nothing.  
2. Once all the neighbors of the current node have been explored, the current node is marked as visited (not checked further), and the evaluated distances are recoded as the final and minimal distances.  
3. Select one unvisited node with smallest distance as the next current node;  
**Until** goal state reached;  
**End.**

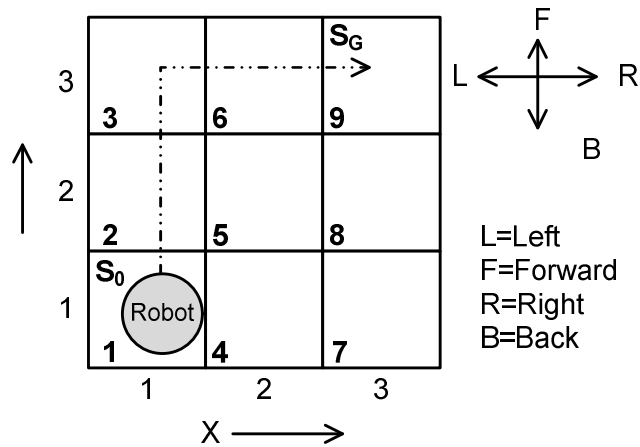


Fig. 1.10 Two dimensional 3×3 grid environment

The Dijkstra's algorithm is explained for the 3×3 grid shown in Fig. 1.10. In Fig. 1.10, there are nine states (nodes). State 1 is the source node and state 9 is the goal node. From each node there are maximum four possible paths as shown in the graph (Fig. 1.11). Weights of all the edges are 1,  $\infty$  and -1. 1 is assigned for a feasible edge. The self-loop and/or collision between robot and the boundary in Fig. 1.11 signify the penalty with reward of -1.  $\infty$  is assigned for an invalid edge. The steps of Dijkstra's algorithm are given in Algorithm 1.1.

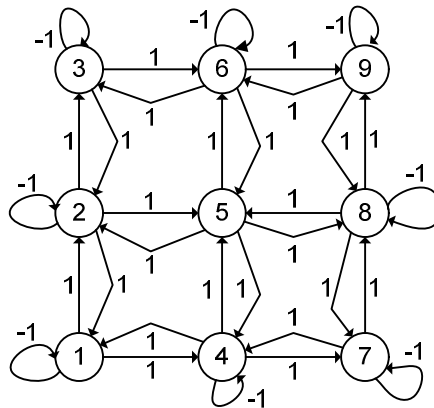


Fig. 1.11 Corresponding graph of Fig. 1.10

The trace of the Dijkstra's algorithm for robot path planning is given in Table 1.1. The bold numbers are the selected node corresponding to the column's node from the current node. The run-time complexity of the Dijkstra's algorithm is  $O(|V| \log |V| + |E|)$ . where  $|V|$  and  $|E|$  are the number of edges and nodes respectively.

Table 1.1 Trace of Dijkstra's algorithm for Fig. 1.11

		Nodes →								
		1	2	3	4	5	6	7	8	9
Visited nodes	{1}	-1	<b>1</b>	∞	1	∞	∞	∞	∞	∞
	{1,2}	1	-1	<b>2</b>	2	2	∞	∞	∞	∞
	{1,2,3}	2	1	-1	3	2	<b>3</b>	∞	∞	∞
	{1,2,3,6}	3	2	1	4	3	-1	5	6	<b>4</b>

### 1.2.2.2 A\* (A-star) Algorithm

A\* is a heuristic search-based algorithm [33]. In A\* algorithm, the quality of a node is measured by introducing two cost functions: one is heuristic cost and another is the generation cost. The heuristic cost, denoted by  $h(x)$  is a measure of distance (here city block distance) between the current node  $x$  to the goal node. The generation cost of a node  $x$  denoted by  $g(x)$ , measures the distance of node  $x$  from the source node. Total cost function at node  $x$  is the summation of  $f(x)$  and  $g(x)$ . The following definitions are required before explaining the A\* algorithm [33].

**Definition 1.9:** A node  $x$  is called *open* if the node  $x$  has been generated and the heuristic cost  $h(x)$  has been computed over it but it has not been expanded yet.

**Definition 1.10:** A node  $x$  is called *closed* if it has been expanded for generating offspring.

The steps of A\* algorithm is given Algorithm 1.2. Example 1.2 is given for better understanding of A\* algorithm in the perspective of robot path planning problem.

**Algorithm 1.2 A\* Algorithm**

**Input:** Generate a search graph  $G$ , including the starting node  $x$ . Mark node  $x$  as an open;  
**Output:** The optimal path;  
**Begin**  
 Initialize: Create a list of *closed* node keeping them initially empty;  
**Repeat**

1. If list of *open* node is empty, then exit with failure;
2. Let node  $n$  is selected from the list of open nodes and removes it from the set. Put the node  $n$  on the closed nodes list;
3. If  $n$  is the goal node, then exit and return the solution obtained to trace a path from the node  $n$  to node  $x$  in the search graph  $G$ ;
4. Expand node  $n$  and generate the set  $M$ , which contains its successors that are not already the ancestors of  $n$  in  $G$ . Add the elements of  $M$  as successors of  $n$  in  $G$ ;
5. Point  $n$  from each members of  $M$ , which does not belong to  $G$  and add them in the open list. If for all the members of  $M$  already belong to open or closed list of nodes, then redirect the pointer to  $n$ , subject to the shortest path is found through  $n$ . If all the members of  $M$  are belong to closed list of nodes, then redirect the pointers of its entire offspring in  $G$ , so that they point toward the back along the best paths found till now to these offspring;
6. Sort the elements of open list in order of increasing cost function (sum of heuristic cost and generation cost);

**Until** goal state reached;  
**End.**


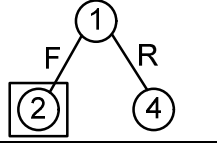
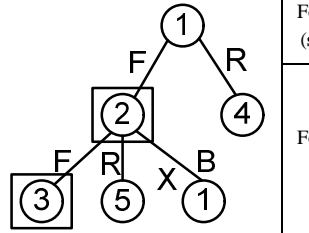
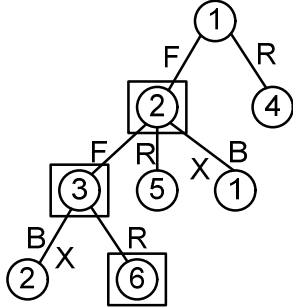
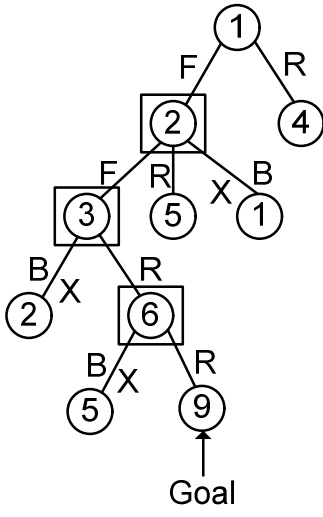


**Example 1.2:** In this example, the A\* algorithm is employed to find the shortest path between source node 1 to the goal node 9 as shown in Fig. 1.10. The heuristic cost  $h(x)$  of node  $x(x_x, x_y)$  is given by the city-block distance and it is defined in (1.5).

$$h(x) = |x_g - x_x| + |y_g - y_x|, \quad (1.5)$$

where,  $(x_g, y_g)$  is the goal coordinate.

Table 1.2 Trace of A\* algorithm from Fig. 1.10

Step	State-space	Heuristic cost	Generation cost	Total cost	
0		4	0	4	
1		For node 2 (selected)	3	1	4
		For node 4	3	1	4
2		For node 3 (selected)	2	2	4
		For node 5	2	2	4
3		1	3	4	
4		0	4	4	

The trace of the A\* algorithm is given in Table 1.2. In step 0, robot starts from node 1 and its heuristic cost is 4 and generation cost is 0. Hence, total cost is 4. In step 1, node 1 is expanded by the action forward (F) to node 2 and by the action right (R) to node 4. The total cost of both the nodes 2 and 4 is 3+1=4. Node 2 is selected and it is expended further by the actions forward (F), right (R), and Back (B) to the nodes 3, 5 and 1 respectively. The total cost of the nodes 3 and 5 is 2+2=4. Node 1 is not selected following Definition 1.10. Node 3 is selected and it is extended further to node 2 and 6 by the action back (B) and right (R) respectively. Here, node 2 is a closed node by Definition 1.10 and hence, it is eradicated. So, node 6 is expanded to node 6 and 9 by the action back (B) and forward (F). Again node 6 is eradicated by Definition 1.10 and node 9 is the goal state. The total cost function of the node 9 is 4 with 0 heuristic cost. Hence, optimal path is generated by sequentially following the nodes 1, 2, 3, 6, and 9.

### 1.2.2.3 D\* (D-star) Algorithm

Unlike, A\* [68], D\* [6] algorithm may be employed to efficiently plan in a dynamic unknown or partially known environments, by adjusting the weights of the edges (arcs). In the present path-planning application, each state is assumed as a node and weight of each edge (arc) connecting two nodes represents the cost of moving from one node to another. Initially, a path is planned from current node to the goal employing the A\* algorithm using the known information. In the journey of the robot towards the goal state, it discovers the presence of obstacles in its path and the graph is modified by adapting the arc weight. The robot again computes the shortest path from its node position to the goal. The process continues until it reaches its goal position or it concludes that the goal is inaccessible. The Trace of the D\* algorithm is shown below in Table 1.3 by adding an obstacle in state 3 of Fig. 1.10 as shown in Fig. 1.12. Step 0 and 1 are same as A\* algorithm. In step 2, node 3 is expanded to node 1 and 5 by action right (R) and back (B) respectively.

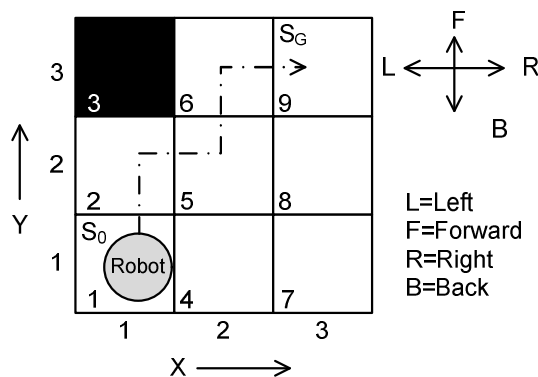

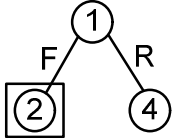
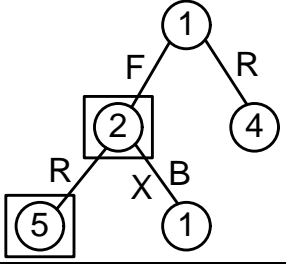
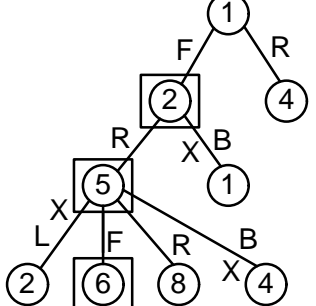
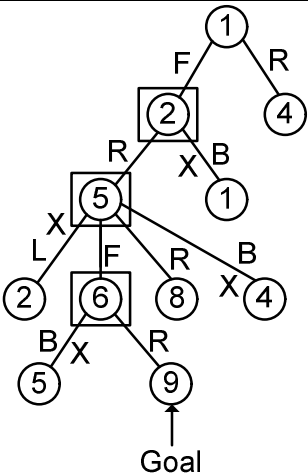


Fig. 1.12 Two dimensional 3×3 grid environment with an obstacle

Table 1.3 Trace of D\* algorithm from Fig. 1.12

Step	State-space	Heuristic cost	Generation cost	Total cost	
0		4	0	4	
1		For node 2 <b>(selected)</b>	<b>3</b>	<b>1</b>	<b>4</b>
		For node 4	3	1	4
2		For node 5	2	2	4
3		For node 6 <b>(selected)</b>	<b>1</b>	<b>3</b>	<b>4</b>
		For node 8	1	3	4
4		0	4	4	

Node 3 is not accessible as there is an obstacle at node 3. So, node 5 is expanded by left (L), forward (F), right (R) and back (B) actions to nodes 2, 6, 8 and 4 respectively. Node 2 and 4 are closed nodes following Definition 1.10. Selecting node 6 and expanding it to node 5 and 9

by actions back (B) and right (R) respectively, the goal node 9 is reached. So, optimal path is generated by sequentially following the nodes 1, 2, 5, 6, and 9.

#### 1.2.2.4 Planning by STRIPS-like language

Representation is the main bottleneck of the earlier explained search-based planning techniques due to enormous state-space. To address such representation problem, STRIPS-like language [31] is proposed by the Stanford Research Institute Problem Solver group, which is expressive enough to characterize a planning problem logically. STRIPS stands for Stanford Research Institute Problem Solver. STRIPS is the first well-known logic-based representation of the discrete planning algorithm, which is the extension of first order logic (propositional logic). Following representations are employed in STRIPS.

**State:** In STRIPS, agent decomposes the environment into logical conditions (TRUE or FALSE), and then the state is represented by conjunctions of function-free ground literals. Ground literal refers to the predicates, which cannot break any more. Suppose, a home service robot is instructed to bring a cup of tea with a biscuit and a magazine. So, in STRIPS, the initial state is formed using the following predicates “at(home),” “ $\neg$  have(tea),” “ $\neg$  have(biscuit),” “ $\neg$  have(magazine).” Here, home represents the initial position. Now initial state is the conjunctions of the function free predicates (ground literals), i.e., “at(home) $\wedge$  $\neg$  have(tea) $\wedge$   $\neg$  have(biscuit) $\wedge$  $\neg$  have(magazine).” However, the goal state is the “at(home) $\wedge$ have(tea) $\wedge$ have(biscuit) $\wedge$ have(magazine)”. Now, the task is to find out the sequence of actions to reach from initial state to the goal state.

**Action:** An action follows following two conditions: preconditions and effect. In precondition, an agent needs to satisfy certain feasibility condition before executing an action. For example, for “have(tea)” the agent must go to a nearby tea stall because tea is not available at(home). Also the preconditions are always positive ground literals. On the other hand, effects are the conjunction of positive and negative ground literals. For example, if there is an action “go(tea stall)” from “at(home)”, then the precondition is “at(home) $\wedge$ path(here, there)” and the effect is “at(tea stall)  $\wedge$  $\neg$  at(home)”. Hence, to reach the goal state “at(home) $\wedge$ have(tea) $\wedge$ have(biscuit) $\wedge$ have(magazine)” an agent must satisfy all the preconditions and effects.

### 1.2.3 Single agent reinforcement learning

In single agent RL [36], (Fig. 1.13) an agent receives a reward/penalty as a feedback due to an action at its present (current) state or situation from the surrounding (environment). Such scalar feedback measures the quality of the action in that state. In the literature of RL, this quality value is well-known as state-action value. The robot remembers or stores the <state, action, reward> profile as an experience for future reference. Once robot learns all possible

<state, action, reward> profiles, it plans optimally in terms of time and/or energy, from any state within the environment it learns. The single agent RL can be explained by the well-known multi-armed bandit problem [75], [76].

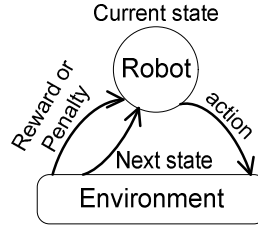


Fig. 1.13 Structure of reinforcement learning

### 1.2.3.1 Multi-Armed Bandit Problem

In the literature of English, a bandit refers to a robber or gambler, who belongs to a gang typically isolated from the human society. If a bandit has only one arm, then the bandit is called one-armed bandit. The one-armed bandit is also well-known as a slot machine, because a slot machine is operated by a button located on the front panel of the machine. A *slot machine* is a casino gambling machine, which rolls three or more times once the button is pushed. As the slot machine stops rolling, it pays off the bandit based on the pattern formed by the symbols visible on the front side of the machine. A multi-armed bandit consists of a series of slot machines arranged in a row. In multi-armed bandit problem [75], [76], the bandit has to decide which slot machine to play and for how many times to maximize the sum of the rewards earned.

The gambler starts playing the multi-armed bandit problem without any knowledge about the slot machines. In each trial, the gambler faces a trade-off between the “exploration” of a new slot machine to obtain better reward than the present rewards, and “exploitation” of a slot machine that has already obtained highest expected rewards. Similar trade-off is experienced by a reinforcement learner in RL. Hence, the multi-armed bandit is employed to manage several projects in a large organization, where initially the properties of the projects are partially known or unknown, but as time passes the properties becomes fully known to the bandit.

Suppose, a multi-armed bandit [75], [76] has  $N$ -slot machines (or  $N$ -arms), which are being played by the bandit, and the bandit receives different reward for each arm, with an aim to determine the arm having the maximum reward. To choose the best arm, i.e., an arm corresponding to the maximum reward (or greedy reward); the agent (or bandit) may compute the running average of rewards of all the arms given in (1.6).

$$Q_t(a) = \frac{r_1 + r_2 + \dots + r_k}{k}, \quad (1.6)$$

where  $Q_t(a)$  refers to the estimated value of the action  $a$  in  $t$  trials (play). We assume that action  $a$  was played  $k$  times in  $t$  trials and  $r_k$  was the reward of choosing the action (arm)  $a$  at  $k^{th}$  time step. As choosing an arm is analogous to choosing an action, the value of each arm may also be defined as the expected reward of the arm. Let the expected optimal reward of the action  $a$  is  $Q^*(a)$ . Based on the greedy action selection policy, the optimal action  $a^*$  is chosen by (1.7).

$$a^* = \arg \max_a Q^*(a) \quad (1.7)$$

The said greedy action selection may trap the agent (bandit) in local minima. To overcome the problem of trapping in local minima, an agent has to explore a new arm to receive new reward (well-known as exploration), which might be better than the present reward. Randomization of the probability of choosing an arm, which is not the greedy one, is referred to as the exploration. In RL, always there is a trade-off between the exploration and exploitation. For example, let  $N=10$  in the said multi-armed bandit problem. Each arm (analogous to an action)  $a \in [1,10]$  has a random reward given in (1.8) drawn from a normal random distribution with mean zero and variance one,  $N(0,1)$ . (1.8) represents the true value or expected reward of the ten arms.

$$Q^*(a) = [0.0325, 0.8530, 0.1341, 0.0620, -0.2040, 0.6525, 0.8927, -0.9418, -1.4122, 0.8089] \quad (1.8)$$

By (1.7) and (1.8)

$$\begin{aligned} a^* &= \arg \max_a Q^*(a) \\ &= \arg \max [0.0325, 0.8530, 0.1341, 0.0620, -0.2040, 0.6525, 0.8927, -0.9418, -1.4122, 0.8089] \\ &= 7. \end{aligned} \quad (1.9)$$

By (1.9) seventh action is the optimal action denoted by  $a^*$ . The learning process is started by estimating the true values from the earlier distribution of  $N(0,1)$  setting the exploration parameter  $\varepsilon$  to 0.2 the first estimate is given in (1.10).

$$Q_{est}^0(a) = [0.6761, -1.4321, -0.1824, 3.1140, -1.5285, -2.4264, -1.6687, -0.5252, -0.1021, -0.7124] \quad (1.10)$$

By (1.7), (1.10) and assuming  $Q^*(a) = Q_{est}^0(a)$

$$\begin{aligned} a^* &= \arg \max_a Q^*(a) \\ &= \arg \max [0.6761, -1.4321, -0.1824, 3.1140, -1.5285, -2.4264, -1.6687, -0.5252, -0.1021, -0.7124] \\ &= 4. \end{aligned} \quad (1.11)$$

By (1.11) bandit should choose the fourth action but by (1.9) the optimal action is the seventh one. So, the greedy choice is misleading the action selection. Several estimations are done to update the  $Q_{est}(a)$  vector using (1.6). The learning process continues until the agent recognizes the seventh action as its best choice among the ten actions. The variation of average reward with the number of trial for different  $\varepsilon$  is given in Fig. 1.14.

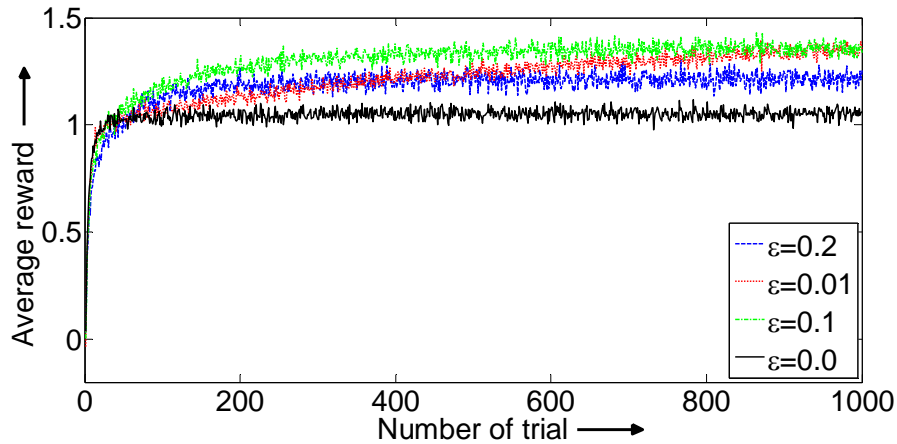


Fig. 1.14 variation of average reward with the number of trial for different  $\varepsilon$  in 10-Armed Bandit Problem

### 1.2.3.2 Dynamic programming and Bellman equation

DP [45] is an optimization technique, which transforms a large complex problem into a sequence of simple problems, by dividing it into finite overlapping sub problems; where overlapping indicates that the sub problems can recursively form the actual large complex problem. Breaking a large complex problem into finite overlapping sub problems is the condition of applying DP upon the large complex problem. In DP, there is a relation between the solution offered by the large problem and solutions offered by the sub problems. In the literature of optimization, this relationship is well-known as the Bellman equation (BE) or DP equation.

Each sub problem is solved by a DP algorithm and the solution is stored in a database. In the subsequent iterations, if a sub problem already addressed reappears, then that sub problem is not readdressed, but its solution is exploited from the database. Finally, one optimal solution is chosen from the evaluated value functions. The basic four steps for a DP algorithm are given below [45].

1. Divide the large complex problem into finite overlapping sub problems.
2. A value function is defined recursively based on the overlapping sub problems.
3. Compute and memorize the value functions of the overlapping sub problems to avoid repetition.
4. Obtain an optimal solution from the evaluated value functions.

Value function is the heart of DP, as it expresses the quality of a state because of an optimal action in terms of numerical value. If one needs to maximize the value function  $v(s)$  at a state  $s \in \{s\}$ , then using the principle of DP, the problem can be expressed in the BE as given in (1.12).

$$v(s) = \max_a [r(s,a) + \gamma v(s')], \gamma \in (0,1) \quad (1.12)$$

where,  $r(s,a)$  refers to the reward received at state  $s$  because of an action  $a$  and  $v(s')$  denotes the value function at next state  $s'$ .

### 1.2.3.3 Correlation between reinforcement learning and Dynamic programming

It is apparent from the earlier sections that the RL works on the principle of reward/penalty received by the agents as a feedback from the environment, DP is nothing but an optimization technique, which optimizes the BE [71], [72]. Fig. 1.15 indicates that the single agent Q-learning (QL) is the combination of the RL and the DP. The details of single agent Q-learning are given in the next section.

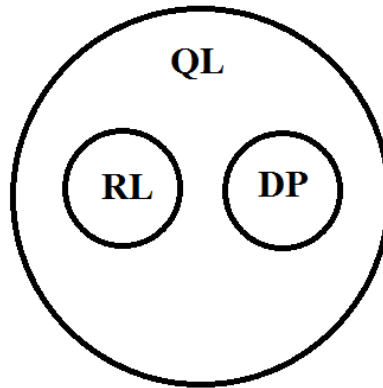


Fig. 1.15 Correlation between the RL and DP

### 1.2.3.4 Single agent Q-learning

Q-learning is one well-known paradigm among the RL techniques coined by Watkins and Dayan [77] in 1989. In Q-learning, an agent (robot) adapts in an unknown environment, and receives two types of reward due to an action at a given state within the environment. One reward is immediate reward received as a feedback from the environment as explained earlier in section 1.2.3. Another reward is evaluated at the next state. The evaluated reward at the next state is of two types based on the nature of the environment. If the environment is deterministic, then best (or optimal) future reward is evaluated at the next state, shown in Fig. 1.16. Since, in deterministic environment, an agent can move from a given state to the next state with probability one due to an action. On the other hand, in stochastic environment, a robot moves from a given state to the next one by assigning a probability in  $[0, 1]$  due to an action. Hence, in stochastic environment, robot evaluates the expected best (or optimal) future reward at the next state. The expected best future reward in the next state is the expectation of selecting the best action in the next state in terms of numerical value. The mechanism of evaluating the expected best future reward in Q-learning is shown in Fig. 1.17.



In Fig. 1.16, initially all the Q-values at Q-table are set to zero. At the current state 1 robot executes an action right (R) and receives an immediate reward  $r(1, R)=0$  from the environment. In the next state 3, maximum future reward is evaluated from the Q-table. Until  $(3, F)$  is not explored, in the next state,  $Q(3, L)=81$  is the best future reward and updated Q-value at  $(1, R)$  is  $Q(1, R)=72.9$ . On the other hand, once  $(3, F)$  is explored in the next state,  $Q(3, F)=100$  is the best future reward and updated Q-value at  $(1, R)$  is  $Q(1, R)=90$ .

In Fig. 1.17, 'R' inside the circle symbolizes a robot. Here, each state 1 to 3 has distinct frictional properties. In such stochastic environment, any one next state among the three possible next states may be reached due to left action executed by R at 1. So, there is a probability of moving to the next state from the current state due to an action. In the literature, such probability is well-known as the state-transition probability and the expected future reward is evaluated thereof.

In Q-learning, the future reward prediction depends on the current state-action pair. It is apparent, that the future reward prediction in Q-learning of an agent depends exclusively upon the current state but not on the past state-action pairs, which is the Markov property. The Markov property is also well-known as the memory less property. This idea is framed inside the Markov Decision Process (MDP). In Q-learning, MDP plays a significant role in finding the optimal value function corresponding to the optimal policy  $\pi^*$ . The definition of MDP is given in Definition 1.11 [78].

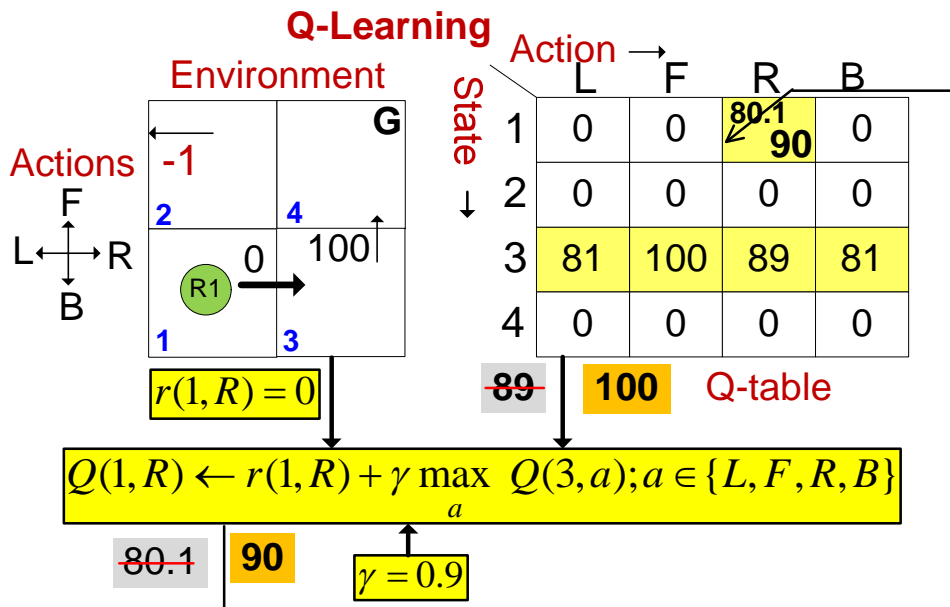


Fig. 1.16 Single agent Q-learning

**Definition 1.11:** A Markov decision process (MDP) is a 4-tuple  $\langle S, A, r, p \rangle$  [79], [80] where,  $S$  refers to a finite set of states,  $A$  denotes a finite set of actions,  $r : S \times A \rightarrow \mathbb{R}$  refers to the reward function of the agent, and  $p : S \times A \rightarrow [0,1]$  indicates the state transition probability.

$$v(s, \pi^*) = \max_a [r(s, a) + \gamma \sum_{s'} p[s' | (s, a)] v(s', \pi^*)] \quad (1.13)$$

where,  $v(s, \pi^*)$  and  $v(s', \pi^*)$  represent the value at current state  $s$  and next state  $s'$  due to optimal policy  $\pi^*$ ,  $\gamma$  denotes the discounting factor,  $p[s' | (s, a)]$  is the state-transition probability to reach next state  $s'$  from current state  $s$  due to action  $a \in A$ ,  $r(s, a)$  is the immediate reward at state  $s$  due to action  $a$ .

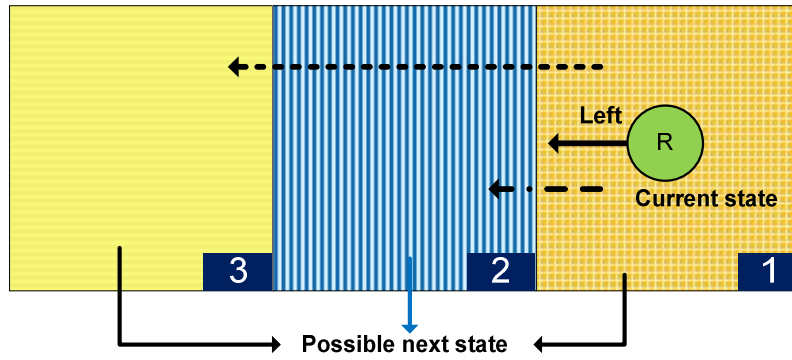


Fig. 1.17 Possible next state in stochastic situation

If an agent directly learns its optimal strategy without knowing either reward function or the state-transition probability, then the learning policy is called model-free RL [81]. Q-learning is one such model-free learning, involving the basic equation given in (1.14).

$$Q^*(s, a) = [r(s, a) + \gamma \sum_{s'} p[s' | (s, a)] v(s', \pi^*)]. \quad (1.14)$$

Here,  $Q^*(s, a)$  is the optimal Q-value. After infinite revisit of state  $S$  due to action  $a$ ,  $Q(s, a)$  turns to  $Q^*(s, a)$ . If next state is deterministically known for each action, then the Q-learning is called deterministic. In deterministic situation  $p[s' | (s, a)] = 1, \forall s'$ .

Combining (1.13) and (1.14) we can write,

$$v(s, \pi^*) = \max_a [Q^*(s, a)]. \quad (1.15)$$

Hence, the problem transforms to determining  $Q^*(s, a)$  for all  $(s, a)$ . If  $Q^*(s, a)$  is found, one can identify the action which maximizes the  $v(s, \pi^*)$ . So, the Q-learning update rule becomes,

$$Q(s, a) = r(s, a) + \gamma \max_{a'} [Q[\delta(s, a), a']] \quad (1.16)$$

$$\text{where, } s' \leftarrow \delta(s, a) \quad (1.17)$$

be the state-transition function. Hence,  $Q(s', a') = Q(\delta(s, a), a')$ . By combining (1.16) and (1.17) we obtain (1.18).

$$Q(s, a) = r(s, a) + \gamma \max_{a'} Q(s', a'). \quad (1.18)$$

where,  $\max_{a'} Q(s', a')$  indicates the action  $a' \in A$  for which maximum Q-value,  $Q(s', a')$  is received at next state  $s'$ . Now, the Q-learning update rule with learning rate  $\alpha \in (0, 1]$  is given by (1.19).

$$Q(s, a) \leftarrow (1 - \alpha)Q(s, a) + \alpha[r(s, a) + \gamma Q(s', a^*)] \quad (1.19)$$

However, in the stochastic situation, the state-transition probability to reach the next state  $s' \in \{s\}$  from the state  $s$  because of action  $a$ , is  $P[s' | (s, a)] \neq 1$ . So, Q-value adaption rule in the stochastic situation is given by (1.20): [81]

$$Q(s, a) = (1 - \alpha)Q(s, a) + \alpha[r(s, a) + \gamma \sum_{s'} P[s' | (s, a)] \max_{a'} Q(s', a')]. \quad (1.20)$$

After infinite revisit of  $(s, a)$ , Q-value,  $Q(s, a)$  turns to the optimal Q-value  $Q^*(s, a)$ . The convergence proof of (1.20) is given in [2]. Single agent Q-learning steps are given in Algorithm 1.3.

<b>Algorithm 1.3: Single agent Q-learning</b>
<b>Input:</b> Current state $s$ and action set $A$ ;
<b>Output:</b> Optimal Q-value $Q^*(s, a), \forall s, \forall a$ ;
<b>Begin</b>
Initialize: $Q(s, a) \leftarrow 0, \forall s, \forall a$ and $\gamma \in [0, 1]$ ;
<b>Repeat</b>
Select an action $a \in A$ randomly and execute it;
Receive an immediate reward $r(s, a)$ ;
Evaluate next state $s' \leftarrow \delta(s, a)$ ;
Update: $Q(s, a)$ by (1.19) for deterministic situation, by (1.20) for stochastic situation and $s \leftarrow s'$ ;
<b>Until</b> $Q(s, a)$ converges;
<b>Obtain:</b> $Q^*(s, a) \leftarrow Q(s, a), \forall s, \forall a$ ;
<b>End.</b>

### 1.2.3.5 Single agent planning using Q-learning

Fig. 1.18 explains the single robot planning mechanism. At first the robot (R1) observes its current state 3 and its corresponding Q-values from the Q-table. Then at 3 the robot evaluates the action corresponding to the maximum Q-value using the learned Q-table. In Q-table, at the row of 3 the action R corresponds to the maximum Q-value. Robot executes the action R

and moves to the next state 4. The above steps are repeated until the robot reaches its goal state.

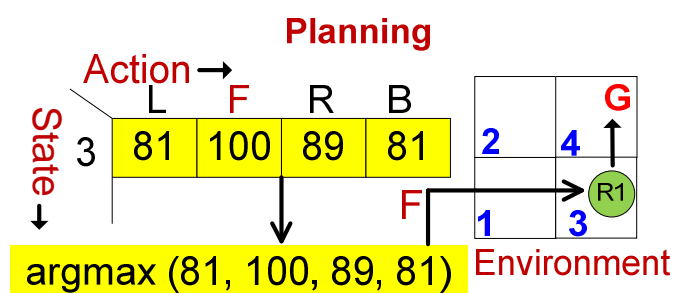


Fig. 1.18 Single agent planning

### 1.3 Multi-agent planning and coordination

Multi-agent planning and coordination are two almost similar terminologies both belong to the multi-agent systems. Multi-agent planning refers to determining the sequence of feasible actions of the agents to achieve individual goals maintaining optimality. However, coordination refers to skilful and effective interaction among the agents to serve the purpose of all the agents. The section describes the multi-agent planning and coordination terminologies with corresponding algorithms.

#### 1.3.1 Terminologies related to multi-agent coordination

A multi-agent system includes several agents. Naturally, action of an agent influences the rewards received by the other agents. This calls for special arrangement for learning and planning in a multi-agent system. Fig. 1.19 outlines the architecture of a multi-agent system.

In multi-agent system, a state-space is a collective set of states of an agent. The definition of the state-space is required a priori, to address a planning problem. Such description of the state-space is problem specific. In a multi-robot coordination problem, instead of states, the joint state is defined.

**Definition 1.12:** A *joint state* is the collection or union of individual states in a fixed order following the ascending order of the agents.

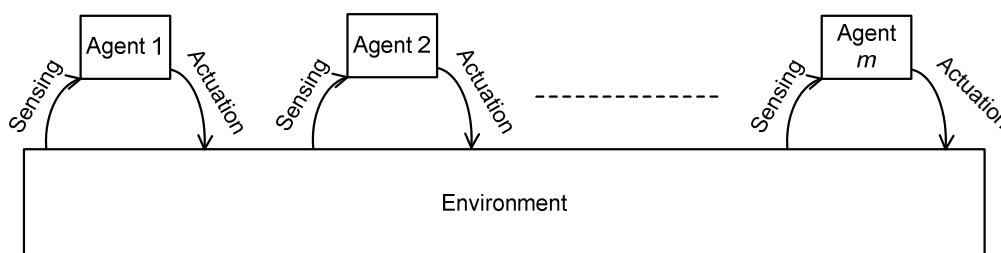


Fig. 1.19 Multi-agent system with  $m$  agents

Suppose,  $s_i$  is the individual state of agent  $i \in [1, m]$ , then the joint state for  $m$  agents system is given by  $S = \langle s_i \rangle_{i=1}^m$ .

**Definition 1.13:** The phrase: *joint-action* is a widely used term in multi-agent system and is defined by the collection or union of individual actions in a fixed order following the ascending order of the agents.

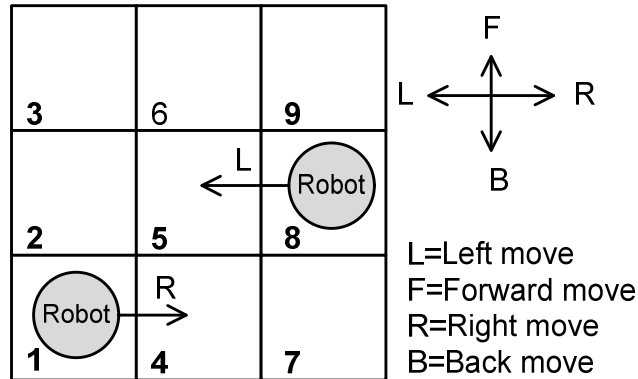


Fig. 1.20 Robots executing joint action  $\langle R, L \rangle$  at joint state  $\langle 1, 8 \rangle$  and move to the next state  $\langle 4, 5 \rangle$

Suppose,  $a_i$  is the individual action of agent  $i \in [1, m]$ , then the joint action for the  $m$  agent system is given by  $A = \langle a_i \rangle_{i=1}^m$ . In Fig. 1.20, due to joint action  $\langle R, L \rangle$  at  $\langle 1, 8 \rangle$  robots move to the joint next state  $\langle 4, 5 \rangle$  as shown in Fig. 1.20.

### 1.3.2 Classification of multi-agent system

There exist several state-of-the-art attributes, based on which the multi-agent system (MAS) is classified [1], [37]. Attributes relevant to the present thesis are employed here to classify the MAS. Basically MAS is of two types: cooperative and competitive. Like any social living beings an agent belongs to cooperative MAS cooperate with remaining agents. However, in the competitive MAS, agents do compete among themselves to acquire limited resources required for livelihood. In this chapter, we consider only the cooperative MAS.

**Classification based on cooperation:** Classification of MAS based on the cooperative aspect of the agents is done by measuring the ability of an agent to cooperate with remaining agents while performing a task. Agents cooperate with the remaining agents are well-known as cooperative agent and those do not cooperate rather they compete with others are distinguished as non-cooperative agent. The goal of cooperative agents is to achieve a common objective. On the other hand, the non-cooperative agents have always conflicting objectives. Fig. 1.21 provides a detailed classification of cooperative agents only based on the knowledge level.

**Classification based on knowledge level:** Further classification of cooperative MAS can be done based on the knowledge level of an agent about the remaining agents in the same team. In Fig. 1.21, one is aware agent, which has knowledge about its teammates and the unaware agent does not have such knowledge about the remaining agents in the environment.

**Classification based on coordination:** Next the aware agents are classified based on the coordination procedure employed by the agents. There are three types of coordination. In strong (weak) coordination, agents strictly (do not strictly) follow the coordination protocols. In the third type, i.e., not coordinated, agents do not coordinate with other agents. Classification based upon the coordination is shown in Fig. 1.21.

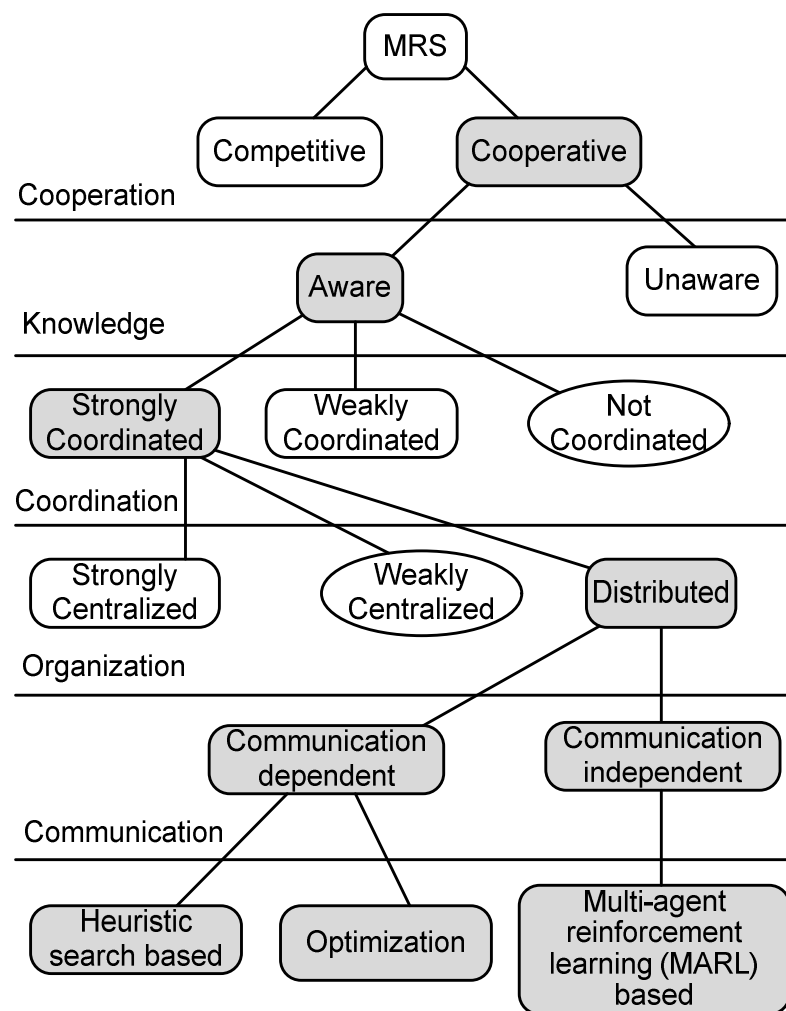


Fig. 1.21 Classification of multi-robot systems

**Classification based on organization:** Strongly coordinated agents are further classified based on the responsibilities of the agents in a team (or organization) as shown in Fig. 1.21.

By this aspect, the centralized approaches are distinguished from the distributed approaches. In centralized approach, an agent is elected as a leader for the entire team. The leader is responsible to distribute the task among all the agents in the team. The remaining agents (follower) act according to the instructions provided by the leader. However, in the distributed system, agents are completely autonomous in view of the decision making process, as there is no leader in the team. On the other hand, the centralized system can further be classified based upon the way the leader is elected among the team members. If only one robot leads the complete mission, then such centralized system is known as strongly centralized. However, in a weakly centralized system, more than one agent is allowed to lead the team towards the completion of the mission.

**Classification based on communication:** Distributed robots are classified based upon the dependency on communication among the agents as shown in Fig. 1.21. There are two types of distributed agents, one is communication dependent, and another is communication independent.

Besides the above classification, MAS can further be classified considering “Team Composition” (combination of heterogeneous and homogeneous robots), “System Architecture,” and “Team Size.”

Several approaches are available in the literature of multi-robot coordination. Among them coordination by MAQL and EO algorithms are described in this chapter. To improve readability GT and DP are briefly described below.

### 1.3.3 Game theory for multi-agent coordination

GT formally analyzes the strategic situation of the multi-agent system, where each agent potentially affects the interests of other agents in the environment [38], [41], [42]. Two types of game are considered in the present thesis: static and dynamic. The definitions of static and dynamic games are given below.

**Definition 1.14:** A *static game* with  $m$  player is defined by a tuple  $\langle m, A_1, A_2, \dots, A_m, r_1, r_2, \dots, r_m \rangle$  [42],

where,

$A_i, i \in [1, m]$  is the set of finite actions of player  $i$

and  $r_i : \times_{i=1}^m A_i \rightarrow \mathbb{R}, i \in [1, m]$  refers to the reward function of player  $i$ , where,  $\times$  denotes the Cartesian product.

**Definition 1.15:** If a static game is played repeatedly, then the game is well-known as *repeated game*.

In static game, multiple agents execute their actions at a joint state and agents do not have any state-transition. Hence, a static game is also known as state-less game. Now, to handle the games with state-transitions, another version of game called dynamic game is defined below.

**Definition 1.16:** A *dynamic game* with  $m$  number of agents is defined as a 5-tuple  $\langle m, \{S\}, \{A\}, r_i, p_i \rangle$

where,  $\{S\} = \times_{i=1}^m S_i$  is the joint state-space,

$\{A\} = \times_{i=1}^m A_i$  is the joint action-space,

$r_i = \{S\} \times \{A\} \rightarrow \mathbb{R}$  is the reward function at joint state-action of agent  $i$ ,

and  $P_i = \{S\} \times \{A\} \rightarrow [0,1]$  is state-transition function of agent  $i$ .

Suppose, an agent  $i \in [1, m]$  selects an action  $a_i$  from the pool of its action set  $A_i$  and plays the repeated game. The conjunction of the individually chosen actions for all the agents form a joint action  $A \in \times_{i=1}^m A_i$ . Let  $\pi_i(a_i)$  refers to the probability of selecting an action  $a_i \in A_i$  by agent  $i$ , where

$$\pi_i : A_i \rightarrow [0,1]. \quad (1.21)$$

If  $\pi_i(a_i) = 1$ , then the strategy of agent  $i$ ,  $\pi_i$  is deterministic for  $a_i \in A_i$ . The strategy profile for  $m$  agents is given by

$$\Pi = \{\pi_i : i \in [1, m]\}, \quad (1.22)$$

The strategy profile

$$\Pi_{-i} = \{\pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \dots, \pi_m\} \quad (1.23)$$

denotes the strategy of all the agents except the strategy of agent  $i$ ,  $\pi_i$ , where

$$\Pi = \Pi_{-i} \cup \{\pi_i\}. \quad (1.24)$$

It is apparent from Definition 1.14 and Definition 1.16 that a dynamic game is also a static game with state-transitions. In a static game, agents look for a balanced condition or equilibrium among them, such that no one would receive any incentive by unilateral deviation. In the literature, there are two well-known equilibria exist: Nash equilibrium (NE) and correlated equilibrium (CE).

Before understanding equilibrium, let at a given state  $s$  an agent (here robot) have an action set  $A$ . An action  $a^* \in A$  corresponding to the maximum reward at state  $s$  refers to the optimal or greedy action. Collection of such optimal actions executed at each state is termed as optimal policy or strategy. In a particular state, if a robot executes an action, then the action is well-known as a *pure strategy*. However, the *mixed strategy* is the randomization over the pure strategies. To understand mixed strategy rock-paper-scissor game is given in Example 1.3.

**Example 1.3:** Rock-paper-scissor [41], [42] is a two player hand game, played for fun by kids and sometimes for decision making by adults. Each player has three options: rock, paper or scissor and a player can choose one in a trial. The player expresses his/her choice to another player by using a hand to form one of the shapers as shown in Fig. 1.22. With these options



this game can have three possible outcomes excluding a tie. The three possible outcomes are given one by one.

1. One rock crushes scissor. Here, player playing rock beats the player playing scissor.
2. But if paper covers rock, then the player chooses to play paper beats the player playing rock.
3. On the other hand, if scissors cut paper; then the play of paper is defeated by the play of scissor.

However, if the choices of both the players are same, then a tie occurs and the game is replayed until the tie is broken.

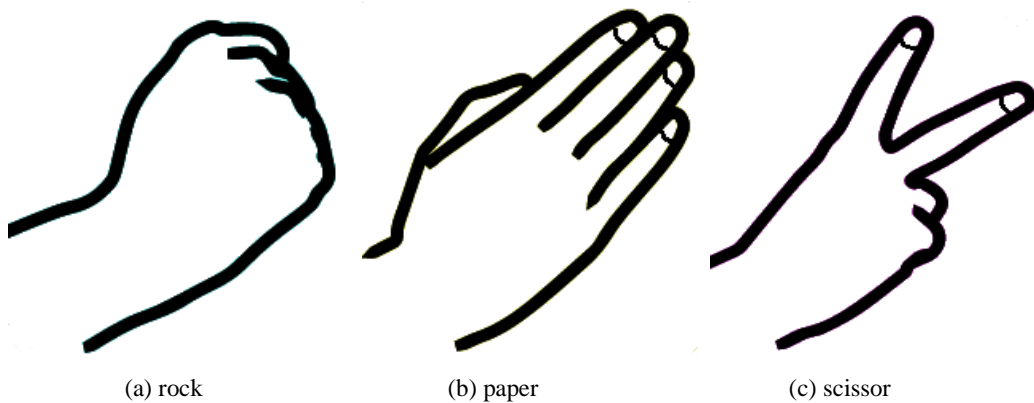


Fig. 1.22 Hands gestures in rock-paper-scissor game

After finite trials of the game the rewards of both the players are given in the reward matrix as shown in Fig. 1.23. In Fig. 1.23, one cell contains two rewards. The first reward is for Player 1 and second one is for Player 2. In case of a tie, both the players receive zero (0) reward. If a player wins the game, then the player is rewarded by one (1). On the other hand, if the player loses, then the player is penalized by negated one (-1).

		Player 2		
		rock	paper	scissor
Player 1	rock	(0,0)	(-1,1)	(1,-1)
	paper	(1,-1)	(0,0)	(-1,1)
	Scissor	(-1,1)	<b>(1,-1)</b>	(0,0)

Fig. 1.23 rock-paper-scissor game

It is apparent that in the rock-paper-scissor game (Fig. 1.22 and Fig. 1.23), optimal mixed strategy of Player 1 and 2 is to execute each action with a probability 1/3. Now, suppose,

Player 1 knows in advance that the Player 2 is playing the pure strategy "paper," then optimal pure strategy for Player 1 is "scissor" as it provides maximum reward to Player 1.

### 1.3.3.1 Nash equilibrium (NE)

NE is a solution concept of the multi-agent interactive system from where no player deviates to maintain its current reward, which is the maximum one. The Definition of NE is given in Definition 1.17. NE is of two types: pure strategy NE (PSNE) and mixed strategy NE (MSNE). To evaluate PSNE at a joint state, an agent selects an action from its own action set, which corresponds to its maximum reward due to joint action, where remaining agents' actions are kept fixed. The joint action at a joint state for which all the agents receive maximum reward and no one has any selfish intension to deviate from its chosen action is well-known as PSNE at that joint state. An example is considered in Fig. 1.24 and Fig. 1.25 to evaluate PSNE in a static game or state-less or one-stage or normal-form game [82].

**Definition 1.17:** *Nash equilibrium* is a stable joint action (or strategy) at a given joint state ( $S$ ) of a system that involves  $m$  interacting agents, such that no unilateral deviation (deviation of an agent independently) can occur as long as all the agents follow the same optimal joint action  $A_N = \langle a_i^* \rangle_{i=1}^m$  at a joint state  $S \in \{S\}$  for *pure strategy NE*. Further, for a *mixed strategy NE* agents perform the joint action  $A = \langle a_i \rangle_{i=1}^m$  with a probability  $p^*(A) = \prod_{i=1}^m p_i^*(a_i)$ , where  $p_i^* : \{a_i\} \rightarrow [0,1]$ ,  $p^* : \{A\} \rightarrow [0,1]$ .

Let  $a_i^* \in \{a_i\}$  be the optimal action of agent  $i$  at  $s_i$  and  $A_{-i}^* \subseteq A$  be the optimal joint action profile of all agents except agent  $i$  at joint state  $S = \langle s_j \rangle_{j=1, j \neq i}^m$  and  $Q_i(S, A)$  be the joint Q-value of agent  $i$  at  $S$  because of joint action  $A \in \{A\}$ . Then the condition of pure strategy NE at  $S$  is

$$\begin{aligned} Q_i(S, a_i^*, A_{-i}^*) &\geq Q_i(S, a_i, A_{-i}^*), \quad \forall i \\ \Rightarrow Q_i(S, A_N) &\geq Q_i(S, A'), \quad \forall i \text{ [where } A_N = \langle a_i^*, A_{-i}^* \rangle \text{ and } A' = \langle a_i, A_{-i}^* \rangle] \end{aligned} \quad (1.25)$$

and condition of mixed strategy NE at  $S$  is

$$Q_i(S, p_i^*, p_{-i}^*) \geq Q_i(S, p_i, p_{-i}^*), \quad \forall i, \quad (1.26)$$

where,  $Q_i(S, p) = \sum_{A \in \{A\}} p(A) Q_i(S, A)$  and  $p_{-i}^*(A_{-i}) = \prod_{j=1, j \neq i}^m p_j^*(a_j)$  be the joint probability of selecting joint action profile of all agents except agent  $i$  denoted by  $A_{-i} \subseteq A$ .

Agents follow Fig. 1.25 to evaluate PSNE  $A_N = \langle a_i^*, A_{-i}^* \rangle$  and Fig. (1.26) for mixed strategy NE  $\langle p_i^*(a_i), p_{-i}^*(A_{-i}) \rangle$  respectively at joint state  $S$ .

### 1.3.3.1.1 Pure strategy NE (PSNE)

Assuming in the one-stage game there are two robots R1, R2 and each has two actions. Robot 1 (R1) selects one action from the set  $\{L, F\}$  and robot 2 (R2) selects one from  $\{L, F\}$ . As a result there is a joint action set. The joint action set  $\{LL, LF, FL, FF\}$  is the all possible combinations (the Cartesian product) of  $\{L, F\}$  and  $\{L, F\}$ . Fig. 1.24(a) and (b) provide the reward tables of R1 and R2 at joint state-action space respectively. The rows of the state-action tables indicate the joint state. Joint state is the conjunction of individual states, here  $S$  and  $S'$ . Each column corresponds to a joint action  $A$ .  $A$  can be any one from the set  $\{LL, LF, FL, FF\}$ . The entries for each joint state-action pair are called joint state-action value. To evaluate PSNE at joint state  $S'$  the rewards at  $S'$  due to joint actions are mapped in the reward matrix, as shown in Fig. 1.24(c) and (d). In Fig. 1.24(c) and (d), each cell of the reward matrices displays the rewards at a joint state  $S'$  due to individual actions respectively for R1 and R2. In Fig. 1.24(c) to (e), rows indicate the actions of R1 and columns indicate the actions of R2. However, in Fig. 1.24(e), each cell shows the rewards of both the agent. First entry is for R1 and the second one is for R2.

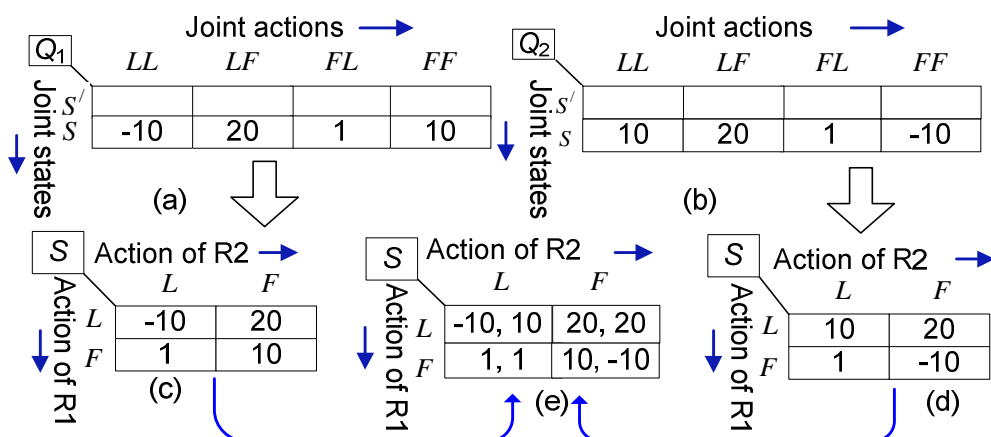


Fig. 1.24 Reward mapping from joint Q-table to reward matrix

Fig. 1.25(a) to (d) show the reward matrices of R1 and R2 at joint state  $S'$ . In Fig. 1.25(a), R1 selects its best action assuming that R2 has been selected  $L$  indicated by solid black arrows. In this situation, R1 prefers action  $F$  and receives 1 as a reward indicated in Fig. 1.25(a). Similarly, R1 receives 20 as a reward assuming that R2 has been selected  $F$  as shown in Fig. 1.25(b). Similarly, R2 earns 20 and 1, when R1 selects  $L$  and  $F$  respectively as indicated in Fig. 1.25(c) and (d). Finally, Fig. 1.25(e) shows the common solution producing cells, which are the PSNE. Computation of PSNE for two agents is performed by the

following three steps.

1. Fix the action of R1, then select the best reward of R2, considering all possible actions of itself.
2. Fix the action of R2, then select the best reward of R1, considering all possible actions of itself.
3. If the results of selection fall in the same cell, then PSNE = joint actions corresponding to the selected common grid.

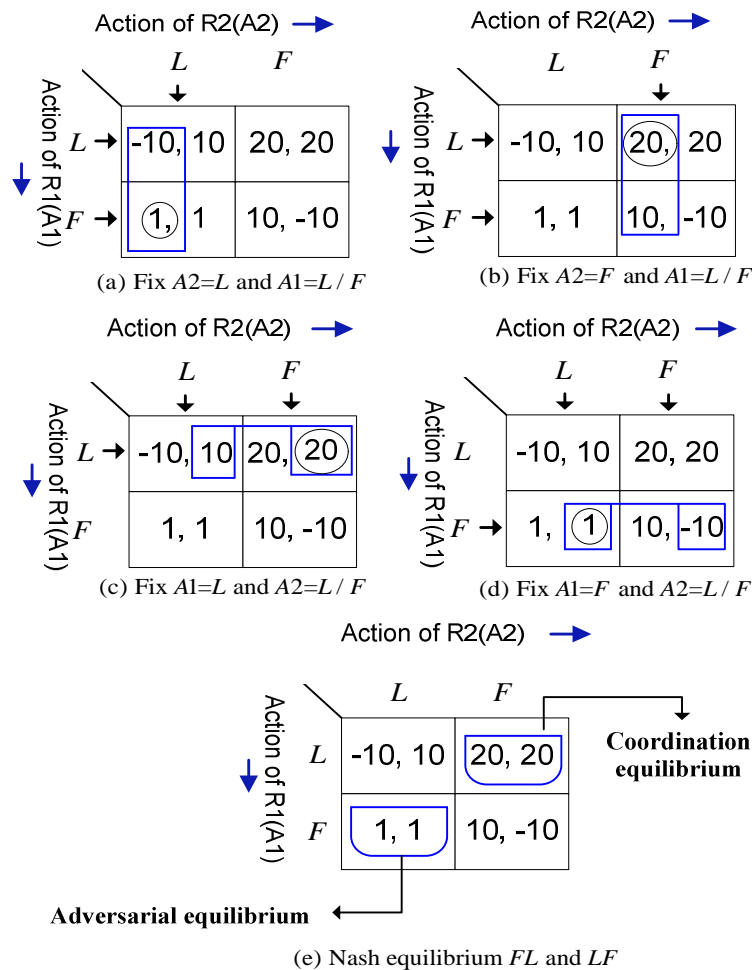


Fig. 1.25 Pure strategy Nash equilibrium evaluation

Here, two PSNE are obtained:  $\langle F, L \rangle$  and  $\langle L, F \rangle$ . Let us examine them one by one. For the  $\langle F, L \rangle$  both the robots receive 1. Now, if any one robot selfishly attempts to change its action aiming at maximizing its own reward, then the robot, which changes its action, causes to decrease its reward from 1 to -10. Besides, if R1 changes its action from  $F$  to  $L$  then R2's

reward improves from 1 to 10. Again, if R2 changes its action from  $F$  to  $L$  then R1's reward improves from 1 to 10. So, the joint action  $\langle F, L \rangle$  is an adversarial equilibrium. Adversarial equilibrium is a PSNE in competitive situation. Now, the joint action  $\langle L, F \rangle$  is coordination equilibrium, where both the robots receive maximum reward selflessly i.e., 20. Coordination equilibrium is a PSNE in cooperative situation. The present thesis considers coordination equilibrium only.

### 1.3.3.1.2 Mixed strategy NE (MSNE)

MSNE is stochastic. In MSNE, each robot randomizes its own pure strategies by assigning a probability in between zero and one for each pure strategy. Let R1 selects its actions  $L$  and  $F$  with a probability  $p$  and  $(1-p)$  respectively. Also, let R2 selects its actions  $L$  and  $F$  with a probability  $q$  and  $(1-q)$  respectively. The summary of rewards at MSNE is given in Fig. 1.26. At MSNE the expected reward of  $L$  and  $F$  against  $q$  are equal. Equating these two expected rewards yield  $p$  and  $(1-p)$  as shown in Fig. 1.26(a). Similarly, one can find  $q$  and  $(1-q)$  as shown in Fig. 1.26(b). The expected reward of a mixed strategy is the weighted sum of the expected rewards of all the pure strategies in the mix. Finally, the expected reward of R1 by employing  $p$  against  $q$  and the expected reward of R2 by employing  $q$  against  $p$  are given in Table 1.4. Also it is listed in Fig. 1.26(d). Finally, the MSNE is  $\langle (p, (1-p)); (q, (1-q)) \rangle$  given in Fig. 1.26(c). Table 1.4 provides the expected reward at MSNE for two players. Example 1.4 provides an example of MSNE for a two player tennis game. Example 1.4 is given below to illuminate MSNE.

Table 1.4 Expected reward of R1 and R2 at MSNE

expected reward of R1 by employing $p$ against $q$	$p[-10q + 20(1-q)] + (1-p)[1q + 10(1-q)]$
expected reward of R2 by employing $q$ against $p$	$q[10p + 1(1-p)] + (1-q)[20p - 10(1-p)]$

**Example 1.4:** Fig. 1.27 shows the reward matrix for a two player tennis game between Venus and Serena. Let in Fig. 1.27, Venus is the row player and Serena is the column player. If Venus chooses Left ( $L$ ), then she attempts to pass Serena to Serena's left ( $l$ ). If Venus decides Right ( $R$ ), then she is attempting to pass Serena to Serena's right ( $r$ ). Serena chooses  $l$ , means that she bends slightly towards her  $l$ . Similarly, Serena chooses  $r$  means she slightly bends towards her  $r$ . There is no PSNE in Fig. 1.27. Let's find MSNE for the tennis game. In MSNE, each agent's mix should be the best for the remaining agents' mix. To find Serena's NE mix  $(q, 1-q)$  look at Venus's rewards. Now, Venus's rewards against  $q$  while choosing  $L$  and  $R$  is given by  $50q + 80(1-q)$  and  $90q + 20(1-q)$  respectively. In MSNE,  $L$  and  $R$  both themselves must be the best response against  $q$ . So,

$$50q + 80(1-q) = 90q + 20(1-q). \quad (1.27)$$

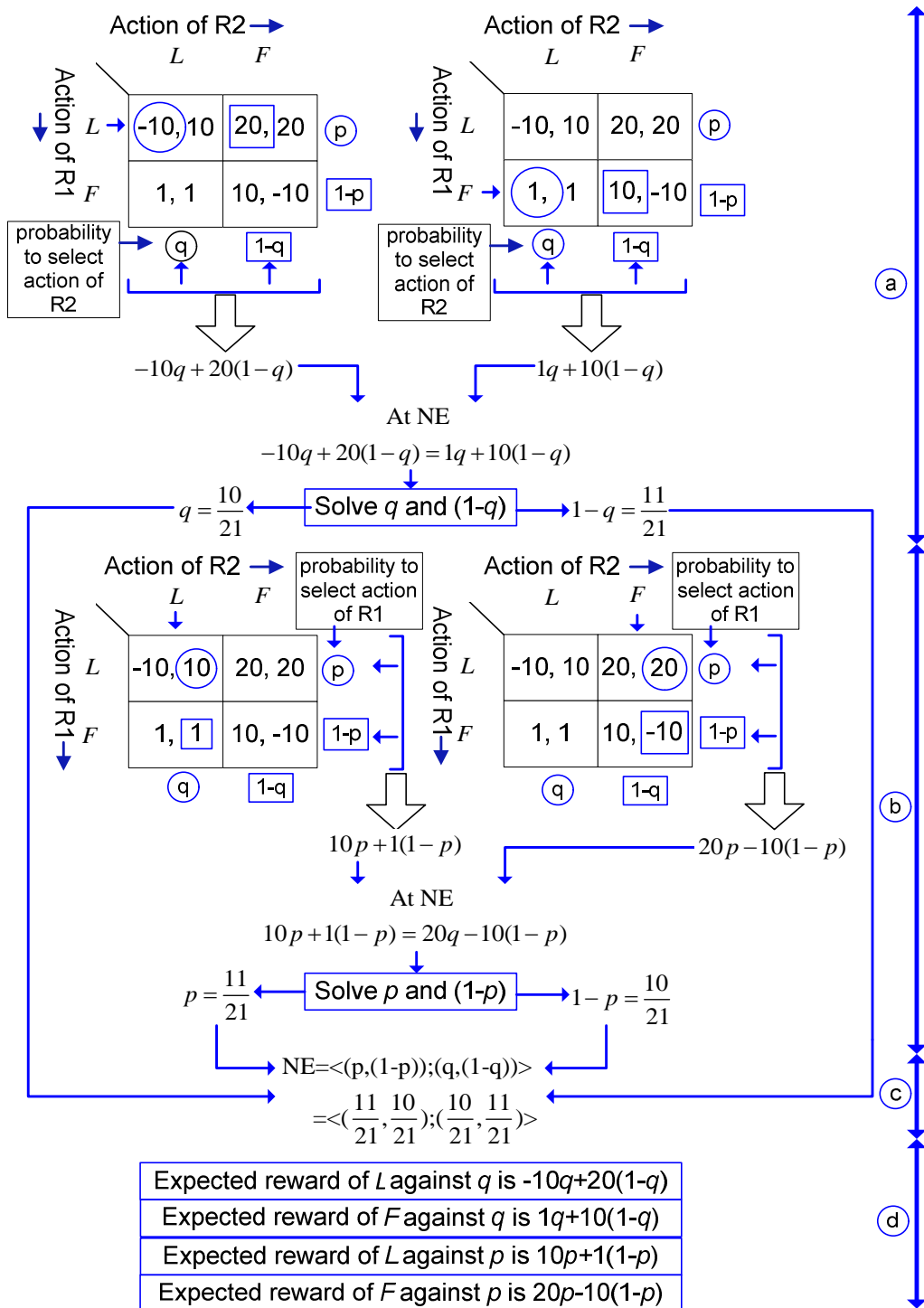


Fig. 1.26 Evaluation of mixed strategy Nash equilibrium

Therefore, solving (1.27) we obtain  $q = 0.6$  and  $1 - q = 0.4$ . Now, to find Venus's NE mix  $(p, 1 - p)$  look at Serena's rewards. Serena's reward against  $p$  while choosing  $l$  and  $r$  is given by  $50p + 10(1 - p)$  and  $10p + 80(1 - p)$  respectively. In MSNE,  $l$  and  $r$  both themselves must be the best response against  $p$ . So,

$$50p + 10(1 - p) = 10p + 80(1 - p) \quad (1.28)$$

Therefore, solving (1.28) we obtain  $p = 0.7$  and  $1 - p = 0.3$ . Hence, the MSNE is given by  $[(p, 1 - p); (q, 1 - q)] = [(0.7, 0.3); (0.6, 0.4)]$ .

		Serena →		
		$l$	$r$	
Venus ↓	$L$	50, 50	80, 20	$p$
	$R$	90, 10	20, 80	$1 - p$
		$q$	$1 - q$	

Fig. 1.27 reward matrix for tennis game

For multi-agent coordination without any communication among the agents, agents face coordination problem in the presence of multiple coordination equilibria [72]. Here, coordination problem refers to the problem of selecting unique equilibrium by all the robots. Such problem can be resolved by selecting a joint action based on a signal (e.g., traffic signal), which is commonly accessible by all the robots. Before discussing about the remedies of equilibrium selection Example 1.5 is provided to realize the problem.

**Example 1.5:** The reward matrix for a common reward two-agent static game is given in Fig. 1.28, where both  $a$  and  $b$  are the rewards. There are two action sets  $\{x_0, x_1\}$  and  $\{y_0, y_1\}$  for agent X and Y respectively. Now, if  $a > b > 0$ , then there are two equilibria  $\langle x_0, y_0 \rangle$  and  $\langle x_1, y_1 \rangle$ . But, only  $\langle x_0, y_0 \rangle$  is the optimal and hence, one would expect that the agents play  $\langle x_0, y_0 \rangle$ . If  $a = b > 0$ , then none of the agents' have any reason to prefer any one action.

		Agent Y →	
		$y_0$	$y_1$
Agent X ↓	$x_0$	$a$	$0$
	$x_1$	$0$	$b$

Fig. 1.28 Reward matrix of in a common reward two-agent static game

In such situation, there exist multiple equilibria. Choosing one equilibrium among multiple equilibria by random selection or by focusing personal basing may leads to suboptimal (or uncoordinated) equilibrium.

A robot can resolve the problem of equilibrium selection [38] in a coordinated game by repeatedly playing a game by the same robot. In the literature, correlated equilibrium (CE) addresses the problem of equilibrium selection.

### 1.3.3.2 Correlated equilibrium (CE)

CE is more general than the NE [72]. There are four variants of CE: Utilitarian, Egalitarian, Republican, and Libertarian equilibria [72]. In each variant, a numerical value is maximized and its corresponding index (joint action) is well-known as CE. The former numerical value may be evaluated by one of the following techniques.

1. In Utilitarian equilibrium, the numerical value to be maximized is evaluated by adding all robots' rewards.
2. The least efficient robot's reward is maximized in the Egalitarian equilibrium.
3. Most efficient robot's reward maximized in Republican equilibrium.
4. In Libertarian equilibrium, the numerical value to be maximized is evaluated by multiplying all the robots' rewards.

Like NE in CE, there are pure strategy and mixed strategy CE. The definition of CE is given in Definition 1.18. The pure strategy Egalitarian equilibrium evaluation is shown in Fig. 1.29.

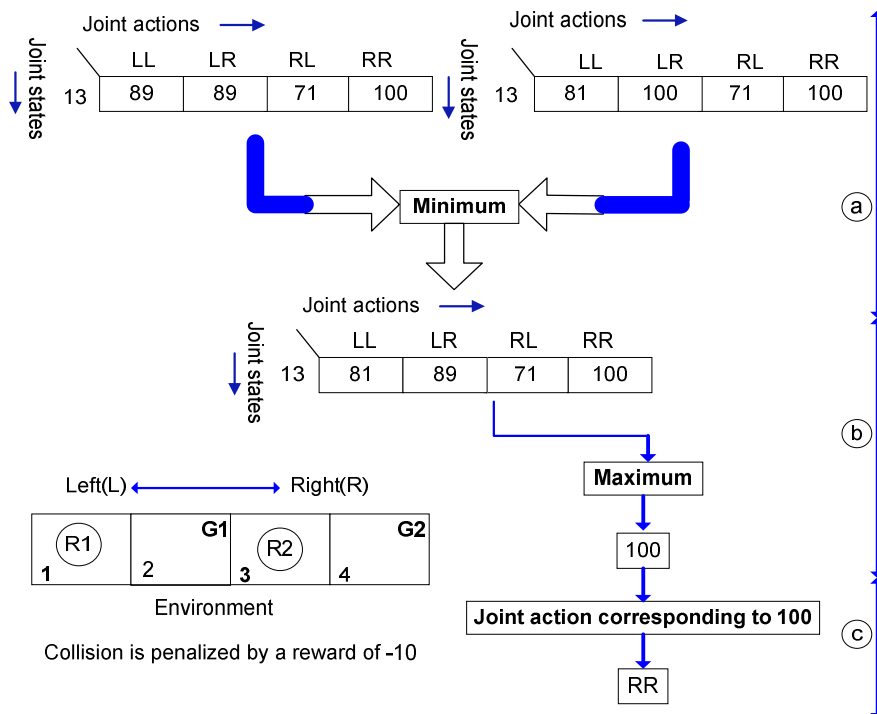


Fig. 1.29 Pure strategy Egalitarian equilibrium, which is one variant of CE



**Definition 1.18:** *Correlated equilibrium (CE)* at a joint state,  $S = \langle s_i \rangle_{i=1}^m$  with  $m$  interacting agents is the pure strategy CE,  $A_C$  and mixed strategy CE,  $p^*(A_C)$  if agents follow (1.29) and (1.30) respectively.

$$A_C = \arg \max_A [\Phi(Q_i(S, A))] \quad (1.29)$$

$$p^*(A_C) = \arg \max_{p(A)} [\Phi[\sum_A p(A)(Q_i(S, A))] \quad (1.30)$$

where,  $\Phi \in \{ \sum_{i=1}^m, \text{Min}_{i=1}^m, \text{Max}_{i=1}^m, \prod_{i=1}^m \}$ . (1.31)

Game of chicken reflects the idea of CE and is illustrated in Example 1.6.

**Example 1.6:** In game of chicken, two players play by heading toward each other as shown in Fig. 1.30. If both the players move (M) on the same way, then they collide, which results in penalty for both the agents. If one player moves and another player waits (cooperate (C)), then both the players are rewarded. The player which successfully moves receives more reward, than the player which cooperates. Both the players receive zero reward if none of them move. In Fig. 1.31, both the joint action  $(M, C)$  and  $(C, M)$  is the PSNE. To achieve PSNE without establishing any communication among the players, they should follow a signal (like traffic signal), which is commonly accessible for both the players.

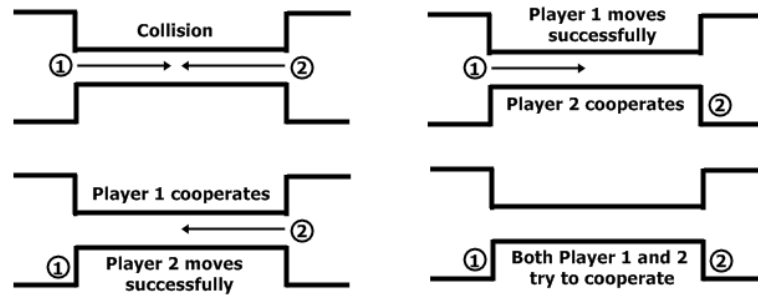


Fig. 1.30 Game of Chicken

		Player 2 →	
		M	C
↓ Player 1	M	-5, -5	10, 5
	C	5, 10	0, 0

Fig. 1.31 Reward matrix in Game of Chicken

### 1.3.3.3 Static game examples

A few examples of static games are given below.

**Constant-sum-game:** In constant-sum-game [41], summation of the two players' rewards is constant as shown in Fig. 1.32, where  $\langle a, b \rangle$  and  $\langle x, y \rangle$  are the action sets of player 1 and 2 respectively. In Fig. 1.32, the value of the constant is 1.

		Player 2 →	
		x	y
Player 1 ↓	a	5, -4	-7, 8
	b	-2, 3	4, -3

Fig. 1.32 Constant-sum-game

**Zero-sum-game:** Zero-sum-game is a special case of constant-sum-game [41]. It is a two player game, where the summation of the two players' reward is always zero. This indicates that one player's gain is equivalent to another player's loss. Hence, net change in reward is zero. Chess and tennis are the examples of zero-sum-game, where there is a winner and a loser. Financial market is also an example of zero-sum-game. In the literature of GT, matching pennies and rock-paper-scissor (given in Example 1.3) are the well-known examples of zero-sum-game. Example 1.7 illustrates the game of matching pennies.

**Example 1.7:** In matching pennies, two pennies are thrown by two players simultaneously. The rewards of the players depend on whether the pennies match or not. If both pennies result in head (H) or tail (T), then player 1 wins and rewarded by player 2's penny. If there is a mismatch, then player 2 wins and rewarded by player 1's penny. As one player's gain is other player's loss, hence, matching pennies is a zero-sum-game as shown in Fig. 1.33. In matching pennies, there is no PSNE instead there exists MSNE.

		Player 2 →	
		H	T
Player 1 ↓	H	1, -1	-1, 1
	T	-1, 1	1, -1

Fig. 1.33 matching pennies

In some situation, a game does not have a PSNE but every game have a MSNE [42]. For example in rock-paper-scissor game, there is no PSNE but there exists MSNE.

**General-sum-stochastic game:** In general-sum-stochastic game, the summation of all the players' rewards is neither zero nor constant. Prisoner's Dilemma is an example of general-sum-stochastic game and is illustrated in Example 1.8.

**Example 1.8:** In Prisoner's Dilemma two criminals are suspected of committing a crime and are being interrogated in two separate cells. From human physiology both the criminals want to minimize their jail sentence. Both of them face the same scenarios as follows (Fig. 1.34):

- If Player 1 and 2 each Deny (D) each other, then each of them are sentenced by 9 year jails.
  - If Player 1 Deny (D) but Player 2 remains Confess (C), then Player 1 will be set free and Player 2 will serve 10 years jails (and vice versa).
  - If both Player 1 and 2 remain Confess (C), then both of them will only serve 1 years jail.
- Hence, in Prisoner's Dilemma game, (C, C) is the PSNE.

		Player 2 →	
		<i>C</i>	<i>D</i>
Player 1 ↓	<i>C</i>	-1, -1	-10, 0
	<i>D</i>	0, -10	-9, -9

Fig. 1.34 Reward matrix in Prisoner's Dilemma Game

### 1.3.4 Correlation among RL, DP, and GT

It is apparent from the earlier sections that the RL works on the principle of reward/penalty received by the agents as a feedback from the environment. DP is nothing but an optimization technique, which optimizes the BE. On the other hand, GT helps in analyzing the strategic situation of the agents in multi-agent system, where an agent significantly affects the interests of other agents in the environment. Fig. 1.35 indicates that the multi-agent Q-learning (MAQL) comprising of the MARL, GT and DP. However, in the literature, MARL is well-known as MAQL for simplicity.

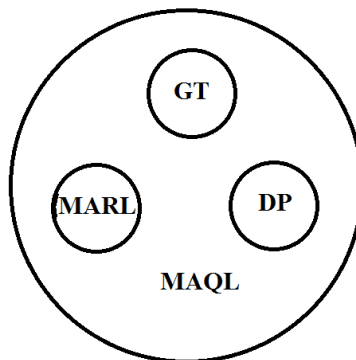


Fig. 1.35 Correlation among the MARL, DP, and GT

### 1.3.5 Classification of MARL

Based on the task type, MARL is classified as cooperative, competitive, and mixed as shown in Fig. 1.36 [82]. Now, the cooperative and mixed algorithms may be designed for static

(stateless) games or for stagewise (dynamic) games. However, there are only two competitive algorithms for two agents namely Minimax-Q and heuristically accelerated multi-agent RL (HAMRL).

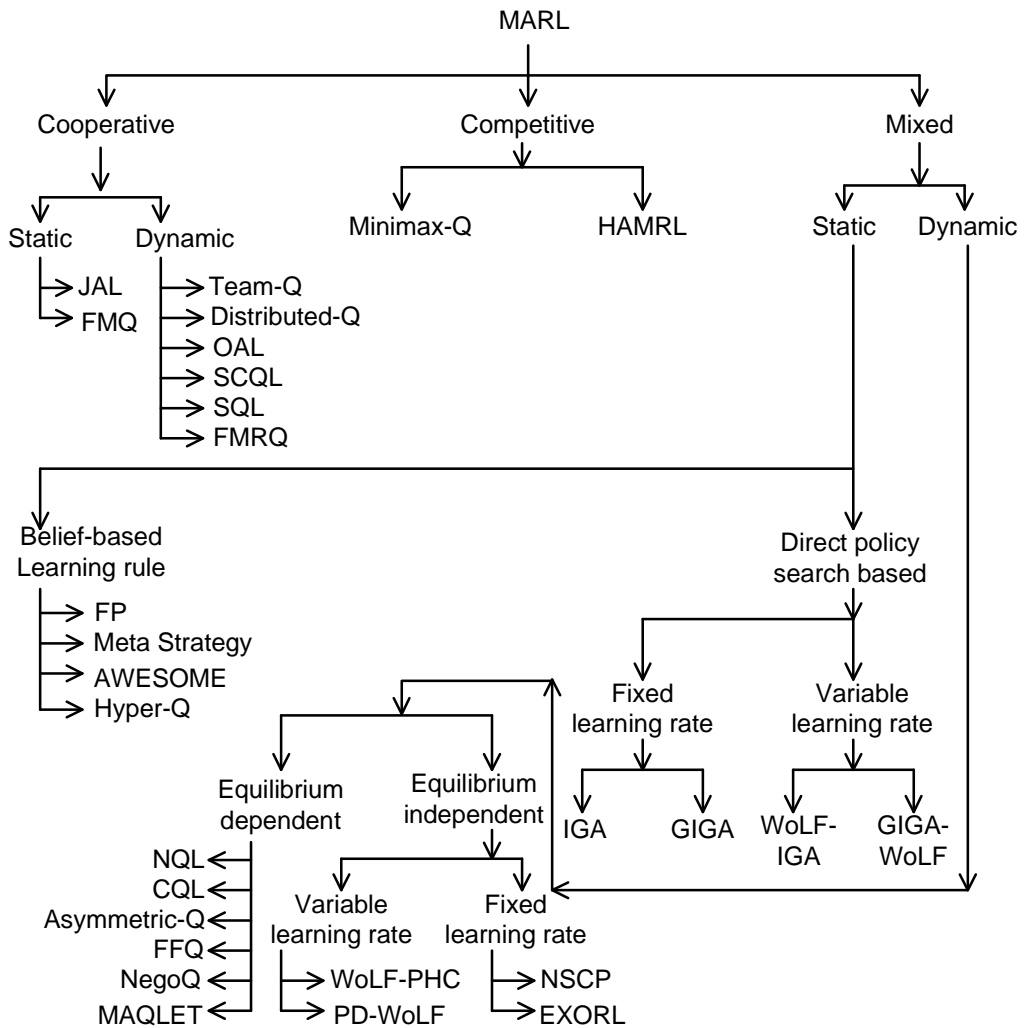


Fig. 1.36 Classification of multi-agent reinforcement learning

Joint Action Learners (JAL) and Frequency Maximum Q-value (FMQ) heuristic are classified as cooperative static algorithm. Team-Q, Distributed-Q, Optimal Adaptive Learning (OAL), Sparse Cooperative Q-learning (SCQL), Sequential Q-learning (SQL), and Frequency of the maximum reward Q-learning (FMRQ) fall within the scope of cooperative dynamic algorithms.

The mixed static algorithms are classified based on the belief on the other agents' policy of an agent and the steps required in searching the optimal policy (direct policy search). Belief based learning include Fictitious play (FP), Meta Strategy, *AWESOME* and Hyper-Q. The direct policy search based algorithms are classified based on variation of the learning rate: fixed learning rate and variable learning rate. Fixed learning rate includes Infinitesimal Gradient Ascent (IGA) and Generalized IGA (GIGA). Win-or-Learn-Fast-IGA (WoLF-IGA) and GIGA-WoLF are under the variable learning rate. The dynamic mixed strategy algorithms are classified as equilibrium dependent and equilibrium independent. Equilibrium dependent algorithms are Nash Q-Learning (NQL), Correlated Q-Learning (CQL), Asymmetric-Q learning, Friend-or-Foe Q-Learning (FFQ) (for more than two agents), Negotiation-based Q-learning (NegoQ) and MAQL with equilibrium transfer (MAQLET). Again equilibrium independent learning algorithms are classified as fixed learning rate and variable learning rate. Fixed learning rate includes Nonstationary Converging Policies (NSCP) and Extended Optimal Response (EXORL) heuristic. WoLF Policy Hill-Climbing (WoLF-PHC) and PD-WoLF are under the variable learning rate. The details of all the algorithms are given in the subsequent sections.

### 1.3.5.1 Cooperative multi-agent reinforcement learning

The cooperative MARL algorithms are given below as listed in Fig. 1.36.

#### 1.3.5.1.1 Static

The static MARL does not involve any state transitions as described in Section 1.3.3. The static MARL algorithms are discussed below.

##### 1.3.5.1.1.1 Independent Learner (IL) and Joint Action Learner (JAL)

In [78], Claus and Boutilier proposed two variants of learners. One is the Independent Learner (IL) and another is the Joint Action Learner (JAL). IL learns Q-value at its own action-space employing the classical single agent Q-learning rule ignoring the presence of other agents. For an IL  $i$  the single agent Q-learning rule (1.19) becomes (1.32) with  $\langle a, r \rangle$  as the experience profile. Also the Q-value earned by IL denoted by  $Q_i(a_i)$  converges to the optimal Q-value  $Q_i^*(a_i)$  for all action  $a_i \in A_i$  in the single agent system.

$$Q_i(a_i) \leftarrow Q_i(a_i) + \alpha[r_i(a_i) - Q_i(a_i)] \quad (1.32)$$

In multi-agent system, all the agents are adapting simultaneously and hence, the environment is no longer stationary, which does not ensure the convergence of Q-values any more. Reconsidering the 1.5, in IL, Agent X learns for the actions  $x_0$  and  $x_1$ . However, if Agent X is a JAL, then it learns for the four joint actions. It is interesting that the expected value of selecting  $x_0$  and  $x_1$  exclusively depends on the strategy played by Y. In 1.5, if  $a = b = 10$ , then Agent X's expected Q-value for  $x_0$  is

$$\begin{aligned}
\hat{Q}_x(x_0) &= \hat{Q}_x(x_0, y_0) \times P(y_0 | x_0) + \hat{Q}_x(x_0, y_1) \times P(y_1 | x_0) \\
&= 10 \times P(y_0 | x_0) + 0 \times P(y_1 | x_0) \text{ [by 1.5 } \hat{Q}_x(x_0, y_0) = 10 \text{ and } \hat{Q}_x(x_0, y_1) = 0 \text{]} \\
&= 10 \times 0.5 + 0 \times 0.5 \\
&= 5,
\end{aligned} \tag{1.33}$$

where  $P(y_0 | x_0)$  and  $P(y_1 | x_0)$  refer to the probability of  $y_0$  and  $y_1$  are being executed respectively by agent Y subject to  $x_0$  is being selected by agent X. To handle the above explained dynamics in multi-agent systems, the JAL maintains a belief about the other agents' strategies and the expected value of action  $a_i$  by agent  $i$  is given below.

$$\hat{Q}_i(a_i) = \sum_{a_{-i} \in A_{-i}} Q_i(a_{-i}, a_i) \prod_{j \neq i} P_{a_{-i}[j]}^i \tag{1.34}$$

$$\text{where, } Q_i(a_{-i}, a_i) \leftarrow Q_i(a_{-i}, a_i) + \alpha [r_i(a_{-i}, a_i) - Q_i(a_{-i}, a_i)] \tag{1.35}$$

The experience tuple of JAL is denoted by  $\langle a_i, a_{-i}, r_i \rangle$ .

So, JAL learns the Q-value at joint action-space considering the presence of other agents by synergistically combining the RL and equilibrium (or coordination) learning methods [83]-[86]. Learning equilibrium depends on the rewards corresponding to the joint actions at a given joint state and these rewards are obtained by the well-known RL more especially by Q-learning. The convergence of Q-learning does depend on the already explained trade-off between the exploration and exploitation. If an agent  $i$  chooses an action  $a_i$  with probability  $P_i(a_i)$ , then probability of choosing remaining actions is  $1 - P_i(a_i)$ . The said trade-off can be balanced by tuning the temperature parameter  $T$  of the Boltzmann strategy given by (1.36). The variation of  $T$  is done in such a way so that the convergence is guaranteed [134].

$$P_i(a_i) = \frac{e^{Q_i(a_i)/T}}{\sum_{\forall \tilde{a}_i} e^{Q_i(\tilde{a}_i)/T}} \tag{1.36}$$

Following conditions are required to satisfy for convergence of both the IL and JAL [78]:

1. The learning rate  $\alpha$  decreases with respect to time  
i.e.,  $\sum_{\alpha=0}^t \alpha = \infty$  and  $\sum_{\alpha=0}^1 \alpha^2 < \infty$ .
2. Each agent selects each of its actions infinitely.
3. The probability of choosing action  $a$  by agent  $i$ ,  $P_t^i(a) \neq 0$ .
4. All the agent's exploration strategy is exploitive. That is,  $\lim_{t \rightarrow \infty} P_t^i(X_t) = 0$ , where  $X_t$  is a random variable denoting the event that some non-optimal action was taken based on  $i$ 's estimated value at time  $t$ .

Finally, myopic heuristic based optimistic exploration strategies are proposed in [78] for optimal action selection.

**1 Optimistic Boltzmann (OB):** Choose the action  $a_{-i}$  using the Boltzmann strategy, assuming  $\text{Max } Q_i(a_i) = \text{Max } Q_i(a_i, a_{-i})$ ,

**2 Weight OB (WOB):** Explore using the Boltzmann strategy using the factor  $P_i$  (optimal match  $a_{-i}$  for  $a_i$ ) i.e.,  $\text{Max } Q_i(a_i) \cdot P_i$  (optimal match  $a_{-i}$  for  $a_i$ ).

**3 Combined:** Employ the Boltzmann strategy, assuming  $V(a_i) = \rho \text{Max } Q_i(a_i) + (1 - \rho)EV(a_i)$  as the value of action  $a_i$ , where  $\rho \in [0,1]$ .

Considering the biasing  $\rho = 0.5$  in [78], it is shown that combined exploration strategy outperforms the OB, WOB and the Boltzmann strategy in terms of the average accumulated reward. The algorithm for IL and JAL are given in Algorithm 1.4 and 1.5 respectively.

<b>Algorithm 1.4: Independent Learners</b>
<p><b>Input:</b> Action set of agent <math>i, A_i, \alpha \in [0,1]</math>;</p> <p><b>Output:</b> Optimal Q-value of agent <math>i, Q_i^*(a_i), a_i \in A_i</math>;</p> <p>Initialize: <math>Q_i(a_i) \leftarrow 0</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Execute an action <math>a_i</math> by agent <math>i</math> employing the Boltzmann strategy;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(a_i)</math>;</p> <p style="padding-left: 40px;">Update: <math>Q_i(a_i) \leftarrow Q_i(a_i) + \alpha[r_i(a_i) - Q_i(a_i)]</math>;</p> <p style="padding-left: 40px;"><math>Q_i^*(a_i) \leftarrow Q_i(a_i)</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>Q_i(a_i)</math> converges;</p> <p><b>End.</b></p>

<b>Algorithm 1.5: Joint Action Learners</b>
<p><b>Input:</b> Action set <math>A_i, \forall i, \alpha \in [0,1]</math>;</p> <p><b>Output:</b> Optimal joint Q-value <math>Q_i^*(a_i, a_{-i}), \forall i</math>;</p> <p>Initialize: <math>Q_i(a_i, a_{-i}) \leftarrow 0</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Execute an action <math>a_i</math> by agent <math>i</math> employing the Boltzmann strategy;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(a_i, a_{-i})</math> by observing other agents' rewards;</p> <p style="padding-left: 40px;">Update: <math>Q_i(a_i, a_{-i}) \leftarrow Q_i(a_i, a_{-i}) + \alpha[r_i(a_i, a_{-i}) - Q_i(a_i, a_{-i})]</math> and</p> <p style="padding-left: 40px;"><math>P_{a_j}^i, \hat{Q}_i(a_i)</math> by (1.50), (1.51) respectively;</p> <p style="padding-left: 40px;"><math>Q_i^*(a_i, a_{-i}) \leftarrow Q_i(a_i, a_{-i})</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>Q_i(a_i, a_{-i})</math> converges;</p> <p><b>End.</b></p>

Unfortunately, the above conditions do not guarantee convergence to equilibrium in the practical and complicated games such as in the climbing game [3], [78] and the penalty game [3], [78].

### 1.3.5.1.1.2 Frequency maximum Q-value (FMQ) heuristic

The independent agent in [87] and [2] including the JAL in [78], does not guarantee convergence to the optimal joint action in the absence of coordination with high penalties. In the Frequency Maximum Q-value (FMQ) heuristic [3], a novel action selection strategy is proposed assuming agents can observe other agents actions and are tested in two coordination problems mentioned in [78]: the climbing game and the penalty game. The said games are repeated cooperative single-stage games and they provide suitable platforms for studying the multi-agent coordination problem.

**Climbing game:** It is apparent from Fig. 1.37 that in the climbing game [3], [78],  $(x, x)$  is the optimal joint action and both the agent should go for it. Now, if Agent 1 plays  $x$  and Agent 2 plays  $y$ , then both the agents receive negative reward (-30). After learning this situation both the agents avoid joint action  $(x, y)$ . Later, if Agent 1 plays action  $z$ , then Agent 2 plays either  $y$  or  $z$  as due to both the joint action  $(z, y)$  and  $(z, z)$  agents receive positive rewards of 6 and 5 respectively. Suppose, Agent 2 is playing  $x$  but Agent 1 does not play  $x$  as it receives negative reward in the past due to  $x$ , and also Agent 1 does not play  $y$  as it provides negative reward. Hence, Agent 1 plays  $z$  and both the agents receive reward of 0. Similarly, if Agent 2 plays  $z$ , then agents receive at least 0 independent of Agent 1's choice. From the above analysis it is apparent that in the climbing game, agents always move away from the optimal joint action.

		Agent 2 →		
		$x$	$y$	$z$
↓ Agent 1	$x$	11	-30	0
	$y$	-30	7	0
	$z$	0	6	5

Fig. 1.37 The climbing game reward matrix

**Penalty game:** Similar to the climbing game presence of multiple equilibria in the penalty game [3], [78] is also challenging to check the performance of the coordination in multi-agent system. In penalty game (Fig. 1.38), both the agents should avoid the joint actions  $(x, z)$  and  $(z, x)$  to avoid the negative reward of -10. Now, in penalty game, there are two optimal joint actions  $(x, x)$  and  $(z, z)$ . Agents can play for any one of them. Suppose, Agent 1 plays  $x$  with an expectation that Agent 2 also plays  $x$  to receive maximum reward of 10. In this situation, if Agent 2 plays  $z$ , expecting Agent 1 plays  $z$  to receive maximum reward of 10. In the above circumstances,  $y$  is the safe choice for both the agents regardless of what other agent's



play and is guaranteed to receive a reward of 0 or 2. Hence, it is challenging to identify the optimal joint action in penalty game for multi-agent coordination.

		Agent 2 →		
		x	y	z
↓ Agent 1	x	10	0	-10
	y	0	2	0
	z	-10	0	10

Fig. 1.38 The penalty game reward matrix

From the climbing game and the penalty game it is apparent that an agent should select its action wisely for convergence. Maintaining a balance between the exploration and exploitation is an intelligent approach for action selection. Balancing the exploration/exploitation is a trade-off and is addressed by the well-known Boltzmann strategy given in (1.36). In (1.36), the probability of selecting an action  $a_i$  for agent  $i$  is evaluated by utilizing the Q-value and the tuning parameter temperature ( $T$ ). If  $T \rightarrow \infty$ , then each action has an equal probability to execute and hence, pure exploration occur. If  $T \rightarrow 0$ , then the action has a probability of one to execute and hence, exploitation occur. In [3],  $T$  is given by

$$T(t) = e^{-st} \times T_{\max} + 1, \quad (1.37)$$

where  $t$  is the learning epoch,  $s$  is a parameter to control the exploration rate and  $T_{\max}$  is initial value of temperature.

In [92], an optimistic assumption based algorithm is proposed. By optimistic assumption an agent updates its Q-value only if the new value is greater than the current one. Unfortunately, the optimistic assumption fails to converge to the optimal joint action due to misleading maximum reward. FMQ heuristic is based on the experience of the agent. Agent counts the frequency of the action which yields the best reward. Instead of optimistic assumption an agent  $i$  uses the Boltzmann strategy with the modified Q-value  $\tilde{Q}_i(A)$  given in (1.38).

$$\tilde{Q}_i(A) = Q_i(A) + f \times \frac{c_{\max}(A)}{c(A)} \times r_{\max}(A), \quad (1.38)$$

		Agent 1 →		
		x	y	z
↓ Agent 2	x	11	-30	0
	y	-30	<b>14/0</b>	6
	z	0	0	5

Fig. 1.39 The penalty game reward matrix

where  $c_{\max}(A)$  is the number of times agent  $i$  receives maximum reward  $r_{\max}(A)$  after executing the action  $A$   $c(A)$  times.  $f$  refers to the control parameter to control the

importance of the FMQ heuristic. The value of  $f$  increases proportionally with the increase in problem difficulty.

<b>Algorithm 1.6: FMQ heuristic</b>
<p><b>Input:</b> Action set <math>A_i, \forall i, \gamma \in [0,1), \alpha \in [0,1), f</math>;</p> <p><b>Output:</b> Optimal joint Q-value <math>Q_i^*(A), \forall i, i \in [1, m]</math>;</p> <p>Initialize: <math>Q_i(A) \leftarrow 0, \forall i</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Execute action <math>a_i \in A_i, \forall i</math> employing FMQ heuristic;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(A), \forall i</math>;</p> <p style="padding-left: 40px;">Update: <math>Q_i(A) \leftarrow Q_i(A) + \alpha[r_i(A) - Q_i(A)]</math> and modify Q-value <math>\tilde{Q}_i(a_i), \forall i</math> by (1.54) for modified Boltzmann strategy (FMQ heuristic);</p> <p style="padding-left: 40px;"><math>Q_i^*(A) \leftarrow Q_i(A)</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>Q_i(A), \forall i</math> converge;</p> <p><b>End.</b></p>

It is observed from the experiments that the FMQ heuristic outperforms the baseline experiments in terms of the convergence to the optimal joint action both in the climbing game and the penalty game [3]. To compare the FMQ heuristic with optimistic assumption a partially stochastic version of the climbing game is given in Fig. 1.39. In the partially stochastic climbing game, at least one of the rewards is stochastic as shown in Fig. 1.39. In Fig. 1.39, the joint action  $(y, y)$  yields a reward of 14 or 0 with probability 0.5. So, in the long run both the agents receive a reward of 7 due to joint action  $(y, y)$ . Hence, the reward matrix given in Fig. 1.37 and 1.39 are equivalent in the long run. The FMQ heuristic also outperforms the baseline experiment and the optimistic assumption in the partially stochastic climbing game, in terms of the convergence to optimal joint action. Unfortunately, the FMQ heuristic fails to convergence to optimal joint action in the fully stochastic penalty game and climbing game. The algorithm for FMQ heuristic is given in Algorithm 1.6.

### 1.3.5.1.2 Dynamic

Dynamic RL is stochastic Markov game with more than one joint state.

#### 1.3.5.1.2.1 Team-Q

Team-Q is a cooperative dynamic Q-learning algorithm. Dynamic indicates the existence of state transitions. In [88], Littman proposed Team-Q learning designed for team games in the framework of team Markov games (Coordination game). In Team-Q learning, the value function  $VQ_i(S')$  of agent  $i \in [1, m]$  at joint next state  $S'$  for the  $m$  agents' team is given in (1.39).

$$VQ_i(S') = \underset{a_1, a_2, \dots, a_m}{\text{Max}} Q_i(S; a_1, a_2, \dots, a_m) \quad (1.39)$$

The update rule in Team-Q learning for agent  $i$  is given in (1.40), without using remaining agents' model like in [78].

$$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma VQ_i(S')], \quad (1.40)$$

where  $A = \langle a_1, a_2, \dots, a_m \rangle$  be the joint action at joint state  $S = \langle s_1, s_2, \dots, s_m \rangle$ . The Team-Q learning is convergent following the generalized Q-learning algorithm [90], [91]. Team-Q learning is similar to Nash Q-Learning (NQL) [89] for the coordination games. Still there exists a challenge regarding the equilibrium selection among multiple equilibria in noisy environment. The algorithm for Team-Q learning is given in Algorithm 1.7.

<b>Algorithm 1.7: Team-Q</b>
<p><b>Input:</b> Action set <math>A_i, \forall i, \gamma \in [0,1), \alpha \in [0,1)</math>;</p> <p><b>Output:</b> Optimal joint Q-value <math>Q_i^*(S, A), \forall i, i \in [1, m]</math>;</p> <p>Initialize: <math>Q_i(S, A) \leftarrow 0, \forall i</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Execute action <math>a_i \in A_i, \forall i</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(S, A), \forall i</math>;</p> <p style="padding-left: 40px;">Update: <math>Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \max_A Q_i(S; A)]</math></p> <p style="padding-left: 40px;">and <math>S \leftarrow S'</math>;</p> <p style="padding-left: 40px;"><math>Q_i^*(S, A) \leftarrow Q_i(S, A)</math>;</p> <p><b>Until</b> <math>Q_i(S, A), \forall i</math> converge;</p> <p><b>End.</b></p>

### 1.3.5.1.2.2 Distributed –Q

In [92], model-free Distributed Q-learning is proposed for cooperative multi-agent system in deterministic situation with a motivation to compute an optimal policy in a cooperative multi-agent environment. The Distributed Q-learner solves two problems. The first problem is concerned with determination of the optimal policy. The second problem deals with selection of one optimal policy among alternatives, which is optimal for the entire team.

To handle multi-agent dynamics MDP is extended to Multi-agent MDP (MMDP), where each agent maximizes its own reward having different goals (i.e. reward-function). However, in the cooperative MMDP, all the agents have identical reward function. Such identical reward-functions are advantageous in finding an equilibrium point, which is an optimal joint action and it maximizes the reward of all the agents. In cooperative MMDP, the learning algorithm is responsible in making cooperation among the agents. Here, also two types of agents are considered: one is JAL and another is IL as mentioned in [78]. IL cannot distinguish the difference between the individual (elementary) action [92] and joint action.

Hence, IL maintains a Q-table of smaller size i.e.,  $S \times A$ , instead of maintaining the Q-table at joint state-action space,  $S \times A^m$ . In [92], the smaller Q-tables are assumed as the projection from the larger central Q-table with a conjecture about the strategies of teammates. So, in [92], a projection approach is proposed by evaluating the individual Q-table in a distributed way without adapting Q-table in joint state-action space by weighting the Q-values from larger Q-table given in (1.41).

$$q_i(S, a_i) \leftarrow \sum_{\forall A = \langle a_i \rangle_{i=1}^m} [P(S, A | a_i) \cdot [r_i(S, a_i) + \gamma \max_{a_i'} q_i(S', a_i')]], \quad (1.41)$$

where,  $P(S, A | a_i)$  refers to the probability of joint action  $A$  to be executed at joint state  $S$  including the action of agent  $i$ ,  $a_i$ .

Another way of projection is the ‘pessimistic assumption.’ By pessimistic assumption the individual smaller Q-value is the least efficient agent’s Q-value obtained from larger central Q-value. Such approach creates robust policies but is not extended in [92], because of its cautious nature. Instead of pessimistic assumption its dual form is utilized to obtain the smaller Q-value from the central Q-table as given in (1.42).

$$q_i(S, a_i) \leftarrow \max_A Q(S, A) \quad (1.42)$$

It can also be written in terms of small Q-table given in (1.43).

$$q_i(S, a_i) = \max_{a_i \in A} q_i(S, a_i) \quad (1.43)$$

<b>Algorithm 1.8: Distributed Q-learning</b>
<p><b>Input:</b> Action set <math>A_i, \forall i, \gamma \in [0, 1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>q_i^*(S, a_i), \forall i, i \in [1, m], a_i \in A</math>;</p> <p>Initialize: <math>q_i(S, a_i) \leftarrow 0, \forall i</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Execute action <math>a_i \in A_i, \forall i</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(S, a_i), \forall i</math>;</p> <p style="padding-left: 40px;">Update: <math>q_i(S, a_i) \leftarrow \max\{q_i(S, a_i), r_i(S, a_i) + \gamma \max_{a_i'} q_i(S', a_i')\}</math> and <math>S \leftarrow S'</math>;</p> <p style="padding-left: 40px;"><math>q_i^*(S, a_i) \leftarrow q_i(S, a_i)</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>q_i(S, a_i), \forall i</math> converge;</p> <p><b>End.</b></p>

The projection technique introduced in (1.43) is also known as optimistic assumption. It is assumed that all agents are acting optimally and the conjunction of the individual optimal actions is also an optimal joint action. However, such assumption is necessarily not true. This inspires the researchers in [92], to propose a Proposition, which states that in cooperative deterministic MMDP

$$q_i^t(S, a_i) = \max_{A=\langle a_i \rangle_{i=1}^m} Q^t(S, A) \quad (1.44)$$

holds and also its prove is given in [92], where  $t$  is the learning epoch. The steps of the Distributed Q-learning are given in Algorithm 1.8. The climbing game and the penalty game are extended for the Distributed Q-learning respectively in Example 1.9 and Example 1.10.

**Example 1.9:** To extend the climbing game for Distributed Q-learning as shown in Fig. 1.40, let both the agents are at joint state  $S$  and for brevity discount factor  $\gamma$  is set to 0. The reward function  $q_i(S, a_i)$  is evaluated employing Algorithm 1.8. Such greedy approach of Algorithm 1.8 yields highest Q-value for both the agents as shown in Fig. 1.40. In Fig. 1.40, the optimal joint action is  $(x, x)$ . However, in the IL, JAL, and the FMQ-heuristic algorithm agents supposed to find the sub-optimal joint action  $(y, y)$  as explained in Fig. 1.37.

		Actions →		
		x	y	z
↓ Individual Q-value	$q_1(S, a_1)$	11	7	5
	$q_2(S, a_2)$	11	7	5

Fig. 1.40 Individual Q-values obtained in the climbing game reward matrix by Distributed Q-learning

**Example 1.10:** Like Example 1.9 to extend the penalty game for Distributed Q-learning as shown in Fig. 1.42, the discount factor  $\gamma$  is set to 0 and the Algorithm 1.8 is employed to evaluate distributed rewards. In the FMQ-heuristic algorithm (Fig. 1.41), there are four optimal joint actions:  $(x, x)$ ,  $(x, z)$ ,  $(z, x)$  and  $(z, z)$  but only  $(x, x)$  and  $(z, z)$  are optimal joint actions with reward 10 as shown in Fig. 1.42 offered by Algorithm 1.8. Unfortunately application of the Distributed Q-learning is limited to the deterministic system only.

		Agent 2 →		
		x	y	z
↓ Agent 1	x	10	0	9
	y	0	2	0
	z	9	0	10

Fig. 1.41 The penalty game reward matrix

		Actions →		
		x	y	z
↓ Individual Q-value	$q_1(S, a_1)$	10	2	10
	$q_2(S, a_2)$	10	2	10

Fig. 1.42 Individual Q-values obtained in the penalty game reward matrix by Distributed Q-learning

### 1.3.5.1.2.3 Optimal Adaptive Learning

There are many straight forward solutions to choose optimal equilibrium among multiple equilibrium solutions, like enforce convention [93] and fictitious play [78], [39]. In [78], the JAL guarantees the convergence to NE in a team game. However, it is not guaranteed that the selected NE is the optimal one. Similar problem arises in game theory like Adaptive play (AP) [94] and evolutionary model proposed in [40].

In model-free RL, agents do not have any idea about the environment; in addition they may receive noisy rewards. Hence, it is impossible to converge properly. In [92], [95], the MDP is extended to Team Markov Game (cooperative MMDP) with an aim to find a deterministic joint strategy to maximize the expected sum of discounted rewards. In [95], Optimal Adaptive Learning (OAL) algorithm is proposed with convergence proof where agents learn to choose the optimal NE among multiple NE with probability one. Let in a three-player coordination game,  $\langle a_1, a_2 \rangle$ ,  $\langle b_1, b_2 \rangle$  and  $\langle c_1, c_2 \rangle$  be the individual action sets of agent 1, 2, and 3 respectively. The reward matrix of this coordination game is shown in Fig. 1.43. It is apparent from Fig. 1.43, that there are three pure strategy NEs  $\langle a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3 \rangle$  and six sub-optimal NEs. The rewards corresponding to the sub-optimal NE are italicized.

		Joint actions of Agent 2 and 3 $\rightarrow$								
		$b_1 c_1$	$b_1 c_2$	$b_1 c_3$	$b_2 c_1$	$b_2 c_2$	$b_2 c_3$	$b_3 c_1$	$b_3 c_2$	$b_3 c_3$
$\leftarrow$ Agent 1	$a_1$	<b>10</b>	-20	-20	-20	-20	5	-20	5	-20
	$a_2$	-20	-20	5	-20	<b>10</b>	-20	5	-20	-20
	$a_3$	-20	5	-20	5	-20	-20	-20	-20	<b>10</b>

Fig. 1.43 Reward matrix of a three player coordination game

Before discussing about the OAL algorithm, the AP algorithm [94] is discussed. In AP game, it is assumed that agents know the game before playing it and one virtual game (VG) is designed. In Team Markov Game, to eliminate the sub-optimal NE, the following arrangement is made. Suppose, in cooperative situation  $VG(S, A)$  be the payoff of the agents at joint state  $S$  because of joint action  $A$ . In VG, it is assumed that at optimal NE the reward denoted by  $VG^*(S, A)$  is equal to one and else it is set to zero, e.g. in Fig. 1.43,  $VG^*(S, A)$  is equal to one if  $A$  is an optimal NE, i.e.  $A \in \{a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3\}$  and else it is zero. Considering weakly acyclic game (WAG) [94] as a VG, where each joint action  $A \in \{A\}$  is considered as a vertex. The vertices are connected with the directed edge avoiding self loop, where for an agent  $i$  the action  $a_i \in A_i$  is the best response to  $A_{-i}$ , here  $-i$  stands for all except agent  $i$ . By the principle of WAG represented as a best-response graph, from any starting vertex  $A$  there exists a directed path to some vertex  $A^* \in \{A\}$  and from  $A^*$  there is no outgoing path [94].

To eliminate the sub-optimal NE or tackle the equilibrium selection problem in WAG, Young proposed AP in [94]. In AP, suppose in a  $m$ -player matrix game the joint action at time  $t$  is denoted by  $A^t \in \{A\}$ . Also assume two integers  $k$  and  $n$  such that  $1 \leq k \leq n$  and  $t \leq n$ . After acting randomly agents look at its experience and restart the learning at  $t = n + 1$ . At  $t = n + 1$  each agent looks reverse at their most recent  $n$  experiences and randomly choose  $k$  samples from that. Now, the expected reward of agent  $i$ 's action  $a_i$  is given in (1.45). After evolution of  $ER(a_i)$  randomly an action is chosen from a set of best response given in (1.46).

$$ER(a_i) = \sum_{A_{-i} \in \{A_{-i}\}} u_i(\{a_i\} \cup A_{-i}) \frac{K_{t+1}(A_{-i})}{k}, \quad (1.45)$$

where,  $K_{t+1}(A_{-i})$  refers to the count the joint action  $A_{-i}$  in the  $k$  samples and  $u_i(\{a_i\} \cup A_{-i}) = u_i(A)$  is the reward of agent  $i$  because of joint action  $A$ .

$$BR_i^t = \{a_i \mid a_i = \arg \max_{a_i' \in A_i} ER(a_i')\} \quad (1.46)$$

It is shown in [94], that by AP WAG converges to a strict NE. Unfortunately, all the VGs are not WAG and hence, the AP may not converge to a strict NE for all VGs. To address the said problem the WAG and AP algorithms are modified as follows.

The WAG and AP are modified as WAG with respect to a biased set (WAGB). In WAGB, there is a set  $D$  containing a few Nash equilibria of the WAGB. A game is a WAGB if from any vertex  $A$  one path exists leading to the NE belongs to set  $D$  or a strict NE [95]. In AP, agents randomly select the NE among multiple best responses of the agents. On the other hand, in biased AP (BAP) [95], agents deterministically select the best-response action as a NE belongs to  $D$ . Suppose,  $W_t$  denotes the set of  $k$  samples drawn from the most recent  $n$  joint actions. The following two conditions are satisfied. First condition is that the joint action  $A' \in D$  such that,  $\forall A, A \in W_t, A_{-i} \subset A$  and  $A_{-i} \in A'$ . Second condition is that there must exist at least a joint action  $A \in D$  so that  $A \in W_t$  and  $A \in D$ . If the above two conditions are satisfied, then agent  $i$  chooses its best response action  $a_i$  such that  $a_i \in a_i'$ , where

$$t' = \max\{T \mid a^T \in W_t \wedge a^T \in D\}. \quad (1.47)$$

The philosophy of (1.47) is that the action  $a_i$  is the component of the most recent NE belonging to  $D$ . If the above two conditions are not satisfied, then AP is implemented. Hence, it can be concluded that the BAP on WAGB converges to either a NE belongs to  $D$  or a strict NE. The above techniques are applicable only when the game structure is known. To learn in an unknown game structure multi-agent  $\epsilon$ -optimality is employed. By definition a joint action is  $\epsilon$ -optimality at joint state  $S$  and time  $t$  if  $Q_t(S, A) + \epsilon \geq \max_{A'} Q_t(S, A'), \forall A' \in \{A\}$ .

Let the set of  $\epsilon$ -optimal joint action, which converges  $Q_t$  to  $Q^*$  with slower rate than  $VG_t$

converges to  $VG^*$ . Here,  $\varepsilon$  varies proportionately to the function  $B(N_t) \in [0,1]$ , where  $B(N_t)$  decreases slowly and monotonically to zero with  $N_t$ .  $N_t$  refers to the minimum time required to sample a state-action pair. The algorithm for OAL is given in Algorithm 1.9 [95]. The convergence proof of the OAL algorithm is given in [95].

<b>Algorithm 1.9: Optimal Adaptive Learning</b>
<p><b>Input:</b> Action set <math>A_i, \forall i</math> at joint state <math>S, \gamma \in [0,1]</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q^*(S, A)</math>;</p> <p>Initialize: <math>t = 0, n_t(S, A) = 1, T_t(S'   (S, A)) = \frac{1}{ S }, R_t(S, A) = 0, \varepsilon_t = C, A^{\varepsilon_t}(S) = A, D = A</math>;</p> <p><b>Repeat</b> // <math>n_t(S, A)</math> is the number of times the joint action <math>A</math> has been executed in joint state <math>S</math> up to time <math>t</math></p> <p><b>If</b> <math>t \leq m</math>,</p> <p style="padding-left: 20px;"><b>Then</b> randomly select an action <math>a_i, \forall i</math>;</p> <p><b>Else do</b></p> <p style="padding-left: 20px;"><b>Begin</b></p> <p style="padding-left: 40px;">Update the virtual game <math>VG_t</math> at joint state <math>S</math>;</p> <p style="padding-left: 40px;">Randomly select records from <math>n</math> recent observations of other agents' joint actions played at joint state <math>S</math>;</p> <p style="padding-left: 40px;">Evaluate expected payoff of individual action <math>a_i</math> of the VG at joint state <math>S</math> by (1.45) and construct the best response set by (1.46);</p> <p style="padding-left: 40px;"><b>If</b> condition 1 and 2 in BAP are TRUE</p> <p style="padding-left: 60px;"><b>Then</b> choose best response action with respect to the biased set <math>D</math>;</p> <p style="padding-left: 40px;"><b>Else</b> randomly select a best response action from <math>BR_i^f(S)</math>;</p> <p style="padding-left: 40px;"><b>End If.</b></p> <p style="padding-left: 40px;"><b>End.</b></p> <p style="padding-left: 20px;"><b>End If.</b></p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i^f(S, A)</math>;</p> <p style="padding-left: 40px;"><b>Update:</b> <math>n_t(S, A) \leftarrow n_t(S, A) + 1, R_t(S, A) \leftarrow R_t(S, A) + \frac{1}{n_t(S, A)}(r_i^f(S, A) - R_t(S, A)),</math></p> <p style="padding-left: 40px;"><math>T_t(S'   (S, A)) \leftarrow T_t(S'   (S, A)) + \frac{1}{n_t(S, A)}(1 - T_t(S'   (S, A))),</math></p> <p style="padding-left: 40px;"><math>Q_{t+1}(S, A) \leftarrow R_t(S, A) + \gamma \sum_{\forall S' \in \{S\}} T_t(S'   (S, A)) \times \max_{\forall A' \in \{A\}} Q_t(S', A'), t \leftarrow t + 1, N_t \leftarrow \min_{S, A} n_t(S, A);</math></p> <p style="padding-left: 40px;"><b>If</b> <math>\varepsilon_t &gt; CB(N_t)</math></p> <p style="padding-left: 60px;"><b>Then do</b></p> <p style="padding-left: 80px;"><b>Begin</b></p> <p style="padding-left: 100px;"><math>\varepsilon_t &gt; CB(N_t), Q^*(S, A) \leftarrow Q(S, A), \forall i</math> and <math>A^{\varepsilon_t}(S) \leftarrow \{A   Q_t(S, A) + \varepsilon_t \geq \max_{A' \in \{A\}} Q_t(S, A')\}</math>;</p> <p style="padding-left: 80px;"><b>End;</b></p> <p style="padding-left: 60px;"><b>End If.</b></p> <p><b>Until</b> <math>Q(S, A), \forall i</math> converge;</p>

#### 1.3.5.1.2.4 Sparse cooperative Q-learning (SCQL)

One of the principle bottlenecks of the MAS is the exponential increase in the space and time complexity, with the increase in number of agents. Kok *et al.* [96] observed that in most of the MAS agents are required to coordinate their actions only in a few states and in the



remaining they act independently. In the coordinated joint state  $S$ , the Q-value of an agent  $i$  is denoted by  $Q_i(S, A)$ . However, if  $S$  be the uncoordinated joint state, then the Q-value of agent  $i$  is denoted by  $Q_i(S, a_i)$ . In case of uncoordinated joint state, the global Q-value  $Q(S, A)$  for  $m$  number of agents is defined as the summation of individual Q-values given by (1.48).

$$Q(S, A) = \sum_{i=1}^m Q_i(S, a_i). \quad (1.48)$$

Based on the above observations, in [96], Kok and Vlassis proposed Sparse Cooperative Q-learning (SCQL), where the Q-tables of the agents are sparsely maintained as discussed above.

#### 1.3.5.1.2.5 Sequential Q-learning (SQL)

In [97], Wang and Silva proposed Sequential Q-learning (SQL) to handle conflicting behavior of the agents that arises in tightly coupled multi-robot object transportation. In SQL, robots do not select their actions simultaneously; rather they do it sequentially based on their predefined priorities. In SQL, the problem of behavior conflict is addressed by avoiding the selection of same actions those already selected by the preceding robots. Assuming  $i^{\text{th}}$  robot is denoted by  $R_i, i \in [1, m]$  and all the robots are arranged in a special sequence. The subscript  $i$  in  $R_i$  indicates its position in the sequence. All the robots repeat steps given in Algorithm 1.10 to form a joint action avoiding the conventional steps in the classical steps MAQL. The joint action offered by Algorithm 1.10 avoids the bottleneck of behavior conflict in tightly coupled multi-robot object transportation.

<b>Algorithm 1.10: Joint action formation in SQL</b>
<p><b>Initialize</b> <math>\Psi = \phi</math>; // <math>\phi</math> be the empty set.  Observe current joint state <math>S</math>;  <b>For</b> <math>i = 1</math> to <math>m</math>    Evaluate the currently available action set <math>\Delta_i</math>, where <math>A_i</math> the action set of robot be <math>i</math>.  <math>\Delta_i = (A_i - (A_i \cap \Psi))</math>;    <math>R_i</math> selects the action <math>a_i^j = \Delta_i</math> by probability <math>P(a_i^j) = \frac{e^{Q_i(S, a_i^j)}}{\sum_{r=1}^{ \Delta_i } e^{Q_i(S, a_i^r)}}</math>.      Include the action <math>a_i^j</math> to the set <math>\Psi</math>;  <b>End For</b>  <b>Execute</b> the corresponding selected action <math>a_i^j, \forall i</math>;</p>

#### 1.3.5.1.2.6 Frequency of the maximum reward Q-learning (FMRQ)

In [98], Zhang *et al.* proposed a MARL algorithm for fully cooperative tasks, namely frequency of the maximum reward Q-learning (FMRQ), which aims at achieving the optimal NE to maximize the system performance with respect to the metric of interest. In FMRQ, a modified immediate reward signal is used, which is obtained by identifying the highest global

immediate reward. In FMRQ, an agent needs to share only its state and reward at each learning epoch with remaining agents.

In FMRQ, authors considered two issues: first they investigated whether the NE is good enough for the fully cooperative MAS, secondly the curse of dimensionality of the MARL is considered by storing the Q-value at joint state-individual action space.

To describe the dynamics of the FMRQ differential equations are formulated for the four cases including two-agent two-action repeated game, and a three-agent two action repeated game. In each case, the critical points of the differential equations are analyzed and it is observed that FMRQ converges to equilibrium with maximum global rewards in all the five cases [98]. In case 1, there exists only one global immediate reward. Case 2 and 3 have two maximum immediate rewards in diagonal positions and in the same row respectively. In case 4, three maximum immediate rewards exist and in case 5, only one global immediate reward exists [98].

<b>Algorithm 1.11: FMRQ for an agent <math>i</math> in repeated games</b>
<p><b>Input:</b> Action set <math>a_i \in A_i, \forall i</math> and learning rate <math>\alpha \in [0,1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q^*(a_i)</math>;</p> <p>Initialize: <math>Q(a_i) \leftarrow 0</math>, count of selecting action <math>a_i</math>, <math>n_{a_i} = 0</math>, number of times maximum global immediate reward received by action <math>a_i</math>, <math>n_{\max\_a_i} = 0</math> and frequency of getting maximum immediate reward after selecting action <math>a_i</math>, <math>fre(a_i) = 0</math>;</p> <p><b>Repeat</b></p> <p style="padding-left: 20px;">Select an action <math>a_i</math> by the Boltzmann exploration scheme;</p> <p style="padding-left: 20px;"><math>n_{a_i} = n_{a_i} + 1</math>;</p> <p style="padding-left: 20px;">Execute the action <math>a_i</math> and update <math>n_{\max\_a_i}</math> and <math>rh(a_i)</math>;</p> <p style="padding-left: 20px;"><b>For</b> each action <math>a_i \in A_i</math> <b>do</b> // <math>rh(a_i)</math> refers to history of global immediate reward obtained by action <math>a_i</math></p> <p style="padding-left: 40px;"><b>Begin</b></p> <p style="padding-left: 60px;">Evaluate <math>fre(a_i)</math> by (1.49);</p> <p style="padding-left: 60px;"><math>Q(a_i) = Q(a_i) + \alpha(fre(a_i) - Q(a_i))</math>;</p> <p style="padding-left: 60px;">Set <math>n_{a_i} = 0</math>, <math>n_{\max\_a_i} = 0</math> and <math>fre(a_i) = 0</math>;</p> <p style="padding-left: 40px;"><b>End</b></p> <p style="padding-left: 20px;"><b>End For</b></p> <p><b>Until</b> <math>Q^*(a_i)</math>; converges;</p>

In FMRQ, the size of a Q-table for an agent  $i$  is  $|\{S\}| \times |\{A_i\}|$ . In FMRQ, algorithm (Algorithm 1.11), the immediate reward of an agent  $i$ , denoted by  $r_i(a_i)$  is replaced by the frequency of getting the maximum global immediate reward by the same action  $a_i$ , denoted by  $fre(a_i)$ .

$$fre(a_i) = \frac{n_{\max\_a_i}}{n_{a_i}}, \quad (1.49)$$

where  $n_{a_i}$  refers to the number of times action  $a_i$  is selected by agent  $i$  and  $n_{\max\_a_i}$  is the number of times agent  $i$  achieves the maximum global immediate reward. Moreover, the superiority of the FMRQ algorithm is verified by two case studies: one is the 12-vertex box-pushing by 4-agents and the other one is the distributed sensor network optimization problem. The FMRQ algorithm is provided in Algorithm 1.11 for an agent  $i$  in repeated games.

### 1.3.5.2 Competitive multi-agent reinforcement learning

The competitive MARL algorithms are discussed below. Here two competitive MARL algorithms are discussed. One is Minimax Q-learning for two agents and its extension for general sum game for more than two agents called Heuristically-accelerated multi-agent reinforcement learning.

#### 1.3.5.2.1 Minimax-Q Learning

In [99], Littman proposed a competitive algorithm namely minimax-Q learning for two agents. In minimax-Q learning, both the agents have conflicting goals with an objective of maximizing the sum of its own discounted expected reward. In other words, an agent tries to maximize a reward function and simultaneously the opponent agent tries to minimize it. In [2] and [87] authors realized that an agent must interact with other agents and the environment during the learning phase without proposing any supporting mathematical model. In addition, the theory of MDP [81], [45], which is an extension of game theory, also cannot handle the multi-agent dynamics. Even sometime it is assumed that the environment is stationary. Littman in [99] considered only two-player zero-sum Markov game. In zero-sum game, the summation of the rewards of two agents is zero [41]. In every MDP, there is at least one strategy that is stationary, deterministic, and optimal [99]. But most of the cases the optimal strategies are probabilistic. For example in Fig. 1.23, (rock, paper, and scissor, Example 1.3) selection of a deterministic policy by any one player leads to punishment and hence, the player is defeated. The probabilistic strategy is required to represent the uncertainty about the agents' action choice. Suppose, the opponent agent has an action  $O \in \{O\}$  and Q-value is denoted by  $Q(S, A, O)$  as introduced in (1.50).

$$Q(S, A, O) \leftarrow r(S, A, O) + \gamma \sum_{S'} P(S' | (S, A)) \times V(S'), \quad (1.50)$$

$$\text{where } V(S') = \max_{\pi \in P(\{A\})} \min_{O \in \{O\}} \pi_A \cdot Q(S, A, O). \quad (1.51)$$

(1.51) indicates the expected reward to the agent for playing strategy  $\pi$  against the opponent's choice  $O \in \{O\}$ .  $P(\{A\})$  refers to the probability distribution over the action set  $\{A\}$ . The

algorithm for Minimax Q-learning is given in Algorithm 1.12 [99]. Algorithm 1.12 is tested in a two-player Markov game and it is compared with Q-learning. The convergence of Minimax-Q learning is guaranteed and the strategy offered by it is a safe choice against the opponent even in the worst situation.

<b>Algorithm 1.12: Minimax Q-learning</b>
<p><b>Input:</b> Action <math>A \in \{A\}</math>, opponent's action <math>O \in \{O\}</math> at joint state <math>S</math>, <math>\alpha \in [0,1)</math> and <math>\gamma \in [0,1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q^*(S, A, O)</math>;</p> <p>Initialize: <math>Q(S, A, O) \leftarrow 0</math>, <math>\pi(S, A) = \frac{1}{ A }</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Choose an action to execute by <math>\pi(S, A)</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r(S, A, O)</math>;</p> <p style="padding-left: 40px;"><b>Update:</b> <math>Q(S, A, O) \leftarrow (1-\alpha)Q(S, A, O) + \alpha[r(S, A, O) + \gamma V(S')]</math>, <math>S \leftarrow S'</math>,</p> <p style="padding-left: 40px;"><math>\pi(S, A) = \arg \max_{\pi(S, A)} [\text{Min}_{O \forall A} \sum \pi(S, A) \times Q(S, A, O)]</math> and <math>V(S') = \max_{\pi \in P(\{A\})} \min_{O \in \{O\}} \pi_A \cdot Q(S, A, O)</math>;</p> <p style="padding-left: 40px;"><math>Q^*(S, A, O) \leftarrow Q(S, A, O)</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>Q(S, A, O)</math> converges;</p> <p><b>End.</b></p>

### 1.3.5.2.2 Heuristically-accelerated multi-agent reinforcement learning

In [100], Bianchi *et al.* proposed Heuristically-accelerated multi-agent reinforcement (HAMRL), which attempts to speed up in convergence of MARL, by balancing exploration/exploitation employing a heuristic function for action selection. There exist a series of literature [100]-[103], where heuristic functions are used to increase the convergence speed of the MARL. The work of [100] is the extension of [103], whereas in [103], Littman's Minimax-Q is heuristically accelerated. Bianchi *et al.* defined a heuristic function  $H : \{S\} \times \{A\} \times \{O\} \rightarrow \mathbb{R}$ , which influences the action selection of the agents during the learning phase, when an agent executes an action  $A \in \{A\}$  at state  $S \in \{S\}$  against the opponent's action  $O \in \{O\}$ . In [100], authors employ the modified  $\epsilon$ -greedy learning rule including the heuristic function  $H(S, A, O)$  given by (1.52).

$$\pi^c(S) = \arg \max_A \min_O [Q(S, A, O) + \xi H(S, A, O)^\beta] \quad (1.52)$$

and  $\xi \in \mathbb{R}, \beta \in \mathbb{R}$  are the weightage on the confidence of the heuristic function. In (1.52), if  $\xi = 0$ , then (1.52) becomes (1.53), which is the standard  $\epsilon$ -greedy.

$$\pi(S) = \begin{cases} \pi^c(S), & \text{if } p \geq \epsilon, \epsilon \in [0,1], \\ \text{select an action randomly,} & \text{otherwise} \end{cases} \quad (1.53)$$

where  $p \in [0,1]$  is a random number. Considering  $\xi = \beta = 1$  the heuristic function is given by

$$H(S, A, O) = \begin{cases} \max_i Q(S, i, O) - Q(S, A, O) + \eta, & \text{if } A = \pi^H(S) \\ 0, & \text{otherwise} \end{cases}, \quad (1.54)$$

where  $\eta \in \mathbb{R}$ ,  $\pi^H(S)$  is the heuristic policy. The superiority of the HAMRL (Algorithm 1.13) is validated by conducting the experiments in two robots soccer game.

<b>Algorithm 1.13: HAMRL for Zero sum game</b>
<p><b>Input:</b> Action <math>A \in \{A\}</math>, opponent's action <math>O \in \{O\}</math> at joint state <math>S</math>, <math>\alpha \in [0,1)</math>, <math>\gamma \in [0,1)</math> and <math>\varepsilon \in [0,1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q^*(S, A, O)</math>;</p> <p>Initialize: <math>Q(S, A, O) \leftarrow 0</math>, <math>H(S, A, O), \pi^H, \eta</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Choose an action <math>A \in \{A\}</math> using the modified <math>\varepsilon</math>-greedy rule;</p> <p style="padding-left: 40px;">Execute <math>A \in \{A\}</math>, observe the opponent's action <math>O \in \{O\}</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r(S, A, O)</math>;</p> <p style="padding-left: 40px;"><b>Update:</b> <math>Q(S, A, O) \leftarrow (1 - \alpha)Q(S, A, O) + \alpha[r(S, A, O) + \gamma V(S')]</math>, <math>S \leftarrow S'</math> and <math>H(S, A, O)</math>,</p> <p style="padding-left: 40px;">where <math>V(S') = \max_{A \in \{A\}} \min_{O \in \{O\}} Q(S, A, O)</math>;</p> <p style="padding-left: 40px;"><math>Q^*(S, A, O) \leftarrow Q(S, A, O)</math>;</p> <p style="padding-left: 20px;"><b>Until</b> <math>Q(S, A, O)</math> converges;</p> <p><b>End.</b></p>

### 1.3.5.3 Mixed multi-agent reinforcement learning

Mixed MARL includes the following algorithms. The mixed MARL may be cooperative or competitive. It can be categorized based on the number of joint states involved: static and dynamic.

#### 1.3.5.3.1 Static

The static MARL algorithms are further extended in Fig. 1.36.

##### 1.3.5.3.1.1 Belief-based Learning rule

In belief based learning algorithm, an agent maintains a belief about the remaining agents' strategy. This section illustrates the belief-based learning rule.

##### 1.3.5.3.1.1.1 Fictitious play

Fictitious play (FP) [104] is a belief-based learning rule. Here belief indicates that a player adapts with the strategy about opponent players' and behaves as per the strategy learned. In FP, a robot can resolve the problem of equilibrium selection [38] in a coordinated game by repeatedly playing the game by the same robot. FP is an effective and efficient approach to reach equilibrium in a coordinated game. As per FP, agent  $i$  learns the models of all the other agents  $j \neq i$  by the model given in (1.55).

$$P_{a_j}^i = \frac{C_{a_j}^j}{\sum_{\forall \tilde{a}_j} C_{\tilde{a}_j}^j}, \quad (1.55)$$

where  $P_{a_j}^i$  refers to the model of agent  $j$ 's strategy evaluated by agent  $i$  or agent  $i$ 's assumption of playing  $a_j \in A_j$  by agent  $j$  or  $\Pi_{-i}$  and  $C_{a_j}^j$  be the number of times agent  $i$  observed agent  $j$  executing action  $a_j$ . In cooperative games, the strategy offered by (1.55) leads to an equilibrium, where in case of multiple equilibrium, agents randomly choose any one. Also, in FP, a player does not need to learn about opponent players' reward rather it maintains a belief about the opponents' feature strategy. If a FP converges to  $\Pi^*$ , then  $\Pi^*$  is a NE.

#### 1.3.5.3.1.1.2 Meta strategy

In [105], Powers and Shoham proposed a straight forward MARL algorithm for repeated games, which have the following two requirements. The first requirement is to specify a class of opponents and against them the algorithm yields a reward that approaches the reward corresponding to the best response. Second requirement is that the reward offered by the algorithm fulfills a threshold of security level reward. Constraining the above requirements the algorithm achieves a close to optimal payoff in self-play. Based on the above conditions an algorithm is proposed in [105], for stationary opponents only. However, to learn in a repeated game a learning algorithm is required. In the learning algorithm, an agent plays its best response with a prior probability of its opponent's strategy. GAMUT [59] is employed to test the superiority of the proposed algorithm in [105].

In [106], two properties are presented related to the rationality and convergence. By rationality in a stage game, if the other players' strategies converge to stationary strategy, then the learning algorithm will converge to a stationary strategy and it is the best response to the other players' strategies. Another property is related to the convergence. By this property the learner will necessarily converge to a stationary strategy.

In [106], Bowling and Veloso proposed an algorithm for known repeated game having two players and two actions. Conitzer and Sandholm in [107], extends the work in [106] for all repeated games. It is investigated in [105] that the algorithms considering self-play proposed in [106] and [107] are not convergent against all possible opponents. In Fig. 1.30, 1.31 and 1.34, by Tit-for-Tat algorithm for the Prisoner's Dilemma and game of Chicken offers higher average reward in self-play than the rewards at NE. To avoid encounter the opponent outside the target set, security value  $V_s$  is defined in (1.56).

$$V_s = \max_{\pi_1 \in \{\pi_1\}} \max_{\pi_2 \in \{\pi_2\}} V_e(\pi_1, \pi_2) \quad (1.56)$$

In summary, Powers and Shoham synergistically fuse the FP [39], Bully [108] and Minimax [99] strategy with an aim to create most powerful hybrid algorithm [105].

**Algorithm 1.14: Bully Algorithm**

```

Begin
  An agent  $i$  initiates an election;
  Agent  $i$  sends election message to all agents with higher IDs and waits for feedback;
  If feedback is not OK
    Then agent  $i$  becomes coordinator and sends coordination message to all agents with lower IDs;
  Else
    The agent  $i$  drops out and waits for a coordination message;
  End;
  If an agent receives an election message
    Then immediately sends coordination message subject to that the agent has highest ID;
  Else
    Return OK and starts an election;
  If an agent receives a coordination message
    Then the agent  $i$  treats the sender as the coordination;
End.

```

By FP an agent plays best response against its stationary opponent utilizing the likelihood of other agents to select an action from history. In [105], the best response

$$B_r(\pi) \leftarrow \arg \max_{x \in X} (OV_e(x, \pi)) \quad (1.57)$$

$$\text{where, } X = \{y \in \Pi_1 : EV(y, \pi) \geq \max_{z \in \pi_1} (EV(z, \pi)) - \varepsilon\} \quad (1.58)$$

**Algorithm 1.15: Meta Strategy Algorithm**

```

Begin
  Set strategy = BullyMixed
  Play strategy at time step  $t_1$ ;
  Play strategy at time step  $t_2$ ;
  If strategy=BullyMixed AND  $AVGValue_H < V_{Bully} - \varepsilon_1$  with probability  $P$ 
    Then set strategy=  $B_{r\varepsilon_2}(d_0^t)$  and play;
  End If
  If  $\|d_0^{t_1} - d_{t-t_1}^t\| < \varepsilon_3$ 
    Set best Strategy =  $B_{r\varepsilon_2}(d_0^t)$ ;
  Else if strategy=BullyMixed AND  $AVGValue_H > V_{Bully} - \varepsilon_1$ 
    Then set Best strategy=BullyMixed;
  Else
    Set best Strategy=Best Response;
  End If
  Until end of the game;
  If  $AVGValue_{t-t_0} < V_{security} - \varepsilon_0$ 
    Play Maximin strategy for  $t_3$  time steps
  Else
    Play best Strategy for  $t_3$  time steps;
  End If
End

```

In [105], Bully algorithm (Algorithm 1.14) is extended to handle multiple strategies with equal reward by maximizing opponent's values. In Bully algorithm, a full set of mixed strategies are

$$BullyMixed \leftarrow \arg \max_{x \in X} (OV_e(x, B_r(x))), \quad (1.59)$$

$$X = \{y \in \Pi_1 : V_e(y, B_{r0}(y)) = \max_{z \in \Pi_1} (V_e(z, B_{r0}(z)))\} \quad (1.60)$$

Bully algorithm is one, which is employed to elect a coordinator dynamically among  $m$  number of agents with unique identify (ID) in the field of distributed computing. In distributed artificial intelligence, an algorithm needs to act as a leader (or coordinator). In distributed algorithm, it is assumed that each agent has a unique ID and goal of the algorithm is to find out the agent with highest ID. The Bully algorithm is given in Algorithm 1.14.

Finally, the Minimax strategy is defined as

$$\text{maximin} \leftarrow \arg \max_{\pi_1 \in \Pi_1} \min_{\pi_2} V_e(\pi_1, \pi_2) \quad (1.61)$$

Initial portion of the Algorithm 1.15 is related to coordination/exploration to identify the class of opponent and choose one strategy among three. If neither stationary strategy nor Bully strategy holds, then best response strategy is applied. The algorithm plays with one of the three strategies maintaining the average reward within the security level and improving the maximum strategy when it is too low, where  $d_{t_1}^{t_2}$  refers to the distribution of opponent actions for the period from  $t_1$  to  $t_2$ .  $Avg_n$  represents the average value achieved by the agent during the last  $n$  epoch.  $V_{Bully}$  represents  $V_e(BullyMixed, B_{r0}(BullyMixed))$ .

### ***1.3.5.3.1.1.3 Adapt When Everybody is Stationary, Otherwise Move to Equilibrium (AWESOME)***

As per [109] the minimum requirements of multi-agent system are that agents learn optimally against stationary opponents and converge to a NE when all the agents are playing the same algorithm. WoLF-IGA [110] has been satisfied the above criteria in a two-agent two-action repeated game assuming that the opponents' strategies are observable. In [109], Conitzer and Sandholm proposed *Adapt When Everybody is Stationary, Otherwise Move to Equilibrium* (AWESOME), which is guaranteed to have the above properties for more than two agents and actions assuming that the opponents' actions (not strategies) are observable. In AWESOME, either agents' aim at adapting with the present strategies of the opponent agents or they converge to an already learned NE. Once, both of the above hypotheses are discarded, agents' restart the learning by the AWESOME algorithm.

The basic idea of the AWESOME is straight forward. If other agents' are following stationary strategies, then AWESOME offers its best strategy to the other agents. On the other hand, if other agents' adapt their strategies, then AWESOME follows an already learned



equilibrium. In spite of the above basic idea, the following additional specifications are made before proposing the AWESOME algorithm.

<b>Algorithm 1.16: AWESOME Algorithm</b>
<pre> <b>For</b> <math>i = 1</math> to <math>m</math>     <math>\pi_i^* \leftarrow \text{ComEquStrategy}(i)</math>; //compute equilibrium strategy for agent <math>i</math> <b>End For</b>; <b>Repeat</b>     <b>For</b> <math>i = 1</math> to <math>m</math>         <math>\text{Ini2Empty}(h_i^{\text{prev}})</math>; <math>\text{Ini2Empty}(h_i^{\text{curr}})</math>;     <b>End For</b>;     <math>\text{APPE} \leftarrow \text{true}</math>; // All players playing Equilibrium     <math>\text{APS} \leftarrow \text{true}</math>; // All players stationary     <math>\beta \leftarrow \text{false}</math>; // <math>\beta</math> is true if the equilibrium hypothesis is just rejected     <math>t \leftarrow 0</math>; // denotes the <math>t^{\text{th}}</math> epoch and is initializes to zero in every restart.     <math>\phi \leftarrow \pi_{Me}^*</math>; // refers to the AWESOME player's current strategy     <b>While</b> <math>\text{APPE}</math>         <b>For</b> <math>j = 1</math> to <math>N^t</math>             <math>\text{Play}(\phi)</math>; //Play the strategy <math>\phi</math>             <b>For</b> <math>i = 1</math> to <math>m</math>                 <math>\text{Update}(h_i^{\text{curr}})</math>;             <b>End For</b>;             <b>End For</b>;             <b>If</b> <math>\text{APPE} = \text{false}</math>                 <b>If</b> <math>\beta = \text{false}</math>                     <b>For</b> <math>i = 1</math> to <math>m</math>                         <b>If</b> <math>(\ h_i^{\text{cur}} - h_i^{\text{prev}}\  &gt; \varepsilon_s^t)</math>                             <b>Then</b> <math>\text{APS} \leftarrow \text{false}</math>;                         <b>End If</b>;                     <b>End For</b>;                     <b>Then</b> <math>\beta \leftarrow \text{false}</math>; <math>a \leftarrow \arg \max_a V(a, h_{-Me}^{\text{curr}})</math>;                     <b>If</b> <math>V(a, h_{-Me}^{\text{curr}}) &gt; V(\phi, h_{-Me}^{\text{cur}}) + n  A  \varepsilon_s^{t+1} \mu</math>;                         <b>Then</b> <math>\phi \leftarrow a</math>;                     <b>End If</b>;                 <b>End If</b>;             <b>If</b> <math>\text{APPE} = \text{true}</math>                 <b>For</b> <math>i = 1</math> to <math>m</math>                     <b>If</b> <math>(\ h_i^{\text{cur}} - \pi_i^p\  &gt; \varepsilon_e^t)</math>                         <b>Then</b> <math>\text{APS} \leftarrow \text{false}</math>; <math>\phi \leftarrow \text{RandAct}()</math>; <math>\beta \leftarrow \text{true}</math>;                     <b>End If</b>;                 <b>End For</b>; <b>End If</b>;             <b>For</b> <math>i = 1</math> to <math>m</math>                 <math>h_i^{\text{cur}} \leftarrow h_i^{\text{prev}}</math>; <math>\text{Ini2Empty}(h_i^{\text{curr}})</math>;             <b>End For</b>; <math>t \leftarrow t + 1</math>;         <b>End While</b>; </pre>

- From the beginning it is specified which equilibrium to repeat and restart learning by the AWESOME to avoid confusion.
- After restarting the learning agents forget whatever it learned for simplicity.
- Following one equilibrium strategy among the already computed other equilibrium strategies may lead to divergence from equilibrium. Although, a null hypothesis exists, AWESOME does not reject the hypothesis without sufficient confirmations.
- If an agent selects its own action by its own mixed strategy, then AWESOME rejects the equilibrium strategy to avoid the nonequilibrium strategy.
- After rejecting the equilibrium strategy by AWESOME, randomly an action is chosen from a pool and changes its strategy.
- In AWESOME, except actions the strategies of the remaining agents are not observable. Hence, one needs to specify how to reject an equilibrium strategy which is common to all the agents.

The AWESOME algorithm is given in Algorithm 1.16 [109], and is developed based on the above specifications. It is shown in [109], AWESOME learns best responses against the stationary opponents, and AWESOME converges to NE in self-play.

#### **1.3.5.3.1.1.4 Hyper-Q**

Q-learning is a well-known technique to learn optimal strategies by an agent utilizing the cumulative rewards earned by it in an infinite trial-and-error. Unfortunately, this is not applicable for nonstationary environment with multiple adaptive agents. Most of the multi-agent Q-learner [72], [89], [99] requires knowledge about other agents' rewards and Q-function at each learning epoch. These MAQL algorithms are convergent subject to the following conditions which are not realizable in practice. Firstly, an agent can observe all agents' rewards. Second, all the learning agents follow the same learning algorithms. In [111], Gerald proposed Hyper-Q learning. Hyper-Q learner learns only the mixed strategies and the strategies of the remaining agents are estimated employing the Bayesian inference. Hyper-Q learner aims at overcoming the above limitations of multi-agent system by modeling the environment as repeated stochastic game, where only the remaining agents' actions are observable but the rewards received due to the actions are not observable.

Assuming the Hyper-Q learner is playing in a stochastic Markov game and hence, the reward functions of the agents become the function the available joint actions. Now, instead of choosing the best joint action with probability one (pure strategy), in stochastic Markov game, an agent chooses actions with the best probability (mixed strategy). The Hyper-Q learning update rule is given in (1.62).

$$\Delta Q(S, p_i, p_{-i}) \leftarrow \alpha [r(S, p_i, p_{-i}) + \gamma \max_{a_i} Q(S, p_i', p_{-i}') - Q(S, p_i, p_{-i})], \quad (1.62)$$

where  $p_i$  and  $p'_i$  denote the mixed strategy to select action  $a_i$  and  $a'_i$  at joint state  $S$  and joint next state  $S'$  respectively.  $p_{-i}$  and  $p'_{-i}$  refer to the joint mixed strategy of all the agents except  $i$  to select joint action  $A_{-i}$  and  $A'_{-i}$  at joint state  $S$  and joint next state  $S'$  respectively. It is indicated in [111] that establishing the convergence for the function approximation based Q-learning is more difficult than the same for the Q-learning. If all the agents do explore in a similar exploration strategy, then like Q-learning in Hyper-Q learning, agents may fail to spot the optimal mixed strategy in the strategy space after infinite visit of the joint states. In case of stationary opponent strategy, the stochastic game becomes a MDP with stationary state transitions and stationary rewards. Under the above circumstance Hyper-Q learning converges. Remaining convergence conditions are given in [111]. To estimate opponent strategy Bayesian strategy estimation is done in [111]. By Bayesian estimation one can write

$$P(S|H) = \frac{P(H|S)P(S)}{\sum_{S'} P(H|S')P(S')}, \quad (1.63)$$

where  $H$  refers the history of observed actions,  $S$  and  $S'$  are the discrete state and next state respectively. The outstanding performance of Hyper-Q learning in terms of convergence rate and opponent agent's strategy modeling is tested in the framework of two-player, three-action matrix game Rock-Paper-Scissors game (Example 1.3).

#### **1.3.5.3.1.2 Direct policy search based**

Direct policy search based algorithms are further classified as fixed learning rate and variable learning rate as shown in Fig. 1.36.

##### **1.3.5.3.1.2.1 Fixed learning rate**

The algorithms with fixed learning rates are given below

###### **1.3.5.3.1.2.1.1 Infinitesimal Gradient Ascent (IGA)**

In [112], Singh and Kearns proposed (Infinitesimal Gradient Ascent) IGA based on the positive changes in expected reward of the agents. The IGA is tested in a two player, two-action iterated general-sum-games. It is shown in [112], that agents converge to NE but once they fail to converge to NE they can never reach the NE. Literature shows that agents converge to NE but with restriction and limiting the applicability of the NE [113]. Following the gradient ascent (positive change) is the most common trend in machine learning algorithm. It is not guaranteed that the strategies computed by gradient ascent in two-player, two-action iterated games will converge to NE. However, the average reward is guaranteed to converge NE. For example let there is a two-player, two-action general-sum game. The reward matrix of the row (R) and column (C) player is given in Fig. 1.44.

		$C \rightarrow$	
		$a_1$	$a_2$
$R$	$a_1$	$r_{11}, c_{11}$	$r_{12}, c_{12}$
$\downarrow$	$a_2$	$r_{21}, c_{21}$	$r_{22}, c_{22}$

Fig. 1.44 reward matrix in a two-player two-agent game

Let row player choose action  $a_1$  stochastically with probability  $0 \leq r \leq 1$  and column player choose action  $a_1$  stochastically with probability  $0 \leq c \leq 1$ . The expected payoff of the row and column player is given by (1.64) and (1.65) respectively.

$$V_R(r, c) = r_{11}(rc) + r_{22}((1-r)(1-c)) + r_{12}(r(1-c)) + r_{21}((1-r)c) \quad (1.64)$$

$$V_C(r, c) = c_{11}(rc) + c_{22}((1-r)(1-c)) + c_{12}(r(1-c)) + c_{21}((1-r)c) \quad (1.65)$$

Here, the strategy pair  $(r, c)$  is called NE if and only if, the following two conditions hold.

1) if for any mixed strategy  $r'$  (1.66) holds: i.e.,

$$V_R(r', c) \leq V_R(r, c) \quad (1.66)$$

and 2) for any mixed strategy  $c'$  (1.67) holds: i.e.,

$$V_C(r, c') \leq V_C(r, c). \quad (1.67)$$

Gradient for the row player and column player is given by (1.66) and (1.69) respectively considering  $u = (r_{11} + r_{22}) - (r_{21} + r_{12})$  and  $u' = (c_{11} + c_{22}) - (c_{21} + c_{12})$ .

$$\frac{\delta V_R(r, c)}{\delta r} = cu - (r_{22} - r_{12}) \quad (1.68)$$

$$\frac{\delta V_C(r, c)}{\delta c} = ru' - (c_{22} - c_{12}) \quad (1.69)$$

The mixed strategy update rules are given by (1.70) and (1.71), where  $\eta$  refers to the step size.

$$r \leftarrow r + \eta \frac{\delta V_R(r, c)}{\delta r} \quad (1.70)$$

$$c \leftarrow c + \eta \frac{\delta V_C(r, c)}{\delta c} \quad (1.71)$$

Assuming that the gradient ascent algorithm is a full information game and hence, both the players know the game matrices and the mixed strategies played by the opponent players in the previous step.

By game theory [43] the sequences of strategies over time may never converge to NE. However, in [112], it is shown that the average rewards of both the players always converge. The basic logic behind the analysis of two players acting according to IGA is a two dimensional dynamic system. Considering the infinitesimal step size of  $\eta$  ( $\eta \rightarrow 0$ ), IGA is proposed in [112]. By (1.66)-(1.71) and setting  $\eta \rightarrow 0$  the unconstrained dynamics of the strategy pair can be expressed as a function of time in (1.72).

$$\begin{bmatrix} \frac{\delta r}{\delta t} \\ \frac{\delta c}{\delta t} \end{bmatrix} = \begin{bmatrix} 0 & u \\ u' & 0 \end{bmatrix} \begin{bmatrix} r \\ c \end{bmatrix} + \begin{bmatrix} -(r_{22} - r_{12}) \\ -(c_{22} - c_{21}) \end{bmatrix} \quad (1.72)$$

If the matrix  $U$  given in (1.73) is invertible, then trajectories of the unconstrained strategies of the two-player two-action stochastic game are of having either limit cycle behavior or have divergent nature. The direction and structure of these trajectories depend on the exact values of  $u$  and  $u'$ .

$$U = \begin{bmatrix} 0 & u \\ u' & 0 \end{bmatrix} \quad (1.73)$$

By solving (1.72)  $(r^*, c^*)$  is given in (1.74)

$$(r^*, c^*) = \left[ \frac{c_{22} - c_{21}}{u'}, \frac{r_{22} - r_{12}}{u} \right] \quad (1.74)$$

The average expected reward of the IGA player converges to a NE following one of the following conditions. First condition is that the trajectories of the strategy pair will automatically converge to a NE. The other condition is that the trajectories due to the strategy pair will not converge but the average reward of the two players reward will converge to the NE. To prove these conditions following exclusive and exhaustive cases are considered [114].

1.  $U$  is non-invertible, if  $u \neq 0/u' \neq 0$  or  $u \neq 0, u' \neq 0$ . Such cases can appear in team, zero-sum and general-sum games.

2.  $U$  is invertible, if the Eigen values of (1.75) are imaginary with zero real part, i.e., when  $uu' < 0$ .

$$\begin{bmatrix} 0 & u \\ u' & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \quad (1.75)$$

3.  $U$  is invertible, if its Eigen values are real with zero imaginary part. This condition may appear in team and general-sum games but not in zero-sum games, i.e., when  $uu' > 0$ .

If  $U$  has imaginary Eigen values with zero real part, then based on the location of the center (i.e.,  $(r^*, c^*)$ ) in the two-dimensional plane there are three possibilities.

1. The center  $(r^*, c^*)$  is in the interior of the unit square, 2. Center  $(r^*, c^*)$  is on the boundary of the unit square and 3. Center  $(r^*, c^*)$  is outside of the unit square.

#### 1.3.5.3.1.2.1.2 Generalized Infinitesimal Gradient Ascent (GIGA)

Convex programming is the generalization of the linear programming having several applications in machine learning domain [115]-[117]. The convex programming aims at searching a point  $F$  which maximizes the cost function.

$$c : F \rightarrow \mathbb{R}. \quad (1.76)$$

A convex programming comprises a feasible set  $F \subseteq \mathbb{R}^n$  and a convex cost function given in (1.76). In applications like industrial optimization, nonlinear facility location problems [115], network routing problems [118] and consumer optimization problems [119], the value of the

end product is unknown until end product is created. In [120], an online convex optimization programming is undertaken with identical feasible set but having dissimilar cost functions. An algorithm is proposed in [120], namely generalized IGA (GIGA), which is generally reliable to solve former problems. GIGA is the extension of IGA [112] applicable for more than two agents. Following the definitions of convex, convex programming problem, online convex programming problem and the assumptions made in [110], make clear idea about the online convex optimization. Interestingly, it is shown in [110] that the repeated games are online linear programming. Finally, GIGA tries to minimize regret [120].

### 1.3.5.3.1.2.2 Variable learning rate

The algorithms with variable learning rates are given below.

#### 1.3.5.3.1.2.2.1 Win or Learn Fast-IGA (WoLF-IGA)

Referring from section 1.3.5.3.1.2.1.1 if the center  $(r^*, c^*)$  is inside the unit square with imaginary Eigen values, then the performance of IGA and WoLF-IGA differs in terms of convergence. It is shown in [110], that IGA does not converge if  $(r^*, c^*)$  lies inside the unit square. But WoLF-IGA converges in such situation. The strategy-space where the player wins and loses is also indicated in the proof. In addition, it is shown in [110], that the trajectories due to Eigen values are pricewise elliptical in nature and take a spiral shapes towards the center. In [110], lemmas are proposed assuming that there are only imaginary Eigen values.

By Lemma 6 in [110], if the learning rate for the row player  $(\alpha_r)$  and the learning rate for the column player  $(\alpha_c)$  remain constant, then the trajectory due to strategy pair forms an

ellipse considering  $(r^*, c^*)$  as the center and  $\begin{bmatrix} 0 \\ \frac{\alpha_c |u|}{\sqrt{\alpha_r |u'|}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  are as the axes of the ellipse. In

[110], Lemma 7 concludes that a player is winning if the strategy of the player is moving away from the center. It is also mentioned in [110], that in a two-person, two-action iterated general-sum game both the players follow the WoLF-IGA algorithm with learning rates  $\alpha_{\max}$  and  $\alpha_{\min}$ , then their strategies will converge to a NE subject to

$$\frac{\alpha_{\min}^r \alpha_{\min}^c}{\alpha_{\max}^r \alpha_{\max}^c} < 1. \quad (1.77)$$

#### 1.3.5.3.1.2.2.2 GIGA-Win or Learn Fast (GIGA-WoLF)

The most common problem in MARL, regret and convergence are addressed in gradient-based GIGA-WoLF [121]. GIGA-WoLF is the synergism of GIGA's no-regret property and WoLF-IGA's convergence property [121]. A bound is assigned to test the GIGA-WoLF's regret against the unknown strategy of an opponent agent. For a two-agent, two-action normal-form game, if one agent follows the GIGA-WoLF algorithm and another agent

follows the GIGA algorithm, then their strategies does converge to NE. Both the properties are validated theoretically and experimentally in [121]. In GIGA-WoLF agents must know about the game and should have the model of opponent agent. In almost all the games (except “problematic” Shapley’s game) unlike GIGA’s strategies GIGA-WoLF’s strategies does converge in self-play to equilibrium.

### 1.3.5.3.2 Dynamic

The dynamic algorithms are categorized as equilibrium dependent and independent. The algorithms based on the equilibrium solution concept are listed below.

#### 1.3.5.3.2.1 Equilibrium dependent

The equilibrium dependent MARL algorithms are given below.

##### 1.3.5.3.2.1.1 Nash-Q Learning

Nash Q-learning (NQL) is the extension of Littman's Minimax-Q learning [99]. In other words, it’s the extension of zero-sum stochastic game to the general sum stochastic game. NQL is a MAQL algorithm, which converges under specific conditions. It looks for optimal joint action (NE) in a game. For multiple NEs in the game, the NQL algorithm is fused with other learning techniques to obtain optimal strategies for the entire team. The adopted framework in [89] is stochastic/Markov games. Markov game is the generalization of the MDP with more than two agents. Unlike, zero-sum-game, here in general sum stochastic game, an agent's gain is no longer its opponent agent's loss. In general-sum-game, an agent's reward depends on other agent's choices and hence, the NE is employed. In NE, an agent cannot deviate unilaterally and it is assumed that there is no communication among the agents. Only agents can observe other agents' strategies and rewards. In addition, the state transition probabilities and reward functions are unknown. The NQL algorithm is designed in such a way so that all the agents converge to the NE with restrictions. NQL is guaranteed that all the agents converge to the NE. But for multiple NE solutions it is not guaranteed that all the agents converge the same NE. In [77], Filar and Vrieze proposed that every general-sum discounted stochastic game posses at least one equilibrium point in stationary strategy. Unlike single agent Q-learning [81] and Minimax Q-learning [99], in NQL the Q-learning update rule for agent  $i$  is given in (1.78).

$$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \text{Nash}Q_i(S')], \forall i \quad (1.78)$$

$$\text{where } \text{Nash}Q_i(S') = \pi_1(S') \dots \pi_m(S') \cdot Q_i(S'). \quad (1.79)$$

An online version of NQL and simulation results on Grid game 1 and 2 are given in [71]. The NQL for general sum stochastic game is given in Algorithm 1.17. The convergence proof of Algorithm 1.17 is given [71].

**Algorithm 1.17: NQL in general-sum game**

**Input:** Action  $a_i \in A_i$  at  $s_i \in S_i, \forall i$ , learning rate  $\alpha \in [0,1)$  and discount factor  $\gamma \in [0,1)$ ;  
**Output:** Optimal Q-value  $Q_i^*(S, A), \forall i$ ; //  $S = \{s_i\}_{i=1}^m, A = \{a_i\}_{i=1}^m$ ;  
Initialize:  $Q_i(S, A) \leftarrow 0, \forall i$ ;  
**Begin**  
  **Repeat**  
    Choose an action  $a_i \in A_i, \forall i$ ;  
    Receive immediate reward  $r_i(S, A), \forall i$ ;  
    **Update:**  $Q_i(S, A) \leftarrow (1-\alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \text{Nash}Q_i(S')], \forall i$  and  $S \leftarrow S'$ ;  
     $Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i$ ; //  $\text{Nash}Q_i(S') = \pi_1(S') \dots \pi_m(S') \cdot Q_i(S')$   
  **Until**  $Q_i(S, A), \forall i$  converge;  
**End.**

**1.3.5.3.2.1.2 Correlated-Q Learning (CQL)**

In [72], Greenwald and Hall introduced a MAQL algorithm namely Correlated-Q Learning (CQL). In CQL, Q-value of an agent updates at Correlated equilibrium (CE). CQL generalizes both NQL and FFQ (discussed in section 0) in general-sum stochastic games. If NE and CE do not intersect, then the agent receives less reward at NE compared to the same at CE. Four variant of CE are defined in [72] and the definition of CE is given in Definition 1.18. The algorithm for CQL is given in Algorithm 1.18. Convergence analysis of the above equilibria in the framework of Markov games are done in [72].

**Algorithm 1.18: Correlated-Q Learning**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha \in [0,1)$  and discount factor  $\gamma \in [0,1)$ ;  
**Output:** Optimal Q-value  $Q_i^*(S, A), \forall i$ ;  
Initialize:  $Q_i(S, A) \leftarrow 0, \forall i$ ;  
**Begin**  
  **Repeat**  
    Choose an action  $a_i \in A, \forall i$ ;  
    Receive immediate reward  $r_i(S, A), \forall i$ ;  
    **Update:**  $Q_i(S, A) \leftarrow (1-\alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma V_i(S')], \forall i$  and  $S \leftarrow S'$ ;  
     $V_i(S') = CE(Q_1(S'), Q_2(S'), \dots, Q_m(S')), \forall i$ ;  
     $Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i$ ;  
  **Until**  $Q_i(S, A), \forall i$ ; converge;  
**End.**

**1.3.5.3.2.1.3 Asymmetric-Q Learning (AQL)**

In [122], Ville proposed Asymmetric-Q Learning (AQL) algorithm, where an agent leads the follower agents by providing the information about the follower agents' strategy to the follower agents. AQL offers the following benefits:



**Algorithm 1.19: Asymmetric-Q learning for the leader**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha \in [0, 1)$  and discount factor  $\gamma \in [0, 1)$ ;

**Output:** Optimal Q-value  $Q_i^*(S, A), \forall i$ ;

Initialize:  $Q_i(S, A) \leftarrow 0, \forall i$ ;

**Begin**

**Repeat**

Choose an action  $a_i \in A, \forall i$ ;

Receive immediate reward  $r_i(S, A), \forall i$ ;

**Update:**  $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma V_i(S')], \forall i$  and  $S \leftarrow S'$ ;

$V_i(S') = SE(Q_1(S'), Q_2(S'), \dots, Q_m(S')), \forall i$ ;

$Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i$ ;

**Until**  $Q_i(S, A), \forall i$ ; converge;

**End.**

- In each state the leader has unique equilibrium point.
- Asymmetric Q-learner always achieves the pure strategy NE very fast. Though mixed strategy NE exists.
- The AQL algorithm enjoys the lower space and computational requirements than conventional algorithms.

**Algorithm 1.20: Asymmetric-Q learning for the follower**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha \in [0, 1)$  and discount factor  $\gamma \in [0, 1)$ ;

**Output:** Optimal Q-value  $Q_i^*(S, A), \forall i$ ;

Initialize:  $Q_i(S, A) \leftarrow 0, \forall i$ ;

**Begin**

**Repeat**

Choose an action  $a_i \in A, \forall i$ ;

Receive immediate reward  $r_i(S, A), \forall i$ ;

**Update:**  $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \max_{A'} Q_i(S', A')]$  and  $S \leftarrow S'$ ;

$Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i$ ;

**Until**  $Q_i(S, A), \forall i$ ; converge;

**End.**

In [122], the existing MAQL algorithms are divided in three clusters. One is the methods utilizing the direct gradients of agents' value function. Second one is the methods that estimate the value functions and then use this estimate to compute equilibrium of the process. Last one is the use of direct policy gradients. The AQL algorithm is developed by Stackelberg equilibrium (SE) [44]. The algorithm for the leader and the follower are given in Algorithm 1.19 and 1.20. The leader agents are capable to maintain all the agents Q-tables. However, the follower agents are not able to maintain all the agents' Q-values and hence, they just

maximize their reward. Experiments are performed in the grid world environment to demonstrate the superiority of the AQL algorithm.

#### 1.3.5.3.2.1.4 Friend-or-Foe Q-learning

In [123], Littman proposed one variant of MAQL algorithm namely Friend-or-Foe Q-learning (FFQ) algorithm with a strong convergence guarantee compared to NE in the framework of general-sum stochastic game, where agents are instructed to consider other agents' either as a friend or foe. Though, FFQ learning is an improvement over the Nash-Q. In FFQ, two variants' of NE are employed. One is adversarial equilibrium and another is coordination equilibrium. In Minimax-Q (zero-sum-game) [99] all the equilibria are adversarial equilibrium. However, in general-sum game all the equilibria are not coordination equilibrium. Coordination equilibrium provides the highest possible reward of agent  $i, i \in [1, m]$  is given in (1.80) [123].

$$R_i(\pi_1, \dots, \pi_m) = \max_{a_1 \in A_1, \dots, a_m \in A_m} R_i(a_1, \dots, a_m) \quad (1.80)$$

Except fully cooperative game, coordination equilibrium need not always exist. The adversarial and coordination equilibria are explained in Fig. 1.25. The difference between the Nash operation and the maximization or minimax operations is that the latter two have unique solutions. However, the Nash operation offers two variant of solutions: adversarial and coordination equilibrium depending on the problem type.

Two Propositions are proposed and proved in [123]. As per the Propositions if a one-stage game has a coordination/adversarial equilibrium, then all of the coordination/adversarial equilibrium have same value. There exist two conditions for convergence [123]. In summary, the conditions statement is that for a game there exists either adversarial/coordination equilibrium. Later two stronger conditions of convergence are proposed in [89], [123]. These conditions can be summarized as follow. There exists an adversarial/coordination equilibrium in a game and every game is defined by the Q-functions adapted during the learning phase. The later conditions are also not sufficient to guarantee convergence. Hu and Wellman in [89], states two theorems that by the later two conditions Nash-Q converges to Nash-Q equilibrium until all the equilibria are adapted during the learning phase are unique. Also by the later two conditions Nash-Q converges to NE, until the required equilibria is employed in (1.82).

Now, in FFQ algorithm  $Nash Q_i(S^l)$  are given in (1.81) and (1.82) for Friend-Q (coordination equilibrium) and Foe-Q (adversarial equilibrium) respectively.

$$Nash Q_i(S^l) = \max_{A'} \sum_{A'} P(A') Q_i(S^l, A') \quad (1.81)$$

$$Nash Q_i(S^l) = \max_{a_1, \dots, a_x} \min_{a_1, \dots, a_y} \sum_{\forall A''} P(A'') Q_i(S^l, A''). \quad (1.82)$$

where,  $A' = \langle a_1, \dots, a_m \rangle$ ,  $A'' = \langle a_1, \dots, a_x, a_1, \dots, a_y \rangle$  and  $y$  refers to the number of foes (opponent agents). The convergence of FFQ learning is subject to that the Nash operator is max or minimax operator [123]. Like NQL, for simulation purpose two grid games are employed [89], [123] in FFQ. Six different variants' of opponents are described in [123]. Though, Nash-Q and FFQ cannot fix the problem of finding equilibria, if neither coordination nor adversarial equilibrium exists. The algorithm for FFQ learning is given in Algorithm 1.21.

<b>Algorithm 1.21: Friend-or Foe-Q Learning</b>
<p><b>Input:</b> Action <math>a_i \in A_i</math> at state <math>s_i \in S_i</math> for all the agents learning rate <math>\alpha \in [0, 1)</math> and discount factor <math>\gamma \in [0, 1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q_i^*(S, A), \forall i</math>;</p> <p>Initialize: <math>Q_i(S, A) \leftarrow 0, \forall i</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Choose an action <math>a_i \in A, \forall i</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(S, A), \forall i</math>;</p> <p style="padding-left: 40px;">Evaluate <i>Nash</i> <math>Q_i(S')</math>, <math>\forall i</math> by (1.81) and (1.82) respectively for Friend-Q and Foe-Q</p> <p style="padding-left: 40px;"><b>Update:</b> <math>Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \text{Nash } Q_i(S')], \forall i</math> and <math>S \leftarrow S'</math>;</p> <p style="padding-left: 40px;"><math>Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i</math>;</p> <p style="padding-left: 40px;"><b>Until</b> <math>Q_i(S, A), \forall i</math>; converge;</p> <p><b>End.</b></p>

#### 1.3.5.3.2.1.5 Negotiation-based Q-learning

In [124], Hu *et al.* proposed a MARL without mutually sharing their value functions. Authors in [124] mentioned that mutual exchange of value function is impractical because of the local restriction of the system and privacy of the agents in case of distributed agents. Doing so appears impossible to evaluate equilibrium in a one short game. In the above circumstances, authors propose a multi-step negotiation process to evaluate three types of pure strategies: PSNE, equilibrium-dominating strategy profile (EDNP) and nonstrict EDNP, instead of computing the computationally expensive MSNE. It is also shown that above mentioned three strategies are symmetric Meta strategies. Fusing the above techniques Hu *et al.* proposed negotiation-based Q-learning (NegoQ) in [124].

NegoQ deals with pure strategy equilibrium. However, in some game (e.g., rock-paper-scissor game, Example 1.3) PSNE does not exist. Another hindrance is that a strategy may be Pareto dominated and so not a PSNE. In Prisoners' Dilemma, only one PSNE (C, C) exist as shown in Fig. 1.34. Though (D, D) is the better choice, but (D, D) is the Pareto optimal and not a PSNE. In this regard, a strategy profile Pareto dominates NE, i.e., EDNP is defined in Definition 1.19.

**Definition 1.19:** In an  $m$ -agent ( $m \geq 2$ ) normal-form game, a joint action  $A \in \{A\}$  is an EDNP if there is a PSNE  $A_N \in \{A\}$  such that

$$Q_i(A) \geq Q_i(A_N), i = [1, m]. \quad (1.83)$$

By Definition 1.19 one can conclude that each agent following EDNP receives more reward than the same of by following PNSE.

**Algorithm 1.22: Negotiation to evaluate the PSNE for agent  $i$  in a Normal-form game**

**Input:** Action  $a_i \in A_i$  only for the agent  $i \in [1, m]$  and  $Q_i(A)$ ; //  $A \in \{A\} = \times_{i=1}^m A_i$

**Output:** PSNE set  $\{A_N\}$ ;

Initialize:  $\{A_N\} \leftarrow \varnothing$  ;

Evaluate maximal reward set for agent  $i$   $MS_i$ ;

**For all**  $A_{-i} \in \{A_{-i}\}$

$a_i = \arg \max_{a_i'} Q(a_i', A_{-i})$ ;

$\{A_N\} \leftarrow \{A_N\} \cup \{a_i, A_{-i}\}$ ;

**End For**

**For all joint action**  $A \in \{A_N\}$

Ask remaining agents that is  $\{A_N\}$  includes  $A$ ; a

**If**  $\{A_N\}$  does not include  $A$  **then**

$\{A_N\} \leftarrow \{A_N\} \setminus \{A\}$ ;

Inform other agents to exclude  $A$  from their  $\{A_N\}$  sets

**End If**

**End For**

**For all joint action**  $A'$  received from remaining agents

**If**  $A'$  belongs to  $MS_i$  **then**

Response as yes to the remaining agents;

**else**

Response as no to the remaining agents;

**End If**

**End For**

Before defining the nonstrict EDSP a normal-form game with the same is given in Fig. 1.45. In Fig. 1.45, there are two PSNE:  $(a_1, b_1)$  and  $(a_2, b_2)$ . It is apparent that the strategy profile  $(a_1, b_3)$  and  $(a_3, b_3)$  provide a greater reward to A than  $(a_1, b_1)$  and a greater reward to B than  $(a_2, b_2)$  respectively. So, the priority of  $(a_1, b_3)$  and  $(a_3, b_3)$  are more than  $(a_1, b_1)$  for A and  $(a_2, b_2)$  for B respectively. Hence, for A and B the nonequilibrium strategy profile  $(a_1, b_3)$  and  $(a_3, b_3)$  partially dominate the existing PSNE. In [124], Hu *et al.* defined them as nonstrict EDSP as given in Definition 1.20.

**Definition 1.20:** In an  $m$ -agent ( $m \geq 2$ ) normal-form game, a joint action  $A \in \{A\}$  is an EDNP if there is a PSNE  $A_N^i \in \{A\}$  such that

$$Q_i(A) \geq Q_i(A_N^i), i = [1, m]. \quad (1.84)$$

		B →		
		$b_1$	$b_2$	$b_3$
A ↓	$a_1$	<b>(20,40)</b>	(4,22)	(29,30)
	$a_2$	(18,9)	<b>(36,19)</b>	(7,4)
	$a_3$	(17,26)	(15,38)	(27,38)

Fig. 1.45 Nonstrict EDNP in Normal-form game

<b>Algorithm 1.23: Negotiation to evaluate the nonstrict EDSP for agent <math>i</math> in a Normal-form game</b>	
<b>Input:</b>	Action $a_i \in A_i$ only for the agent $i \in [1, m]$ , $\{A_N\}$ from Algorithm 1.13 and $Q_i(A)$ ;
	// $A \in \{A\} = \times_{i=1}^m A_i$
<b>Output:</b>	nonstrict EDSP set $\{A_{nP}\}$ ;
	Initialize: $\{A_{nP}\} \leftarrow \varnothing$ ;
	$\{X\} \leftarrow A \setminus \{A_N\}$ ;
<b>For each</b>	PSNE $A_N \in \{A_N\}$
<b>For each</b>	joint action $A \in \{X\}$ ;
<b>If</b>	$Q_i(A) \geq Q_i(A_N)$ <b>then</b>
	$\{X\} \leftarrow \{X\} \setminus \{A\}$ ;
	$\{A_{nP}\} \leftarrow \{A_{nP}\} \cup \{A\}$ ;
<b>End If</b>	
<b>End For</b>	
<b>End For</b>	
<b>For all</b>	joint action $A \in \{A_{nP}\}$
	Ask remaining agents that is $A_{nP}$ includes $A$ ;
<b>If</b>	answer is no <b>then</b>
	$\{A_{nP}\} \leftarrow \{A_{nP}\} \setminus \{A\}$ ;
<b>End If</b>	
<b>End For</b>	
<b>For all</b>	joint action $A'$ received from remaining agents
<b>If</b>	$A'$ belongs to $A_{nP}$ <b>then</b>
	Response as yes to the remaining agents;
<b>else</b>	
	Response as no to the remaining agents;
<b>End If</b>	
<b>End For</b>	

In the multistep negotiation process of computing the above mentioned three pure strategy profiles, agents exchange their preferences of joint actions among themselves in terms of binary answers. An illustration of the multistep negotiation process is given in Fig. 1.46. In Fig. 1.46, 'Y' and 'N' represent as yes and no respectively. A joint action is pure strategy profile if and only if both the agents' responses are yes. The negotiation process comprises of three types: 1) negotiation for finding the set of PSNE, 2) negotiation for finding the set of nonstrict EDSP and 3) negotiation for choosing equilibrium (joint action) from the sets obtained by the above two steps. Evaluation of EDSP follows from the evaluation of the

nonstrict EDSP, as EDSP is a special case of nonstrict EDSP. The negotiation to evaluate the PSNE for agent  $i$  is given in Algorithm 1.22. The negotiation to evaluate the nonstrict EDSP for agent  $i$  is given in Algorithm 1.23. Based on the Negotiation algorithms (Algorithm 1.22 and 1.23) to evaluate the pure strategy profiles the NegoQ algorithm for a Markov game is given in Algorithm 1.24. The superiority of the Algorithm 1.24 is tested in grid-world maps over the state-of-the-art reference algorithms.

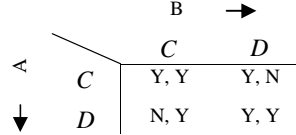


Fig. 1.46 multistep negotiation process between agent A and B

**Algorithm 1.24: Negotiation-Q learning for agent  $i$  in a Markov game**

**Input:** Joint action space  $\{A\}$ , number of agents'  $m$ , state space  $\{S\}$ , learning rate  $\alpha$ , discounting factor  $\gamma$  and exploration rate  $\varepsilon$ ;

**Output:** Optimal joint Q-value  $Q_i^*(S, A)$ ;

Initialize:  $Q_i(S, A) \leftarrow 0$ ;

**Begin**

**Repeat**

Negotiate with remaining agents employing Algorithm 1.13 and 1.14;

Select the pure strategy equilibrium  $A'$  using  $\varepsilon$ -greedy;

Receive experience tuple  $\langle S, A, r_i(S, A), S' \rangle$ ; //  $r_i(S, A)$  and  $S'$  are the immediate reward and next joint state

**Update:**  $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma Q_i(S', A')]$ , and  $S \leftarrow S'$ ;

$Q_i^*(S, A) \leftarrow Q_i(S, A)$ ;

**Until**  $Q_i(S, A)$  converges;

**End.**

#### 1.3.5.3.2.1.6 MAQL with equilibrium transfer

Hu *et al.* in [125] identified, that agents' evaluate the same equilibrium (NE or CE) at a joint state for different one-shot games. Here, two equilibria are declared as same if and only if the Euclidian distance between the probability distribution of the strategies is less than a predefined threshold. Reuse of the previously computed equilibrium (or equilibrium transfer) decreases the convergence time of the equilibrium-based MAQL decreases with negligible transfer loss. Suppose  $G$  and  $G'$  are two one-shot games visits the same joint state  $S$ . Now, the Euclidian distance between the equilibrium strategy  $p$  of  $G$  and  $p'$  of  $G'$  are given in (1.85) and (1.86) respectively for NE and CE.

$$d^{NE}(p, p') = \sqrt{\sum_{i=1}^n \sum_{a_i \in A_i} (p_i(a_i) - p'_i(a_i))^2} \quad (1.85)$$

$$d^{CE}(p, p') = \sqrt{\sum_{A \in \{A\}} (q(A) - q'(A))^2} \quad (1.86)$$

If the  $d^{NE}(p, p')$  or  $d^{CE}(p, p')$  is smaller than a threshold, then  $p$  and  $p'$  are considered as identical in  $G$  and  $G'$ . Hence, by equilibrium transfer one can directly use  $p$  in  $G'$ . As computation of equilibrium is more expensive than checking, hence, there is a significant saving in computational cost. Hu *et al.* in [125], measures the equilibrium transfer loss and based on that loss the equilibrium transfer condition is defined. Let  $p^*$  and  $q^*$  denote the NE and CE of  $G$  and  $G'$  respectively. Now, loss because of transferring the equilibrium  $p^*$  and  $q^*$  from  $G$  to  $G'$  is given by (1.87) and (1.88) respectively.

**Algorithm 1.25: Equilibrium transfer-based MAQL**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha \in [0,1)$ , discount factor  $\gamma \in [0,1)$ , exploration factor  $\varepsilon$ , threshold of transfer loss  $\tau$ ,  $G_c$  be the one-shot game at joint state  $S$  and  $p^*$  previously computed equilibrium at  $S$ ;

**Output:** Optimal joint Q-value  $Q_i(S, A), \forall i; // S \in \{S\} = \times_{i=1}^m S_i$  and  $A \in \{A\} = \times_{i=1}^m A_i$

Initialize:  $Q_i(S, A) \leftarrow 0, \forall i$ ;

**Repeat**

**If** joint state  $S$  has been visited

**then** evaluate maximum utility loss  $\varepsilon^\Omega, \Omega \in \{NE, CE\}$  for transferring to  $G_c$ ;

**Else**

$\varepsilon^\Omega \leftarrow +\infty$ ;

**End if**

**If**  $\varepsilon^\Omega > \tau$

Then evaluate  $p^*$  for  $G_c$ ;

**Else**

**Reuse**  $p^*$  in  $G_c$ ;

**End if**

Select joint action,  $A$  sampled from  $p^*$ ;

Receive experience  $(S, A, r_i, S')$ ,  $\forall i$ ;

Evaluate equilibrium  $p'$  for the next joint state  $S'$ ;

**Evaluate**  $V_i(S') \leftarrow$  expected value of  $p'$  in  $S'$ ,  $Q_i(S, A) \leftarrow (1-\alpha)Q_i(S, A) + \alpha(r_i + \gamma V_i(S'))$  and  $S \leftarrow S'$ ;

**Until**  $Q_i(S, A), \forall i$  converge;

$$\varepsilon^{NE} = \max_{i \in N} \max_{a_i \in A_i} (Q_i^{G'}(a_i, p_{-i}^*) - Q_i^{G'}(p^*)) \quad (1.87)$$

$$\varepsilon^{CE} = \max_{i \in N} \max_{a_i \in A_i} \max_{a'_i \in A'_i} \sum_{A_{-i}} q^*(a_i, A_{-i}) \times [Q_i^{G'}(a'_i, A_{-i}) - Q_i^{G'}(a_i, A_{-i})] \quad (1.88)$$

Here,  $Q_i^{G'}$  refers to the Q-value of agent  $i$  in  $G'$ .

Now the transfer loss condition for NE,  $p^*$  for an agent  $i$  is given by

$$\begin{aligned}
Q_i^{G'}(p^*) + \varepsilon^{NE} &\geq Q_i^{G'}(p^*) + \max_{a_i \in A_i} (Q_i^{G'}(a_i, p_{-i}^*) - Q_i^{G'}(p^*)) \\
&= Q_i^{G'}(p^*) + \max_{a_i \in A_i} Q_i^{G'}(a_i, p_{-i}^*) - Q_i^{G'}(p^*) \\
&= \max_{a_i \in A_i} Q_i^{G'}(a_i, p_{-i}^*).
\end{aligned} \tag{1.89}$$

Similarly, for CE following condition can be derived

$$\sum_{A_{-i}} q^*(a_i, A_{-i}) \times Q_i^{G'}(a_i, A_{-i}) + \varepsilon^{CE} \geq \sum_{A_{-i}} q^*(a_i, A_{-i}) \times Q_i^{G'}(a_i', A_{-i}). \tag{1.90}$$

The algorithm for equilibrium transfer-based MAQL is given in Algorithm 1.25. Superiority of Algorithm 1.25 is tested in Grid World game, Wall game and Soccer game.

### 1.3.5.3.2.2 Equilibrium independent

Equilibrium independent MARL algorithms again categorized based on the learning rate selection given below.

#### 1.3.5.3.2.2.1 Variable learning rate

RL Algorithms with variable learning rate are given below.

##### 1.3.5.3.2.2.1.1 Win or Learn Fast Policy hill-climbing (WoLF-PHC)

In [126], Bowling and Veloso proposed WoLF policy hill climbing algorithm for stochastic game in presence of other adaptive agents, satisfying rationality and convergence. *Rationality* indicates that all agents' policies converge to stationary policies and then the learning algorithm will converge to a stationary policy, which is best response to their policies [126]. The convergence property states that agents necessarily converge to a stationary policy. Also, if all agents are rational and convergent, then it is guaranteed to converge NE. The learning algorithms in [32] and [127], either converges to a sub-optimal policy or does not converge. Proposed WoLF is based on the principle of "*learn quickly while losing and learn slowly while winning*".

Policy hill-climbing (PHC) is a straight forward extension of Q-learning to handle mixed strategies. The PHC algorithm is given in Algorithm 1.26. PHC learns the most recent mixed strategy. The updating of the mixed strategy in PHC is done by selecting the highest valued action as per the learning rate  $\delta \in (0,1]$ . For  $\delta = 1$  the algorithm behaves as single agent Q-learning. Both Q-values and the strategy is convergent following single agent Q-learning.

The main contribution of the proposed algorithm in [126] is the extension of PHC algorithm by employing a variable learning rate and the WoLF principle. In variable learning rate, the learning rate is used by the learning algorithm and is tuned in such a way so that the rationality is maintained. The WoLF principle motivates to learn quickly while losing and



**Algorithm 1.26: Policy hill-climbing (PHC)**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha \in [0,1)$  and discount factor  $\gamma \in [0,1)$ ;

**Output:** Optimal policy  $\pi_i^*(S, A)$ ;

Initialize:  $Q_i(S, A) \leftarrow 0$  and  $\pi_i(S, A) \leftarrow \frac{1}{|A_i|}$ ;

**Begin**

**Repeat**

Choose an action  $a_i \in A$  with probability  $\pi_i(S, A)$ ;

Receive immediate reward  $r_i(S, A)$ ;

**Update:**  $Q_i(S, A) \leftarrow (1-\alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \max_{A'} Q_i(S', A')]$ ,  $S \leftarrow S'$  and

$$\pi_i(S, A) \leftarrow \pi_i(S, A) + \begin{cases} \delta, & \text{If } A = \arg \max_{A'} Q(S, A') \\ \frac{-\delta}{|A_i| - 1}, & \text{otherwise} \end{cases}$$

$\pi_i^*(S, A) \leftarrow \pi_i(S, A)$ ;

**Until**  $\pi_i^*(S, A)$  converges;

**End.**

**Algorithm 1.27: Win or Learn Fast-PHC (WoLF-PHC)**

**Input:** Action  $a_i \in A_i$  at state  $s_i \in S_i$  for all the agents learning rate  $\alpha, \delta_l > \delta_w$  and discount factor  $\gamma \in [0,1)$ ;

**Output:** Optimal policy  $\pi_i^*(S, A)$ ;

Initialize:  $C(S) \leftarrow 0$ ,  $Q_i(S, A) \leftarrow 0$  and  $\pi_i(S, A) \leftarrow \frac{1}{|A_i|}$ ;

**Begin**

**Repeat**

Choose an action  $a_i \in A$  with probability  $\pi_i(S, A)$ ;

Receive immediate reward  $r_i(S, A)$ ;

**Update:** average policy  $\bar{\pi}$ ,  $C(S) \leftarrow C(S) + 1$ ,  $S \leftarrow S'$ ,

$$\bar{\pi}(S, A') \leftarrow \bar{\pi}(S, A') + \frac{1}{C(S)} [\pi_i(S, A') - \bar{\pi}(S, A')]$$

$$\text{and } \pi_i(S, A) \leftarrow \pi_i(S, A) + \begin{cases} \delta, & \text{If } A = \arg \max_{A'} Q(S, A'); \\ \frac{-\delta}{|A_i| - 1}, & \text{otherwise;} \end{cases}$$

$$\delta = \begin{cases} \delta_w, & \text{If } \sum_A \pi_i(S, A) Q_i(S, A) > \sum_A \bar{\pi}(S, A) Q_i(S, A) \\ \delta_l, & \text{otherwise} \end{cases};$$

$\pi_i^*(S, A) \leftarrow \pi_i(S, A)$ ;

**Until**  $\pi_i^*(S, A)$  converges;

**End.**

slowly while winning [126]. The WoLF-PHC algorithm employs two learning rate losing learning rate  $\delta_l$  and winning learning rate  $\delta_w$  where  $\delta_l > \delta_w$ . The winning/losing situation of the agent is determined by contrasting the current reward and the average reward taken over the time. If agent is losing, then larger learning rate  $\delta_l$  is employed. The WoLF-PHC [126] algorithm is given in Algorithm 1.27. The convergence and rationality of the WoLF-PHC algorithm is tested in Matrix games, Grid world game, and Soccer game. In all frameworks, WoLF-PHC outperforms reference algorithms.

#### 1.3.5.3.2.2.1.2 Policy Dynamic based Win or Learn Fast (PD-WoLF)

IGA [112] learner converges to NE rationally but they are not convergent to NE for all the general-sum games. Later IGA was extended to WoLF-IGA in [110] and its convergence proof is shown in [110] for a  $2 \times 2$  game assuming agents know the equilibrium policies of other agents. In [128], Banerjee and Peng did experimental based comparisons of the WoLF and PD-WoLF to establish the superiority of the PD-WoLF both in the bimatrix and the general-sum games.

From section 1.3.5.3.1.2.1.1 considering the sub case of purely imaginary Eigen values,  $U$  and the center  $(r^*, c^*)$  is within the unit square. The solution  $r(t)$  of (1.72) for unconstraint dynamics [129] is given in (1.91), where the value of  $B$  and  $\phi$  depends on the initial values of  $\alpha, \beta$ .

$$r(t) = B\sqrt{u} \cos(\sqrt{uu}t + \phi) + r^* \quad (1.91)$$

PD-WoLF criteria for a row player (agent) are given by (1.92).

$$\alpha_r(t) = \begin{cases} \alpha_{\min}, & \text{if } \Delta_t \Delta_t^2 < 0 \\ \alpha_{\max}, & \text{otherwise} \end{cases} \quad (1.92)$$

where  $\Delta_t = r_t - r_{t-1}$  and  $\Delta_t^2 = \Delta_t - \Delta_{t-1}$ . It is apparent that (1.92) is independent of other agents' policies.

#### 1.3.5.3.2.2.2 Fixed learning rate

MARL Algorithms with fixed learning rate are given below.

##### 1.3.5.3.2.2.2.1 Non-Stationary Converging Policies (NSCP)

One major shortcoming of MAQL is the assumption that the environment is stationary. In [130], Michael and Jeffrey proposed the Non-Stationary Converging Policies (NSCP), where agents are not interested in converging to an equilibrium rather they search for the best response policy for the non-stationary opponents. NSCP predicts the opponents' non-stationary strategy with precision and act by its best-response strategy with respect to the opponents in the well-known test bench of general-sum stochastic games (game with multiple joint states) or matrix games (game with one joint state). The MAQL algorithms [71], [72], [78], [89], [99] and [123] either converge to NE or CE. By [131], the equilibrium-based MAQL algorithms are problematic, as the learning stops at the equilibrium point and the

equilibrium point is necessarily not a goal point. Also an additional problem arises in presence of multiple equilibria. The NSCP algorithm aims at adapting an optimal reward considering the presence of other agents. In [132], an agent converges to best response strategy subject to stationary opponents in two-player general-sum stochastic games. The NSCP algorithm is given in Algorithm 1.28. Simulation results validate the superior performance of the NSCP with respect to reference algorithms.

<b>Algorithm 1.28: Non-Stationary Converging Policies</b>
<p><b>Input:</b> Action <math>a_i \in A_i</math> at state <math>s_i \in S_i, \forall i, i \in [1, m]</math>, learning rate <math>\alpha \in [0, 1)</math> and discount factor <math>\gamma \in [0, 1)</math>;</p> <p><b>Output:</b> Optimal Q-value <math>Q_i^*(S, A), \forall i</math>;</p> <p>Initialize: <math>Q_i(S, A) \leftarrow 0, \forall i</math> and <math>\pi^i(S, A) = \frac{1}{ A }</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>Repeat</b></p> <p style="padding-left: 40px;">Observe the actions taken by all the agents <math>A \in \{A\}</math>;</p> <p style="padding-left: 40px;">Receive immediate reward <math>r_i(S, A), \forall i</math>;</p> <p style="padding-left: 40px;"><b>Update:</b> other agents' strategy <math>\pi_{-i}(S, A) = \frac{1}{ A }, \forall i</math>;</p> <p style="padding-left: 40px;">Select best-response strategy <math>\pi_i^{br}(S, A)</math></p> <p style="padding-left: 40px;">that maximizes <math>BR(S') = \sum_{a_1} \sum_{a_2} \dots \sum_{a_m} \pi_i^{br}(S', a_i) \cdot \prod_{\forall -i} \hat{\pi}_i(S', a_i) \cdot Q_i(S', A)</math>;</p> <p style="padding-left: 40px;"><b>Update:</b> Q-values using the following rules</p> <p style="padding-left: 40px;"><math>Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma BR(S')]</math> and <math>S \leftarrow S'</math>;</p> <p style="padding-left: 40px;"><b>Until</b> <math>Q_i(S, A), \forall i</math>; converges;</p> <p style="padding-left: 40px;">Obtain <math>Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i</math>;</p> <p><b>End.</b></p>

#### 1.3.5.3.2.2.2 Extended Optimal Response Learning (EXORL)

The zero-sum stochastic game proposed by Littman in [99] was extended to general sum-stochastic game by Hu and Wellman in [89] and agents converge to NE in stochastic games by these algorithms. On the contrary, in [89] and [99], agents always try to converge to NE ignoring strategies of other agents. Further, all the agents must agree upon to select a NE in presence of multiple NEs. Thus the algorithms proposed in [89] and [99] are not adaptable in the above sense. In [133], Nobuo and Akira extended optimal response to Extended Optimal Response Learning (EXORL), where agents converge to NE subject to adaptability of other agents. Similar to NQL [89], in EXORL, an agent maintains all agents' Q-tables assuming that it can observe other agents' state-action and reward. EXORL aims at realizing a policy which is optimal response to other agents' policies, where remaining agents are adaptable and attain NE. The EXORL algorithm is given in Algorithm 1.29. JAL [78] learns Q-value due to its

own action and estimates teammates' strategy. Let  $\pi_i$  be the strategy of agent  $i$  at state  $S$  which maximizes (1.93).

$$Q_i(S, \pi_i) = (\pi_i)^T Q_i(S) \bar{\pi}_{-i}(S) \quad (1.93)$$

where,  $\bar{\pi}_{-i}(S)$  refers to estimate of all agents' joint policy except agent  $i$ . Now, if a policy diverges from NE, then the policy may not be suitable to estimate the remaining agents' strategy. This problem is addressed in [133] and the update rule is given by (1.94) and (1.95) tuning the value of  $\rho$ .

$$Q_i(S, \pi_i) = (\pi_i)^T Q_i(S) \bar{\pi}_{-i}(S) - \rho \sigma(S, \pi_i), \quad (1.94)$$

$$\text{where } \sigma(S, \pi_i) = \max_{\pi_{-i}} [(\pi_i)^T Q_{-i}(S) \pi_{-i}] - (\pi_i)^T Q_{-i}(S) \bar{\pi}_{-i}(S), \quad (1.95)$$

here  $\sigma(S, \pi_i)$  refers to the possible increase in expected discounted reward of agent  $i$ . Hence, to maximize left part of (1.95) agent  $i$  has to maximize the first component of right part and also minimizes the second component of the right part. Also (1.95) is a piece-wise linear concave function and it has a sole maximal point. It is shown in [133] that by EXORL an agent plays well subject to that the opponent agents play fixed policy considering small value of  $\rho$ . The EXORL is verified in Matching Pennies, Presidency Game [77] and Battle of sexes game in [133].

#### Algorithm 1.29: EXORL for agent $i$

**Input:** Action  $a_i \in A_i$  at  $s_i \in S_i, \forall i$ , learning rate  $\alpha \in [0, 1)$  and discount factor  $\gamma \in [0, 1)$ ;

**Output:** Optimal Q-value  $Q_i^*(S, A), \forall i$ ; //  $S = \{s_i\}_{i=1}^m, A = \{a_i\}_{i=1}^m$ ;

Initialize:  $Q_i(A, A) \leftarrow 0, \pi_i(S, a_i) \leftarrow \frac{1}{|A_i|} \forall i, \bar{\pi}_{-i}(S, A_{-i}) \leftarrow \frac{1}{|\times_{j=1, j \neq i}^m A_j|}$ ;

**Begin**

**Repeat**

Choose an action  $a_i \in A_i, \forall i$ ;

Receive immediate reward  $r_i(S, A), \forall i$ ;

**Update:**  $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma Q_i(S', A)], \forall i, S \leftarrow S'$

and  $\bar{\pi}_{-i}(S) \leftarrow (1 - \beta)\bar{\pi}_{-i}(S) + \beta \cdot \bar{\pi}_{-i}(S')$ ; //  $\bar{\pi}_{-i}(S') = \begin{cases} 1 & \text{if } A_{-i} = A'_{-i} \\ 0 & \text{otherwise} \end{cases}$

**Until**  $Q_i(S, A), \forall i$  converge;

Obtain  $Q_i^*(S, A) \leftarrow Q_i(S, A), \forall i$ ;

**End.**

### 1.3.6 Coordination and planning by MAQL

In the present thesis, for multi-robot coordination and planning without any communication among the agents we focus on the equilibrium-based MAQL as explained in section 1.3.5.3.2.1. Because of the absence of communication among the agents, each agent needs to

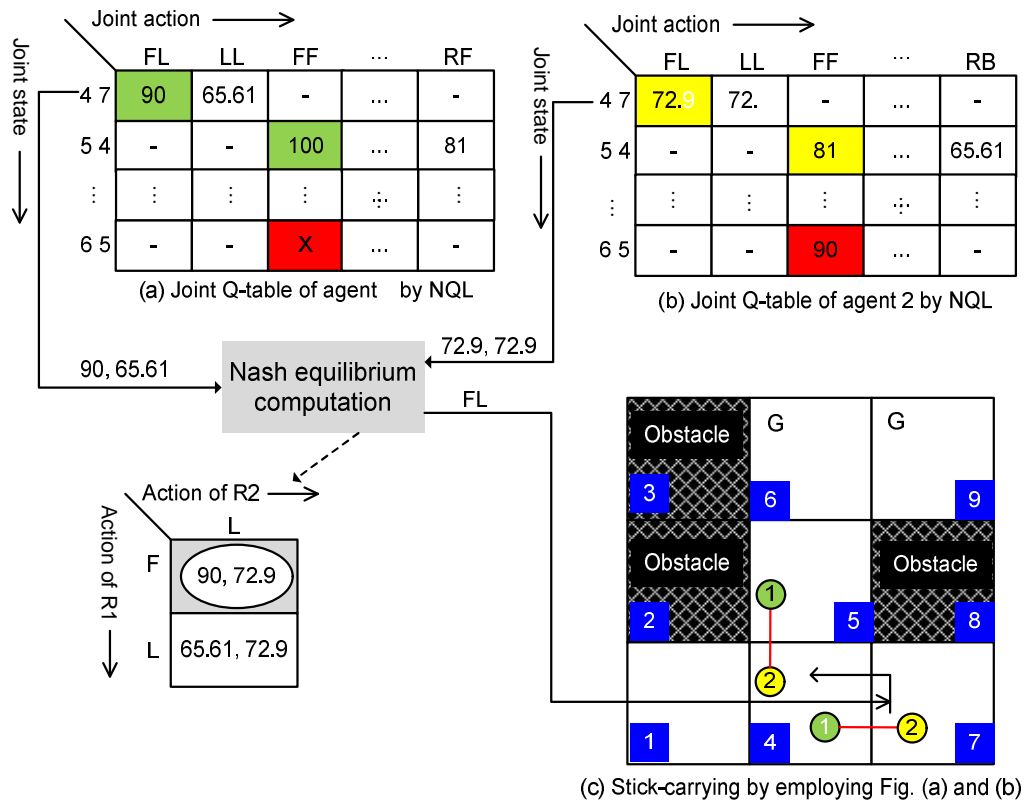


Fig. 1.47 Multi-robot coordination for the well-known stick-carrying problem

maintain all the agents' Q-tables at joint state-action space. Fig. 1.47 explains the multi-robot coordination and planning mechanism for the well-known stick-carrying problem. Stick-carrying problem refers to the transportation of a stick from current positions to the desired destination. Presently twin robots are at a joint state  $\langle 4, 7 \rangle$  with a stick as shown in Fig. 1.47(c). As each robot has both robots Q-tables at joint state-action space, a robot looks for the optimal joint action, i.e., pure strategy NE (PSNE) at  $\langle 4, 7 \rangle$  by evaluating equilibrium. To evaluate equilibrium a robot extracts the information from the joint state  $\langle 4, 7 \rangle$  (Fig. 1.47(a) and 1.47(b)) and PSNE, "FL" is evaluated following the definition of NE as shown in Fig. 1.47. Here both the robots evaluate identical PSNE. Hence, without any communication between the robots coordination occurred and the stick is shifted to the next joint state  $\langle 5, 4 \rangle$  because of the joint action "FL" by the robots.

### 1.3.7 Performance analysis of MAQL and MAQL-based coordination

The MAQL algorithms illustrated above have addressed several challenges of the MAQL. The main challenges of MAQL are suitable action selection for balancing

exploration/exploitation, update policy selection for adaptation of the Q-table in joint state-action space, equilibrium selection among multiple equilibria and the exponential increase in the space and time complexity, with the increase in number of agents. In this regards, to measure the performance of a MAQL over contender MAQL algorithms following metrics are summarized for the above mentioned MAQL.

In JAL [78], the Boltzmann strategy is extended to the optimistic Boltzmann (OB), Weight OB (WOB) and their combination. The superiority of the JAL with the combined method is tested considering the average accumulated reward as the performance metric. The superiority of the FMQ heuristic is measured considering convergence to the optimal joint action as the performance metric. In Team-Q learning [88], the average reward of agents is maximized over the learning epoch. The Distributed Q learner [92] converges to the optimal joint action with less storage and computational cost. Therefore, in Distributed Q-learning computational cost and storage requirement are the performance metrics. In OAL [95] algorithm agents select the optimal NE among multiple NE with probability one. Hence, in OAL, optimal equilibrium selection is the metric. In SCQL [96], the Q-tables are sparsely maintained and performance of the SCQL is measured over reference algorithms in terms of the computational cost and storage requirement. In SQL [97], the metric is the steps required to reach the goal state from the starting state, i.e., selection of the right joint action without any behavior conflict among the agents. In FMRQ [98], agents achieve the coordination type optimal NE to maximize the system performance in terms of average steps per episode for box-pushing problem and average rewards per episode for distributed sensor network problem. In Minimax-Q learning algorithm [99], both the agents learn optimal policies and efficiency of the algorithm is tested in the framework of a two player grid game by measuring the winning percentage of the game by the agent in an episode. Performance of the HAMRL algorithm [100] is measured in terms of the convergence speed. FP [104] addressed the equilibrium selection problem in coordination game. The performance of the Meta strategy [105] is measured in terms of the average reward achieved by the agents. AWESOME [109] learns the best response (NE) considering a stationary opponent and its performance is measured against FP in terms of the distance to equilibrium and distance to the best response. In Hyper-Q learning [111], online Bellman error and average reward variation with respect to the learning epoch are considered as the performance metrics. In [112], IGA proposed a scheme by which agents conditionally converges to the NE. Performance of the GIGA [120], WoLF-IGA [110] and GIGA-WoLF [121] algorithms are measured in terms of the convergence rate. In NQL [89], percentage of NE achieved in a game is considered as the performance metric. In CQL [72], mean Q-value difference is the performance metric. In AQL [122], change in Q-values of the agents with the learning epoch is considered as the

performance metric. The FFQ [123] always converges to a NE and converging to a NE is a metric. Average reward with the episode and number of learning epoch required per episode are the metrics in Negotiation-based Q-learning. In the equilibrium transfer-based MAQL [125], three metrics are considered. First one is the learning speed, second one is the improved average reward, and finally the last one is the reduction in the space complexity. In WoLF-PHC [126] the policy either converges to NE or to a sub-optimal NE and percentage of winning a game by an agent is considered as the performance metric. In PD-WoLF [112], average reward is the performance metric during the learning phase. Average time required to complete a task is considered the performance metric during learning in case of NSCP [130]. In EXORL [99], policy and Q-value learned with the learning epoch are considered as the performance metric.

In MAQL-based coordination, agents re-evaluate the NE/CE as explained in section 1.3.6. As the computational cost of evaluating the NE/CE is very high, run-time complexity is one performance metric in the MAQL-based coordination. On the other hand, space-complexity, successful completion of the task, system resource utilization, and the like are considered as the performance metric during the MAQL-based coordination [97].

## **1.4 Coordination by Optimization Algorithm**

One common bottleneck of the search-based coordination and MARL-based algorithms is the memory requirement and suboptimal solution. Such, bottlenecks are addressed by the Swarm Intelligence (SI) [60], [61] and Evolutionary algorithm (EA) [61]. The advantages of the SI algorithms are Scalability, Adaptability, Collective Robustness, and Individual Simplicity. The scalability of the SI algorithms are remarkable, as the control mechanism adopted by the SI algorithms does not depend upon the swarm size, until the swarm size is not too small [84]. The SI algorithm has very fast response to the rapidly changing environment by employing the auto-configuration and self-organization capabilities, which allow the swarms to adapt online with the dynamic environment [66]. Collective robustness indicates that the SI algorithms are distributed and hence, there is no possibility of single point failure [67]. In spite of very simple behavior of every swarm in any SI algorithm, the group of a swarm can achieve sophisticated group behavior [67]. Particle Swarm Optimization (PSO) algorithm and Firefly algorithm (FA) are two examples of SI algorithms. In PSO, the fitness function is not differentiable and is employed to obtain quality solution for high dimensional problems faster than other alternatives. However, there is a high probability to be trapped in local optima in high-dimensional problems. On the other hand, the FA has a very high probability of exploring the global optima. The advantages of EAs are that they can cope with discontinuities, non-linear constraints, multi-modalities and multi-objective optimization problems.

However, the EAs do not provide any guarantee to provide optimal solutions within finite amount of time. Differential evolution is an example of EA. Stability is a very good attribute of DE over the GA. Another is Imperialist Competitive Algorithm (ICA) [67], which is a socio-political based algorithm. ICA has neighborhood movements both in continuous and discrete search-space. However, the solutions provided by the ICA does not guarantee for optimal solution. In addition, the ICA requires tuning more number of parameters as compared to the PSO, FA, and DE. In the above circumstance, hybridization is a good approach. By hybridization the efficient attributes of two or more algorithms are fused to produce a powerful algorithm. One approach for multi-robot stick-carrying problem is shown in [92], where the hybridization of the motion dynamics of fireflies of the Firefly Algorithm (FA) [62] into a socio-political evolution-based meta-heuristic search algorithm is done and is named as Imperialist Competitive Firefly Algorithm (ICFA). The above mentioned algorithms are implemented for multi-robot coordination following scheme as shown in Fig. 1.48. Brief description of the above mentioned algorithms are given below.

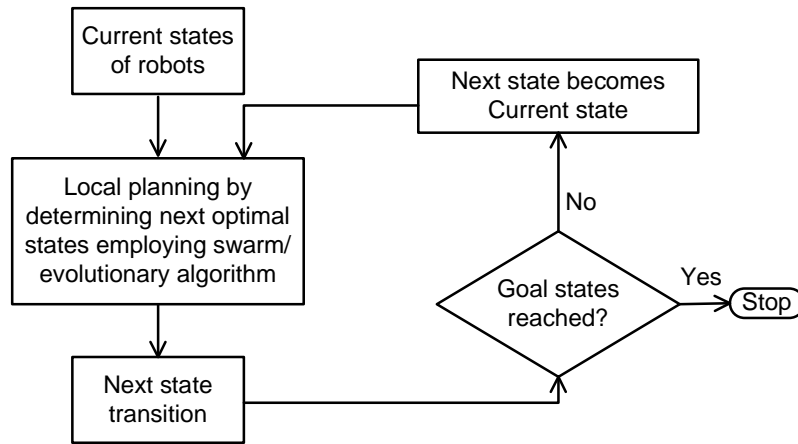


Fig. 1.48 Multi-robot local planning by swarm/evolutionary algorithm

### 1.4.1 Particle Swarm Optimization (PSO) Algorithm

In [60], Kennedy and Eberhart proposed a nonlinear function optimization technique following the behavior of flocking birds namely Particle Swarm Optimization (PSO). Let an  $n$ -dimensional nonlinear function given by (1.96) to be optimized. The PSO aims at finding such a  $\bar{X}$  so that (1.96) is either maximized or minimized depending upon the problem requirement. So, one can say that the solution of (1.96) is an  $n$ -dimensional hyperspace.

$$f(\bar{X}) = f(x_1, \dots, x_n) \quad (1.96)$$

Let us consider a two dimensional problem as given in (1.97) [62]. In (1.97),  $x \in [-10, 10]$  and  $y \in [-10, 10]$  and the plot of (1.97) is given in Fig. 1.49. It is apparent from the Fig. 1.49



that  $(0,0)$  is the only solution in the  $xy$  plane for which the  $f(x, y)$  attains a minimum value of zero. It is quiet easy to identify the minima for the function (1.96) compared to the same for (1.98) [62]. The plot of (1.98) is shown in Fig. 1.50. Unlike Fig. 1.49, in Fig. 1.50, there are multiple optimal points. It is difficult to identify the global optima among them. PSO employs the multi-agent parallel search technique and each agent starts from different initial positions and explores the landscape until a global optima is reached. It is assumed that in PSO, agents can communicate among themselves and share the values of fitness function explored by them.

$$f(x, y) = x^2 + y^2 \quad (1.97)$$

$$f(x, y) = x \sin(4\pi y) + y \sin(4\pi x + \pi) + 1 \quad (1.98)$$

In PSO, each agent flies through the multi-dimensional landscape with a unique position and velocity at each landscape. The population is initialized with random positions denoted by  $\vec{X} = \{x_i\}_{i=1}^S$  each having a random velocity  $\vec{V} = \{v_i\}_{i=1}^S$ . The position and velocity of the  $d$ -th dimension's  $i$ -th particle is given by (1.99) and (1.100) respectively.

$$x_{id}(t+1) = x_{id}(t) + v_{id}(t+1) \quad (1.99)$$

$$v_{id}(t+1) = \omega \cdot v_{id}(t) + C_1 \cdot \phi_1 \cdot (P_{id}(t) - x_{id}(t)) + C_2 \cdot \phi_2 \cdot (g_{id}(t) - x_{id}(t)) \quad (1.100)$$

In (1.100), the first component is the initial velocity of the  $i$ -th particle.  $\omega$  refers to the inertial weight factor.  $C_1$  and  $C_2$  are the constant multiplier termed as self-confidence and swarm confidence respectively. Two random numbers  $\phi_1 \in [0,1]$  and  $\phi_2 \in [0,1]$  introduced in (1.100), which determine the influence of  $\vec{p}(t)$  and  $\vec{g}(t)$  on (1.100).  $\vec{p}(t)$ ,  $\vec{g}(t)$  and  $\vec{x}(t)$  are initialized to zero at  $t=0$ , i.e., and  $\vec{p}(0) = \vec{g}(0) = \vec{x}(0)$ . After that the velocity and position of each particle updates following (1.99) and (1.100). The algorithm for PSO is given in Algorithm 1.30 [62].

<b>Algorithm 1.30: Particle Swarm Optimization (PSO)</b>
<b>Input:</b> Enter the Swarm size ( $S$ ), values of $C_1$ , $C_2$ , $\phi_1 \in [0,1]$ , $\phi_2 \in [0,1]$ , $\omega$ and $V_{\max}$ ;
<b>Output:</b> Approximate global optimal position $\vec{X}^*$ ;
Initialize: Initialized the position and velocity vectors: $\vec{X}_i(0)$ and $\vec{V}_i(0)$ ;
<b>Begin</b>
<b>While</b> termination condition is not reached <b>do</b>
<b>For</b> $i = 1$ to $S$
Evaluate the fitness $f(\vec{X}_i)$ ;
<b>Update</b> $\vec{p}_i$ and $\vec{g}_i$ ;
Adapt position and velocity of the partial by (1.99) and (1.100) respectively.
<b>End For</b> ;
<b>End While</b> .
<b>End</b> .

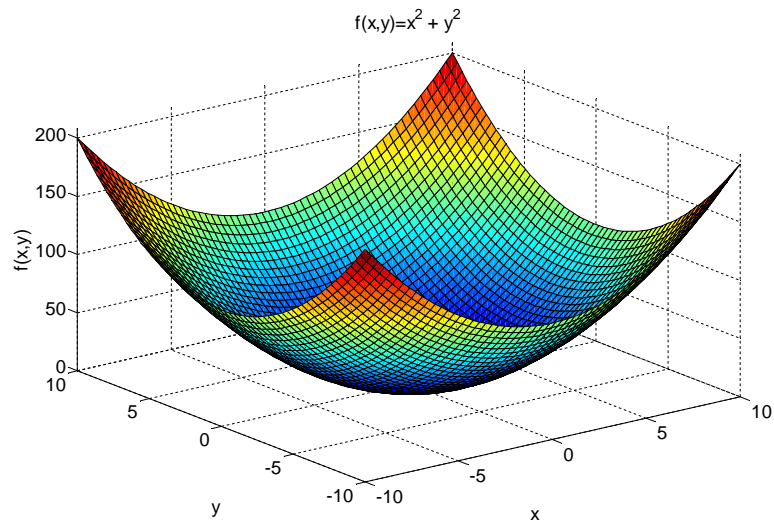


Fig. 1.49 Surface plot of (1.97)

In [69], Pugh *et al.* proposed the noise-resistance PSO for obstacle avoidance in multi-robot systems. In [70], Pugh modified the noise-resistance PSO [69] by setting

$$x_i^{*'} = x_i^{*''}, \text{ if } \text{fitness}(x_i^{*''}) > \text{fitness}(x_i^{*'}), \quad (1.101)$$

where  $x_i^{*'}$  refers to the neighborhood best for particle (here robot)  $i$  and  $x_i^{*''}$  denotes the new neighborhood best particle.

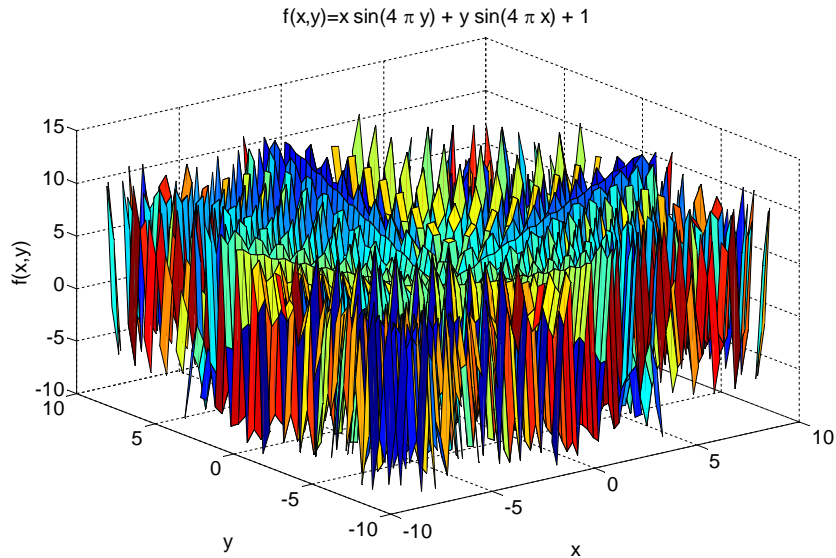


Fig. 1.50 Surface plot of (1.98)

## 1.4.2 Firefly Algorithm (FA)

In Firefly Algorithm (FA) [90], a potential solution to an optimization problem is encoded by the position of a firefly in the search space and the light intensity at the position of the firefly corresponds to the fitness of the associated solution. Each firefly changes its position iteratively by flying towards brighter fireflies at more attractive location in the fitness landscape to obtain optimal solutions.

### 1.4.2.1 Initialization

FA commences with a population  $P_t$  of  $NP$ ,  $D$ -dimensional firefly positions,  $\vec{X}_i(t) = \{x_{i,1}(t), x_{i,2}(t), x_{i,3}(t), \dots, x_{i,D}(t)\}$  for  $i = [1, NP]$  by randomly initializing in the search range  $[\vec{X}^{\min}, \vec{X}^{\max}]$  where  $\vec{X}^{\min} = \{x_1^{\min}, x_2^{\min}, \dots, x_D^{\min}\}$  and  $\vec{X}^{\max} = \{x_1^{\max}, x_2^{\max}, \dots, x_D^{\max}\}$  at the current generation  $t = 0$ . Thus the  $d$ -th component of the  $i$ -th firefly at  $t=0$  is given by (1.102).

$$x_{i,d}(0) = x_d^{\min} + rand(0,1) \times (x_d^{\max} - x_d^{\min}) \quad (1.102)$$

where  $rand(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $d = [1, D]$ . The objective function value  $f(\vec{X}_i(0))$  (which is inversely proportional to the light intensity for minimization problem) of the  $i$ -th firefly is evaluated for  $i = [1, NP]$ .

### 1.4.2.2 Attraction to Brighter Fireflies

Now the firefly  $\vec{X}_i(t)$  is attracted towards the positions of the brighter fireflies  $\vec{X}_j(t)$  for  $i, j = [1, NP]$  but  $i \neq j$  such that  $f(\vec{X}_j(t)) < f(\vec{X}_i(t))$  for minimization problem. Now the attractiveness  $\beta_{i,j}$  of  $\vec{X}_i(t)$  towards  $\vec{X}_j(t)$  is proportional to the light intensity seen by adjacent fireflies. However attractiveness  $\beta_{i,j}$  decreases exponentially with the distance between the fireflies, denoted by  $r_{i,j}$  as given in (1.103).

$$\beta_{i,j} = \beta_0 \exp(-\gamma \times r_{i,j}^m), \quad m \geq 1 \quad (1.103)$$

where  $\beta_0$  denotes the maximum attractiveness experienced by the  $i$ -th firefly at its own position (i.e. at  $r_{i,j} = r_{i,i} = 0$ ) and  $\gamma$  is the light absorption coefficient, which controls the variation of  $\beta_{i,j}$  with  $r_{i,j}$ . This parameter is responsible for the convergence speed of FA. A setting of  $\gamma=0$  leads to constant attractiveness while  $\gamma$  approaching infinity is equivalent to the complete random search [62]. In (1.103)  $m$  is a positive constant representing a non-linear modulation index. The distance between  $\vec{X}_i(t)$  and  $\vec{X}_j(t)$  is computed using the Euclidean norm as follows.

$$r_{i,j} = \|\vec{X}_i(t) - \vec{X}_j(t)\| \quad (1.104)$$

This step is repeated for  $i, j = [1, N]$ .

### 1.4.2.3 Movement of Fireflies

The firefly at position  $\vec{X}_i(t)$  moves towards a more attractive position  $\vec{X}_j(t)$  occupied by a brighter firefly (i.e.,  $f(\vec{X}_j(t)) < f(\vec{X}_i(t))$ ) for  $j = [1, N]$  but  $i \neq j$  following the dynamic given in (1.105).

$$\vec{X}_i(t+1) = \vec{X}_i(t) + \beta_{i,j} \times (\vec{X}_j(t) - \vec{X}_i(t)) + \alpha \times (\text{rand}(0,1) - 0.5) \quad (1.105)$$

The first term in the position updating formula (1.105) represents the  $i$ -th firefly's current position. The second term in (1.105) denotes the change in the position of the firefly at  $\vec{X}_i(t)$  due to the attraction towards the brighter firefly at  $\vec{X}_j(t)$ . Hence it is apparent that the brightest firefly with no more attractive firefly in the current sorted population  $P_t$  will have no motion due to the second term and may get stuck at the local optima. To circumvent the problem, the last term is introduced in (1.105) for the random movement of the fireflies with a step-size of  $\alpha \in (0, 1)$ . Here  $\text{rand}(0,1)$  is a random number generator uniformly distributed in the range  $(0, 1)$ . This step is repeated for  $i = [1, NP]$ . After completion of its journey mediated by the brighter ones, the updated position of the  $i$ -th firefly is represented by  $\vec{X}_i(t+1)$  for  $i = [1, NP]$ .

<b>Algorithm 1.31: Traditional Firefly Algorithm (FA)</b>
<p><b>Input:</b> <math>\vec{X} = (x_1, x_2, \dots, x_D)</math>, fitness function <math>f(X)</math>; // <math>D</math> dimension of the firefly</p> <p><b>Output:</b> <math>\vec{X}_i, i \in [1, n]</math>;</p> <p><b>Initialize:</b> Generate population <math>\vec{X}_i, i \in [1, n]</math>, <math>\alpha \in (0, 1)</math>, <math>\beta_0 = 1</math> and <math>\gamma \in [0.1, 10]</math>;</p> <p><b>While</b> (<math>t &lt; \text{MaxGeneration}</math>)</p> <p style="padding-left: 20px;"><b>For</b> <math>k=1</math> to <math>D</math></p> <p style="padding-left: 40px;"><b>For</b> <math>i=1</math> to <math>n</math></p> <p style="padding-left: 60px;"><b>For</b> <math>j=1</math> to <math>n</math></p> <p style="padding-left: 80px;"><b>If</b> <math>f(\vec{X}_i(t)) &lt; f(\vec{X}_j(t))</math></p> <p style="padding-left: 100px;"><b>then</b> Move <math>\vec{X}_i(t)</math> towards <math>\vec{X}_j(t)</math> in all <math>D</math> dimensions;</p> <p style="padding-left: 80px;"><b>End If;</b></p> <p style="padding-left: 60px;"><b>Update</b> <math>x_{ik}(t+1) = x_{ik}(t) + \beta r_{ij} \times [x_{jk}(t) - x_{ik}(t)] + \alpha(\text{rand} - 0.5)</math>;</p> <p style="padding-left: 40px;"><b>End For;</b></p> <p style="padding-left: 20px;"><b>End For;</b></p> <p style="padding-left: 20px;"><b>End For;</b></p> <p style="padding-left: 20px;">Rank the fireflies based on current fitness and find the current best one;</p> <p><b>End While.</b></p>

After each evolution, the steps 1.4.2.2 and 1.4.2.3 are repeated until one of the following conditions for convergence is satisfied. These conditions include restraining the number of iterations, maintaining error limits, or the both, whichever occurs earlier. In Algorithm 1.31, the number of iterations is considered as the condition of convergence.

### 1.4.3 Imperialist Competitive Algorithm (ICA)

Imperialist Competitive Algorithm (ICA) is a population-based stochastic algorithm, which is inspired by the socio-political evolution and the imperialistic competitive policy of a government to extend its power beyond its boundaries. It has earned wide popularity because of its noticeable performance in computational optimization with respect to the quality of solutions [134]. Like any other evolutionary algorithms, ICA starts with an initial population of solutions, called countries. The countries are classified into two groups—imperialists and colonies, based on their ruling power (which is inversely proportional to their objective function values). The colonies (weaker countries) with their relevant imperialist (stronger country) form some empires. In each empire, the imperialist pursues an assimilation policy to improve the economy, culture, and political situation of its colonies, thus winning their loyalty. Moreover, the empires take part in the imperialistic competition in an attempt to gain more colonies. In ICA, the assimilation of colonies towards their respective imperialists along with the competition among empires eventually results in just one empire in the world with all the other countries as colonies of that unique empire. An overview of the main steps of the ICA is presented next.

#### 1.4.3.1 Initialization

ICA starts with a population  $P_t$  of  $NP$ ,  $D$ -dimensional countries,  $\vec{X}_i(t) = \{x_{i,1}(t), x_{i,2}(t), x_{i,3}(t), \dots, x_{i,D}(t)\}$  for  $i = [1, NP]$  representing the candidate solutions, at the current generation  $t=0$  by randomly initializing in the range  $[\vec{X}^{\min}, \vec{X}^{\max}]$  where  $\vec{X}^{\min} = \{x_1^{\min}, x_2^{\min}, \dots, x_D^{\min}\}$  and  $\vec{X}^{\max} = \{x_1^{\max}, x_2^{\max}, \dots, x_D^{\max}\}$ . Thus the  $d$ -th component (socio-political feature) of the  $i$ -th country at  $t=0$  is given by

$$x_{i,d}(0) = x_d^{\min} + rand(0,1) \times (x_d^{\max} - x_d^{\min}) \quad (1.106)$$

where  $rand(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $d = [1, D]$ . The objective function value  $f(\vec{X}_i(0))$  of the country  $\vec{X}_i(0)$  is evaluated for  $i = [1, NP]$ .

#### 1.4.3.2 Selection of Imperialists and Colonies

The population  $P_0$  is sorted in ascending order of  $f(\vec{X}_i(0))$  for minimization problem with  $i = [1, NP]$ . The first  $N$  countries with less cost function values are selected as imperialists while the remaining  $M = NP - N$  countries are declared as colonies. Hence the population individuals are categorized into two groups of countries — imperialists and colonies.

#### 1.4.3.3 Formation of Empires

The empire under the  $j$ -th imperialist is constructed based on its ruling power. To accomplish this, first the normalized power of the  $j$ -th imperialist country,  $p_j$ , is evaluated by (1.107) with

$f(\bar{X}_{NP}(0))$  representing the objective function value of the weakest country in the current sorted population  $P_0$ .

$$p_j = \frac{f(\bar{X}_{NP}(0)) - f(\bar{X}_j(0))}{\sum_{l=1}^N f(\bar{X}_{NP}(0)) - f(\bar{X}_l(0))} \quad (1.107)$$

It is evident from (1.107) that better the  $j$ -th imperialist (i.e., less objective function value  $f(\bar{X}_j(0))$  for minimization problem), higher is the difference  $f(\bar{X}_{NP}(0)) - f(\bar{X}_j(0))$  leading to the enhancement of its corresponding ruling power,  $p_j$ . Now the initial number of colonies under in the  $j$ -th empire, denoted by  $n_j$  is computed by (1.108).

$$n_j = \lfloor M \times p_j \rfloor \quad (1.108)$$

$$\text{such that } \sum_{j=1}^N n_j = M \quad (1.109)$$

Here  $\lfloor \cdot \rfloor$  represents the floor function. According to (1.108) the stronger imperialists with higher ruling power now possess larger empires. Hence  $p_j$  symbolizes the fraction of the colonies occupied by the  $j$ -th imperialist. Subsequently the  $j$ -th empire is formed by randomly selecting  $n_j$  countries from  $M$  colonies provided that there will be no common colony between two different empires. Hence the number of countries within the  $j$ -th empire including its imperialist is  $n_j+1$ . Let the  $k$ -th country belonging to the  $j$ -th empire be denoted by  $\bar{X}_k^j(t)$  (at generation  $t=0$ ) for  $k= [1, n_j+1]$ . The countries within the  $j$ -th empire are now sorted in ascending order of their objective function values such that the imperialist  $\bar{X}_1^j(t)$  in the  $j$ -th empire attains the first rank. This step is repeated for  $j= [1, N]$ .

#### 1.4.3.4 Assimilation of Colonies

Each imperialist country now attempts to improve its empire by enhancing the socio-political influences of its colonies. To accomplish this, each country  $\bar{X}_k^j(t)$  in the  $j$ -th empire now moves towards its corresponding imperialist  $\bar{X}_1^j(t)$  by changing its characteristic features following (1.110) for  $k= [2, n_j+1]$ .

$$\bar{X}_k^j(t+1) = \bar{X}_k^j(t) + \beta \times \text{rand}(0,1) \times (\bar{X}_1^j(t) - \bar{X}_k^j(t)) \quad (1.110)$$

Here  $\text{rand}(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $\beta$  is the assimilation coefficient. The objective function value of the modified colony  $f(\bar{X}_k^j(t+1))$  is evaluated for  $k= [2, n_j+1]$ . After assimilation, all the countries in the  $j$ -th empire are sorted in ascending order of the objective function values and the first ranked country is declared as the imperialist  $\bar{X}_1^j(t+1)$  of the same empire for the next generation (i.e.,  $t = t+1$ ). The step is repeated for  $j = [1, N]$ .

### 1.4.3.5 Revolution

Revolution creates sudden fluctuation in the economic, cultural, and political aspects of countries in an empire. The colonies in an empire are now equipped with the power of randomly changing their socio-political attributes instead of being assimilated by their corresponding imperialist. It resembles the mutation of trial solutions in the traditional EA. The revolution rate  $\eta$  in the algorithm indicates the percentage of colonies in each empire which will undergo the revolution process. A high value of revolution rate therefore fortifies the explorative power at a cost of poor exploitation capability. Hence a moderate value of revolution rate is favored. Revolution is implemented by randomly selecting  $\eta \times n_j$  countries (including the imperialist) in the  $j$ -th empire (for  $j = [1, N]$ ) and then they are replaced by randomly initialized countries characterized by new socio-political nature. After revolution, as in case of assimilation, all the countries in each empire are sorted in ascending order of the objective function values so that its imperialist is at the first position. The step is repeated for all empires.

### 1.4.3.6 Imperialistic Competition

All the  $N$  empires now participate in an imperialistic competition to take possession of colonies of other weaker empires based on their ruling power. The colonies of the weaker empires will be gradually eluded from the ruling power of their corresponding imperialists and will be thereafter controlled by some other stronger empires. Consequently, the weaker empires will be losing their power and ultimately may be eradicated from the competition. The imperialistic competition along with the collapse mechanism will progressively result in an increment in the power of more dominant empires and diminish the power of weaker ones. The imperialistic competition encompasses the following steps.

#### 1.4.3.6.1 Total Empire Power Evaluation

Once an empire is constructed under the dominance of the  $j$ -th imperialist  $\vec{X}_1^j(t+1)$ , the power of the respective empire is compositely influenced by the objective function value of  $\vec{X}_1^j(t+1)$  as well as the constituent colonies  $\vec{X}_k^j(t+1)$  (after assimilation) under the respective  $j$ -th empire for  $k = [2, n_j+1]$ . The total objective function value of the  $j$ -th empire is evaluated as follows.

$$tc_j = f(\vec{X}_1^j(t+1)) + \xi \cdot \frac{1}{n_j} \sum_{k=2}^{n_j+1} \vec{X}_k^j(t+1) \quad (1.111)$$

Here  $\xi < 1$  is a positive number which regulates the influence of the constituent colonies to control the ruling power of the empire. A tiny value of  $\xi$  causes the total power of the  $j$ -th empire to be determined by its imperialist  $\vec{X}_1^j(t+1)$  only, while increasing the value of  $\xi$  accentuates the importance of the colonies in deciding the total power of the respective

empire. The  $N$  empires now are sorted in ascending order of  $tc_j$  for  $j = [1, N]$ . Then the normalized possession power of the  $j$ -th empire,  $pp_j$ , is evaluated by (1.112) with  $tc_N$  representing the total objective function value of the weakest empire in the current population  $P_t$ .

$$pp_j = \frac{tc_N - tc_j}{\sum_{l=1}^N tc_N - tc_l} \quad (1.112)$$

It is evident from (1.112) that stronger the  $j$ -th empire (i.e., less the total objective function value  $tc_j$  for minimization problem), higher is the possession power,  $pp_j$ , which consecutively increases its probability of seizing colonies from weaker empires. This step is repeated for  $j = [1, N]$ .

#### 1.4.3.6.2 Reassignment of Colonies and Removal of Empire

The empire with least possession power is interpreted as being defeated in the competition. Let the weakest colony of this weakest empire be denoted as  $\vec{X}_{worst}$ , which is now removed from the dominance of its currently ruling imperialist and reassigned as a new colony to one of the stronger empires based on their possession probabilities. It is noteworthy that  $\vec{X}_{worst}$  will not be possessed by the most powerful empires, but stronger the empire, more likely to possess  $\vec{X}_{worst}$ . To accomplish this, the possession probability of the  $j$ -th empire is computed as follows for  $j = [1, N]$ .

$$prob_j = pp_j - rand(0,1) \quad (1.113)$$

Now  $\vec{X}_{worst}$  is assigned as a new colony to the  $j$ -th empire for which the possession probability  $prob_j$  is maximum. However, if the worst colony consists of only its imperial before exclusion operation, (i.e.,  $\vec{X}_{worst}$  is the imperialist of the weakest empire), the removal of  $\vec{X}_{worst}$  will result in the collapse of the weakest empire.

#### 1.4.3.6.3 Union of Empires

The disagreement between two empires may be assessed by the difference in their respective socio-political features. This dissimilarity between any two empires,  $j$  and  $l$  is evaluated by taking the Euclidean distance between the respective imperialists  $\vec{X}_1^j(t+1)$  and  $\vec{X}_1^l(t+1)$  as in (1.114) for  $j, l = [1, N]$ .

$$Dist_{j,l} = \left\| \vec{X}_1^j(t+1) - \vec{X}_1^l(t+1) \right\| \quad (1.114)$$

If  $Dist_{j,l}$  is less than a predefined threshold,  $Th$ , the two empires are merged into one empire. The stronger country among  $\vec{X}_1^j(t+1)$  and  $\vec{X}_1^l(t+1)$  is declared as the imperialist of the newly formed empire.



After each evolution, we repeat from step 1.4.3.4 until one of the following conditions for convergence is satisfied. Stop criteria include a bound by the number of iterations, achieving a sufficiently low error or aggregations thereof.

#### 1.4.4 Differential evolutionary (DE) algorithm

Differential evolutionary (DE) algorithm is a stochastic, population-based global optimization algorithm, introduced by Storn and Price in 1996, to optimize real parameter, real valued functions [62].

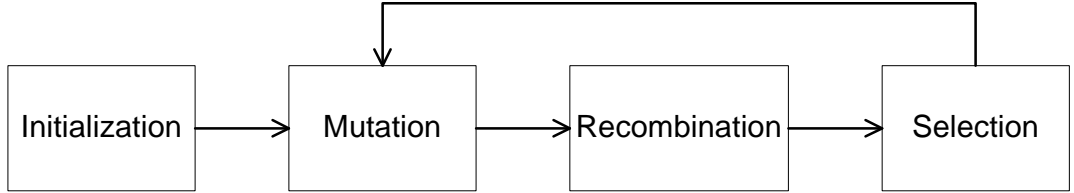


Fig. 1.51 Steps of Differential evolutionary (DE) algorithm

##### 1.4.4.1 Initialization

Range of each parameter i.e., the upper and lower boundaries for each parameter are defined, and then randomly these parameters are initialized.

##### 1.4.4.2 Mutation

The step mutation expands the search-space. Mutation is done by (1.115), where  $F \in [0, 2]$  is the mutation factor.  $x_{r_1,G}$ ,  $x_{r_2,G}$  and  $x_{r_3,G}$  are the randomly selected variables with  $i, r_1, r_2, r_3$  and  $G$  are index.  $v_{i,G+1}$  refers to the donor vector.

$$v_{i,G+1} = x_{r_1,G} + F(x_{r_2,G} - x_{r_3,G}). \quad (1.115)$$

##### 1.4.4.3 Recombination

Employing the target vector  $x_{i,G}$  and the elements of the donor vector  $v_{i,G+1}$  the trial solution vector  $u_{i,G+1}$  is evaluated by following (1.116).

$$u_{i,j,G+1} = \begin{cases} v_{i,j,G+1} & \text{if } \text{rand}() \leq CR \text{ or } j = I_{rand} \\ x_{i,j,G} & \text{if } \text{rand}() > CR \text{ or } j \neq I_{rand} \end{cases}, \quad (1.116)$$

where  $i = [1, N]$ ,  $j = [1, D]$  and  $v_{i,G+1} \neq x_{i,G}$  is checked by  $I_{rand}$ .

##### 1.4.4.4 Selection

The target solution  $x_{i,G}$  is compared with the trial solution vector  $u_{i,G+1}$  and the next generation is selected by (1.117).

$$x_{i,G+1} = \begin{cases} u_{i,G+1} & \text{if } f(u_{i,G+1}) \leq f(x_{i,G}) \\ x_{i,G} & \text{otherwise} \end{cases}, \quad (1.117)$$

where  $i = [1, N]$ .

Step 1.4.4.2. to 1.4.4.4 continue until the termination criterion as explained earlier is reached.

### 1.4.5 Offline optimization

By SI and EO algorithms only the offline optimization is possible due to their huge run-time complexity. In case of multi-robot coordination, robots evaluate the optimal trajectory (collection of coordinates) offline in the sense of system recourse (time and/or energy) utilization. After offline optimization of the trajectory it is executed in the real-robot.

### 1.4.6 Performance analysis of optimization algorithms

The performance of SI, EA and their hybridization can be analyzed by the following performance metrics. *Quality of solution* within a fixed epoch and the *convergence time* are two performances matrices of the SI and EA. In addition, mean best objective function versus function evaluation, accuracy versus function evaluation, and function evaluation versus search space dimensionality can be considered as the performance metrics. In spite of the above mentioned performance metrics, statistical test is conducted over the algorithms for performance measurement.

#### 1.4.6.1 Friedman test

Friedman test [64], which is a non parametrical statistical test, may be carried out on the average objective function values of each of the algorithms for a fixed independent runs, assuming a fixed dimension. To carry out the Friedman test firstly the average ranking ( $R_i$ ) for each of the considered algorithms is calculated as the mean of the individual ranks obtained by them over all the considered  $N$  number of benchmark functions, as shown in (1.118),

$$R_i = \frac{1}{N} \sum_{j=1}^N r_i^j \quad (1.118)$$

Here,  $r_i^j$  refers to the individual rank attained by the  $i$ -th algorithm for the  $j$ -th benchmark function and the results have been computed considering  $N$  benchmark functions. In the next step, a term formally defining the Friedman statistic, that follows a  $\chi_F^2$  distribution with  $(k-1)$  degrees of freedom, has been evaluated using (1.119),

$$\chi_F^2 = \frac{12N}{k(k+1)} \left[ \sum_{i=1}^k R_i^2 - \frac{k(k+1)^2}{4} \right] \quad (1.119)$$

### 1.4.6.2 Iman–Davenport test

Moreover, Iman–Davenport test [65] can also be conducted in order to substantiate the findings of the former statistical analysis. It is basically a deviation from the Friedman test producing more precise results and the Iman-Davenport statistics is calculated as follows,

$$F_F = \frac{(N-1) \times \chi_F^2}{N \times (k-1) - \chi_F^2}. \quad (1.120)$$

Tabular analysis can be shown in case of Friedman test, which demonstrates that the null hypothesis has been rejected if the computed value of  $\chi_F^2$  is greater than the critical value of the  $\chi_F^2$  distribution with degrees of freedom  $(k-1)$  at probability of  $\alpha$  ( $\chi_{3,\alpha}^2$ ). For, Iman-Davenport test the statistic is distributed with  $(k-1)$  and  $(k-1) \times (N-1)$  degrees of freedom. Likewise, the null hypothesis has been rejected as the calculated value of  $F_F$  is greater than the critical value of the  $F_F$  distribution with degrees of freedom  $(k-1)$  and  $(k-1) \times (N-1)$  at probability of  $\alpha$  ( $F_{(k-1),(N-1),\alpha}$ ). It is obvious, that the proposed algorithm is the most efficient one, hence, in the post-hoc analysis, proposed algorithm is assumed to be the control method.

For multi-robot trajectory (path) planning Average total path deviation, Average Uncovered Target Distance, Average total path traversed, and numbers of steps required are considered as the performance metrics.

## 1.5 Scope of the Thesis

A plan refers to the sequence of actions to achieve a predefined goal. On the other hand, coordination indicates the skilful interaction among the agents to generate a feasible planning step. Therefore, coordination is an important issue in modern mobile robotics to address complex real-world problems. The thesis offers novel multi-robot coordination techniques (algorithms) with or without communication among the robots. Here, coordination among multiple robots is achieved by means of two distinct approaches. The first approach attempts to optimize individual robot's parameters with respect to coordination of the entire team of robots. Any traditional meta-heuristic optimization algorithm would serve the purpose. We, however, employ a hybridization of two algorithms to develop a potential algorithm to overcome the individual limitations. Since parameter sharing of two robots is needed for fruitful coordination, the optimization algorithm used requires communication among agents for such parameter sharing. However, communication among robots has several negative consequences. The most important among these is possibility of network failure, which might affect the robustness of the system.

The second alternative approach for multi-robot coordination is learning-induced planning and coordination. Here, each agent learns the group performance and the knowledge formed thereby is used subsequently for planning and coordination. The learning-induced coordination is free from communication. Although there exists a vast literature on learning

based planning and coordination, there is a dearth of literature on reinforcement learning induced planning. We here propose a game theoretic learning algorithm, which has an excellent team-performance in comparison to the existing works. The said algorithm utilizes an “equilibrium strategy” to optimize the performance of the all the agents. The “equilibrium strategy” is so named as a deviation in action by any one agent from the equilibrium may incur degradation in the performance of the agents. The contributions claimed in the present thesis are framed into seven chapters and are briefly outlined below.

Chapter 1 provides an introduction to the multi-robot coordination algorithms for complex real-world problems, including transportation of a box/stick, formation control for defense applications and soccer playing by multiple robots utilizing the principles of reinforcement learning, theory of games, dynamic programming, and/or evolutionary algorithm. Naturally, this chapter provides a thorough survey of the exiting literature of reinforcement learning with a brief overview of the evolutionary optimization to examine the role of the algorithms in the context of multi-agent coordination. Chapter 1 includes multi-robot coordination employing evolutionary optimization, and specially reinforcement learning for cooperative, competitive, and their composition for application to static and dynamic games. The later part of the chapter deals with an overview of the metrics used to compare the performance of the algorithms while coordinating. Fundamental metrics for performance analysis are defined to study the learning and planning algorithms.

Chapter 2 offers learning-based planning algorithms, by extending the traditional multi-agent Q-learning algorithms (Nash Q-Learning and Corelated Q-Learning) for multi-robot coordination and planning. This extension is achieved by employing two interesting properties. The first property deals with the exploration of the team-goal (simultaneous success of all the robots) and the other property is related to the selection of joint action at a given joint state. The exploration of team-goal is realized by allowing the agents, capable of reaching their goals, to wait at their individual goal states, until remaining agents explore their individual goals synchronously or asynchronously. Selection of joint action, which is a crucial problem in traditional multi-agent Q-learning, is performed here by taking the intersection of individual preferred joint actions of all the agents. In case the resulting intersection is a null set, the individual actions are selected randomly or otherwise following classical techniques. The superiority of the proposed learning and learning-based planning algorithms are validated over contestant algorithms in terms of the speed of convergence and run-time complexity respectively.

In chapter 3, it is shown that robots may select the suboptimal equilibrium in presence of multiple types of equilibria (here Nash equilibrium or correlated equilibrium). In the above perspective, robots need to adapt with such a strategy, which can select the optimal

equilibrium in each step of the learning and the planning. To address the bottleneck of the optimal equilibrium selection among multiple types, chapter 3 presents a novel consensus Q-learning for multi-robot coordination, by extending the equilibrium-based multi-agent Q-learning algorithms. It is also shown that a consensus (joint action) jointly satisfies the conditions of the coordination type pure strategy Nash equilibrium and the pure strategy correlated equilibrium. The superiority of the proposed consensus Q-learning algorithm over traditional reference algorithms in terms of the average reward collection are shown in the experimental section. In addition, the proposed consensus-based planning algorithm is also verified considering multi-robot stick-carrying problem as the test bed.

Unlike correlated Q-learning, Chapter 4 proposes an attractive approach to adapt composite rewards of all the agents in one Q-table in joint state-action space during learning, and subsequently these rewards are employed to compute correlated equilibrium in the planning phase. Two separate models of multi-agent Q-learning have been proposed. If success of only one agent is enough to make the team successful, then model-I is employed. However, if an agent's success is contingent upon other agents and simultaneous success of the agents is required then model-II is employed. It is also shown that the correlated equilibrium obtained by the proposed algorithms and by the traditional correlated Q-learning are identical. In order to restrict the exploration within the feasible joint states, constraint versions of the said algorithms are also proposed. Complexity analysis and experiments have been undertaken to validate the performance of the proposed algorithms in multi-robot planning on both simulated and real platforms.

Chapter 5 hybridizes the Firefly Algorithm and the Imperialist Competitive Algorithm. The above explained hybridization results in the Imperialist Competitive Firefly Algorithm, which is employed to determine the time-optimal trajectory of a stick, being carried by two robots, from a given starting position to a predefined goal position amidst static obstacles in a robot world-map. The motion dynamics of fireflies of the Firefly Algorithm is embedded into the socio-political evolution-based meta-heuristic Imperialist Competitive Algorithm. Also the trade-off between the exploration and exploitation is balanced by modifying the random walk strategy based on the position of the candidate solutions in the search space. The superiority of the proposed Imperialist Competitive Firefly Algorithm is studied considering run-time and accuracy as the performance metrics. Finally, the proposed algorithm has been verified in real-time multi-robot stick-carrying problem.

Chapter 6 concludes the thesis based on the analysis made, experimental and simulation results obtained from chapter 1-6. The chapter also examines the prospects of the present thesis in view of the future research trends.

## 1.6 Summary

Chapter 1 introduces multi-robot coordination algorithms for complex real-world problems employing the principles of reinforcement learning, theory of games, dynamic programming, and/or evolutionary algorithm. As expected, chapter 1 includes a thorough survey of the exiting literature of reinforcement learning with a brief overview of the evolutionary optimization to examine the role of the algorithms in view of the multi-agent coordination. Here multi-robot coordination is achieved by employing the evolutionary optimization, and specially reinforcement learning for cooperative, competitive, and their composition for application to static and dynamic games. The remainder of the chapter provides an overview of the metrics used to compare the performance of the algorithms while coordinating.

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## **Chapter 2**

# **Improving Speed of Convergence of Multi-Agent Q-learning for Cooperative Task-Planning by a Robot-Team**

Learning-based planning algorithms are currently gaining popularity for their increasing applications in real-time planning and cooperation of robots. The chapter aims at extending traditional multi-agent Q-learning algorithms to improve their speed of convergence by incorporating two interesting properties, concerning i) exploration of the team-goal and ii) selection of joint action at a given joint state. The exploration of team-goal is realized by allowing the agents, capable of reaching their goals, to wait at their individual goal states, until remaining agents explore their individual goals synchronously or asynchronously. To avoid unwanted never-ending wait-loops, an upper bound to wait-interval, obtained empirically for the waiting team members, is introduced. Selection of joint action, which is a crucial problem in traditional multi-agent Q-learning, is performed here by taking the intersection of individual preferred joint actions of all the agents. In case the resulting intersection is a null set, the individual actions are selected randomly or otherwise following classical multi-agent Q-learning. It is shown both theoretically and experimentally that the extended algorithms outperform its traditional counterpart with respect to speed of convergence. To ensure selection of right joint action at each step of planning, we offer high rewards to exploration of the team-goal and zero rewards to exploration of individual goals during the learning phase. The introduction of the above strategy results in an enriched joint Q-table, the consultation of which during the multi-agent planning yields significant improvement in the performance of cooperative planning of robots. Hardwired realization of the proposed learning based planning algorithm, designed for object-transportation application, confirms the relative merits of the proposed technique over contestant algorithms.

Table 2.1 List of acronyms

<b>Full form</b>	<b>Acronyms</b>
Multi-agent Q-learning :	MAQL
Traditional MAQL :	TMAQL
Nash equilibrium :	NE
Nash Q-Learning :	NQL
NQL with equilibrium transfer :	NQLET
NQL with Property 1 :	NQLP1
NQL with Property 2 :	NQLP2
NQL with Property 1 and 2 :	NQLP12
Correlated equilibrium :	CE
Correlated Q-learning :	CQL
Utilitarian Q-Learning :	UQL
UQL with equilibrium transfer :	UQLET
UQL with Property 1 :	UQLP1
UQL with Property 2 :	UQLP2
UQL with Property 1 and 2 :	UQLP12
Egalitarian Q-Learning :	EQL
EQL with equilibrium transfer :	EQLET
EQL with Property 1 :	EQLP1
EQL with Property 2 :	EQLP2
EQL with Property 1 and 2 :	EQLP12
Republican Q-Learning :	RQL
RQL with equilibrium transfer :	RQLET
RQL with Property 1 :	RQLP1
RQL with Property 2 :	RQLP2
RQL with Property 1 and 2 :	RQLP12
Libertarian Q-Learning :	LQL
LQL with equilibrium transfer :	LQLET
LQL with Property 1 :	LQL1
LQL with Property 2 :	LQL2
LQL with Property 1 and 2 :	LQLP12
Frequency of the maximum reward Q-learning :	FMRQ
FMRQ with Property 1 :	FMRQP1
FMRQ with Property 2 :	FMRQP2
FMRQ with Property 1 and 2 :	FMRQP12
Fast cooperative multi-agent Q-learning :	FCMQL
Nash Q Induced multi-agent planning :	NQIMP
Correlated Q induced multi-agent planning :	CQIMP
Contributed reward by agent $i$ for the team :	$CR_i$
Imperialist Competitive Firefly Algorithm :	ICFA
modified noise-resistant Particle Swarm Optimization :	MNPSO
Differential Evolutionary :	DE
multi-robot joint action learning by demonstration :	MLbD

## 2.1 Introduction

Reinforcement learning (RL) [1-10] refers to a real-time learning paradigm, where an agent learns its environment with respect to a fixed goal by receiving reward/penalty [6] for its actions on the environment. The reward/penalty obtained by the agent for its sequence of actions is used to adapt its effective reward in a given state-action space [11-14], [15]. The motivation of RL is to derive the optimal action at a given environmental state for which the



agent would be able to derive the maximum reward. Such formulation of deriving optimal action at a given state based on the learned experience of interaction with the environment has plenty of interesting applications, including generating moves in a game [16-17], complex task-planning and motion-planning of a mobile robot in a constrained environment [18]. In RL, the environment is typically represented by a Markov Decision Process (MDP) with unknown state-transition probabilities and an unknown reward model [6]. MDP provides the basic mathematical model of a discrete-event system [19].

Among the RL algorithms, Q-learning is most popular. Q-learning does not require any background knowledge of the agents' environment and thus is called model-free. This characteristic of Q-learning is advantageous [20] as learning can be performed without the knowledge of the environment. In Q-learning, optimal policy for each state-action pair is estimated through an iterative process using Dynamic Programming (DP) [21], realized with the well-known Bellman Equation (BE) [21]. In single agent Q-learning, the state-transitions are controlled by the agent itself. However, in a multi-agent environment, all the agents participate to select their individual actions and form a joint action in a joint state-space. Because of joint actions by the agents on the environment, the environment in multi-agent Q-learning (MAQL) [2], [12], [18], [22-40], [42-52], [59] appears as dynamic to an individual agent. Like single agent Q-learning, a MAQL too is described by a MDP, called Multi-agent MDP (MMDP) [23]. [33].

Several extensions of the single agent Q-learning for multi-agent applications is available. The fundamental problems in MAQL, by which it significantly differs from its single-agent counterpart, include [2], [18], [22-40], [42-52], [59] i) joint action selection, ii) update policy selection for adaptation of the Q-table in joint state-action space and iii) exploration of the team-goal. Although the first two problems have been addressed in the literature, the last one remains unattended. In this paper, we provide a solution to MAQL with a motivation to deal with exploration of the team-goal and demonstrate its scope of applications in tight cooperative multi-agent planning.

Several approaches to action selection in a single agent are available. A few of these that deserve special mention includes  $\epsilon$  – greedy exploration [6], Boltzmann strategy [15], [53] the extended Boltzmann strategy for Frequency Maximum Q-value (FMQ) heuristic [45] and also random selection. Selection of a joint action traditionally is done in two phases. First, individual actions are selected by any one of the above techniques. Next, the individual actions are combined to form a joint action. However, there are situations when the joint actions thus obtained are infeasible for a given environment. In Traditional MAQL (TMAQL), the researchers do not check the possibility of infeasible actions, as infeasible actions are penalized and thus automatically get forbidden in subsequent learning epochs.

Wang *et al.* introduced a novel technique for joint action selection in their proposed Sequential Q-Learning (SQL) using a two step procedure [18]. In the first step, they employed the Boltzmann strategy for the individual action selection, and in the second step they designed a specialized selection operation to avoid the same actions repeatedly. Besides joint action selection, there exists extensive literature on update policy in the Q-table. A few of these include Nash Q-learning (NQL) [27-28], correlated Q-learning (CQL) [26], SQL [18], sparse Q-learning [50], heuristically accelerated MAQL [51], MAQL with equilibrium transfer (MAQLET) [48], and Frequency of the maximum reward Q-learning (FMRQ) [52]. These techniques have their individual merits depending on the nature of the problem selected.

Of the three problems in MAQL, we discussed above the major works on the first two. Unfortunately, there is hardly any work on the last problem on exploration of the team-goal. In many real world problems, particularly where tight cooperation of the members is required, such as carrying a stick [63] /pushing a box [52] by two (or more) robots in an environment with plenty of obstacles. Here, the moves that ensure reaching of one or more (but not all) agents to their individual goals are no longer useful. Such moves, if executed during planning, may not allow the agents, who have reached their goals, to perform any further actions. Thus there is an apparent deadlock as no team effort can keep the agents continue changing their states.

In this paper, we overcome the above problem by realization of the following strategy in the learning phase. The strategy includes allowing one or more agents, who could manage to reach their individual goals, to wait in their individual goal states for a significantly large time to give the remaining agents a chance to move to their respective (individual) goals synchronously or asynchronously. Such multi-phase state-transition to the team-goal offers one way to overcome the limitation of single-phase goal transition. Here, the goal transition in the last phase only accumulates high (immediate) reward contributed by an agent for the team, thereby improving the entries in the Q-table for state-transitions for the team-goal. The Q-table thus obtained offers the team the additional benefit to identify the joint action leading to transition to the team-goal.

One question that may be raised is how long the agents, who could manage to reach their goals, wait for the other agents to reach their subsequent goals. A small waiting interval may not be enough to allow all agents to reach their goals. On the other hand, a large interval may keep the entire team waiting at their team-goal. Thus selection of the right time-interval for the agents waiting at their individual goals is a crucial parameter, which in turn determines both the speed and planning performance of the agents.

The other important issue addressed in the chapter is the joint action selection. Here, the agents identify their preferred individual actions in combination with all possible actions by the other agents with an aim to determine the preferred joint action(s) of the team by taking the meet operation of such combinations. The joint action selection introduced above is useful for agents acting synchronously. As agents act synchronously, they do not require setting any priority to them like in [18]. In case no feasible joint action by the above method is available, the agents select individual actions randomly or by standard techniques (Boltzmann strategy and  $\epsilon$ -greedy) used in traditional Q-learning [6], [15], [37], [45] to construct the joint action.

The incorporation of the above two strategies in the MAQL enhances the planning performance of multi-agent systems as the Q-tables, defined in joint state-action space, is enriched with high reward values for state-transitions concerning exploration of the team-goal in the next joint state. The reward values stored in the joint Q-table are also adapted with greater rewards for next joint states geographically closer to the team-goal. The resulting joint Q-table would offer the right selection of next joint state, helping the agents reach the team-goal by an optimal/near-optimal path for transition from the starting joint state to the goal state.

A fast cooperative multi-agent Q-learning (FCMQL) and its associated multi-agent planning algorithm have been developed using the above two strategies in both deterministic and stochastic environment. Experiments undertaken confirm that the proposed algorithms outperform their existing competitors with respect to convergence time in learning and successful team-task in planning. In addition, the joint action selection employing Imperialist Competitive Firefly Algorithm (ICFA) [63], modified noise-resistant Particle Swarm Optimization (MNPSO) [54-55], Differential Evolutionary (DE) [56-57], and multi-robot joint action learning by demonstration (MLbD) [58] algorithms are compared separately with the FCMQL based multi-robot planning algorithms. The merits of the present work are now outlined below.

- 1) Two useful properties have been developed to speed-up the convergence of MAQL algorithms. Property 2.1 establishes the principles used to overcome the exploration of the team-goal. Property 2.2 directs an alternative approach to speed-up the convergence of MAQL by identifying the preferred joint action for the team.

- 2) Incorporation of the above two properties in TMAQL (including NQL, variants of CQL, MAQLET and FMRQ) results in significant improvement in speed of convergence.

- 3) In addition, because of an enriched Q-table to handle transitions to goal states, the proposed FCMQL induced planning algorithm can successfully complete the plan to reach the team-goal, where TMAQL based planning stops inadvertently.

4) Experiments have been developed to validate the performance of the proposed FCMQL with the contender algorithms in terms of the convergence speed and the run-time complexity as the performance metrics.

The rest of the chapter is structured as follows. Preliminaries of RL are reviewed in Section 2.3. Section 2.4 and 2.5 introduces the proposed FCMQL algorithms and Section 2.6 deals with multi-agent cooperative planning algorithms. Section 2.7 includes experiments and results. The conclusions are listed in Section 2.8.

## 2.2 Literature review

Quite a few interesting works on the MAQL have been reported in the literature [18], [22-40], [42-52], [59]. Among the state-of-the-art MAQL algorithms, the following need special mentions. In [24], Claus and Boutilier, aimed at solving the coordination problem using two types of reinforcement learners. The first one, called independent learner (IL) [24], takes care of the learning behavior of individual agents by ignoring the presence of other agents. The second one, called joint action learner (JAL) [24], considers all agents including the self to learn at joint action-space. Unlike JAL, in Team Q-learning [59] proposed by Littman, an agent updates its Q-value at a joint state-action pair without utilizing associated agents' reward; rather the value function of the agent at the next joint state is evaluated by obtaining the maximum Q-value among the joint actions at the next joint state. In [37], Ville proposed Asymmetric-Q learning (AQL) algorithm, where the leader agents are capable of maintaining all the agents Q-tables. However, the follower agents are not allowed to maintain all the agents' Q-tables and hence, they just maximize their own rewards. In AQL, agents always achieve the pure strategy Nash equilibrium (NE), although there does exist mixed strategy NE [27-28]. In [27], Hu and Wellman extended the Littman's Minimax Q-learning [40] to general-sum stochastic game (where summation of all agents' payoff is neither zero nor constant) [16], [41] by taking into account of other agents' dynamics using NE [27], [28], [60]. They also offered a proof of convergence of their algorithm [42]. In [43] and [44], the authors selected one NE optimally in case of its multiple occurrences. In [30], Littman proposed, Friend-or-Foe Q-learning algorithm for general-sum games. In this algorithm, the learner is instructed to treat each other agent either as a friend in Friend Q-learning (FQL), or as a foe in Foe Q-learning. Friend-or-Foe Q-learning provides a stronger convergence guarantee in comparison to that of the existing NE based learning rule [27-28]. In [26], Greenwald and Hall proposed CQL employing correlated equilibrium (CE) [26] to generalize both NQL [27] and Friend-or-Foe Q-learning [30]. The bottlenecks of the above MAQL algorithms are update policy selection for adaptation of the Q-tables in joint state-action space and the curse of dimensionality with the increase in number of learning agents. Several attempts have been made to handle the curse of dimensionality in MAQL. Jelle and Nikos

proposed Sparse Cooperative Q-learning in [50], where a sparse representation of the joint state-action space of the agents is done by identifying the need of coordination among the agents at a joint state. In [50], agents undertake coordination by their actions only in a few joint states. Hence, each agent maintains two Q-tables: one is the individual-action Q-table for un-coordinated joint states and other one is the joint action Q-table to represent the coordinated joint states. In case of uncoordinated states, a global Q-value is evaluated by adding the individual Q-values. In [49], authors offer a neural network based approach for generalized representation of the state-space for multi-agent coordination. By such generalization, agents (here robots) can avoid collision with an obstacle or other robots by collecting minimum information from the sensors. In [51], Reinaldo *et al.* proposed a novel algorithm to heuristically accelerate the TMAQL algorithms. In the literature of MAQL [18], [22-40], [42-52], [59] agents either converge to NE or CE. The equilibrium-based MAQL algorithms [26], [27] are most popular for their inherent ability to determine optimal strategy (equilibrium) at a given joint state. In [48], Hu *et al.* identified the phenomenon of similar equilibria in different joint states and introduced the concept of equilibrium transfer to accelerate the state-of-the-art equilibrium-based MAQL (NQL and CQL). In equilibrium transfer, agents recycle the previously computed equilibria having very small transfer-loss. Recently in [52], Zhang *et al.* attempted to reduce the dimension of the Q-tables in NQL. The reduction is done by allowing the agents to store the Q-values in joint state-individual action space, instead of joint state-action space. However, with the best of our knowledge there is no work in the literature, which considers simultaneous exploration of the individual goals (i.e., team-goal) of the agents.

In the state-of-the-art MAQL (NQL [27], [28] and CQL [26]), balancing exploration/exploitation during the learning phase is an important issue. Traditional approaches used to balance exploration/exploitation in MAQL are summarized here. The  $\epsilon$  – greedy exploration [6], although has wide publicity, needs to tune the value of  $\epsilon$ , which is time-costly. In the Boltzmann strategy [15], the action selection probability is controlled by tuning a control parameter (temperature) [15] and by utilizing the Q-values due to all actions at a given state. Here, the setting of temperature to infinity (zero) implies pure exploration (exploitation). Unfortunately, the Boltzmann strategy antagonistically affects the speed of learning [45]. Evolution of the Boltzmann strategy towards better performance is observed in [38] and [45]. However, the above selection mechanisms are not suitable for selecting a joint action preferred for the team (all the agents) because of the dissimilar joint Q-values offered by the agents at a common joint state-action pair. There are traces of literature concerning joint action selection at a joint state during learning. In [54], Jim and Alcherio employ a MNPSO, where each agent is considered as a swarm and they can communicate with each

other. In [58], the joint action for multi-robot cooperation is selected by learning simultaneous demonstration [58].

## 2.3 Preliminaries

The section presents preliminaries of RL, single agent Q-learning and MAQL concisely. In RL [6], [8], an agent interacts with the environment, by means of a 3-tuple <state ( $s$ ), action ( $a$ ), reward ( $r$ )>. A state refers to the current position of an agent (here robot) within an environment. By executing an action in the current state, the agent receives a scalar reward from the environment and moves to the next state. The scalar reward acts as a feedback for the agent on its immediate performance. Fig. 2.1 provides a schematic overview of RL.

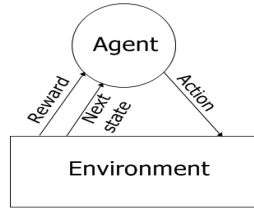


Fig. 2.1 Block diagram of reinforcement learning (RL)

### 2.3.1 Single agent Q-learning

Single agent Q-learning, proposed by Watkins and Dayan, is one of the most widely used RL techniques [20]. In single agent Q-learning, the environment is divided into a finite number of states. At any state, an agent has a finite set of actions, from which it can choose one according to a given policy. The agent learns optimal state-action value (Q-value) for each state-action pair using the principle of DP and BE [21]. In single agent Q-learning, the agent attempts to determine the optimal policy in order to maximize the sum of discounted expected rewards [11]. The single agent Q-learning update rule is given by (2.1) [20]:

$$Q(s, a) \leftarrow (1 - \alpha)Q(s, a) + \alpha[r(s, a) + \gamma \sum_{s'} P[s' | (s, a)] \text{Max}_{a'} Q(s', a')]. \quad (2.1)$$

where,  $Q(s, a)$  and  $r(s, a)$  are the Q-value and immediate reward respectively at state  $s$  due to action  $a$ ,  $a' \in \{a\}$  is the action in the next state  $s' \in \{s\}$ ,  $s' \leftarrow \delta(s, a)$  is the state-transition function,  $\gamma \in [0, 1)$  denotes the discounting factor and  $\alpha \in [0, 1)$  refers to the learning rate. However, in the deterministic situation, the state-transition probability  $P[s' | (s, a)]$  to reach the next state  $s' \in \{s\}$  from the state  $s$  because of action  $a$  is unity. After infinite revisit of  $(s, a)$ , Q-value  $Q(s, a)$  turns to the optimal Q-value  $Q^*(s, a)$ .

### 2.3.2 Multi-agent Q-learning

Unlike single agent Q-learning presented above, in MAQL, the joint Q-value depends on the other agents' actions too. In MAQL, MDP is extended to MMDP [23], [32]. The definition of MMDP is given below.

**Definition 2.1:** A *multi-agent MDP (MMDP)* for  $m$  number of agents can be defined as a 5-tuple  $\langle \{S\}, m, \{A\}, P_i, R_i \rangle$ ,

where,

$\{S\} = \times_{i=1}^m \{s_i\}$  is the joint state space,  $S \in \{S\}$  and  $s_i \in \{s_i\}$  is the state of agent  $i$ ,  $\times$

denotes the Cartesian product,

$\{A\} = \times_{i=1}^m \{a_i\}$  is the joint action space,  $A \in \{A\}$  and  $a_i \in \{a_i\}$  is the action of agent  $i$ ,

$P_i : \{S\} \times \{A\} \times \{S\} \rightarrow [0,1]$  is the joint state-transition probability of agent  $i$ ,

and  $R_i : \{S\} \times \{A\} \rightarrow \mathbb{R}$  is the reward function at a joint state-action pair of agent  $i$ ,

where  $\mathbb{R}$  is the set of real numbers.

MAQL algorithms [18], [22-40], [42-52], [59] usually are of three types: cooperative, competitive and mixed [47]. In this paper, we deal with cooperative MAQL algorithms, where all the agents adapt Q-tables in a common environment. Because of the adaption in a common environment, the environment becomes dynamic, and an agreement is needed among the agents to attain optimal performance of the team. Such agreement is attained by adapting the joint Q-values in equilibrium, e.g., NE [60] and CE [26]. Both NE and CE employ a) pure strategy and b) mixed strategy. The definitions of NE [60] and CE [26] are given below.

**Definition 2.2:** *Nash equilibrium* is a stable joint action (or strategy) at a given joint state ( $S$ ) of a system that involves  $m$  interacting agents, such that no unilateral deviation (deviation of an agent independently) can occur as long as all the agents follow the same optimal joint action  $A_N = \langle a_i^* \rangle_{i=1}^m$  at a joint state  $S \in \{S\}$  for *pure strategy NE*. Further, for a *mixed strategy NE*, agents perform the joint action  $A = \langle a_i \rangle_{i=1}^m$  with a probability

$$p^*(A) = \prod_{i=1}^m p_i^*(a_i), \text{ where } p_i^* : \{a_i\} \rightarrow [0,1], p^* : \{A\} \rightarrow [0,1].$$

Let  $a_i^* \in \{a_i\}$  be the optimal action of agent  $i$  at  $s_i$  and  $A_{-i}^* \subseteq A$  be the optimal joint action profile of all agents except agent  $i$  at joint state  $S = \langle s_j \rangle_{j=1, j \neq i}^m$  and  $Q_i(S, A)$  be the joint Q-value of agent  $i$  at  $S$  because of joint action  $A \in \{A\}$ . Then the condition of pure strategy NE at  $S$  is [60]

$$Q_i(S, a_i^*, A_{-i}^*) \geq Q_i(S, a_i, A_{-i}^*), \forall i$$

$$\Rightarrow Q_i(S, A_N) \geq Q_i(S, A'), \quad \forall i \quad [\text{where } A_N = \langle a_i^*, A_{-i}^* \rangle \text{ and } A' = \langle a_i, A_{-i}^* \rangle] \quad (2.2)$$

and condition of mixed strategy NE at  $S$  is [60]

$$Q_i(S, p_i^*, p_{-i}^*) \geq Q_i(S, p_i, p_{-i}^*), \quad \forall i, \quad (2.3)$$

where,  $Q_i(S, p) = \sum_{\forall A} p(A)Q_i(S, A)$  and  $p_{-i}^*(A_{-i}) = \prod_{j=1, j \neq i}^m p_j^*(a_j)$  be the joint probability of selecting joint action profile of all agents except agent  $i$  denoted by  $A_{-i} \subseteq A$ .

Agents follow (2.2) to evaluate pure strategy NE  $A_N = \langle a_i^*, A_{-i}^* \rangle$  and (2.3) for mixed strategy NE  $\langle p_i^*(a_i), p_{-i}^*(A_{-i}) \rangle$  respectively at joint state  $S$ . Evaluation of NE employing Lemke-Howson method [65] is quiet efficient but limited to two agents problem only. In this chapter, to evaluate NE a simple search method following [66] has been employed.

In NE, agents are allowed to maximize its own reward. However, in CE, the composite benefits of the agents are considered by selecting the individual actions jointly. In [26], authors outline four variants of CE: Utilitarian equilibrium (UE) (representing sum of all the agents' rewards), Egalitarian equilibrium (EE) (computed by taking minimum of all the agents' rewards), Republican equilibrium (RE) (obtained by taking maximum of all the agents' rewards) and Libertarian equilibrium (LE) (which multiplies all the agents' rewards) to evaluate a joint strategy (action).

**Definition 2.3:** *Correlated equilibrium (CE)* at a joint state,  $S = \langle s_i \rangle_{i=1}^m$  with  $m$  interacting agents is the pure strategy CE,  $A_C$  and mixed strategy CE,  $p^*(A_C)$  if agents follow (2.4) and (2.5) respectively [26].

$$A_C = \arg \max_A [\Phi(Q_i(S, A))] \quad (2.4)$$

$$p^*(A_C) = \arg \max_{p(A)} [\Phi(\sum_A p(A)Q_i(S, A))] \quad (2.5)$$

$$\text{where, } \Phi \in \left\{ \sum_{i=1}^m, \min_{i=1}^m, \max_{i=1}^m, \prod_{i=1}^m \right\}. \quad (2.6)$$

In [27], Hu and Wellman proposed NQL with the help of NE to update the reward of the agent at joint state-action space. Similarly, in [26], Greenwald and Hall proposed CQL with the help of CE to update the reward of the agent at joint state-action space. Later in [48], Hu *et al.* attempted to accelerate the NQL and CQL by equilibrium transfer. Recently in [52], Zhang *et al.* attempted to reduce the dimension of the Q-tables in NQL. The reduction is done by allowing the agents to store the Q-values in joint state-individual action space, instead of storing them in joint state-action space. The above mentioned NE/CE-based algorithms are summarized in Algorithm 2.1. In Algorithm 2.1,  $r_i(S, A)$  refers to the immediate reward of agent  $i$  given by (2.7), where  $r_{\max}$  and  $r_{\min}$  are the maximum and minimum immediate rewards respectively.



$$\left. \begin{aligned}
r_i(S, A) &= r_{\max}, \text{ if agent } i \text{ reaches its individual goal,} \\
&= r_{\min}, \text{ if agent } i \text{ does not reach its individual goal,} \\
&= -r, \text{ if agent } i \text{ violates constraint, } r \in \mathbb{R}^+.
\end{aligned} \right\} \quad (2.7)$$

**Algorithm 2.1: NE/CE-based Multi-agent-Q Learning**

**Input:** Current state  $s_i, \forall i$ , action set  $\{a_i\}$ ,  $\mu$  is a small positive threshold to stop the algorithm,  $\gamma \in [0, 1)$  and  $\alpha \in [0, 1)$ ;

**Output:** Joint Q-value of agent  $i$   $Q_i^*(S, A), \forall S, \forall A, \forall i$ ;

**Initialize:**  $Q_i(S, A) \leftarrow 0, \forall S, \forall A, \forall i$ ;

**Repeat**

Observe the current state  $s_i, \forall i$ ;

Randomly select an action  $a_i \in \{a_i\}$  at  $s_i$  and execute it  $\forall i$ ;

Receive  $r_i(S, A), \forall i$ , evaluate next state  $s_i' \leftarrow \delta_i(s_i, a_i), \forall i$  to obtain next joint state  $S' = \langle s_i' \rangle_{i=1}^m$ ;

$Q_i'(S, A) \leftarrow Q_i(S, A), \forall i$ ;

**Update:**

$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \sum_{S'} P_i[S' | (S, A)] \Psi Q_i(S')], \forall i$  // for stochastic

$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \Psi Q_i(S')], \forall i$  //for deterministic,  $\Psi \in \{NE, CE\}$   
and  $S \leftarrow S'$ ; //  $\Psi Q_i(S')$  is the Q-value of agent  $i$  due to  $\Psi \in \{NE, CE\}$  at joint state  $S'$

**Until**  $|Q_i'(S, A) - Q_i(S, A)| < \mu, \forall S, \forall A, \forall i$ ;

Obtain  $Q_i^*(S, A) \leftarrow Q_i(S, A), \forall S, \forall A, \forall i$ .

**Complexity analysis:** To analyze complexities of the Algorithm 2.1 [27], [52] let for  $m$  number of agents  $\{S\}$  be the set of joint states and  $\{A\}$  be the set of joint actions from each joint state  $S \in \{S\}$ . In NQL, CQL and MAQLET, an agent maintains Q-table at joint state-action space. In the absence of communication [27], an agent has to maintain all the agents' Q-tables at joint state-action space. So, the *space complexity* of the NQL, CQL, and MAQLET algorithms is  $m|\{S\}||\{A\}|$ . However, in FMRQ, an agents adapts Q-values at a joint state for each individual actions. Therefore, the *space complexity* of the FMRQ algorithms is  $m|\{S\}||\{a\}|$ , where  $\{a_1\} = \{a_2\} = \dots = \{a_m\} = \{a\}$ . Also in the TMAQL, an agent updates all the agents' Q-values at the current joint state-action pair by selecting the Q-values in the next joint state at NE/CE in each learning epoch. So, the *time complexity* to evaluate NE (considering pure strategy NE) is  $(|\{A\}|-1) \cdot |\{A\}|^{m-1} = O(|\{A\}|)^m$  and *time complexity* to evaluate the pure strategy CE is  $(m-1)(|\{A\}|-1) = O(m|\{A\}|)$ .

## 2.4 Proposed multi-agent Q-learning

The Algorithm 2.1 presented above, suffers from two limitations: i) exploration of the team-goal and ii) joint action selection. In addition, overcoming these limitations derive additional benefit in subsequent planning stage to optimally select the team-goal. We here briefly outline the possible ways to overcome the limitations stated above.

In this section, we propose two important properties to overcome the above limitations and subsequently increase the speed of convergence of MAQL algorithms. In the first property, when one agent reaches its goal, it would remain idle, while its teammates continue exploration for their respective goals. The second property ascertains selection of a joint action at a joint state corresponding to the least reward of all the agents. It is shown that such selection accelerates the learning of the Q-table in MAQL algorithms. It is shown that the convergence speed of the proposed FCMQL algorithms is more than the same of the TMAQL (NQL, CQL, MAQLET, and FMRQ) algorithms. We now define a new term, called contributed reward by agent  $i$  for the team denoted by  $CR_i$ .

**Definition 2.4:** The contributed reward by agent  $i$  to achieve the team-goal is a scalar quantity:  $CR_i(S, A)$ , defined at joint state  $S$  due to joint action  $A$ , is given by

$$\left. \begin{aligned} CR_i(S, A) &= r_{\max}, \text{ if all agents reach their goals simultaneously,} \\ &= r_{\min}, \text{ if atleast one agent is left to reach its goal,} \\ &= -r, \text{ if atleast one agent violates constraint, } r \in \mathbb{R}^+. \end{aligned} \right\} \quad (2.8)$$

where,  $r_{\max}$  and  $r_{\min}$  are the maximum and minimum immediate rewards respectively.

Violation of constraints generally indicates collision among the team-mates.

$$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[CR_i(S, A) + \gamma \sum_{S'} P_i[S' | (S, A)] \Psi Q_i(S')] \quad (2.9)$$

where,  $\gamma \in [0,1)$  and  $\alpha \in [0,1)$  refer to the discounting factor and learning rate respectively.

The joint state-transition function designed following TMAQL is given below:

$$\left. \begin{aligned} \delta(S, A) &= \langle \delta_i(s_i, a_i) \rangle_{i=1}^m \\ &= \langle s'_i \rangle_{i=1}^m \\ &= S' \\ &\in \{S\} \end{aligned} \right\} \quad (2.10)$$

and  $P_i[S' | (S, A)]$  denotes the joint state-transition probability of agent  $i$  to reach the next joint state  $S' \in \{S\}$  from the joint state  $S$  because of joint action  $A$ .  $\Psi Q_i(S')$  is the Q-value of agent  $i$  because of  $\Psi \in \{NE, CE\}$  at next joint state  $S'$  and is evaluated by a simple search method [66].

### 2.4.1 Two useful properties

The properties based on which the FCMQL algorithms are being developed are discussed below. The properties are valid both in deterministic and stochastic situations. Property 2.1 is derived using Statute 2.1 given below.

**Statute 2.1:** *Unlike the TMAQL in the proposed FCMQL, when an agent moves to its goal state, it will not restart the learning process by randomly selecting a state (excluding its goal state); rather it waits in its goal state, and will restart learning along with all other agents, when the last agent moves to its individual goal state.*

**Property 2.1:** *In MAQL, if all the agents follow Statute 2.1, then the probability of exploring the team-goal monotonically increases with  $k$  in a learning episode, where  $k$  refers to the number of agents explore their individual goals.*

**Proof:** See Appendix 2.1 (Section 2.10).

Besides exploration of the team-goal, to speed-up learning further, Property 2.2 is proposed. For the sake of convenience of the readers, the definition of preferred joint action is given below.

**Definition 2.5:** If  $Q_i(S,A)$  refers to the Q-value of agent  $i$  at joint state  $S$  because of joint action  $A$ , then the set of *preferred joint actions*  $\{A_i^P\}$  of agent  $i$ , is obtained by (2.11).

$$\{A_i^P\} = \arg \max_A [Q_i(S,A)]. \quad (2.11)$$

We also define the common preferred set of joint actions of  $m$  agents as

$$\{A^P\} = \bigcap_{i=1}^m \{A_i^P\}. \quad (2.12)$$

In case  $\{A^P\} = \emptyset$ , the agents would select their individual preferred actions randomly or by traditional selection techniques [6], [15], [53], [45].

In Property 2.2 introduced below, if at a joint state  $S$  only one joint action  $A$  remains non-utilized, then the probability to execute  $A$  at  $S$  becomes one, i.e., the joint action  $A$  is selected certainly for execution. This, in other words, indicates that the joint actions already taken in a joint state should not be repeated, until all the joint actions at that joint state have been explored.

**Property 2.2:** *In MAQL, if  $\{A^P\}$  is the set of equally preferred joint action for the team at the joint state  $S$  where  $\{A^P\} = \bigcap_{\forall i} \{A_i^P\} = \bigcap_{\forall i} \{\arg \min_A [Q_i(S,A)]\} \neq \emptyset$ , then  $P(S, A^{P'}) > P(S, A^P)$*

where  $A^{P'}$  refers to the preferred joint action in the next iteration

**Proof:** See Appendix 2.1 (Section 2.10).

In Property 2.2, if an agent  $i$  receives a penalty (reward of  $-r$  by (2.8)) at a joint state  $S$  because of a joint action  $A$  before improving its joint Q-value  $Q_i(S,A)$  from the initialized value (generally zero), then the agent is trapped at the former joint state  $S$ . To overcome such

problem  $Q_i(S,A)$  is re-initialized to zero. Such re-initialization improves the speed of convergence of the proposed FCMQL by avoiding the trapping at local minima.

## 2.5 Proposed FCMQL algorithms and their convergence analysis

In this Section, we propose FCMQL algorithms with their convergence analysis, where FCMQL refer to a set of algorithms given by {NQLP12, EQLP12, UQLP12, RQLP12, LQLP12, FMRQP12}.

### 2.5.1 Proposed FCMQL algorithms

In Algorithm 2.2, the proposed FCMQL algorithms enjoy the benefits of the proposed Property 2.1 and Property 2.2, which are responsible for exploring the team-goal rapidly and speeding-up learning process respectively. In Algorithm 2.2, we compute pure strategy NE, if it exists; otherwise mixed strategy NE is evaluated. However, it may please be noted that in the proposed FMRQP12 and its associated variants, agents maintain Q-tables at joint state-individual action space denoted by  $Q_i(S,a_j), \forall S, \forall a_j, \forall i$  [52].

**Complexity analysis:** To analyze the complexity of the proposed Algorithm 2.2, let there be  $m$  number of agents. Let  $\{S\}$  and  $\{A\}$  be the set of joint states and joint actions respectively.

The *space complexity* of the proposed FCMQL algorithms, except FMRQP12, is given by  $m|\{S\}| \cdot |\{A\}|$ , where for the latter the complexity is  $m|\{S\}| \cdot |\{a\}|$ , where  $\{a_1\} = \{a_2\} = \dots = \{a_m\} = \{a\}$ . Now, referring to Property 2.1, the best- and worst- case time complexities in one learning epoch for an agent in the proposed FCMQL algorithms, in the absence of communication [28] are given by  $(|\{a\}|-1) = O(|\{a\}|)$  and  $(|\{A\}|-1)|\{A\}|^{m-1} = O(|\{A\}|^m)$  respectively.

### Algorithm 2.2: Fast Cooperative Multi-agent Q-learning (FCMQL)

**A. Input:** Current state  $s_i, \forall i$ , joint action set  $\{A\}$ ,  $\mu$  is a small positive threshold to stop the algorithm,  $\gamma \in [0,1)$  and  $\alpha \in [0,1)$ ;

**B. Output:** Joint Q-value of agent  $i$   $Q_i^*(S, A), \forall S, \forall A, \forall i$ ;

**C. Initialize:**  $Q_i(S, A) \leftarrow 0, \forall S, \forall A, \forall i$ ;

**Repeat**

1. Observe the current state  $s_i, \forall i$ ;
2. **If**  $\bigcap_{\forall i} \{\arg \min_A [Q_i(S, A)]\} \neq \emptyset$ 
  - Then** select a joint action  $A \in \{A\}$  employing Property 2.2; // by Property 2.2
  - Else** Randomly select an action  $a_i \in \{a_i\}, \forall i$ ;
- End If**;
3. Receive immediate reward  $r_i(s_i, a_i), \forall i$  and evaluate  $CR_i$  by Definition 2.4;
4. Evaluate next state  $s_i' \leftarrow \delta_i(s_i, a_i), \forall i$  and joint next state  $S' = \langle s_i' \rangle_{i=1}^m$ ;
5. **If**  $s_i = g_i$  holds for  $i < m$  //  $g_i$  is the goal state of agent  $i$ 
  - Then** the agent  $i$  waits at  $g_i$ , until  $S_{-i} = G_{-i}$  //where  $-i$  indicates all except agent  $i$  or up to a finite time  $T_f$  obtained empirically;
  - Else** select a joint action  $A \in \{A\}$  by step 2; // by Property 2.1
- End If**;
6. **If**  $s_i = g_i, \forall i$ 
  - Then** restart learning by randomly selecting a joint state (except team-goal state);
- End If**;
7.  $Q_i'(S, A) \leftarrow Q_i(S, A), \forall i$ ;
8. **Update:**  $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[CR_i(S, A) + \gamma \Psi Q_i(S')]$ ; //for deterministic  
 $Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[CR_i(S, A) + \gamma \sum_{S'} P_i[S' | (S, A)] \Psi Q_i(S')]$ ; //for stochastic  
and  $S \leftarrow S'$  //  $\Psi Q_i(S')$  be the Q-value of agent  $i$  due to  $\Psi \in \{NE, CE\}$  at joint state  $S'$

**Until**  $|Q_i'(S, A) - Q_i(S, A)| < \mu, \forall S, \forall A, \forall i$ ;

Obtain  $Q_i^*(S, A) \leftarrow Q_i(S, A), \forall S, \forall A, \forall i$ .

## 2.5.2 Convergence analysis of the proposed FCMQL algorithms

The convergence of the proposed FCMQL is compared with the TMAQL in Theorem 2.1 and is given below.

**Theorem 2.1:** *The expected time of convergence of the proposed FCMQL is less than the same of the TMAQL.*

**Proof:** See Appendix 2.1 (Section 2.10).

## 2.6 FCMQL-based cooperative multi-agent planning

In this section, the proposed FCMQL based cooperative multi-agent planning algorithms are discussed. In the proposed FCMQL based multi-agent planning, agents move from the current joint state to the next joint state following the principle of pure strategy NE/CE, which is evaluated by utilizing the joint Q-tables adapted by the FCMQL and satisfying the task-constraint. Here, task-constraint refers to the constraint which agents have to satisfy during the planning phase in the deterministic and/or the stochastic environment. Consider the problem of object-carrying, where an object (stick, triangle or square) needs to be transported to a desired location with the help of multiple robotic agents that hold the stick at its two extremities [63], triangle at its vertexes and rectangle at its corners. Robots maintain a fixed distance between them to avoid falling off of the object carried by them. Holding the object without a fall is considered as a task-constraint. This is a problem, where cooperation of two (or more) robots is needed for the required transportation problem.

In TMAQL, the immediate reward is given in (2.7), and is designed to measure the individual agent's performance. However, in the proposed FCMQL, the immediate reward is given by (2.8) and is designed to measure the team performance. The benefit of such proposed reward function during the planning phase is realized and analyzed in the proposed FCMQL based planning (Algorithm 2.3) and Theorem 2.2 respectively, in terms of optimal team performance, measured by the number of joint state-transitions required to reach the team-goal. The Definition of the optimal team performance is given in Definition 2.6. The FCMQL based cooperative multi-agent planning is given in Algorithm 2.3.

**Definition 2.6:** If the planning algorithm evaluates joint state-transitions following NE/CE, and the terminal state-transition ends at the team-goal, then the agents are called to have the *optimal team performance* considering the number of joint state-transitions required to reach the terminal (team-goal) joint state as the performance metric.

Theorem 2.2 shows that the non-team goal state transitions cannot be a NE in the proposed NQLP12 based multi-agent planning. However, it does a NE in the TMAQL based multi-agent planning.

**Theorem 2.2:** *If all excluding at least one agent explores its individual goal state employing the proposed NQLP12 or TMAQL due to joint action  $A_N$  at joint state  $S$ , then in the NQLP12 induced planning,  $A_N$  is not a NE, but in TMAQL induced planning,  $A_N$  is a NE at  $S$ .*

**Proof:** See Appendix 2.1 (Section 2.10).

By Theorem 2.2 one can confirm that in NQLP12 based cooperative multi-agent planning (Algorithm 2.3), an agent never executes a joint action, which results in a goal state-transition of at least one agent. However, in the TMAQL induced planning, agents do prefer such joint

actions. The agent which reaches its individual goal in the planning phase cannot move any more, resulting in low probability to reach the team-goal. Trivially it can also be shown that in the other variants of the proposed FCMQL induced multi-agent planning, agents do not prefer a joint action which leads to a non team-goal state transition, except in the proposed RQLP12.

**Algorithm 2.3 FCMQL based cooperative multi-agent planning**

**Input:**  $Q_i, \forall i$ , feasible joint state  $S_F$ ;

**Output:** NE (or CE)  $A_N$  (or  $A_C$ ) at  $S_F$ ;

**Repeat**

- Observe current state  $s_i, \forall i$ ;
- Evaluate NE (or CE),  $A_N$  (or  $A_C$ ) following (2.2) (or (2.4)) and satisfying task constraint;
- Execute  $A_N$  (or  $A_C$ ) at  $S_F$  and go to next feasible joint state  $S'_F$  and  $S_F \leftarrow S'_F$ ;
- For multiple  $A_N$  (or  $A_C$ ) solutions at  $S_F$  select the first one;

**Until** the task is complete.

## 2.7 Experiments and results

This section includes four experiments. The first experiment is designed to examine the relative performance of the proposed FCMQL algorithms over the reference algorithms in view of the team-goal exploration, considering convergence speed as the performance metric. The second experiment is designed to compare the performance of the proposed Algorithm 2.3 over the reference algorithms. The third experiment examines the merits of the proposed

<b>S1</b>	<b>S2</b>									
10	20	30	40	50	60	70	80	90	100	
9	19	29	39	49	59	69	79	89	99	
8	18	28	38	48	58	68	78	88	98	
7	17	27	37	47	57	67	77	87	97	
6	16	26	36	46	56	66	76	86	96	
5	15	25	35	45	55	65	75	85	95	
4	14	24	34	44	54	64	74	84	94	
3	13	23	33	43	53	63	73	83	93	
2	12	22	32	42	52	62	72	82	92	
1	11	21	31	41	51	61	71	81	91	
								<b>G2</b>	<b>G1</b>	

Fig. 2.2 Experimental workspace for two agents during the learning phase

Algorithm 2.3 over the existing ones, including joint action selection by MLbD [58], MNPSO [54], [55] DE [56], [57] and ICFA [63] algorithms using run-time complexity as the metric. The computer simulations undertaken for the experiments are coded and tested with an Intel(R) Core(TM) i7-3770 CPU with a clock speed of 3.40GHz. Finally, in the last experiment we examine the performance of the proposed Algorithm 2.3 in real environment with twin Khepera-II mobile robots [64].

Table 2.2 Details of 10×10 grid maps

Number of agents	situation	map	team-goal	Joint starting state	number of obstacles	obstacle state number
2	Deterministic	1	81, 91	10, 20	12	9, 27, 40, 46, 52, 54, 58, 61, 63, 67, 82, 84
2		2	55, 65	45, 55	6	25, 48, 53, 57, 68, 75
2		3	55, 65	45, 55	8	25, 46, 48, 53, 57, 66, 68, 75
3		7	9, 20, 19	72, 81, 71	6	8, 28, 45, 49, 73, 86
4		8	81, 82, 92, 91	10, 20, 19, 9	7	16, 29, 33, 41, 47, 64, 83
2	Stochastic	4	81, 91	10, 20	12	9, 27, 40, 46, 52, 54, 58, 61, 63, 67, 82, 84
2		5	55, 65	45, 55	6	25, 48, 53, 57, 68, 75
2		6	55, 65	45, 55	8	25, 46, 48, 53, 57, 66, 68, 75
3		9	9, 20, 19	72, 81, 71	6	8, 28, 45, 49, 73, 86
4		10	81, 82, 92, 91	10, 20, 19, 9	7	16, 29, 33, 41, 47, 64, 83

All the experiments are studied in ten different 10×10 grid world maps given in Table 2.2. Fig. 2.2 indicates map 1 (for deterministic) or 4 (for stochastic) in a two agent system during the learning phase with twelve obstacles (marked as a black rectangle) and a team-goal  $\langle G1, G2 \rangle$ . Each agent can perform four actions: such as moving Left (L), Forward (F), Right (R) and Back (B). In case of stochastic environment, the state-transition probabilities are assigned as randomly generated constant values, satisfying the property of a Markovian matrix, where the sum of state-transition probabilities at each state is unity. Like the TMAQL algorithms, the starting positions can be selected randomly for a fixed team-goal state, which is predefined during the learning. To maintain uniformity, all the algorithms are initialized with



identical joint starting states. Each grid in the multi-robots' workspace is assigned a positive integer to indicate its identity in the workspace with the help of mapping functions defined in Appendix 2.2 (Section 2.11) [64].

The discounting factor  $\gamma$  is chosen as 0.9 and learning rate  $\alpha$  is set to 0.1. On exploration of the team-goal, agents are awarded by (2.8), where maximum immediate reward  $r_{\max} = 100$ , minimum immediate reward  $r_{\min} = 0$ . In addition, the violation of constraint is penalized by a reward of  $r = -1$ .

**Experiment 2.1 (Study of convergence speed):** This experiment aims at examining the relative superiority in convergence speed of the proposed FCMQL over the existing algorithms. The study includes: a) convergence of state-action pairs with learning epochs, b) determining the number of times a given team-goal is explored within a fixed number of learning epochs, c) average reward of  $m$  agents, where the reward of an agent is the average of the entries in the Q-table, d) convergence in state-action pairs with learning epochs, where joint action selection is done by Property 2.2.

The results of the first study are given in Fig. 2.3, developed for NQL and in Fig. A2. 2.1-A2. 2.3 in Appendix 2.2 (Section 2.11) [64]. It is apparent from Fig. 2.3 and Fig. A2. 2.1-A2. 2.3 that the FCMQL outperforms reference algorithms with respect to number of joint state-action pairs converged ( $N_c$ ). Here, the FMRQP12 is not compared with reference algorithms. As in FMRQ, an agent does not adapt its Q-value in joint state-action space; instead the Q-values are adapted in joint state-individual action space. However, FMRQP12 is compared in the later part of the same experiment.

It is interesting that in the second to seventh columns of the Table A2. 2.1 and A2. 2.2 [64], the proposed FCMQL algorithms designed with Property 2.1 outperform the realization of FCMQL with Property 2.2 in the measure of  $N_c$ . But gradually at higher learning epochs (eighth, ninth and tenth columns of Table A2. 2.1 and A2. 2.2 [64]), the superiority of the proposed FCMQL algorithms realized with Property 2.2 is observed over the FCMQL designed with Property 2.1.

The result of the second study emphasizes that the proposed FCMQL, when developed with Property 2.1, yields a high count in the team-goal exploration in comparison to its traditional counterpart, including those developed with Property 2.2 (see Fig. A2. 2.4, Table A2. 2.3-A2. 2.4 [64]).

The third study reveals that the average reward of  $m$  agents, denoted by average of average reward (AAR) (where reward of an agent being measured by its average Q-table value) evaluated for the proposed FCMQL exceeds that of the TMAQL. The high value in AAR of the agents indicates that early convergence in FCMQL in comparison to TMAQL (see Fig. 2.4 and Fig. A2. 2.5-A2. 2.7 [64]).

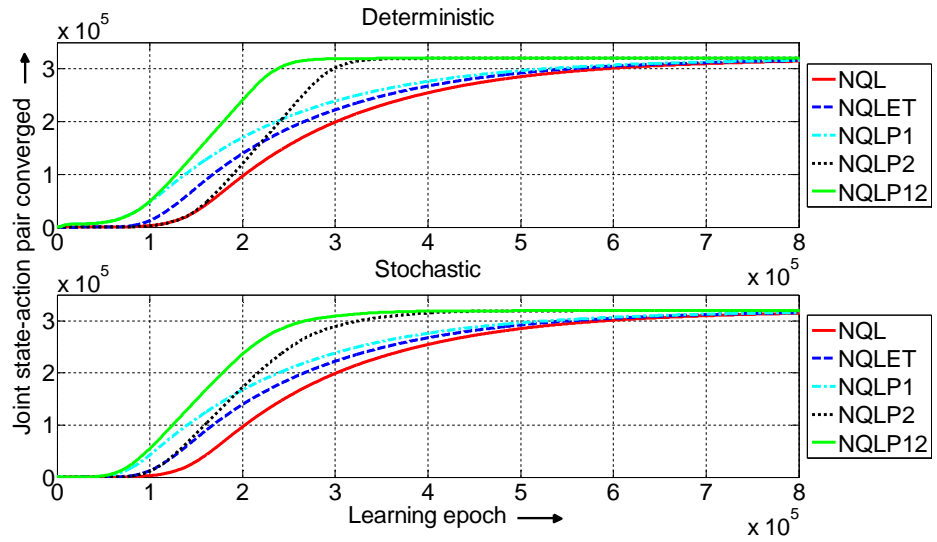


Fig. 2.3 Convergence plot of NQLP12 and reference algorithms for two agents

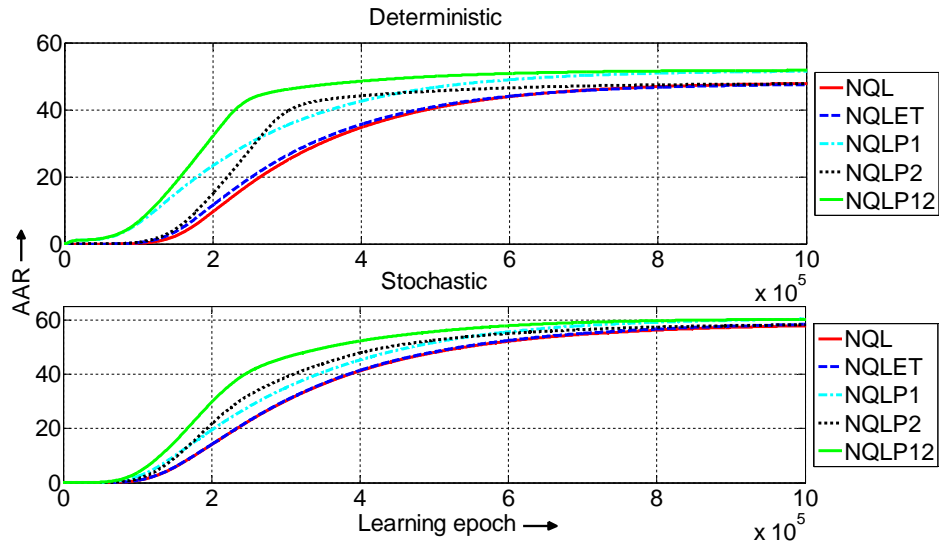


Fig. 2.4 Average of average reward (AAR) plot of NQLP12 and reference algorithms for two agents

The last study dealing with convergence of joint state-action pairs with learning epochs, while embedding Property 2.2 in designing FCMQL, yields larger value in convergence of joint state-action pairs than same obtained by the TMAQL (see Fig. 2.5 and Fig. A2. 2.8-A2. 2.9 [64]).

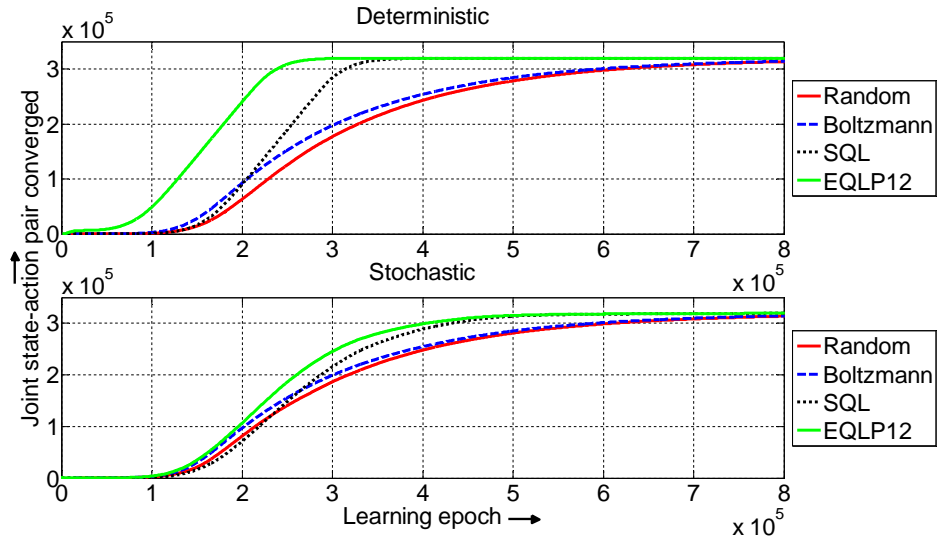
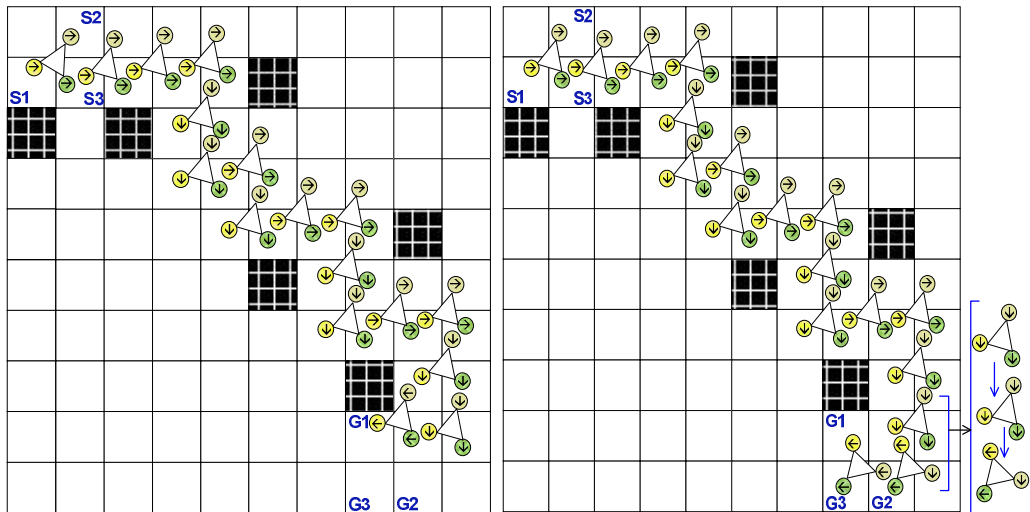


Fig. 2.5 Joint action selection strategy in EQLP12 and reference algorithms for two agents

**Experiment 2.2 (Planning performance):** The motivation of the present study is to examine the completion of a task in planning and is tested with the well-known object-carrying problem on a  $10 \times 10$  grid-map by 2, 3 and 4 agents. Fig. 2.6 and Fig. A2. 2.10-A2. 2.14 in [64] offer the planned paths for robot team toward the predefined team-goal. It is worthwhile to



a. by NQIMP algorithm

b. by NQLP12 based cooperative multi-agent planning

Fig. 2.6 Cooperative path planning to carry a triangle by three robots in deterministic situation

note that the TMAQL induced multi-agent planning fails to reach their team-goal, while the FCMQL included multi-agent planning is successful to complete the task. The reason behind the success of FCMQL lies in the enrichment of the Q-table because of incorporation of

Property 2.1 in the learning phase. In Fig. 2.6.b, the arrows outside the environment indicate rotation of the triangle by robots for successful the team-goal state-transition.

**Experiment 2.3 (Run-time complexity):** This study includes run-time analysis of the proposed FCMQL induced multi-agent planning along with a set of well-known algorithms from different domains. The algorithms used for comparison include: 1) ICFA [63], 2) MNPSO [54], [55], 3) DE [56], [57], and 4) MLbD [58]. The run-time analysis reveals, that the proposed FCMQL has the least run-time complexity in comparison to its contenders (see Table 2.3 and 2.4).

Table 2.3 Run-time complexity of Algorithm 2.3 over reference algorithms in deterministic situation

Algorithms	map 1 (stick-carrying)		map 7 (triangle-carrying)			map 8 (square-carrying)			
	Run-time (minute)		Run-time (minute) for Agent			Run-time (minute) for Agent			
	1	2	1	2	3	1	2	3	4
Algorithm 2.3	<b>0.195</b>	<b>0.191</b>	<b>0.244</b>	<b>0.246</b>	<b>0.242</b>	<b>0.309</b>	<b>0.313</b>	<b>0.310</b>	<b>0.304</b>
MRLbD	14.24	14.54	20.57	21.05	20.56	27.56	28.06	27.39	28.10
ICFA	51.16	51.01	60.56	61.10	60.51	64.56	64.10	64.51	64.56
MNPSO	70.43	70.58	50.54	50.38	51.01	40.26	40.45	40.42	40.56
DE	90.45	90.56	86.54	86.34	86.38	79.45	79.34	79.34	79.04

Table 2.4 Run-time complexity of Algorithm 2.3 over reference algorithms in stochastic situation

Algorithms	map 4 (stick-carrying)		map 9 (triangle-carrying)			map 10 (square-carrying)			
	Run-time (minute)		Run-time (minute) for Agent			Run-time (minute) for Agent			
	1	2	1	2	3	1	2	3	4
Algorithm 2.3	<b>0.184</b>	<b>0.182</b>	<b>0.218</b>	<b>0.220</b>	<b>0.221</b>	<b>0.293</b>	<b>0.302</b>	<b>0.301</b>	<b>0.302</b>
MRLbD	18.34	18.27	25.28	24.58	25.49	33.56	33.59	33.34	33.09
ICFA	52.27	53.10	60.76	60.25	60.17	63.59	63.55	63.45	63.57
MNPSO	71.54	71.52	51.34	51.65	52.04	39.45	39.34	39.32	39.12
DE	91.04	89.52	83.34	84.58	83.32	80.34	78.26	79.28	80.12

For ICFA and DE the objective functions used for the object-transportation task are as given in [63] and [57] respectively. In MNPSO, population size is equal to the number of agents. Here, agents learn in a parallel and distributed fashion, so as to reduce the run-time requirement by the MNPSO with an increase in the number of agents. By varying the number of robots in MNPSO algorithm, different objects are transported. On the other hand, little progress is attained in the field of learning joint action by simultaneous demonstrations. MLbD is a novel technique to learn multi-robot joint action from simultaneous demonstrations as given in [58]. Here also agents learn sequence of individual actions obtained from demonstration with the help of HAMMER architecture [58]. The joint action plan is then identified by spatio-temporal clustering algorithm. Here we compare the MLbD

from [58] with the proposed Algorithm 2.3. In MLbD, agents need to communicate among themselves, which require an extra cost in terms of time and energy.

**Experiment 2.4 (Real-time planning):** This experiment is concerned with examining planning performance of the proposed FCMQL algorithms in the stick-carrying problem realized with twin Khepera-II robots [64]. The stick-carrying problem refers to determining the pathways to transfer stick from a given starting position to a fixed destination, where the robots hold the stick at its two ends. We consider a grid-world map for the robots with  $6 \times 6$  square grids.

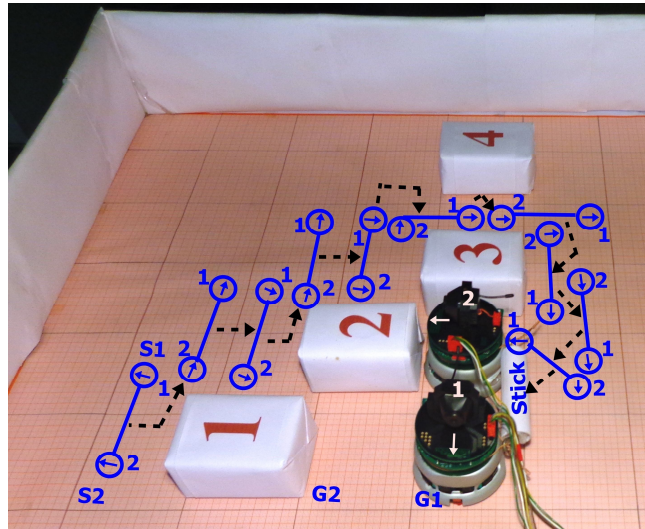


Fig. 2.7 Cooperative path planning to carry a stick by two Khepera-II mobile robots using NQIMP algorithm (bigger dimension in Fig. A2. 2.15 [64])

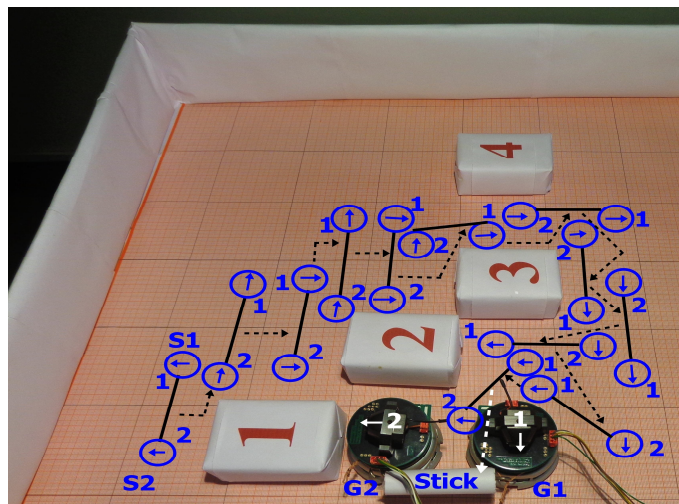


Fig. 2.8 Cooperative path planning to carry a stick by two Khepera-II mobile robots using Algorithm 3 (bigger dimension in A2.15 [64])

Fig. 2.8 provides a snapshot of the experimental instance, when the robots reach the goal positions using the proposed FCMQL induced planning. The path followed by the robots employing the reference algorithms also is shown in Fig. 2.7. The experiments presented indicate that the simulated results presented earlier are realized in hardware.

Table 2.5 Time taken by Khepera-II mobile robots to reach a team-goal with different speeds in

Algorithm 2.3

Run-time obtained	Speed (unit)	Run-time (sec)	
		Agent 1	Agent 2
Theoretically	2	8.75	9.14
	3	5.83	6.09
	5	3.50	3.66
Experimentally	2	11.71	12.43
	3	9.45	10.23
	5	8.28	9.36

## 2.8 Conclusions

The chapter aims at extending the TMAQL with two useful characteristic properties: exploration of the team goal and the joint action selection. The incorporation of the first property ensures exploration of the team-goal by multi-phased transitions of the agents asynchronously or synchronously to finally reach the team-goal, and thereby offer high reward values to such pre-goal state to the goal state transitions. The second property helps in identifying common preferred joint actions for the team, thus avoiding same joint actions at the same joint states and thereby enhancing the learning speed of the agents. The Q-table obtained in joint state-action space using the proposed FCMQL algorithms have been employed in the multi-agent planning algorithm to autonomously select goal state-transitions from the pre-goal states based on their high reward values stored in the Q-table. TMAQL-induced planners occasionally fail to reach the team-goal as such state-transitions which might result in due to follow-up actions of Property 2.1 in FCMQL are missing from the Q-table obtained by TMAQL.

The convergence of the proposed FCMQL is shown in Theorem 2.1, that the expected convergence time of the proposed FCMQL algorithms is less than the same of TMAQL algorithms. The complexity analysis reveals the superiority of the proposed FCMQL algorithms over the TMAQL algorithms.

Four different experiments have been conducted to validate the performance of FCMQL and the FCMQL based planning algorithms over the contender algorithms. In Experiment 2.1, the FCMQL algorithms outperform reference algorithms in terms of convergence rate, exploration of the team-goal and the Average of average reward (AAR) parameter. In

Experiment 2.2, Algorithm 2.3 outperforms reference algorithms considering successfully completion of a task as the performance metric. In Experiment 2.3, the merit of the Algorithm 2.3 is verified considering the run-time requirement as the performance metric over the reference algorithms: ICFA [63], MNPSO [58], [59], DE [56], [57], and MLbD [52] algorithms with respect to the well-known object-transportation problems. In Experiment 2.4, the superiority of Algorithm 2.3 is verified over contender algorithms utilizing in a real-time planning problem using twin Khepera-II mobile robots.

## 2.9 Summary

Chapter 2 offers learning-based planning algorithms, by extending the traditional multi-agent Q-learning algorithms (Nash Q-Learning and Corelated Q-Learning) for multi-robot coordination and planning. This extension is achieved by employing two interesting properties. The first property deals with the exploration of the team-goal (simultaneous success of all the robots) and the other property is related to the selection of joint action at a given joint state. The exploration of team-goal is realized by allowing the agents, capable of reaching their goals, to wait at their individual goal states, until remaining agents explore their individual goals synchronously or asynchronously. Selection of joint action, which is a crucial problem in traditional multi-agent Q-learning, is performed here by taking the intersection of individual preferred joint actions of all the agents. In case the resulting intersection is a null set, the individual actions are selected randomly or otherwise following classical techniques. The superiority of the proposed learning and learning-based planning algorithms are validated over contestant algorithms in terms of the speed of convergence and run-time complexity respectively.

## 2.10 Appendix 2.1

### Proof of Property 2.1

Let,

$l$  be the number of states in a given environment,

$m$  be the number of agents learning cooperatively in a given environment,

$j$  be the feasible actions for each agent.

The proof is segregated into the following three components.

**1.** Here, agent 1 can occupy any one of  $l$  states in the next iteration. Consequently, agent 2 would occupy any one of  $(l-1)$  possible next states. In the similar manner, it can be shown that agent  $m$  can occupy any one of  $(l-m)$  possible next states. Thus there would be as many as  $l(l-1)\dots(l-m+1) = {}^l P_m$  possible next joint states, where  $P$  denotes the permutation operator.

2. Thus probability that the next joint state ( $S'$ ) is equal to the team-goal ( $G$ ) due to a joint action ( $A$ ) at joint state ( $S$ ) is given by (A1 2.1).

$$\Pr((S' = G) | (S, A)) = \frac{1}{|P_m|} \quad (\text{A1 2.1})$$

3. Now, each agent can have  $j$  feasible actions. So, two agents would have  $j \times j = j^2$  joint actions. Proceeding similarly,  $m$  agents would have  $j^m$  possible joint actions. Thus the probability of randomly selecting a joint action ( $A$ ) at a joint state ( $S$ ) from the joint action set ( $\{A\}$ ) is given by (A1 2.2).

$$\Pr(S, A) = \frac{1}{j^m} \quad (\text{A1 2.2})$$

Now, probability that the next joint state ( $S'$ ) is the team-goal ( $G$ ) after executing a joint action ( $A$ ) randomly from the joint action set ( $\{A\}$ ) at a joint state ( $S$ ) given in (A1 2.3) by conditional probability.

$$\begin{aligned} \Pr((S, A) \cap (S' = G) : A \in \times_{i=1}^m \{a_i\}; S, S' \in \times_{i=1}^m \{s_i\}; G = \langle g_i \rangle_{i=1}^m) \quad [g_i \text{ is the goal of agent } i] \\ = \Pr(S, A) \times \Pr((S' = G) | (S, A)) \quad [\text{since } \Pr(C \cap D) = \Pr(C) \cdot \Pr(D | C)] \\ = \frac{1}{j^m} \times \frac{1}{|P_m|} \quad [\text{by (A1 2.1) and (A1 2.2)}] \quad (\text{A1 2.3}) \end{aligned}$$

In general, suppose  $k$  agents have already reached their individual goals. Then by Statute 2.1, the number of active learning agents become  $(m - k)$ . So, one can rewrite (A1 2.3) as in (A1 2.4). In (A1 2.4),  $-k$  in suffix indicates that all except the  $k$  number of agents and  $\{S\} = \{S_{-k}\} \cup \{S_{m-k}\}$ , etc., where  $(m - k)$  in the suffix indicates the joint state for  $(m - k)$  agents.

$$\begin{aligned} \Pr((S_{-k}, A_{-k}) \cap (S'_{-k} = G_{-k}) : A_{-k} \in \times_{n=1}^m \{a_n\}; S_{-k}, S'_{-k} \in \times_{n=1}^m \{s_n\}; G_{-k} = \langle g_n \rangle_{n=1}^m; n \notin [1, k]) \\ = \Pr(S_{-k}, A_{-k}) \times \Pr((S'_{-k} = G_{-k}) | (S_{-k}, A_{-k})) \quad [\text{since } \Pr(C \cap D) = \Pr(C) \cdot \Pr(D | C)] \\ = \frac{1}{j^{m-k}} \times \frac{1}{|P_{m-k}|} \quad [\text{since } m \leftarrow m - k] \quad (\text{A1 2.4}) \end{aligned}$$

Since the result obtained in (A1 2.4) is a monotonically increasing function of  $k$ , with increase in  $k$ , the probability of exploring the team-goal monotonically increases.  $\square$

### Proof of Property 2.2

Let at each state there exist  $j$  number of feasible actions for each of the  $m$  agents. Therefore, at a joint state ( $S$ ) there are  $j^m$  number of feasible joint actions. Let joint action set for all the agents at joint state  $S$  are

$$\left. \begin{aligned} \{A_i\} &= \{A\}, \forall i \\ \text{and } |\{A\}| &= j^m, \forall i. \end{aligned} \right\} \quad (\text{A1 2.5})$$



In MAQL, the initial joint Q-values usually are assumed to be zero, i.e.,  $Q_i(S, A) = 0, \forall S, \forall A, \forall i$  at iteration  $t = 0$ . So, by Definition 2.5, preferred joint action set of agent  $i = [1, m]$  at joint state  $S$  is

$$\begin{aligned} \{A_i^p\} &= \arg \min_A [Q_i(S, A)], \forall i \\ &= \{A\}, \forall i. \\ &\quad [\text{since } Q_i(S, A) = 0, \forall S, \forall A, \forall i, \therefore \arg \min_A [Q_i(S, A)] \text{ returns all } A \text{ for agent } i \text{ at } S] \\ &= \{A\}, \forall i. \quad [\text{by (A1 2.5)}] \end{aligned} \tag{A1 2.6}$$

Now, by the given statement of Property 2.2, preferred joint action set of the team ( $m$  agents) at joint state  $S$  is

$$\begin{aligned} \{A^p\} &= \bigcap_{i=1}^m \{A_i^p\} \\ &= \bigcap_{i=1}^m \{A\} \quad [\text{by (A1 2.6)}] \\ &= \{A\} \end{aligned} \tag{A1 2.7}$$

Therefore, probability to execute joint action  $A^p$  at the joint state  $S$  is

$$\begin{aligned} \Pr(S, A^p) &= \frac{1}{|\{A^p\}|} \\ &= \frac{1}{|\{A\}|}. \quad [\text{by (A1 2.7)}] \end{aligned} \tag{A1 2.8}$$

Let after first iteration the joint Q-value of agent  $x \in [1, m]$  has been improved because of preferred joint action  $A_x^p \in \{A_x^p\} = \langle a_x, A_{-x} \rangle$  at the joint state  $S \in \{S\}$ . So, joint Q-value of agent  $x$  at joint state  $S$  is

$$Q_x(S; a_x, A_{-x}) > Q_x(S; A), \forall A, A \neq \langle a_x, A_{-x} \rangle, \tag{A1 2.9}$$

where,  $A_{-x}$  be the joint action except the action of agent  $x$ ,  $a_x \in \{a_x\}$ . Therefore, updated preferred joint action set of agent  $x \in [1, m]$  at joint state  $S$  by Definition 2.5 is

$$\begin{aligned} \{A_x^{p'}\} &= \arg \min_A [Q_x(S, A)] \\ &= \{A_x^p\} - \{A_x^j\} \quad [\text{where, } \{A_x^j\} \text{ be the joint action set of agent } x \text{ containing action } a_x] \\ &\subset \{A_x^p\}. \end{aligned} \tag{A1 2.10}$$

Therefore, by (A1 2.5), (A1 2.10) and the given statement of the Property 2.2, updated preferred joint action set  $\{A^{p'}\}$  of the team ( $m$  agents) at joint state  $S$  is

$$\begin{aligned} \{A^{p'}\} &= \bigcap_{\substack{i=1, \\ i \neq x}}^m \{A_i^p\} \cap \{A_x^{p'}\} \\ &= \bigcap_{\substack{i=1, \\ i \neq x}}^m \{A\} \cap \{A_x^{p'}\} \quad [\text{by (A1 2.7)}] \\ &= \{A\} \cap \{A_x^{p'}\} \end{aligned}$$

$$= \{A_x^{p'}\}. \quad [\text{since } \{A_x^{p'}\} \subset \{A\}, x \in [1, m]] \quad (\text{A1 2.11})$$

Therefore, probability to execute joint action  $A_x^{p'}$  at joint state  $S$  is

$$\begin{aligned} \Pr(S, A^{p'}) &= \frac{1}{|\{A^{p'}\}|} \\ &= \frac{1}{|\{A_x^{p'}\}|} \quad [\text{by (A1 2.11)}] \\ &> \frac{1}{|\{A_x^p\}|} \quad [\text{since } |\{A_x^{p'}\}| < |\{A_x^p\}| \text{ by (A1 2.10)}] \\ &= \frac{1}{|\{A\}|} \quad [\text{by (A1 2.5)}] \\ &= P(S, A^p). \quad [\text{by (A1 2.8)}] \end{aligned} \quad (\text{A1 2.12})$$

Hence, the Property is proved.  $\square$

### Proof of Theorem 2.1

The expected time of convergence of FCMQL,  $T_e^F$ , decreases with an increase in the probability of exploring the team-goal, given by  $\Pr(S, A) \times \Pr((S' = G) | (S, A))$  and also the joint action selection probability  $\Pr(S, A)$ . Thus,  $T_e^F$  can be modeled by an exponentially decreasing function of the joint probability of exploration of the team-goal and probability of joint action selection, i.e.,

$$T_e^F = \exp\{-\Pr(S, A)^2 \times \Pr((S' = G) | (S, A))\}. \quad (\text{A1 2.13})$$

It is important to note that the expression (A1 2.13) is equally good for TMAQL. However, in TMAQL, the probabilities:  $\Pr(S, A)$  and  $\Pr((S' = G) | (S, A))$  both remain constant over the learning epochs, whereas in FCMQL the above two probabilities increase with increase in learning epochs. Thus, (A1 2.13) transforms to (A1 2.14) and (A1 2.15) for TMAQL and FCMQL respectively.

$$T_e^{TM} = \exp(-k) \quad (\text{A1 2.14})$$

for any positive real number  $k$ .

$$T_e^F = \exp(-k - \delta k(T_e^F)) \quad (\text{A1 2.15})$$

where  $\delta k(T_e^F)$  is a linearly increasing function of  $T_e^F$ . A little algebra, given below, returns

$T_e^F$  as a non-linear function of  $k$

$$T_e^F \cdot \exp(\delta k(T_e^F)) = \exp(-k)$$

$$\Rightarrow T_e^F \cdot \delta k(T_e^F) = \exp(-k) \quad [\text{as } \delta k(T_e^F) \rightarrow 0.]$$

$$\Rightarrow T_e^F \cdot \{\beta \cdot T_e^F\} = \exp(-k) \quad [\text{by linear approximation of } \delta k(T_e^F)]$$

$$\Rightarrow T_e^F = \sqrt{\frac{1}{\beta} \exp(-k)} \quad (\text{A1 2.16})$$

It is apparent from (A1 2.14) and (A1 2.16) that  $T_e^F < T_e^{TM}$ . Thus the theorem follows.  $\square$

### Proof of Theorem 2.2

Let the Q-values of agent  $i \in [1, m]$  at joint state  $S$  due to all joint actions  $A \in \{A\}$  is denoted by the set  $\{Q_i(S, A) : A \in \{A\}\}$ . Now, one can write,

$$r_{\min} \leq Q_i(S, A) \leq r_{\max}, \forall i. \quad (\text{A1 2.17})$$

where,  $r_{\min}$  and  $r_{\max}$  are the minimum and maximum immediate rewards respectively of an agent. Let agent  $x \in [1, m]$  explores its individual goal due to  $A_N$  at  $S$  and the subsequent joint Q-value of agent  $x$  adapted by the proposed NQLP12 is given by

$$Q_x(S, A_N) = r_{\min}. \quad [\text{by (2.8)}] \quad (\text{A1 2.18})$$

Again, by (A1 2.17)

$$r_{\min} \leq Q_x(S, A'). \quad [\text{where } x \in [1, m] \text{ and } A' \in \{A\}] \quad (\text{A1 2.19})$$

Combining (A1 2.18) and (A1 2.19) we obtain

$$Q_x(S, A_N) \leq Q_x(S, A'), A' \in \{A\}. \quad (\text{A1 2.20})$$

However, in the above situation joint Q-value of agent  $x$  adapted by TMAQL is

$$Q_x(S, A_N) = r_{\max}. \quad [\text{by (2.7)}] \quad (\text{A1 2.21})$$

Again, by (A1 2.17)

$$r_{\max} \geq Q_x(S, A'), A' \in \{A\}. \quad [\text{where } x \in [1, m] \text{ and } A' \in \{A\}] \quad (\text{A1 2.22})$$

Combining (A1 2.21) and (A1 2.22) we obtain

$$Q_x(S, A_N) \geq Q_x(S, A'), A' \in \{A\}. \quad (\text{A1 2.23})$$

Now, by the principle of multi-agent planning algorithm, let maximum joint Q-value of all except agent  $x$  at joint state  $S$  because of joint action  $A_N$  is  $Q_{-x}(S, A_N) \in \{Q_{-x}(S, A) : A \in \{A\}\}, \forall -x$ . Therefore,

$$Q_{-x}(S, A_N) \geq Q_{-x}(S, A'), A' \in \{A\}, \forall -x. \quad (\text{A1 2.24})$$

By (A1 2.20), (A1 2.24) we can conclude that  $Q_i(S, A_N) \leq Q_i(S, A')$  for  $i = x$  and  $Q_i(S, A_N) \geq Q_i(S, A'), \forall i, i \neq x$ . Hence, by Definition 2.2, we say that in the proposed NQLP12 induced planning, the joint action  $A_N$  is not a NE at  $S$ . Again by (A1 2.23), (A1 2.24) we conclude that  $Q_i(S, A_N) \geq Q_i(S, A'), \forall i$ . So, by Definition 2.2, we say that in TMAQL induced planning, the joint action  $A_N$  is a NE at  $S$ .  $\square$

## 2.11 Appendix 2.2

### Additional details of Experiment 2.1

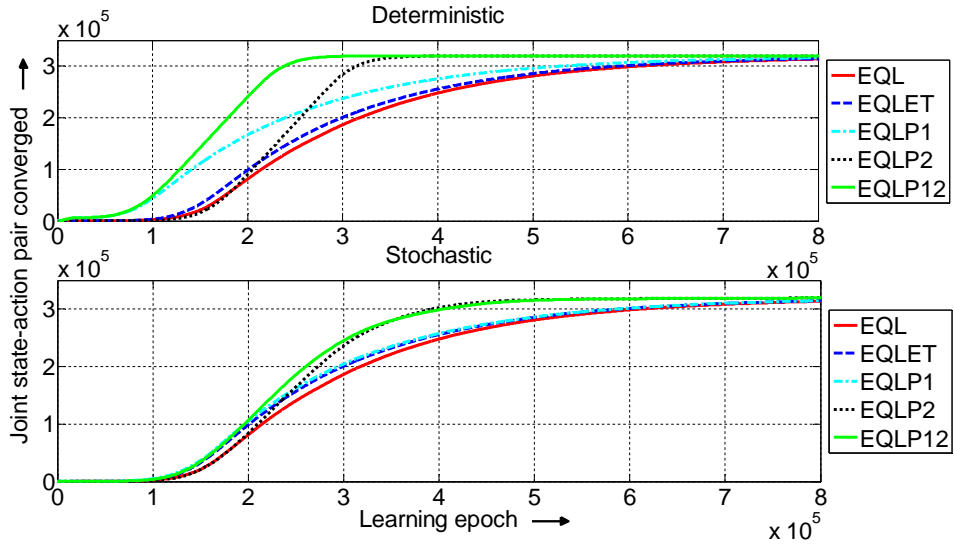
Two agents' individual state numbers  $s_1$  and  $s_2$  are mapped into a single integer  $S$  (joint state) by the mapping function:

$$S = (s_2 - 1) \times n + s_1 \quad (\text{A2.2.1})$$

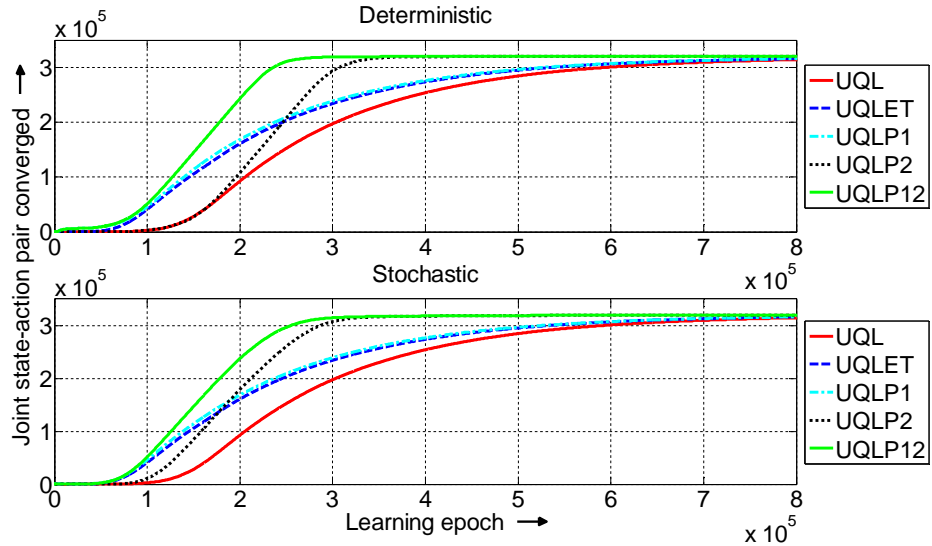
for  $n \times n$  grid map. For three and four agents the mapping functions are given by (A2.2.2) and (A2.2.3) respectively, where  $s_3$  and  $s_4$  are the state of third and fourth agent respectively.

$$S = (s_3 - 1) \times n^2 + (s_2 - 1) \times n + s_1 \quad (\text{A2.2.2})$$

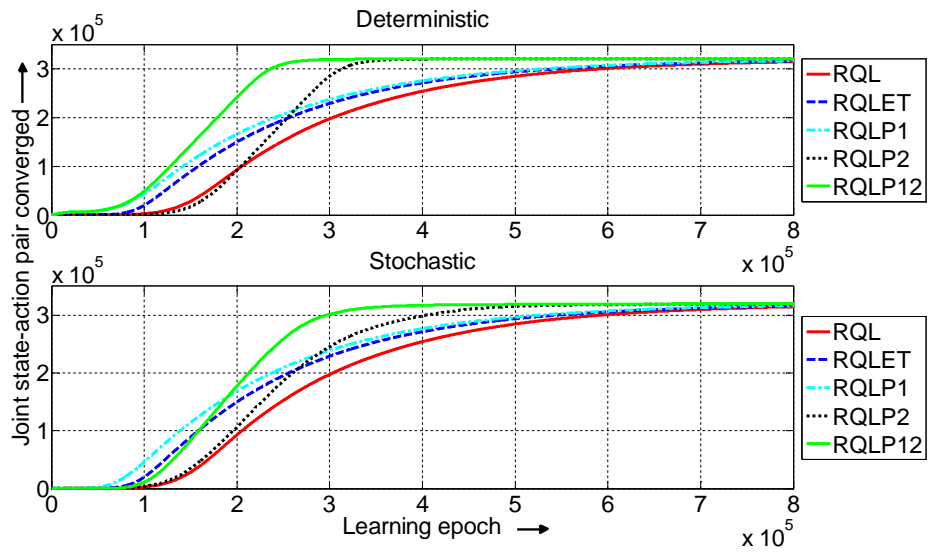
$$S = (s_4 - 1) \times n^3 + (s_3 - 1) \times n^2 + (s_2 - 1) \times n + s_1 \quad (\text{A2.2.3})$$



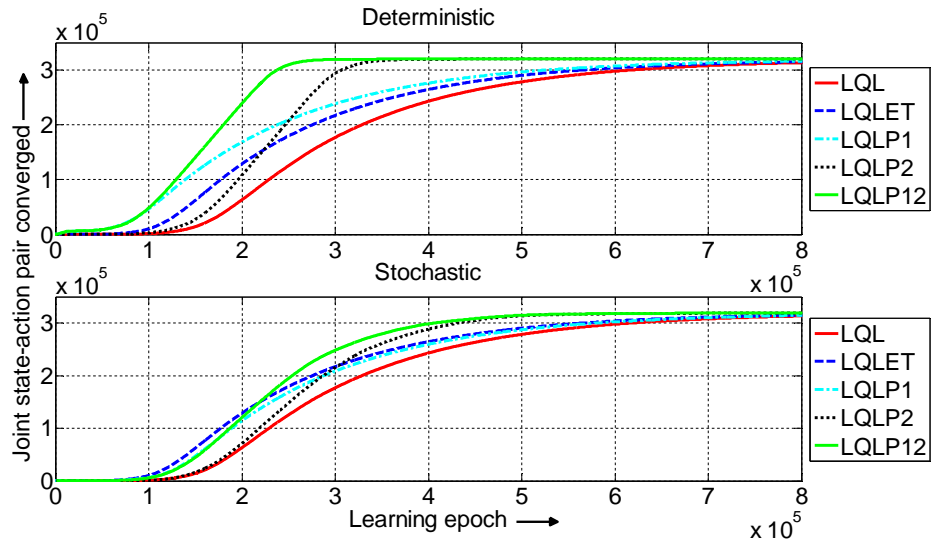
(b) EQLP12 and reference algorithms for two agents



(c) UQLP12 and reference algorithms for two agents



(d) RQLP12 and reference algorithms for two agents



(e) LQLP12 and reference algorithms for two agents

Fig. A2. 2.1 Convergence plot of FCMQL and reference algorithms for two agents

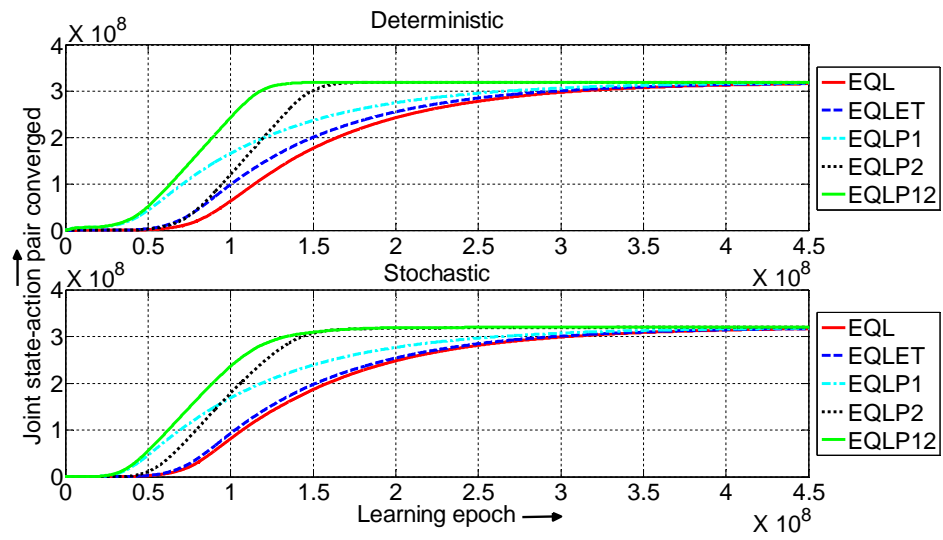


Fig. A2. 2.2 Convergence plot of EQLP12 and reference algorithms for three agents

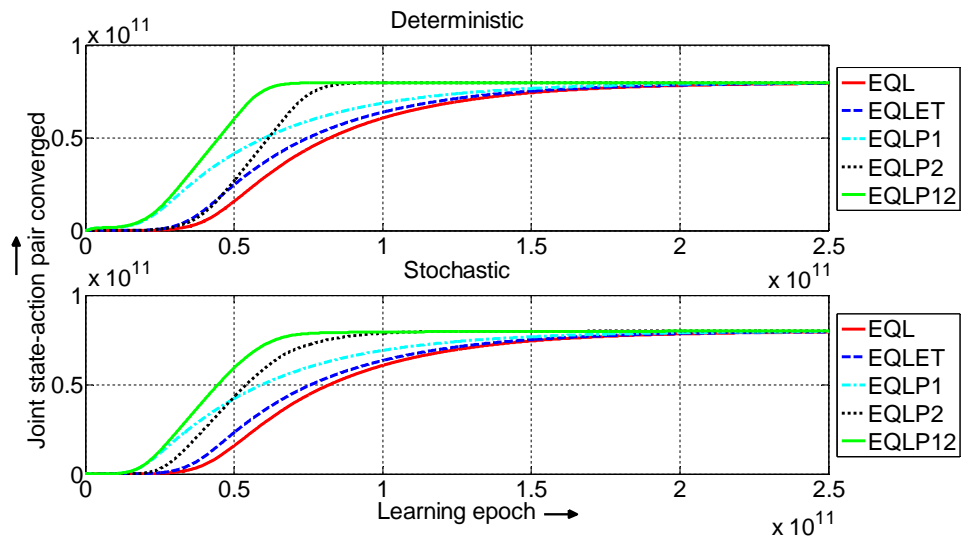


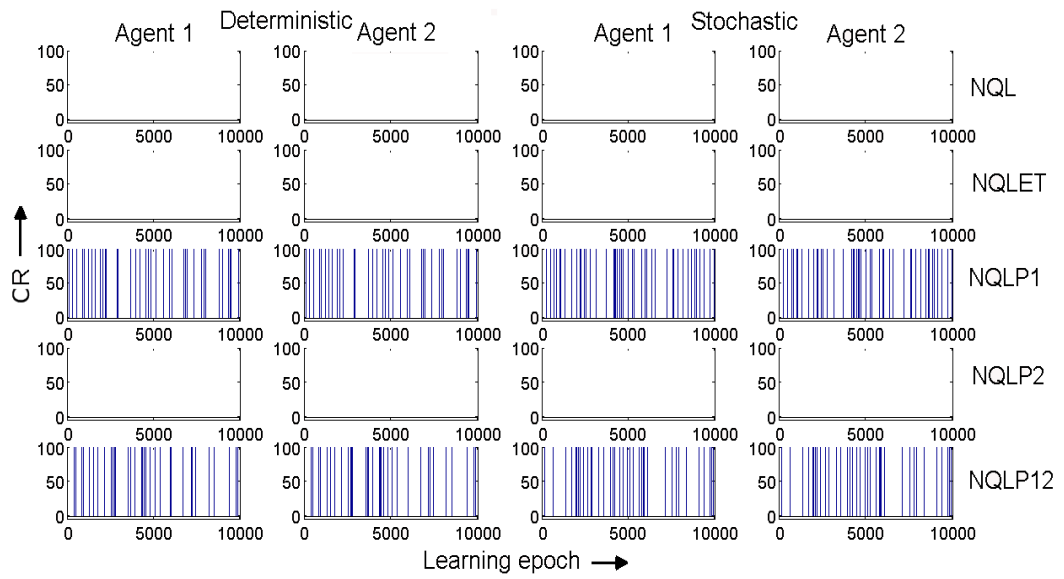
Fig. A2. 2.3 Convergence plot of EQLP12 and reference algorithms for four agents

Table A2. 2.1 Number of joint state-action pair converged in deterministic situation for two agents

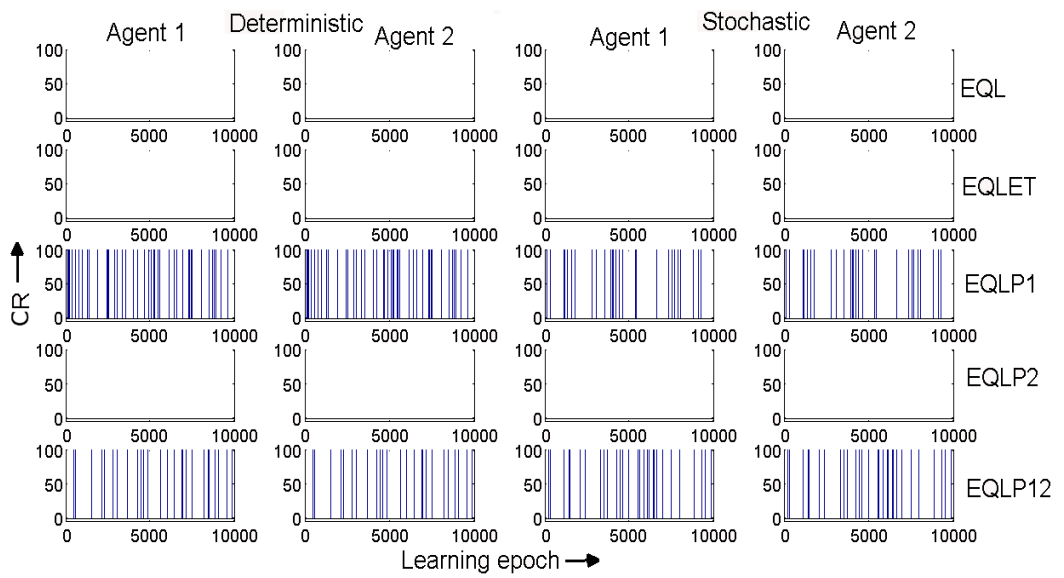
Algorithm ms	Number of joint state-action pair has been converged within								
	10 <sup>5</sup> epochs in			15×10 <sup>5</sup> epochs in			10 <sup>6</sup> epochs in		
	map			map			map		
	1	2	3	1	2	3	1	2	3
NQL	2976	2965	2979	30060	30057	30061	318314	318312	318317
NQLET	12365	12367	12366	74299	74302	74298	318723	318725	318724
NQLP1	47566	47567	47568	115164	115168	115167	318878	318879	318877
<b>NQLP2</b>	2600	2602	2599	31522	31524	31521	<b>319968</b>	<b>319968</b>	<b>319968</b>
<b>NQLP12</b>	<b>48846</b>	<b>48847</b>	<b>48844</b>	<b>142010</b>	<b>142011</b>	<b>142014</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>
UQL	2732	2734	2733	27824	27826	27823	318296	318299	318297
UQLET	40726	40727	40729	105388	105387	105390	318795	318798	318799
UQLP1	45834	45833	45836	113614	113615	113618	318858	318861	318860
<b>UQLP2</b>	2910	2909	2912	27140	27141	27143	<b>319968</b>	<b>319968</b>	<b>319968</b>
<b>UQLP12</b>	<b>50934</b>	<b>50932</b>	<b>50935</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>
EQL	2388	2390	2389	20834	20835	20834	318230	318229	318228
EQLET	3582	3580	3581	32630	32630	32632	318342	318340	318341
EQLP1	43518	43517	43516	111101	111102	111100	318863	318861	318862
<b>EQLP2</b>	894	896	894	16092	16093	16091	<b>319968</b>	<b>319968</b>	<b>319968</b>
<b>EQLP12</b>	<b>48524</b>	<b>48526</b>	<b>48525</b>	<b>141596</b>	<b>141597</b>	<b>141599</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>
RQL	2732	2732	2733	27824	27826	27824	318299	318298	318300
RQLET	19648	19649	19648	88577	88575	88576	318667	318668	318666
RQLP1	43256	43254	43255	110322	110321	110323	318954	318955	318953
<b>RQLP2</b>	1096	1097	1098	16820	16821	16822	<b>319968</b>	<b>319968</b>	<b>319968</b>
<b>RQLP12</b>	<b>48748</b>	<b>48748</b>	<b>48749</b>	<b>142642</b>	<b>142641</b>	<b>142642</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>
LQL	1377	1379	1378	14067	14067	14068	317953	317954	317955
LQLET	10019	10020	10018	62028	62027	62029	318502	318503	318501
LQLP1	46126	46125	46127	113724	113725	113726	318786	318786	318786
<b>LQLP2</b>	2910	2911	2912	27140	27141	27143	<b>319968</b>	<b>319968</b>	<b>319968</b>
<b>LQLP12</b>	<b>47954</b>	<b>47955</b>	<b>47954</b>	<b>140408</b>	<b>140408</b>	<b>140410</b>	<b>319968</b>	<b>319968</b>	<b>319968</b>

Table A2. 2.2 Number of joint state-action pair converged in stochastic situation for two agents

Algorithm ms	Number of joint state-action pair has been converged within								
	10 <sup>5</sup> epochs in			15×10 <sup>5</sup> epochs in			10 <sup>6</sup> epochs in		
	map			map			map		
	4	5	6	4	5	6	4	5	6
NQL	12365	12366	12363	74299	74302	74301	318723	318724	318723
NQLET	2976	2978	2977	30060	30062	30061	318314	318313	318315
NQLP1	42371	42372	42373	111814	111817	111816	318823	318823	318825
<b>NQLP2</b>	11472	11471	11475	82937	82939	82938	<b>319937</b>	<b>319938</b>	<b>319939</b>
<b>NQLP12</b>	<b>54705</b>	<b>54708</b>	<b>54706</b>	<b>147816</b>	<b>147815</b>	<b>147818</b>	<b>319948</b>	<b>319951</b>	<b>319950</b>
UQL	40726	40727	40728	105388	105386	105389	318795	318796	318798
UQLET	2732	2733	2735	27824	27827	27826	318296	318295	318299
UQLP1	45768	45765	45769	113985	113986	113987	318822	318824	318821
<b>UQLP2</b>	10877	10876	10879	84040	84042	84039	<b>319896</b>	<b>319900</b>	<b>319899</b>
<b>UQLP12</b>	<b>51191</b>	<b>51193</b>	<b>51196</b>	<b>146529</b>	<b>146531</b>	<b>146532</b>	<b>319893</b>	<b>319895</b>	<b>319896</b>
EQL	2388	2390	2387	20834	20833	20836	318228	318230	318231
EQLET	3582	3581	3583	32630	3263	32634	318342	318341	318342
EQLP1	4928	4927	4929	36492	36493	36494	318438	318440	318439
<b>EQLP2</b>	1738	1740	1739	20824	20823	20825	<b>319810</b>	<b>319812</b>	<b>319811</b>
<b>EQLP12</b>	<b>3759</b>	<b>3761</b>	<b>3760</b>	<b>34658</b>	<b>34660</b>	<b>34661</b>	<b>319803</b>	<b>319806</b>	<b>319804</b>
RQL	2732	2733	2731	27824	27825	27823	318298	318302	318299
RQLET	19648	19647	19649	88575	88577	88576	318667	318669	318668
RQLP1	45653	45654	45655	113840	113839	113842	318820	318819	318822
<b>RQLP2</b>	10751	10750	10749	83343	83341	83344	<b>319882</b>	<b>319883</b>	<b>319881</b>
<b>RQLP12</b>	<b>3759</b>	<b>3760</b>	<b>3761</b>	<b>34658</b>	<b>34657</b>	<b>34659</b>	<b>319803</b>	<b>319802</b>	<b>319804</b>
LQL	1377	1379	1378	14067	14068	14066	317953	317952	317954
LQLET	10018	10020	10019	62028	62027	62029	318502	318501	318504
LQLP1	6362	6364	6363	46961	46960	46963	318486	318485	318487
<b>LQLP2</b>	1042	1040	1043	16178	16180	16179	<b>319766</b>	<b>319765</b>	<b>319764</b>
<b>LQLP12</b>	<b>5721</b>	<b>5720</b>	<b>5722</b>	<b>44249</b>	<b>44249</b>	<b>44250</b>	<b>319783</b>	<b>319784</b>	<b>319782</b>

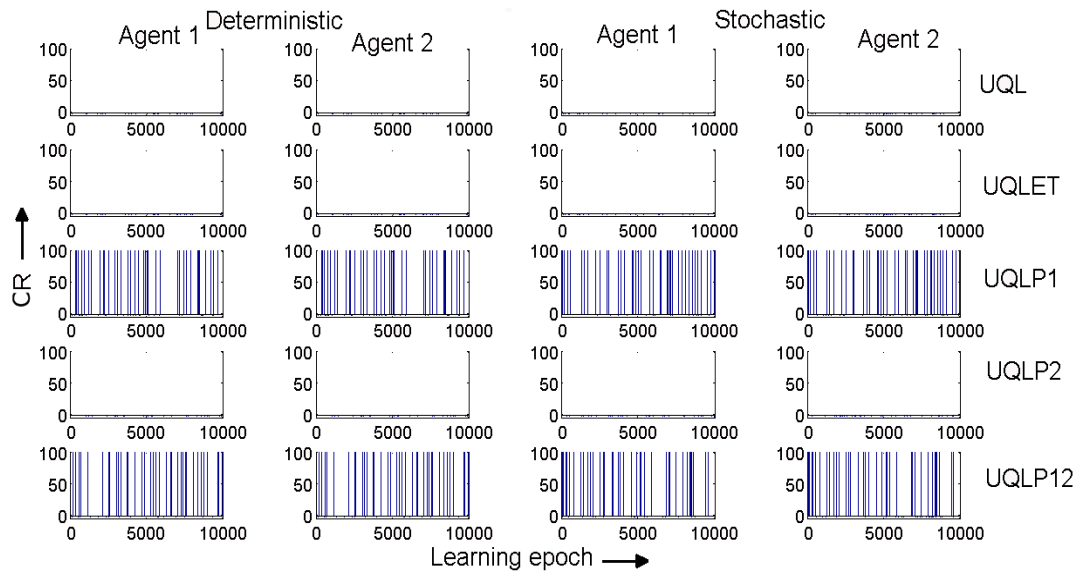


(a) NQLP12 and reference algorithms for two agents

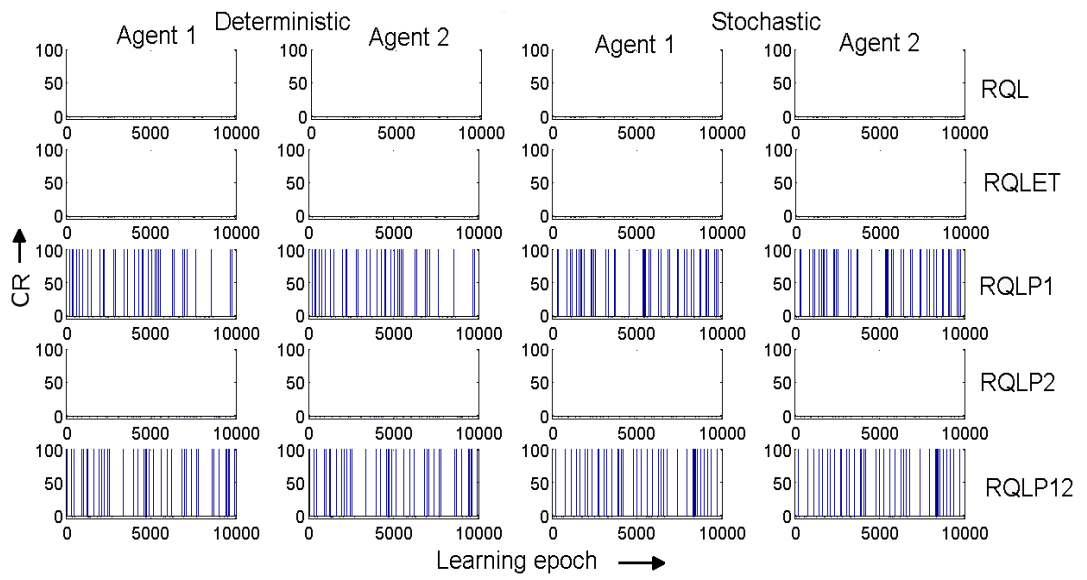


(b) EQLP12 and reference algorithms for two agents

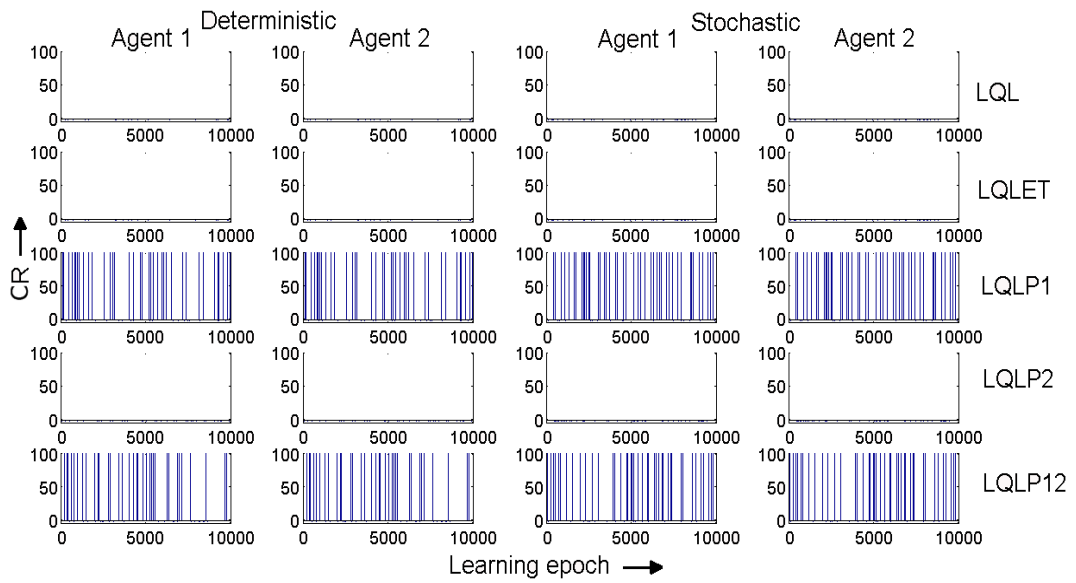




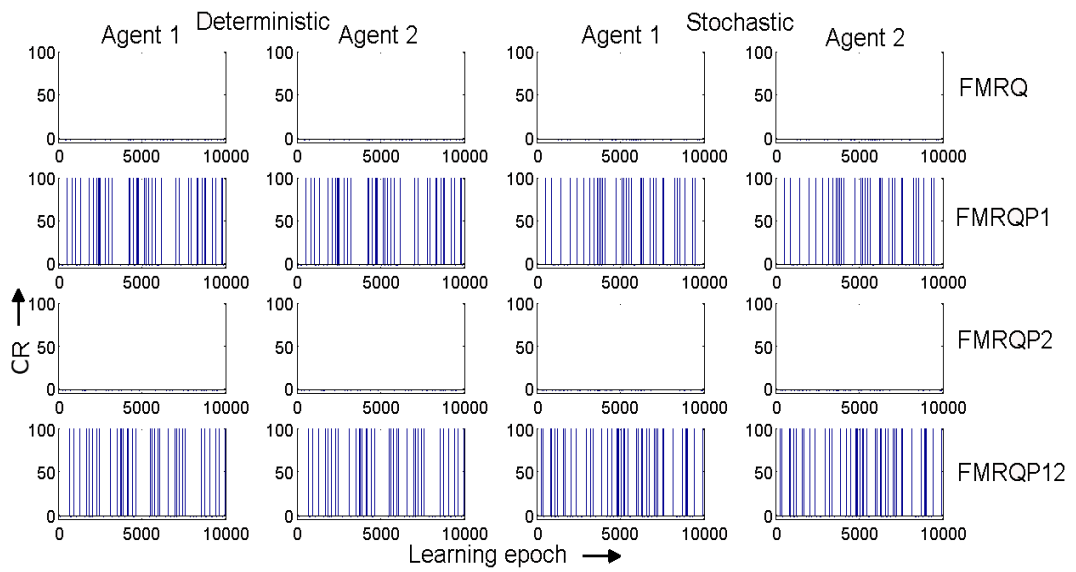
(c) UQLP12 and reference algorithms for two agents



(d) RQLP12 and reference algorithms for two agents



(e) LQLP12 and reference algorithms for two agents



(f) FMRQP12 and reference algorithms for two agents

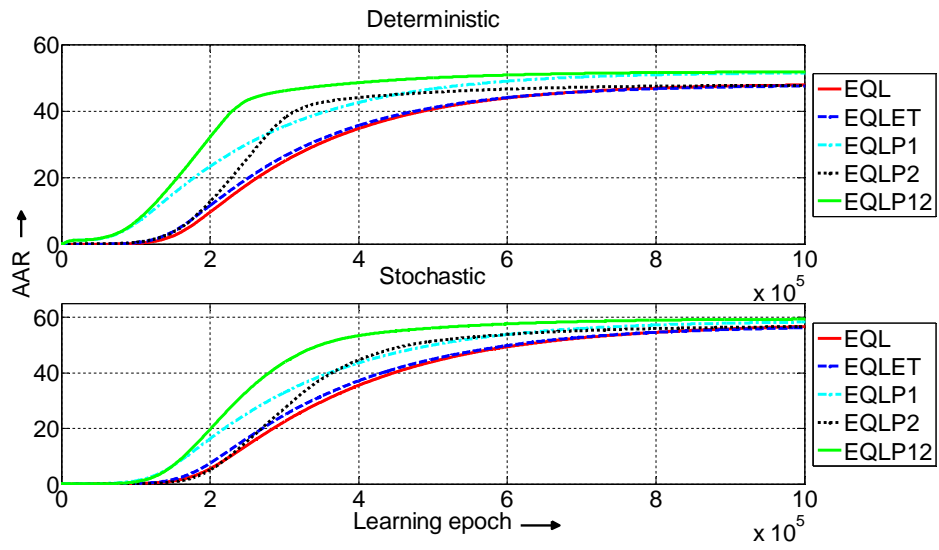
Fig. A2. 2.4 CR versus learning epoch plot for FCMQL and reference algorithms for two agents

Table A2. 2.3 Count of team-goal explored in the deterministic situation for two agents within

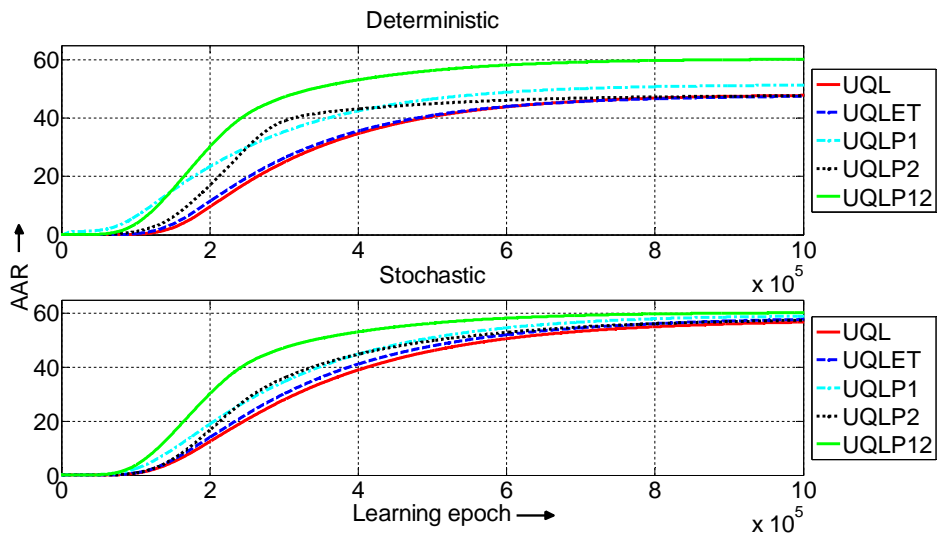
Algorithms	15000 epochs			10000 epochs			5000 epochs		
	map			map			map		
	1	2	3	1	2	3	1	2	3
NQL	0	1	0	0	0	0	0	0	0
NQLET	1	0	0	0	0	0	0	0	0
<b>NQLP1</b>	<b>55</b>	<b>65</b>	<b>70</b>	<b>37</b>	<b>38</b>	<b>41</b>	<b>16</b>	<b>20</b>	<b>27</b>
NQLP2	0	1	1	0	0	0	0	0	0
<b>NQLP12</b>	<b>57</b>	<b>66</b>	<b>68</b>	<b>38</b>	<b>39</b>	<b>40</b>	<b>18</b>	<b>22</b>	<b>26</b>
UQL	1	1	0	0	0	0	0	0	0
UQLET	0	1	1	0	0	0	0	0	0
<b>UQLP1</b>	<b>56</b>	<b>68</b>	<b>69</b>	<b>37</b>	<b>38</b>	<b>41</b>	<b>21</b>	<b>23</b>	<b>27</b>
UQLP2	1	0	1	0	0	0	0	0	0
<b>UQLP12</b>	<b>55</b>	<b>65</b>	<b>67</b>	<b>38</b>	<b>39</b>	<b>39</b>	<b>23</b>	<b>25</b>	<b>26</b>
EQL	0	1	0	0	0	0	0	0	0
EQLET	1	0	2	0	0	0	0	0	0
<b>EQLP1</b>	<b>57</b>	<b>69</b>	<b>70</b>	<b>38</b>	<b>39</b>	<b>41</b>	<b>18</b>	<b>23</b>	<b>25</b>
EQLP2	1	2	0	0	0	0	0	0	0
<b>EQLP12</b>	<b>56</b>	<b>62</b>	<b>68</b>	<b>38</b>	<b>39</b>	<b>40</b>	<b>20</b>	<b>22</b>	<b>26</b>
RQL	1	0	1	0	0	0	0	0	0
RQLET	0	1	0	0	0	0	0	0	0
<b>RQLP1</b>	<b>56</b>	<b>68</b>	<b>69</b>	<b>37</b>	<b>38</b>	<b>41</b>	<b>21</b>	<b>23</b>	<b>27</b>
RQLP2	0	1	1	0	0	0	0	0	0
<b>RQLP12</b>	<b>57</b>	<b>63</b>	<b>69</b>	<b>38</b>	<b>39</b>	<b>40</b>	<b>19</b>	<b>21</b>	<b>25</b>
LQL	0	2	1	0	0	0	0	0	0
LQLET	1	0	1	0	0	0	0	0	0
<b>LQLP1</b>	<b>55</b>	<b>62</b>	<b>68</b>	<b>37</b>	<b>38</b>	<b>40</b>	<b>20</b>	<b>24</b>	<b>27</b>
LQLP2	1	0	1	0	0	0	0	0	0
<b>LQLP12</b>	<b>56</b>	<b>61</b>	<b>68</b>	<b>38</b>	<b>39</b>	<b>41</b>	<b>22</b>	<b>23</b>	<b>26</b>
FMRQ	0	1	0	0	0	0	0	0	0
<b>FMRQP1</b>	<b>60</b>	<b>65</b>	<b>70</b>	<b>37</b>	<b>38</b>	<b>41</b>	<b>21</b>	<b>26</b>	<b>27</b>
FMRQP2	2	1	0	0	0	0	0	0	0
<b>FMQRP12</b>	<b>56</b>	<b>61</b>	<b>68</b>	<b>38</b>	<b>39</b>	<b>41</b>	<b>23</b>	<b>24</b>	<b>26</b>

Table A2. 2.4 Count of team-goal explored in the stochastic situation for two agents within

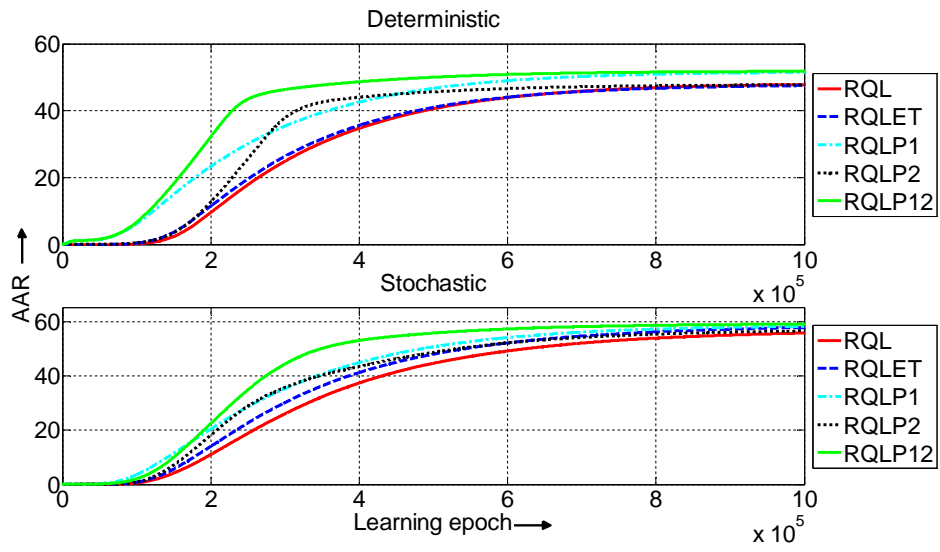
Algorithms	15000 epochs			10000 epochs			5000 epochs		
	map			map			map		
	4	5	6	4	5	6	4	5	6
NQL	1	0	0	0	0	0	0	0	0
NQLET	0	2	1	0	0	0	0	0	0
<b>NQLP1</b>	<b>55</b>	<b>59</b>	<b>66</b>	<b>35</b>	<b>38</b>	<b>41</b>	<b>12</b>	<b>18</b>	<b>28</b>
NQLP2	0	1	0	0	0	0	0	0	0
<b>NQLP12</b>	<b>56</b>	<b>59</b>	<b>65</b>	<b>36</b>	<b>37</b>	<b>40</b>	<b>16</b>	<b>19</b>	<b>26</b>
UQL	0	1	1	0	0	0	0	0	0
UQLET	1	0	0	0	0	0	0	0	0
<b>UQLP1</b>	<b>57</b>	<b>58</b>	<b>63</b>	<b>37</b>	<b>38</b>	<b>41</b>	<b>18</b>	<b>22</b>	<b>25</b>
UQLP2	0	1	1	0	0	0	0	0	0
<b>UQLP12</b>	<b>55</b>	<b>57</b>	<b>65</b>	<b>36</b>	<b>39</b>	<b>40</b>	<b>15</b>	<b>21</b>	<b>27</b>
EQL	0	2	1	0	0	0	0	0	0
EQLET	1	0	0	0	0	0	0	0	0
<b>EQLP1</b>	<b>57</b>	<b>58</b>	<b>65</b>	<b>37</b>	<b>39</b>	<b>41</b>	<b>16</b>	<b>23</b>	<b>27</b>
EQLP2	0	0	1	0	0	0	0	0	0
<b>EQLP12</b>	<b>56</b>	<b>58</b>	<b>64</b>	<b>35</b>	<b>37</b>	<b>40</b>	<b>19</b>	<b>20</b>	<b>26</b>
RQL	0	1	0	0	0	0	0	0	0
RQLET	1	0	2	0	0	0	0	0	0
<b>RQLP1</b>	<b>55</b>	<b>59</b>	<b>65</b>	<b>34</b>	<b>37</b>	<b>39</b>	<b>21</b>	<b>22</b>	<b>25</b>
RQLP2	0	2	1	0	0	0	0	0	0
<b>RQLP12</b>	<b>56</b>	<b>58</b>	<b>65</b>	<b>35</b>	<b>39</b>	<b>41</b>	<b>17</b>	<b>21</b>	<b>26</b>
LQL	0	2	1	0	0	0	0	0	0
LQLET	1	1	0	0	0	0	0	0	0
<b>LQLP1</b>	<b>57</b>	<b>58</b>	<b>65</b>	<b>36</b>	<b>37</b>	<b>40</b>	<b>16</b>	<b>19</b>	<b>25</b>
LQLP2	0	1	0	0	0	0	0	0	0
<b>LQLP12</b>	<b>55</b>	<b>58</b>	<b>66</b>	<b>35</b>	<b>37</b>	<b>41</b>	<b>21</b>	<b>23</b>	<b>27</b>
FMRQ	2	0	1	0	0	0	0	0	0
<b>FMRQP1</b>	<b>56</b>	<b>58</b>	<b>64</b>	<b>37</b>	<b>38</b>	<b>40</b>	<b>18</b>	<b>22</b>	<b>26</b>
FMRQP2	1	0	1	0	0	0	0	0	0
<b>FMQRP12</b>	<b>56</b>	<b>57</b>	<b>65</b>	<b>35</b>	<b>38</b>	<b>41</b>	<b>21</b>	<b>24</b>	<b>26</b>



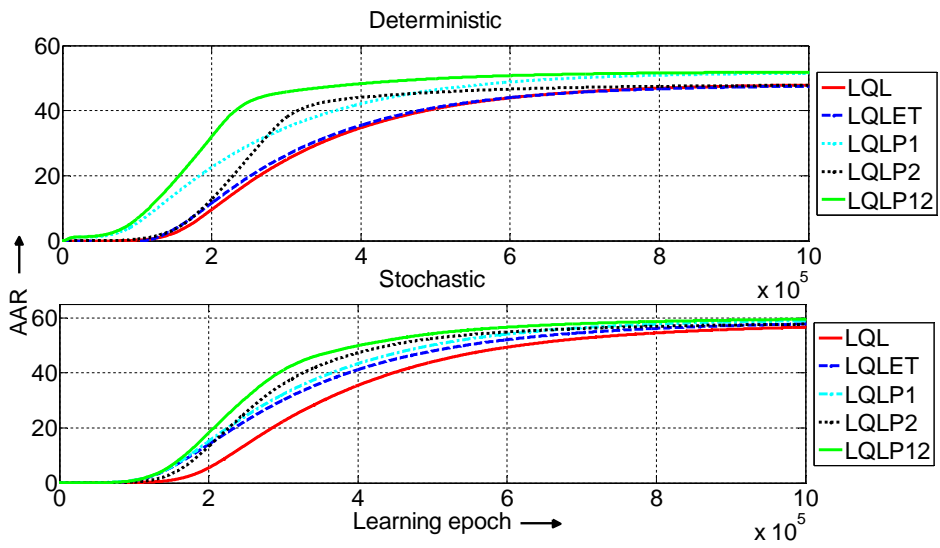
(b) EQLP12 and reference algorithms for two agents



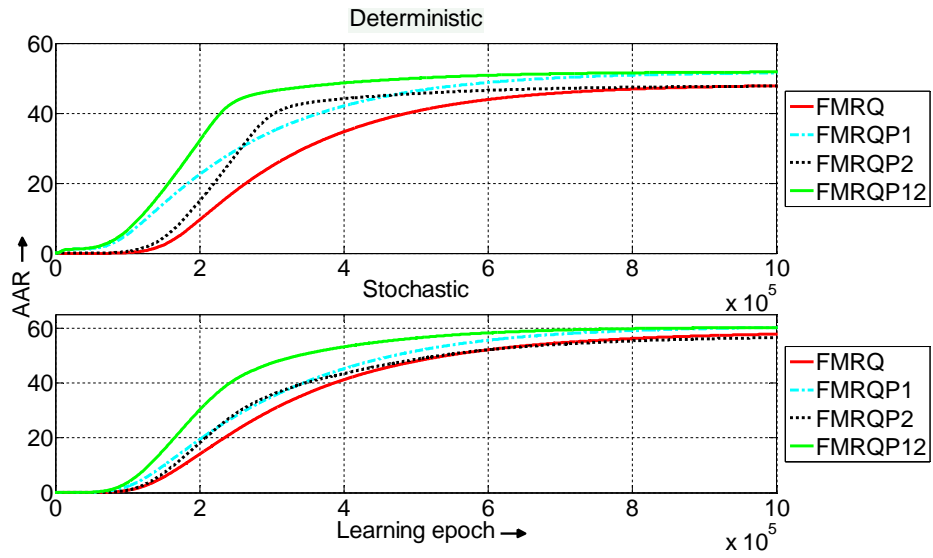
(c) UQLP12 and reference algorithms for two agents



(d) RQLP12 and reference algorithms for two agents



(e) LQLP12 and reference algorithms for two agents



(e) FMRQP12 and reference algorithms for two agents

Fig. A2. 2.5 Average of average reward (AAR) plot of FCMQL and reference algorithms for two agents

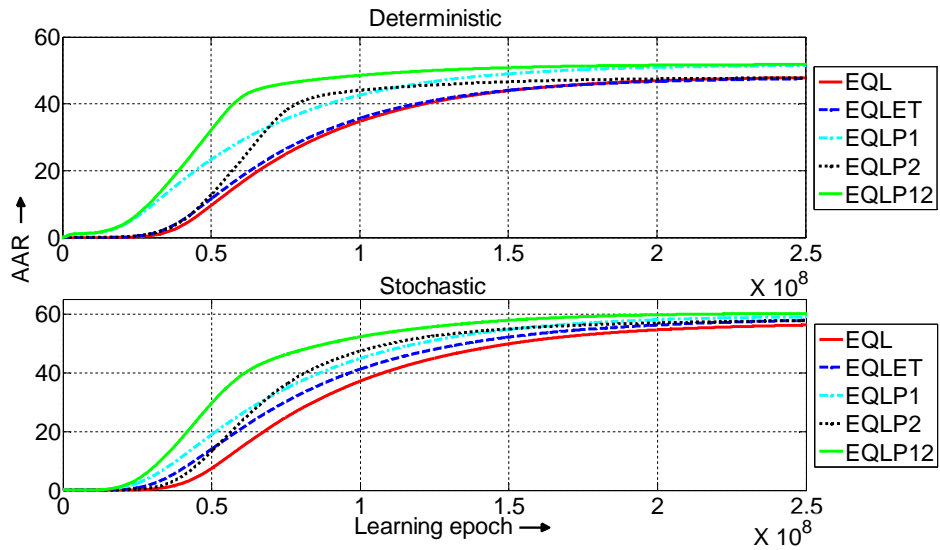


Fig. A2. 2.6 Average of average reward (AAR) plot of EQLP12 and reference algorithms for three agents

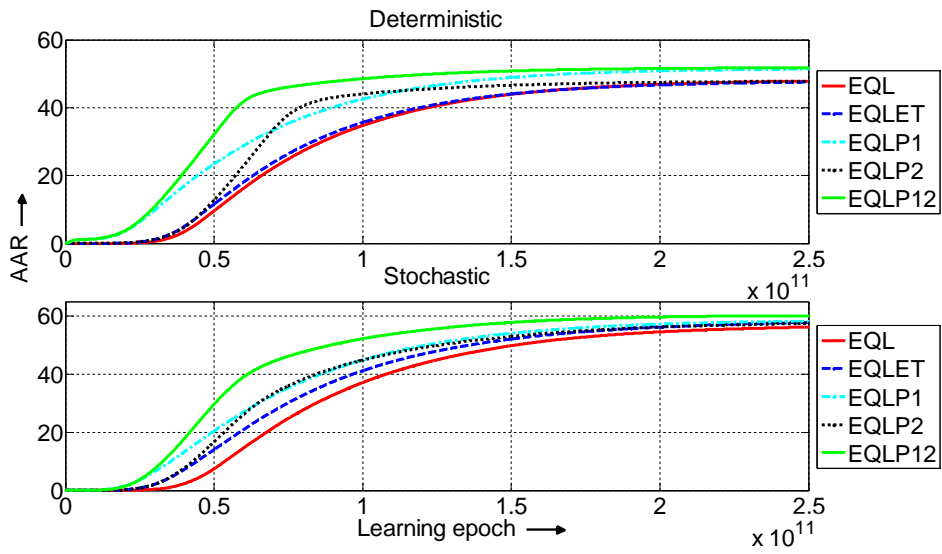


Fig. A2. 2.7 Average of average reward (AAR) plot of EQLP12 and reference algorithms for four agents

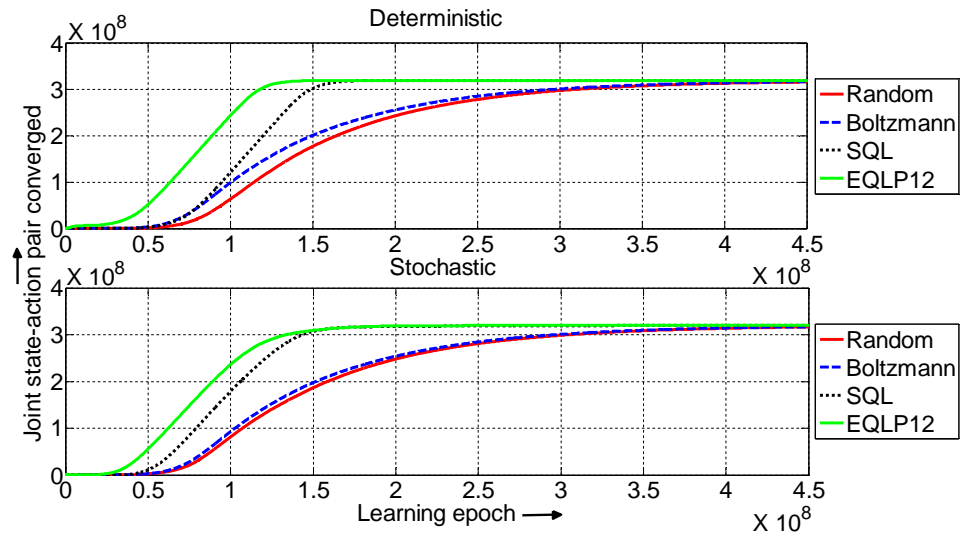


Fig. A2. 2.8 Joint action selection strategy in EQLP12 and reference algorithms for three agents

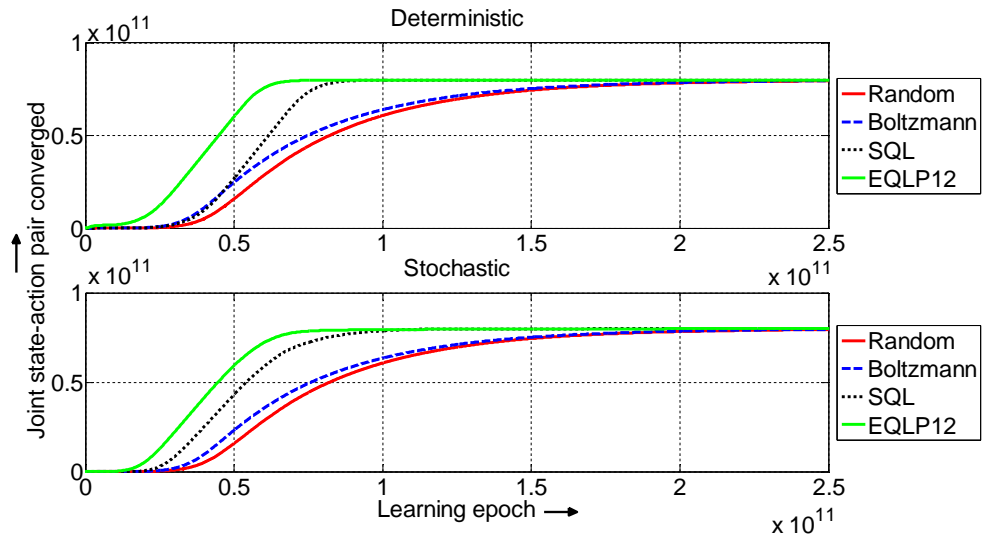
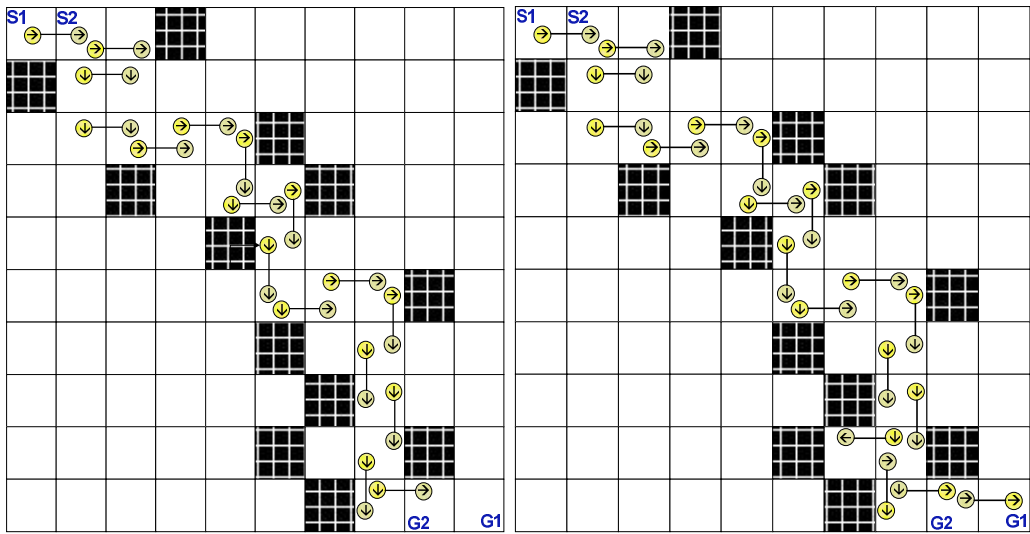


Fig. A2. 2.9 Joint action selection strategy in EQLP12 and reference algorithms for four agents

**Additional details of Experiment 2.2**

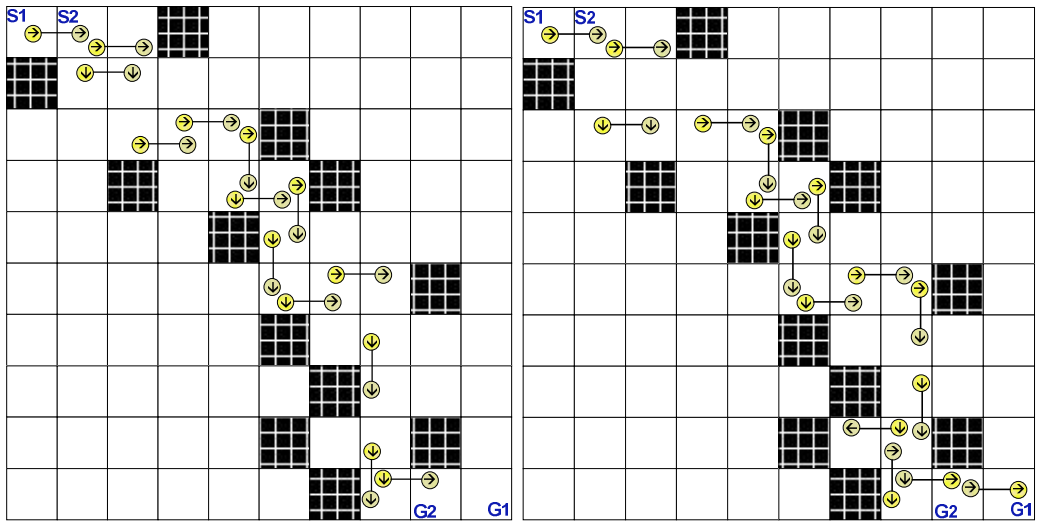


a. by NQIMP algorithm

b. by NQLP12 based cooperative multi-agent Planning

Fig. A2. 2.10 Path planning with stick in deterministic situation

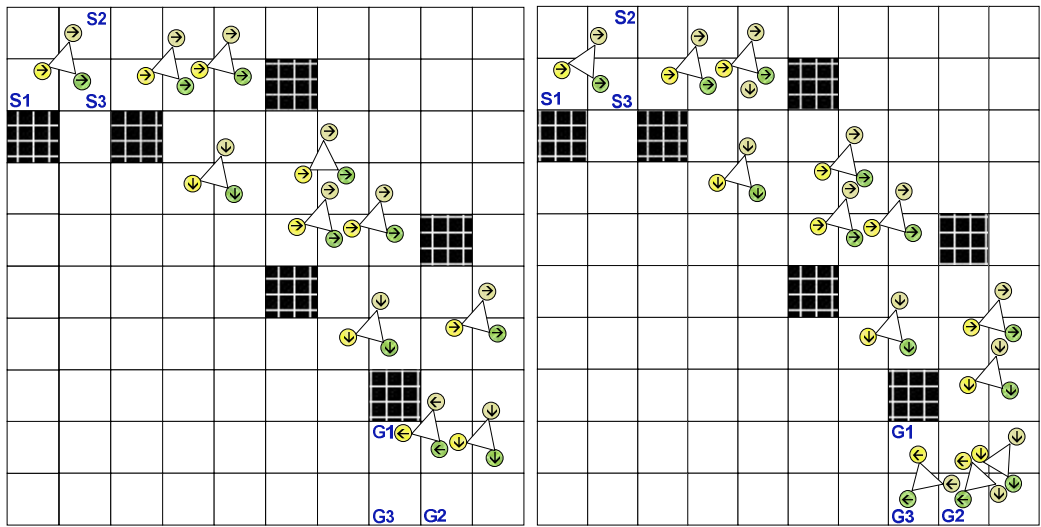




a. by NQIMP algorithm

b. by NQLP12 based cooperative multi-agent Planning

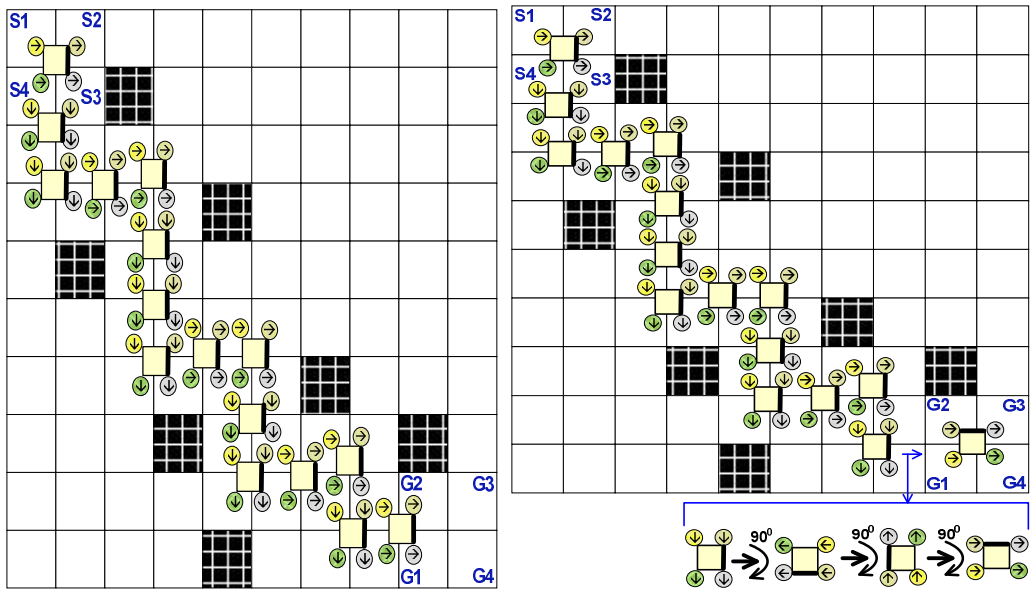
Fig. A2. 2.11 Path planning with stick in stochastic situation



a. by NQIMP algorithm

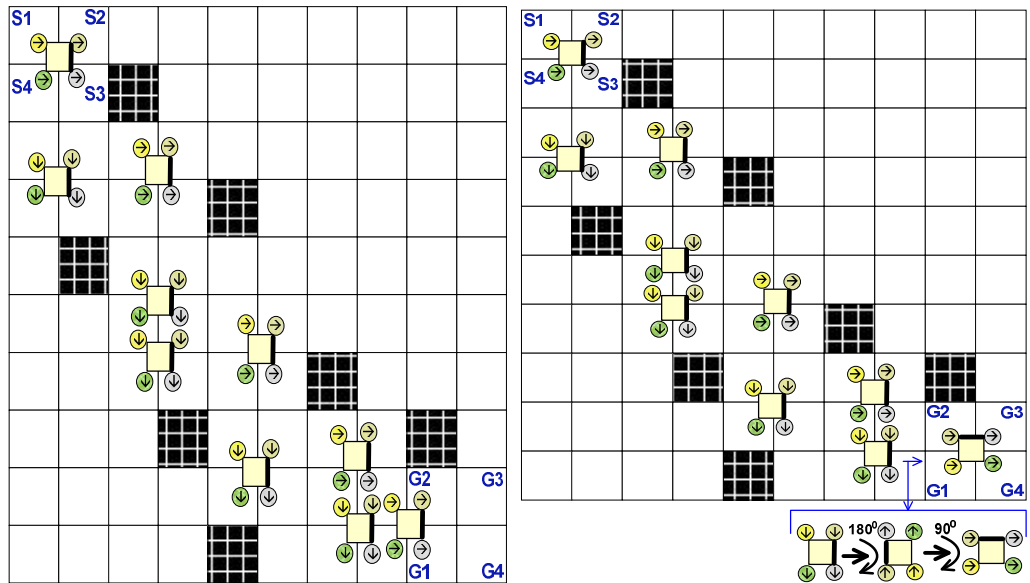
b. by NQLP12 based cooperative multi-agent planning

Fig. A2. 2.12 Path planning with triangle in stochastic situation



a. by NQIMP algorithm                      b. by NQLP12 based cooperative multi-agent planning

Fig. A2. 2.13 Path planning with square in stochastic situation



a. by NQIMP algorithm                      b. by NQLP12 based cooperative multi-agent planning

Fig. A2. 2.14 Path planning with square in deterministic situation

### Additional details of Experiment 2.4

**Details of Khepera-II mobile robot:** Khepera-II is a miniature robot [61], [62], equipped with an onboard Microcontroller (Motorola 68331), and includes a flash memory of 512 KB and clock speed of 25MHz, having 8 inbuilt infrared proximity sensors. 1 unit speed of

Khepera II mobile robot is 0.08mm/10ms. Selected speeds in this experiment are 2 unit (0.16 mm/10ms), 3 unit (0.24 mm/10ms) and 5 unit (0.4 mm/10ms). Considering one grid length of 80 mm (square grid), theoretically time taken by an agent to cover one grid length, with 2 unit, 3 unit and 5 unit speed are 500 ms, 333.33 ms and 200 ms respectively [61], [62]. Assuming a circle of 40 mm radius inside each grid, Khepera-II has to cover  $20\pi$  mm of the total circumference of the circle for 90 degree rotation. Therefore theoretically time taken by an agent for one 90 degree rotation with 2 unit, 3 unit and 5 unit speed are 392.7 ms, 261.8 ms and 157 ms respectively.

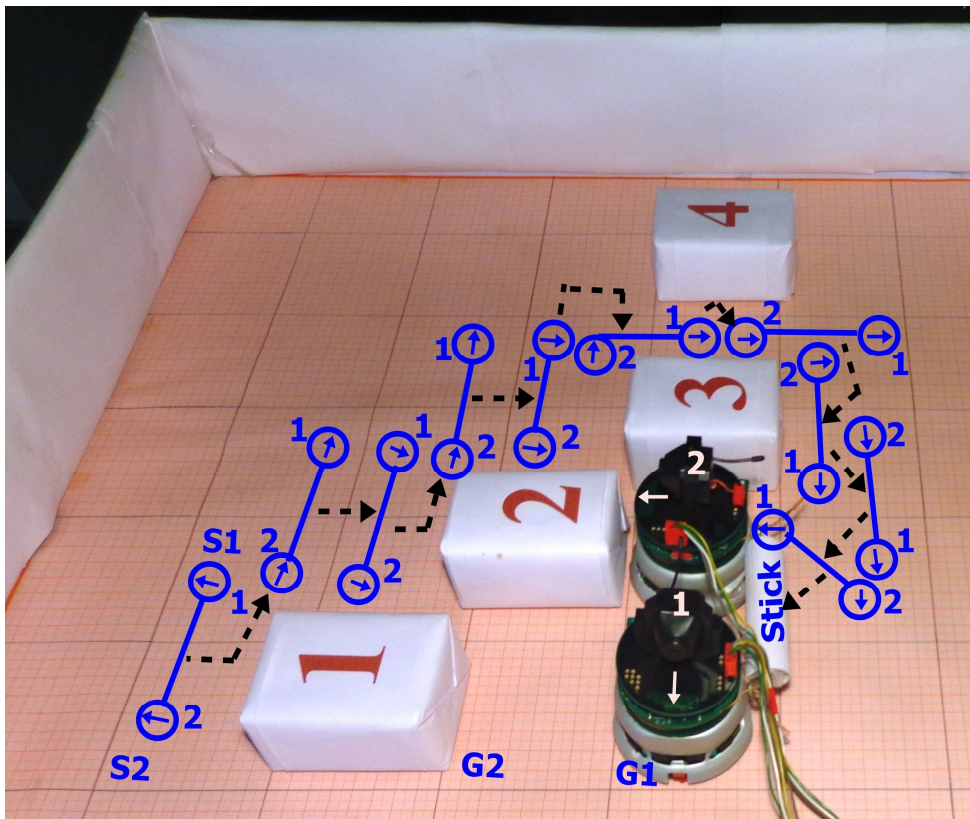


Fig. A2. 2.15 Cooperative path planning to carry a stick by two Khepera-II mobile robots using NQIMP algorithm

The stick-carrying problem has been realized in Fig. 2.7 and 2.8 by controlling two Khepera-II mobile robots using pre-learned joint Q-tables. The stick length is one grid width and two robots can carry the stick if they occupy neighborhood cells. Each Khepera-II mobile robots (agents) are connected by wires to two different Pentium IV machines through serial port connections. Agents do not communicate between them while transporting the stick. The next joint states of the robots are determined by evaluating the NE employing the learned joint Q-tables, stored in the attached Pentium IV machine.

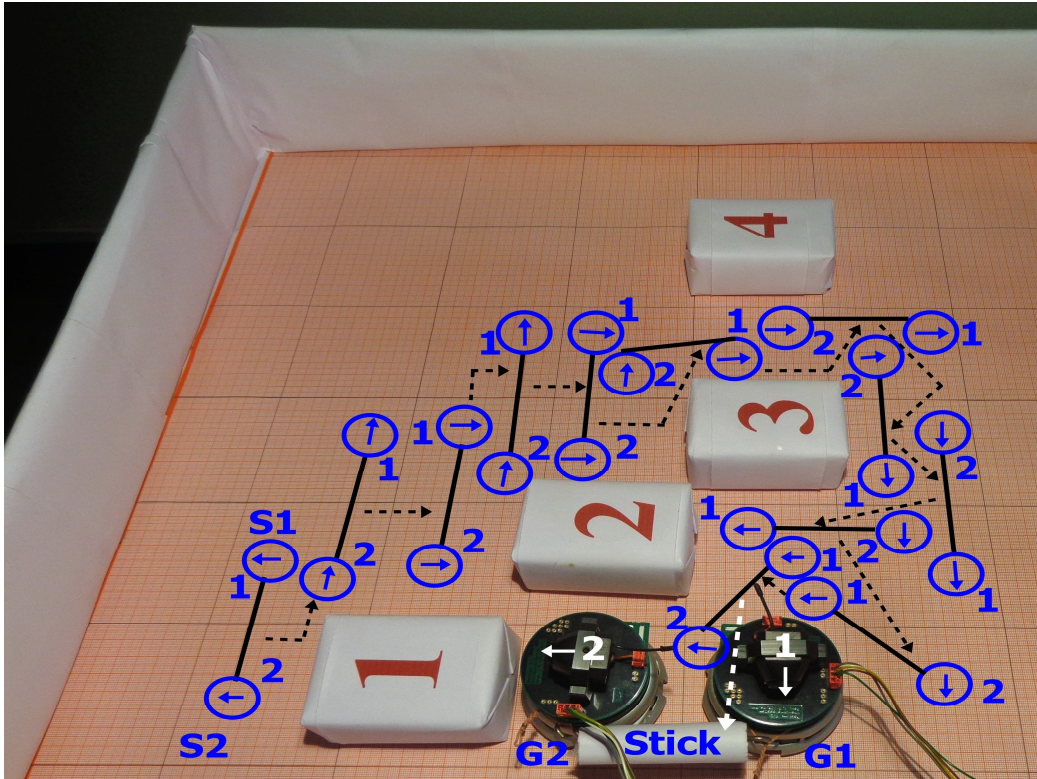


Fig. A2. 2.16 Cooperative path planning to carry a stick by two Khepera-II mobile robots using NQIMP algorithm

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# **Chapter 3**

## **Multi-robot Cooperative Planning by Consensus Q-learning**

Multi-robot cooperation entails planning by multiple robots for a common objective, where each robot/agent actuates upon the environment based on the sensory information received from the environment. Multi-robot cooperation employing equilibrium-based reinforcement learning is optimal in the sense of system resource (time and/or energy) utilization, because of the prior adaption of the environment by the robots. Unfortunately, robots cannot enjoy such benefit of reinforcement learning in presence of multiple types of equilibria (here Nash equilibrium or correlated equilibrium). In the above perspective, robots need to adapt with a strategy, so that robots can select the optimal equilibrium in each step of the learning. The chapter proposes consensus-based multi-agent Q-learning to address the bottleneck of the optimal equilibrium selection among multiple types. An analysis reveals that a consensus (joint action) is coordination type pure strategy Nash equilibrium as well as pure strategy correlated equilibrium. The superiority of the proposed consensus-based multi-agent Q-learning algorithm over the traditional reference algorithms in terms of the average reward collection is shown in the experimental section. In addition, the proposed consensus-based planning algorithm is also verified considering multi-robot stick-carrying problem as a benchmark.

Table 3.1 List of acronyms

Full form	Acronyms
Multi-agent Q-learning :	MAQL
Nash equilibrium :	NE
Nash Q-Learning :	NQL
Correlated equilibrium :	CE
Correlated Q-learning :	CQL
Consensus Q-learning :	CoQL
Utilitarian Q-Learning :	UQL
Egalitarian Q-Learning :	EQL
Republican Q-Learning :	RQL
Libertarian Q-Learning :	LQL

### 3.1 Introduction

Planning [1] refers to the execution of an action sequence, with an aim to achieve a predefined goal by optimally employing the system resource (time and/or energy). An agent (here robot) can plan individually or in a group. While planning in a group, the agent may be cooperative or competitive towards the remaining group members. In this paper, only the cooperative robots are considered and analyzed.

Several techniques are available for planning in the literature, including Graphs [2], Voronoi diagrams [3], potential field [4], adaptive action selection [5], intention inference [6], cooperative conveyance [7], perceptual cues [8], and the like. All these require information about the environment, and thus are unable to function when the information about the environment is absent. Reinforcement Learning (RL) [9]-[18] fills this void.

RL is the model-free approach and hence, it is preferred over other traditional planning approaches. There exist a number of RL algorithms [9]-[18]. Based on the number of agents involved, RL is of two types: single agent and multi-agent. Multi-Agent Reinforcement Learning (MARL) algorithms are of three types: cooperative, competitive and mixed [19]. In this paper, we focus only on cooperative MARL. Among the cooperative MARL algorithm equilibrium-based MARL is one type [19], where each agent updates its joint Q-value at equilibrium. By equilibrium, an agent attains a balance condition among the agents. In the literature, there are two types of equilibria: Nash equilibrium (NE) [20] and correlated equilibrium (CE) [21]. In Nash Q-learning (NQL) [22]-[23] and correlated Q-learning (CQL) [21], agents' update Q-values at joint state-action space employing the NE and CE respectively. As we are dealing only with the cooperative MAQL, so here only the coordination type NE is considered. It is difficult to find out that which equilibrium (NE or CE) is optimal at a joint state. If agents are instructed to select any one type of equilibrium (either NE or CE) *a priori*, then there is a chance of missing the optimal solution.

To address the above problem we introduce the concept of consensus [24]-[25] in the domain of MARL, from the field of cooperative control [26] and potential games (PGs) [26].

A consensus is a coordination type pure strategy NE [26]. In this paper, agents' are instructed to update Q-values at consensus and proposed a novel consensus Q-learning (CoQL) algorithm. In addition, an analysis reveals that that a consensus jointly satisfies the criterion of pure strategy NE as well as pure strategy CE. Experimental result demonstrates the superiority of the proposed CoQL algorithms over the reference algorithms in terms of the average of the average rewards (AAR) earned by the agents. The consensus-based multi-robot cooperative planning algorithm is also proposed and its superiority is shown over the reference algorithms considering path length and torque requirement by the robots as the performance metric. The merits of the chapter are:

1. The problem of selecting the equilibrium type in multi-agent system is addressed by proposing the CoQL algorithm.
2. Agents evaluate consensus (joint action) [73-76] in each step of learning and planning phases.
3. It is shown by an analysis that a consensus at a joint state is a coordination type pure strategy NE as well as pure strategy CE.

The rest of the chapter is structured as follows. Preliminaries of the Q-learning are given in Section 3.2. Section 3.3 introduces the concept of consensus. Section 3.4 proposes the consensus-based Q-learning and planning. Section 3.5 includes experiments and results. The conclusions are listed in Section 3.6.

## 3.2 Preliminaries

In RL, the learner works on the principle of reward/penalty received from the environment. Q-learning is an example of RL. This section briefly introduces the adaption mechanism of single agent Q-learning [27] and the state-of-the-art equilibrium-based multi-agent Q-learning (MAQL) algorithms. The state-of-the-art equilibrium-based MAQL includes NQL [22]–[23] and the four variants of CQL [21].

### 3.2.1 Single agent Q-learning

The single agent Q-learning is proposed by Watkins and Dayna [27]. The recursive update rule of the single agent Q-learning for an agent, denoted by  $i$  is given by (1) [27].

$$Q_i(s_i, a_i) \leftarrow (1 - \alpha)Q_i(s_i, a_i) + \alpha[r_i(s_i, a_i) + \gamma \sum_{\forall s'_i} P_i[s'_i | (s_i, a_i)] \max_{a'_i} Q_i(s'_i, a'_i)], \quad (3.1)$$

where,  $\alpha \in (0,1]$  refers to the learning rate,  $\gamma \in (0,1]$  denotes the discounting factor,  $r_i(s_i, a_i)$  is the immediate reward received by the agent  $i$  because of action  $a_i \in \{a_i\}$  at the current state  $s_i \in \{s_i\}$ .  $Q_i(s_i, a_i)$  refers to the sum of long term discounted rewards or Q-value of agent  $i$  at state  $s_i$  because of action  $a_i$ .  $P_i[s'_i | (s_i, a_i)]$  is the probability of moving to the next state

$s'_i \in \{s_i\}$  from current state  $s_i$  because of  $a_i$ . In the literature of Q-learning,  $P_i[s'_i | (s_i, a_i)]$  is well-known as the state transition probability. On completion of the learning an agent (here robot) begins to plan. During the planning phase it selects the action corresponding to the maximum Q-value in the current state, at each step of planning.

### 3.2.2 Equilibrium-based multi-agent Q-learning

The Q-value adaption mechanism of single agent Q-learning is not applicable for MAQL. As in multi-agent system, each agent learns in a common environment, which resulting in a dynamic environment. Several attempts have been made to address such multi-agent dynamics [22]-[23]. In cooperative MAQL, each agent attempts to maximize its own reward as well as the reward of the team. Such requirement can be attained by achieving a balanced condition among the agents, where no agent has any selfish intension to deviate from the balanced condition. In the literature of cooperative MAQL, the above mentioned balanced condition is achieved following equilibrium, e.g. NE or CE, where each agent updates its optimal expected future reward at equilibrium. Equilibrium-based cooperative MAQL is one of the interesting learning-based multi-robot planning algorithms, where each robot has inherent capabilities to adapt equilibrium at the current joint state.

In this paper, we are interested only with the pure strategy NE/CE (or joint action). The definitions of pure strategy NE and pure strategy CE are respectively given in Definition 3.1 and 3.2.

**Definition 3.1:** With  $m$  interacting agents' *pure strategy Nash equilibrium* at a joint state  $S \in \{S\}$  is a joint action  $A_N = \langle a_i^* \rangle_{i=1}^m$ , such that no unilateral deviation (selfish deviation of an agent) can occur as long as all the agents follow the same optimal joint action  $A_N = \langle a_i^* \rangle_{i=1}^m$  at  $S$ .

Assuming  $a_i^* \in \{a_i\}$  be the optimal action of agent  $i$  at  $s_i$  and  $A_{-i}^* \subseteq A$  be the optimal joint action profile of all agents except agent  $i$  at  $S = \langle s_i \rangle_{i=1}^m$  and  $Q_i(S, A)$  be the joint Q-value of agent  $i$  at  $S$  because of joint action  $A \in \{A\}$ . Then the condition of pure strategy NE

$$A_N = \langle a_i^*, A_{-i}^* \rangle \text{ at } S \text{ is} \\ Q_i(S, A_N) \geq Q_i(S, a_i, A_{-i}^*), \forall i \quad (3.2)$$

**Definition 3.2:** With  $m$  interacting agents *pure strategy Correlated equilibrium* at a joint state ( $S$ ) is the optimal pure strategy profile  $A_{CE} = \langle a_1, a_2, \dots, a_m \rangle^*$ , if and only if agents follow (3.3).

$$A_{CE} = \arg \max_A [\Psi(Q_i(S, A))] \quad (3.3)$$

$$\text{where, } \Psi \in \{Min, \sum, Max, \prod\} \quad [21]. \quad (3.4)$$

$$\forall i \quad \forall i \quad \forall i \quad \forall i$$

Here, CE has four variants: Egalitarian equilibrium (EE), Utilitarian equilibrium (UE), Republican equilibrium (RE) Libertarian equilibrium (LE). One problem of equilibrium-

based MAQL is the selection of optimal equilibrium among the multiple types of equilibria. In addition, in the context of multi-robot cooperative planning problem, selection of optimal equilibrium refers to the selection of optimal joint action. In this context, the traditional equilibrium-based MAQL algorithm is given in Algorithm 3.1 [21]-[23].

<b>Algorithm 3.1: Equilibrium-based MAQL</b>
<p><b>Input:</b> Current state <math>s_i, \forall i</math>, action set <math>A_i</math> at <math>s_i, \forall i</math>, <math>\alpha \in [0,1)</math> and <math>\gamma \in [0,1)</math>;</p> <p><b>Output:</b> Optimal joint Q-value <math>Q_i^*(S, A), \forall S, \forall A, \forall i</math>;</p> <p><b>Begin</b>  Initialize: <math>Q_i(S, A) \leftarrow 0, \forall S, \forall A, \forall i</math>;</p> <p><b>Repeat</b>  Select an action <math>a_i \in A_i, \forall i</math> randomly and execute it;  Observe immediate rewards <math>r_i(S, A), \forall i</math>;</p> <p>Evaluate <math>s_i' \leftarrow \delta_i(s_i, a_i), \forall i</math> to obtain <math>S' = \langle s_i' \rangle_{i=1}^m</math>;</p> <p><math>Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \cdot \Psi Q_i(S')], \forall i</math>  and <math>S \leftarrow S'; // \Psi \in \{NE, CE\}</math></p> <p><b>Until</b> <math>Q_i(S, A), \forall S, \forall A, \forall i</math> converges;</p> <p><math>Q_i^*(S, A) \leftarrow Q_i(S, A), \forall S, \forall A, \forall i</math>;</p> <p><b>End.</b></p>

### 3.3 Consensus

In this section, the cooperative control problem employing potential games mainly focusing upon the consensus problem is briefly discussed. Here cooperative control [28]-[30] refers to a planning problem (e.g. object-transportation) by autonomous agents, satisfying all the necessary constraint. One paradigm of cooperative control problem is the consensus problem with plenty of literature in computer science and in the field of distributed computing [31], where the challenge is to design the objective functions of the autonomous agents at a given joint state due to a joint action to realize the team objective amidst obstacles. Alternatively, the cooperative control problem (consensus problem) can also be deciphered by employing the concept of game-theory. In the context of cooperative control, potential game has a big role to play [28]. In potential game, agents require the perfect alignment between the team objective/potential function and the individual objective of the agent. A consensus in the PG is guaranteed to converge to a pure strategy NE with a potential function of increasing nature [28] In Q-learning, individual objective is equivalent to the Q-value. The following definitions are required to understand the later sections of the paper. The definition of consensus is given by utilizing the concept that all the PGs are guaranteed to converge to a pure strategy NE as given in [28].

**Definition 3.3:** In a  $m$  player game, if  $S \in \times_{i=1}^m S_i$  and  $A \in \times_{i=1}^m A_i$  indicate the joint state and joint action respectively, individual objective functions are  $\{Q_i : S \times A \rightarrow \mathbb{R}\}_{i=1}^m$  and potential function is denoted by  $\Phi : S \times A \rightarrow \mathbb{R}$  satisfies

$$\begin{aligned} Q_i(S, a_i', A_{-i}) - Q_i(S, a_i'', A_{-i}) = \\ \Phi(S, a_i', A_{-i}) - \Phi(S, a_i'', A_{-i}) \end{aligned} \quad (3.5)$$

i. e., all players' objective functions are aligned with the potential function, then the game is an *Exact Potential game* (EPG), where  $A_{-i} \in \times_{j=1, j \neq i}^m A_j$  and  $a_i'', a_i' \in A_i$  [28].

**Definition 3.4:** In a  $m$  player game, if  $S \in \times_{i=1}^m S_i$  and  $A \in \times_{i=1}^m A_i$  indicate the joint state and joint action respectively, individual objective functions  $\{Q_i : S \times A \rightarrow \mathbb{R}\}_{i=1}^m$  and potential function denoted by  $\Phi : S \times A \rightarrow \mathbb{R}$  satisfy

$$\left. \begin{aligned} Q_i(S, a_i', A_{-i}) > Q_i(S, a_i'', A_{-i}) \\ \Phi(S, a_i', A_{-i}) > \Phi(S, a_i'', A_{-i}) \end{aligned} \right\} \quad (3.6)$$

i. e., at least one player's objective function is aligned with the potential function, then the game is *weakly acyclic game* (WAG), where  $A_{-i} \in \times_{j=1, j \neq i}^m A_j$  and  $a_i'', a_i' \in A_i$  [28].

**Definition 3.5:** A *consensus* is a joint action  $A^* = \langle a_i^*, A_{-i}^* \rangle \in \{A\}$  at a given joint state  $S \in \{S\}$ , which jointly maximizes the individual objective function  $Q_i(S, a_i, A_{-i}), \forall A, \forall i$  or  $Q_i(S, a_i^*, A_{-i}^*) \geq Q_i(S, a_i, A_{-i})$ ,  $\forall A, \forall i$  and the potential function  $\Phi(S, A)$  or  $\Phi(S, A^*) \geq \Phi(S, A)$ ,  $\forall A$  [28].

## 3.4 Proposed Consensus Q-Learning and Planning

In the section, we proposed a novel Consensus Q-learning (CoQL). Subsequently a consensus-based multi-robot cooperative planning algorithm is proposed.

### 3.4.1 Consensus Q-learning

An example is given in Fig. 3.1 to understand the importance of consensus in multi-robot cooperative planning. Let at a given joint state two robots 1 and 2 are synchronously cooperating with the action set respectively  $A_1 = \{a, b\}$  and  $A_2 = \{x, y\}$  having no communication among the robots. The reward matrixes in two different joint states are given in Fig. 3.1(a) and (b) In Fig. 3.1(a), suppose, robots plan following the CE (UE) and then they have two solutions  $ax$  and  $by$  to cooperate by (3.3). In such situation, in the absence of communication among the robots, they cannot select one joint action to cooperate. But if they evaluate coordination type NE (cooperative NE) by (3.2) and select the joint action  $ax$ , then the above problem can be addressed. However, in Fig. 3.1(b), contains two coordination type NEs ( $ax$  and  $by$ ) by (3.2) and again the same problem arises. Here, robots can go for evaluating the CE (UE or EE) by (3.3) and selects joint action  $ax$  to cooperate.

It is interesting to note that both the robots' receive maximum reward for the joint action, which satisfies the criterion of coordination type NE (or pure strategy NE for brevity) and pure strategy CE jointly. Motivated by this observation, we are interested to find such an equilibrium, which is a pure strategy NE as well as pure strategy CE. To achieve this we borrow the concept of consensus from PGs, which by definition is a pure strategy NE. In this paper, by a Theorem we have shown that a consensus is also a pure strategy CE.

		$A_2 \rightarrow$	
		$x$	$y$
$A_1 \downarrow$	$a$	<b>90, 72.9</b>	81, 72.9
	$b$	72.9, 72.9	<b>72.9, 90</b>

		$A_2 \rightarrow$	
		$x$	$y$
$A_1 \downarrow$	$a$	<b>99, 100</b>	94, 96
	$b$	94, 95	<b>97, 98</b>

(a) Two UE ( $ax$  and  $by$ ) and one NE ( $ax$ ) (b) Two NE ( $ax$  and  $by$ ) and one UE or EE ( $ax$ )

Fig. 3.1 Equilibrium selection in multi-agent system

**Theorem:** In a potential game (PG), if  $A^* = \langle a_i^*, A_{-i}^* \rangle \in \{A\}$  is a consensus point (joint action) at a given joint state  $S \in \{S\}$ , then at joint state  $S$  a consensus ( $A^*$ ) is a pure strategy Nash equilibrium (NE),  $A_N$  as well as a pure strategy Correlated equilibrium (CE),  $A_{CE}$ . Assuming there exist at least one coordination type pure strategy NE.

**Proof:** Since  $A^*$  is a consensus point, by Definition 5, we have:

$$\begin{aligned}
& Q_i(S, a_i^*, A_{-i}^*) \geq Q_i(S, a_i, A_{-i}^*), \forall A, \forall i \\
& \Rightarrow Q_i(S, a_i^*, A_{-i}^*) \geq Q_i(S, a_i, A_{-i}^*), \forall i. \quad (\because a_i, A_{-i}^* \in \{A\}) \\
& \Rightarrow Q_i(S, A^*) \geq Q_i(S, a_i, A_{-i}^*), \forall i. \quad (\because a_i^*, A_{-i}^* \Rightarrow A^*)
\end{aligned} \tag{3.7}$$

By (3.7) and Definition 3.1 we can say that

$$A^* = A_N \tag{3.8}$$

at  $S$ . Again by Definition 3.5 at consensus, the inequality (3.9) holds.

$$\Phi(S, A^*) \geq \Phi(S, A), \forall A \tag{3.9}$$

Now, by Definition 3.3 all players' objective functions are aligned with the potential function in an *Exact Potential game* (EPG) and by Definition 3.4 in WAG, at least one agent's objective function is aligned with the potential function and hence  $\Phi(S, A)$  is assumed as in (3.10).

$$\Phi(S, A) = \Omega[Q_i(S, A)] \tag{3.10}$$

$$\text{where, } \Omega \in \{Min, \sum, Max, \prod\}, \quad \forall i \quad \forall i \quad \forall i \quad \forall i \tag{3.11}$$

Now, by (3.9)

$$\begin{aligned}
& \Phi(S, A^*) \geq \Phi(S, A), \forall A \\
& \Rightarrow \Omega[Q_i(S, A^*)] \geq \Omega[Q_i(S, A)], \forall A \quad [\text{by (3.10) and } \Omega \in \{Min, \sum, Max, \prod\}] \\
& \Rightarrow \Omega[Q_i(S, A^*)] = \max_A [\Omega[Q_i(S, A)]]
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow Q_j(S, A^*) = \max_A [\Omega[Q_i(S, A)]] && [\text{Let } \Omega[Q_i(S, A^*)] = Q_j(S, A^*), j \in [1, m]] \\
&\Rightarrow \arg_A [Q_j(S, A^*)] = \arg \max_A [\Omega[Q_i(S, A)]] \\
&\Rightarrow A^* = \arg \max_A [\Omega[Q_i(S, A)]] \\
&\Rightarrow A^* = A_{CE} \quad [\text{by Definition 3.2}] \tag{3.12}
\end{aligned}$$

So, to hold (3.9) in a PG,  $A^*$  should be a  $A_{CE}$ . Hence, by (3.8) and (3.12) we can say that a consensus  $A^*$  at a given joint state  $S$  is a  $A_N$  as well as  $A_{CE}$ .  $\square$

<b>Algorithm 3.2: Consensus Q-learning (CoQL)</b>	
<b>Input:</b> Current state $s_i, \forall i$ , action set $A_i$ at $s_i, \forall i$ , $\alpha \in [0,1)$ and $\gamma \in [0,1)$ ;	
<b>Output:</b> Optimal joint Q-value $Q_i^*(S, A), \forall S, \forall A, \forall i$ ;	
<b>Begin</b>	
Initialize: $Q_i(S, A) \leftarrow 0, \forall S, \forall A, \forall i$ ;	
<b>Repeat</b>	
Select an action $a_i \in A_i, \forall i$ randomly and execute it;	
Observe immediate rewards $r_i(S, A), \forall i$ ;	
Evaluate $s'_i \leftarrow \delta_i(s_i, a_i), \forall i$ to obtain $S' = \langle s'_i \rangle_{i=1}^m$ ;	
$Q_i(S, A) \leftarrow (1 - \alpha)Q_i(S, A) + \alpha[r_i(S, A) + \gamma \cdot \text{Co}Q_i(S')], \forall i$	} //Proposed
and $S \leftarrow S'$ ; // Co=NE and CE	
<b>Until</b> $Q_i(S, A), \forall S, \forall A, \forall i$ converges;	
$Q_i^*(S, A) \leftarrow Q_i(S, A), \forall S, \forall A, \forall i$ ;	
<b>End.</b>	

So, in the CoQL and planning algorithms, instead of evaluating the pure strategy NE/CE at a joint state, a consensus is evaluated motivated by the cooperative control and PG, as a consensus is a pure strategy NE as well as a pure strategy CE as shown in the proposed Theorem. The proposed CoQL algorithm is given in Algorithm 3.2. The brace in Algorithm 3.2 indicates the difference between Algorithm 3.1 and 3.2.

### 3.4.2 Consensus-based multi-robot planning

In multi-agent planning phase, each agent evaluates consensus by jointly satisfying (3.2) and (3.3) at a feasible joint state. It may be noted that for multiple consensuses at the given joint state, the consensus which appears first is selected. In this paper, we have considered the well-known stick-carrying problem, where each robot needs to reach its individual goal optimally without violating any constraint. Constraint violation refers to the collision with obstacle or the team-mates and falling of stick following Algorithm 3. The brace in Algorithm 3 indicates the key contribution in the planning algorithm.



**Algorithm 3.3:** Consensus-based planning

**Input:** Feasible joint starting state  $S_F$ , joint goal state  $S_G$  and optimal joint Q-value  $Q_i^*(S, A), \forall i$ ;

**Output:** Consensus or joint action which is a NE as well as CE  $A_F^*$  at  $S_F$ ;

**Begin**

**While**  $S_F \neq S_G$  **do Begin**

**For**  $A \in \{A\}$

        Evaluate consensus by jointly following (3.2) and (3.3); } //Proposed

**If** next feasible joint state  $S_F'$  satisfies all the constraints;

**Then**  $A_F^* \leftarrow A$  and  $S_F \leftarrow S_F'$ ; //  $S_F'$  is the next joint state

**End If**;

**End for**;

**End While**;

**End.**

## 3.5 Experiments and Results

Two experiments are presented in this section. The first experiment is designed to study the relative performance of the CoQL over the reference algorithms, considering average reward collection by the agents as a performance metric. Another experiment is framed to study the relative performance of the consensus-based planning algorithm over the reference algorithms, considering multi-robot stick carrying problem as a benchmark in terms of state-transitions required to complete the task.

### 3.5.1 Experimental setup

All the experiments related to learning are performed in ten different  $10 \times 10$  grid world maps for two and three agents. However, for brevity multi-robot planning is conducted for two agents only in  $5 \times 5$  grid world maps. Each agent can execute one among the four possible actions (Left (L), Forward (F), Right (R), and Back (B)) at a state. As an agent reaches its goal state due to an action from a state, it receives maximum immediate reward of 100. Similarly, an agent receives zero (0) immediate reward for a non-goal state transition. The constraint violation is penalized by a negated immediate reward (here -1). In addition, to the above parameter setting the learning rate,  $\alpha$  and discounting factor,  $\gamma$  are set to 0.1 and 0.9 respectively.

### 3.5.2 Experiments for CoQL

In this experiment, at each state an agent selects its action randomly from its individual action pool. In the next step, agent updates its own as well as the remaining agents' Q-values at joint state-action space following Algorithm 3.2. Average of the average rewards (AAR) as given in (3.13) is considered as a performance metric of the learning algorithms for  $m$  number of learning agents.

$$AAR = \left( \sum_{i=1}^m \left( \sum_{\forall S} \left( \sum_{\forall A} (Q_i(S, A)) \times \frac{1}{|\{A\}|} \right) \times \frac{1}{|\{S\}|} \right) \times \frac{1}{m} \right) \quad (3.13)$$

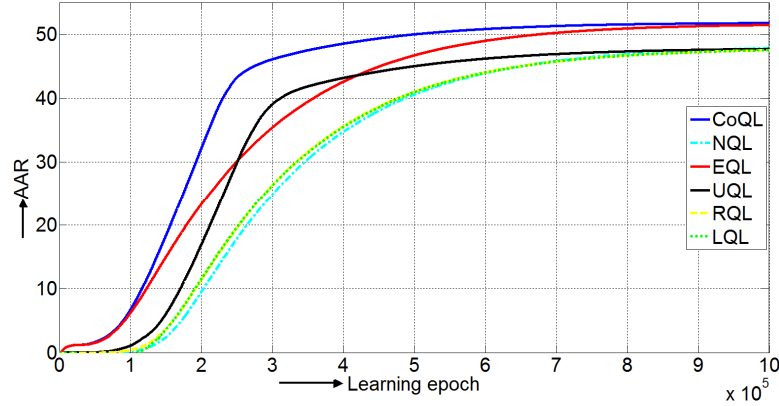


Fig. 3.2 AAR versus learning epoch for two agent system

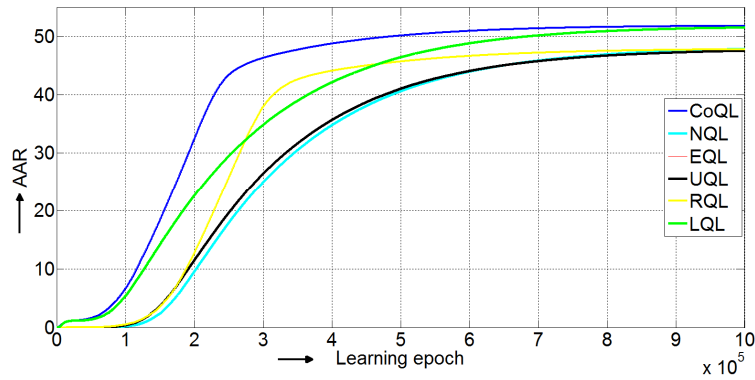


Fig. 3.3 AAR versus learning epoch for three agent system

It is apparent from Fig. 3.2 that the AAR collected by a team of two agents over the learning epoch (iteration) in CoQL is more than the same offered by the traditional NQL and different variants of CQL (EQL, UQL, RQL, and LQL). Similar experiment is conducted for three agents as shown in Fig. 3.3.

### 3.5.3 Experiments for consensus-based planning

In this experiment, performance of the consensus-based multi-robot cooperative planning algorithm has been tested considering the well-known stick-carrying problem as a benchmark. The stick-carrying problem refers to the transportation of a stick from starting position to the fixed destination optimally without violating any constraint. It is apparent from Fig. 3.5 and 3.5

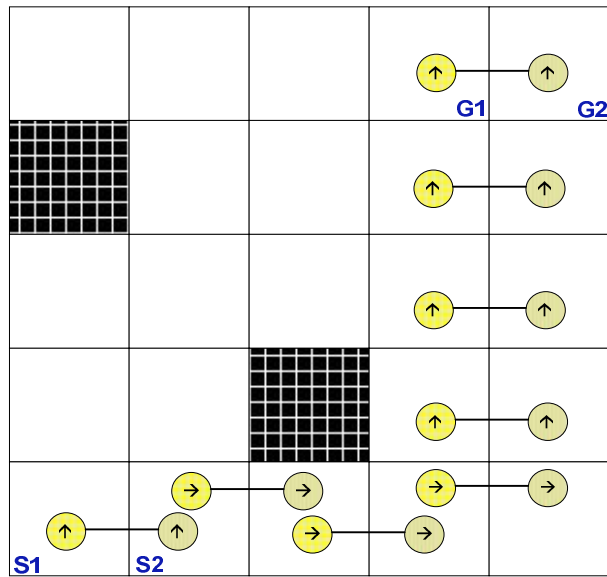


Fig. 3.4 planning path offered by the consensus-based multi-agent planning algorithm

Table 3.2 Planning performance

Planning algorithm	Number of state-transitions required		Number of 90° turns required	
	A1	A2	A1	A2
Consensus-based	7	7	2	2
NQL-based	7	7	3	3

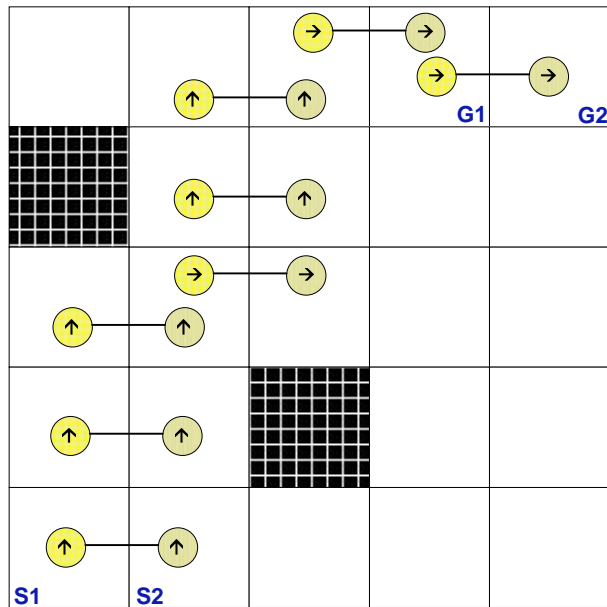


Fig. 3.5 planning path offered by the Nash Q- learning based planning algorithm

that the planning path offered by the consensus-based multi-agent cooperative planning algorithm is better than the same offered by the traditional learning based planning path, in terms of the path length and the number of  $90^\circ$  turns. Minimization of the  $90^\circ$  turns minimizes the torque requirement by the robots, and hence, saving in the energy consumption. Table 3.2 illustrates the planning performance of the proposed Consensus-based planning algorithm over the NQL-based planning algorithm (here NQL) in terms of the above explained metrics.

### **3.6 Conclusions**

The chapter proposes a novel CoQL algorithm for multi-robot cooperative planning. The proposed CoQL algorithm addresses the problem of equilibrium selection among different types of equilibria, by evaluating the consensus (joint action) at the current joint state. An analysis reveals that a consensus at a joint state is a pure strategy NE as well as pure strategy CE. The novelty of the CoQL lies in the adaption of the joint Q-values at consensus. The superiority of the proposed CoQL algorithm is verified over the reference algorithms in terms of the AAR earned by the agents against the learning epoch. In addition, consensus-based multi-robot cooperative planning algorithm is proposed and its superiority is verified over reference algorithms considering path length and torque requirement as the performance metrics.

### **3.7 Summary**

In Chapter 3, it is shown that robots may select the suboptimal equilibrium in presence of multiple types of equilibria (here Nash equilibrium or correlated equilibrium). In the above perspective, robots need to adapt with such a strategy, which can select the optimal equilibrium in each step of the learning and the planning. To address the bottleneck of the optimal equilibrium selection among multiple types, chapter 3 presents a novel consensus Q-learning for multi-robot coordination, by extending the equilibrium-based multi-agent Q-learning algorithms. It is also shown that a consensus (joint action) jointly satisfies the conditions of the coordination type pure strategy Nash equilibrium and the pure strategy correlated equilibrium. The superiority of the proposed consensus Q-learning algorithm over traditional reference algorithms in terms of the average reward collection are shown in the experimental section. In addition, the proposed consensus-based planning algorithm is also verified considering multi-robot stick-carrying problem as the testbed.

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## **Chapter 4**

# **An Efficient Computing of Correlated Equilibrium for Cooperative Q-Learning Based Multi-Robot Planning**

In traditional multi-agent Q-learning induced planning, we need to evaluate Nash/correlated equilibrium at a given joint state during both learning and planning phases. Determination of such equilibrium being computationally expensive prohibits the planning in real-time. The chapter introduces a novel approach to adapt composite rewards of all the agents in one Q-table in joint state-action space during learning, and uses these rewards to compute Correlated equilibrium in the planning phase. Two schemes of multi-agent Q-learning have been proposed. If success of only one agent is enough to make the team successful, then scheme-I is employed. However, if an agent's success is contingent upon other agents and simultaneous success of the agents is mandatory then scheme-II is employed. New algorithms for multi-agent learning/planning have been proposed, centering on the said schemes. It is shown that the correlated equilibrium obtained by the proposed algorithms and the traditional correlated Q-learning are identical. In order to restrict the exploration within the feasible joint states, constraint versions of the said algorithms are also proposed. An analysis is included to demonstrate the significant saving of computational time and space by the proposed algorithms. In addition, convergence analysis of the proposed algorithms is done. Experiments have been undertaken to validate the performance of the proposed algorithms in multi-robot planning on both simulated and real platforms.

## 4.1 Introduction

Reinforcement learning (RL) works on the principle of reward and penalty earned by an agent (robot) [1]-[7], [15] from the environment. An agent is an autonomous body [8]-[11] capable of maintaining state transitions [12], [13] freely in a given environment. In RL, an agent learns a policy  $\pi$  to maximize a *value function*  $V^\pi(s)$  [14], at any environmental state  $s$  to achieve its (fixed) goal. Q-learning belongs to the family of RL algorithms. In Q-learning, the agent learns an optimal policy to select the best (optimal) action  $a$  at state  $s$  to maximize the sum of immediate reward and the value function of the next state  $s'$ , discounted by a factor  $0 \leq \gamma < 1$ .

In single agent Q-learning [16], [17], the environment is stationary because the agent earns immediate reward due to its own action on the environment [18], [19]. However, in multi-agent scenario, the immediate reward obtained by an agent depends also on the other agents' actions on the environment, and thus the environment seems to be non-stationary [18]-[23]. Although the non-stationary behavior of the environment has not yet been modeled directly in multi-agent Q-learning (MAQL), its effect is considered by updating the joint state-action value function of the agents at equilibrium [24].

Several algorithms for MAQL have been proposed for both cooperative and competitive applications [18]-[34]. Among the equilibrium based cooperative MAQL algorithms, Nash-Q learning (NQL) [21], [28] and Friend-Q learning (FQL) [23] algorithms need special mention. Both these algorithms allow each agent to optimize its reward (payoff) in joint state-action space, considering fixed strategies (pure or mixed) of all other agents among possible alternatives. Meanwhile, if an agent selects only one action with unity probability, then the agent is said to use a *pure strategy*. A *mixed strategy* is the assignment of a probability distribution over the available actions, indicating a possibility of being selected by an agent [35]. The strategy profile corresponding to Nash-equilibrium (NE) [35], [36], [52], [53] thus refers to the best joint strategy of all the agents that allow each agent to maximize its payoff, considering fixed strategies of all other agents. Such payoff updating policy offers maximum freedom to an agent to give its best choice. In [22], Greenwald and Hall compared the relative performance of NQL and FQL algorithms with Correlated Q-learning (CQL), where the last one is used as the reference. Different variants of CQL exist in the literature based on the definition of  $\Omega$ -equilibrium [22], where  $\Omega$  usually takes any one of the four types: Utilitarian (U), Egalitarian (E), Republican (R) and Libertarian (L) by which an agent updates its future reward.

Curse of dimensionality is one of the prohibiting factors of the state-of-the-art equilibrium-based MAQL (NQL and CQL). Such bottleneck increases with the increase in number of learning agents while adapting Q-values in joint state-action space [43]-[45] by the state-of-



the-art update policies. To address the curse of dimensionality in MAQL, Jelle *et al.* proposed the Sparse Cooperative Q-learning [44], where each agent maintains two Q-tables based on the requirement of coordination among the agents at a joint state. Zhang *et al.* successfully reduced the dimension of Q-tables in NQL, where unlike the traditional NQL; agents store Q-values in joint state-individual action space [45]. To accelerate the convergence of the state-of-the-art equilibrium-based MAQL (NQL and CQL), Hu *et al.* introduced the concept of equilibrium transfer [43] by exploiting the previously computed equilibria in different joint states with negligible transfer-loss. However, with the best of our knowledge there is no equilibrium-based MAQL in the literature, where agents adapt only one joint Q-table to accumulate the rewards of all agents.

The applicability of RL includes finance sector [47], gaming industry [31], [36], robotics [5], [9], [29], [33] and many more. In this chapter, proposed algorithms are tested in the test bed of multi-robot object-transportation problems [38], [46].

The  $\Omega$  Q-learning ( $\Omega$ QL) algorithms proposed in this chapter has two attractive features, which are not available in the traditional CQL. First, during the learning phase, an agent needs to adapt only one Q-table in joint state-action space unlike adapting  $m$  joint Q-tables for  $m$  agents in CQL. Second, the evaluation of the computationally expensive correlated equilibrium (CE) is avoided, following a tricky approach of computing it partially during the learning and the rest during the planning phases. This offers benefits in real-time planning as computation of a CE, which is time-costly, is avoided here by the proposed technique.

Two schemes of  $\Omega$ QL have been proposed to serve two distinct type of MAQL based planning applications. Scheme-I ensures the success of the team, if only one agent is successful to serve its goal. This is useful for weakly coupled multi-agent systems, where only one agent is active at a time to serve the fixed goal. For example, in a soccer game only one person/agent at a time takes the ball ahead, serving its individual as well the team goal. Scheme-II ensures simultaneous success of all the agents in a tightly coupled multi-agent system, such as long stick/big object carrying by multiple robots. Both the schemes adapt Q-tables in joint state-action space. However, there is a small difference in the adaptation mechanism of Q-values by the two schemes. Scheme-I is used to adapt a Q-table in joint state-action space based on the individual Q-values of the agents and the effect of coordination among the agents, received as feedback from the environment. Scheme-II adapts Q-table in joint state-action space by considering group ( $\Omega$ ) Immediate Reward ( $\Omega$ IR) as a function of individual immediate rewards plus expected group ( $\Omega$ ) future reward as a function of individual expected future rewards discounted by a factor  $\gamma$  in  $[0, 1)$ .

Scheme-I and -II have four variants depending on the functional form used to compute the Q-value in joint state-action space. We here use a general nomenclature

$\Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \}$  for a unified treatment of the four variants, for computing Q-values

following U, E, R and L equilibria respectively.

During the planning phase [37], we obtain one of the four equilibria, depending on the choice of  $\Phi$  in the preceding learning phases. The planned task is then executed following the obtained equilibrium. Sometimes to execute a plan, we need to satisfy certain constraints that appear naturally from the problem under consideration. For example, in a twin robot cooperation to carry a stick held by the robots at the two end-points of the stick [38], the stick-length is a constraint. It acts as a constraint in planning as for all possible next positions of the robots, the separating distance should be equal to the stick length. The constraint can be handled at the planning phase but at the cost of extra time to identify feasible next states for the robots. Alternatively, the feasible joint state-actions can be learned during the learning phase, so that equilibrium obtained in the planning phase always falls in the feasible action-space. Here, we emphasized learning only at the feasible joint state-action space to speed up planning.

The main contributions of the chapter are briefly summarized below.

1. Unlike traditional CQL, where CE is evaluated both in learning and planning phases, here we need to compute CE partly in the learning and the rest in the planning phases, thereby requiring CE computation once only when learning based planning is employed.
2. It has been proved that the CE obtained by the proposed schemes is same as that obtained by the traditional CQL algorithms.
3. The computational cost to evaluate CE here is much smaller than that obtained by traditional CQL algorithms for the following reasons. Computation of CE in CQL requires consulting  $m$  Q-tables in joint state-action space for  $m$  agents, whereas in the present context we use a single Q-table in the joint state-action space for evaluation of CE.
4. Complexity analysis undertaken here confirms the last point. Both time-and space-complexity-wise the proposed algorithms are less expensive than traditional CQL algorithms.
5. Problem-specific constraints are taken care of in the proposed  $\Omega$ QL to avoid unwanted exploration of the infeasible state-space during the learning phase, thereby saving additional run-time complexity during the planning phase.
6. Experiments are undertaken to validate the proposed concepts in simulated and practical multi-agent robotic platform (here Khepera-environment).

The rest of the chapter is organized as follows. In Section II, an overview of the single agent Q-learning and equilibrium based MAQL algorithms are given. Proposed cooperative multi-agent Q-learning and corresponding planning algorithms are given in Section III. Section IV offers complexity analysis. Simulation and experimental results are presented in Section V. Conclusions are given in Section VI.

## 4.2 Single-Agent Q-Learning and Equilibrium Based MAQL

This section discusses the preliminary ideas concisely on single agent Q-learning and equilibrium-based MAQL algorithm for better understanding of the proposed methods.

### 4.2.1 Single Agent Q learning

In 1989, Watkins and Dayan [17] coined the single agent Q-learning, which is one of the most widely used RL techniques. It works by continuously updating an agent's state-action value (Q-value) by a fixed policy and earns a reward (or penalty) from the environment in each step of learning. The Q-learning update rule of an agent  $i$  is given in (4.1), assuming  $\hat{Q}_i^*(s'_i)$  to be the expected (indicated by  $\hat{\phantom{x}}$ ) optimal (indicated by  $*$ ) Q-value [3], [17] of agent  $i$  at next state  $s'_i \in S_i$  and the expression of  $\hat{Q}_i^*(s'_i)$  is given in the Appendix [51] following the traditional representation [14], [21].

$$Q_i(s_i, a_i) \leftarrow (1-\alpha)Q_i(s_i, a_i) + \alpha[r_i(s_i, a_i) + \gamma \cdot \hat{Q}_i^*(s'_i)] \quad (4.1)$$

where,  $Q_i(s_i, a_i)$  and  $r_i(s_i, a_i)$  are the Q-value and immediate reward respectively at a state  $s_i \in S_i$  because of action  $a_i \in A_i$  of agent  $i$ ,  $\alpha \in [0,1)$  be the learning rate,  $\gamma \in [0,1)$  denotes the discounting factor and  $s'_i \leftarrow \delta_i(s_i, a_i)$  indicates the state-transition from state  $s_i$  to next state  $s'_i$  because of action  $a_i$  with state-transition probability  $p_i(s'_i | (s_i, a_i))$  of agent  $i$ .

### 4.2.2 Equilibrium based MAQL

Depending upon the type of tasks MAQL can be classified into three categories: cooperative, competitive and mixed [10]. Multi-agent cooperative scenario demands formulation of a joint policy, which benefits each agent individually and also the team. The analysis and further enhancement of the cooperative MAQL is carried out in this chapter.

In MAQL algorithm, more specifically in equilibrium-based MAQL, each agent updates Q-values individually in joint state-action space employing one of the following equilibria: NE [28] and CE [22] to update the expected joint Q-value at equilibrium of an agent at the joint next state. CE includes U-equilibrium (UE), E-equilibrium (EE), R-equilibrium (RE) and L-equilibrium (LE) [22].

Suppose in a  $m$  agent system, due to joint action  $K \in \{K\} = A_1 \times A_2 \times \dots \times A_m = \times_{i=1}^m A_i$  at joint state  $G \in \{G\} = \times_{i=1}^m S_i$  the agent  $i$  earns an immediate reward  $r_i(G, K)$  and  $Q_i(G, K)$  be the Q-value of agent  $i$  because of joint action  $K$  at joint state  $G$ , where  $\times$  denotes the Cartesian product. Now, following traditional representation [14], [21] of the expected joint Q-value at a given joint next state,  $G' \in \{G\}$  because of mixed strategy  $\Omega$  – equilibrium,

$$\Omega p^*(K' | G') = \prod_{i=1}^m p_i^*(a'_i | s'_i), \quad (4.2)$$

for an agent  $i$  is given in (4.3),

$$\Omega \hat{Q}_i^*(G') = \sum_{\forall G'} p(G' | (G, K)) \sum_{\forall K'} \Omega p^*(K' | G') Q_i(G', K') \quad (4.3)$$

$$\text{where, } p(G' | (G, K)) = \prod_{i=1}^m p_i(s_i' | (s_i, a_i)) \quad (4.4)$$

be the joint state-transition probability and  $K' \in \{K\}$  be the joint action at joint next state  $G'$ . The definition of *CE* is given below.

**Definition 4.1:** *Correlated equilibrium (CE)* [22], [39] at a joint state  $G = \langle s_i \rangle_{i=1}^m$  with  $m$  interacting agents is the pure strategy CE,  $\Omega K^*$  and mixed strategy CE,  $\Omega p^*(K)$  if agents follow (4.5) and (4.6) respectively for  $\Omega \in \{U, E, R, L\}$ .

$$\Omega K^* = \arg \max_K [\Phi(Q_i(G, K))] \quad (4.5)$$

$$\Omega p^*(K) = \arg \max_{\Omega p(K)} [\Phi[\sum_K \Omega p(K)(Q_i(G, K))]] \quad (4.6)$$

where,  $\Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \}$ .

The CQL update rule is given in (4.7) [22] following the traditional representation [14], [21].

$$Q_i(G, K) \leftarrow (1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \cdot \Omega \hat{Q}_i^*(G')] \quad (4.7)$$

CQL algorithm [22] is given in the supplemental file [51].

## 4.3 Proposed Cooperative Multi-agent Q-Learning and Planning

In CQL algorithm, the entries of the Q-tables of  $m$ -agents at a given joint state because of a joint action need not necessarily be same. However, in the present formulation, an attempt is made to solve the CQL algorithm by efficiently employing the  $\Omega \in \{U, E, R, L\}$ -equilibrium, with the motivation to create single joint Q-table in joint state-action space by considering  $m$  Q-tables at individual state-action space and the environmental feedback about possible penalty due to multi-agent coordination at joint state-action space.

After each learning epoch, the results of adaptation of each agent's individual Q-tables are exploited to update the single Q-table in joint state-action space. Two techniques are proposed, namely scheme-I and -II, by efficiently employing the  $\Omega \in \{U, E, R, L\}$ -equilibrium to evaluate the single joint Q-table. To distinguish the nomenclatures of scheme-I and -II respectively  $\bar{\cdot}$  and  $\sim$  is placed on the top of each symbol. The basis of proposed schemes with their applicability is provided below.

### 4.3.1 Proposed schemes with their applicability

The chapter proposes two distinct schemes for addressing two types of situations that may arise during the MAQL based multi-robot planning. In the first scheme, the success of the team is subject to the success of any one agent. Scheme-I is applicable for the weakly coupled

multi-agent systems, where only one agent is enough at a time to serve the fixed team goal. For example, suppose,  $m$  numbers of agents are assigned to transport a box from one location to another following scheme-I, and let transportation of the box requires only one agent at a time. Once failure of an agent is detected, its nearest agent expresses cooperation toward the failed agent for successful transportation of the box. Here, the success of any one agent is enough to make the team successful. On the other hand, in the second scheme, the success of a team is contingent upon the simultaneous success of all the agents and it is applicable for the tightly coupled multi-agent system, such as long stick/big object carrying by multiple robots [38], [46]. Therefore, in both schemes, agents adapt Q-tables depending upon the task requirement.

In the proposed scheme-I induced Q-learning ( $\Omega$ QL-I), agents adapt the Q-table in joint state-action space exploiting the individual Q-values at individual state-action space and the effect of coordination among the agents, received as a feedback from the environment to be explained in the next section. However, in the proposed scheme-II induced Q-learning ( $\Omega$ QL-II), agents adapt the Q-table in joint state-action space following the traditional MAQL rule by evaluating the  $\Omega$ IR as a function of individual immediate rewards plus expected group ( $\Omega$ ) future reward as a function of individual expected future rewards discounted by a factor  $0 \leq \gamma < 1$ .

It may be notated that in the proposed  $\Omega$ QL-II, at a joint state,  $G$  because of all the joint actions,  $K \in \{K\}$ , if an agent  $i$  receives less than or equal to ( $\leq$ ) reward for  $EE$  (or  $LE$ ) or greater than or equal to ( $\geq$ ) reward for  $UE$  (or  $RE$ ) than the same by the proposed  $\Omega$ QL-I, then for  $EE$  (or  $LE$ ) the joint action  $K \in \{K\}$  corresponding to equal reward ( $\Omega\bar{Q}(G,K) = \Omega\tilde{Q}(G,K)$ ) is preferred, as simultaneous success of all the agents is desired to make the team successful. On the other hand for  $UE$  (or  $RE$ ), also the joint action,  $K \in \{K\}$  corresponding to equal reward ( $\Omega\bar{Q}(G,K) = \Omega\tilde{Q}(G,K)$ ) is preferred as success of any one agent is enough to make the team successful.

### 4.3.2 Immediate rewards in Scheme-I and -II

In the literature of MAQL, agents receive only one type of immediate reward, i.e., immediate reward at joint state-action space. However, it is our observation that the immediate rewards at individual state-action space and immediate rewards at joint state action space are often diverse. Hence, in this chapter we have considered two types of immediate rewards for an agent  $i$ . First one is the immediate rewards at individual state-action space,  $r_i(s_i, a_i)$  and the second one is the immediate rewards at joint state-action space during the multi-agent coordination,  $d_i(G, K)$ . The physical significance of such reward categorization is that an agent should not receive penalty or reward because of remaining agents' actions. For example, if each robot in a group individually receives immediate reward at joint state-action

space and subsequently employs either scheme-I or -II to obtain single Q-table at joint state-action space, then the penalty incurred by a robot due to possible collision with an obstacle, might influence the identical Q-values offered by the scheme-I or -II. The above phenomenon is not desired and hence such immediate reward categorization is done. The definition of the proposed immediate reward is given in Definition 4.2.

**Definition 4.2:**  $\Omega \in \{U, E, R, L\}$  immediate reward ( $\Omega IR$ ),  $R(G, K)$  is given by (4.8), where

$$\Phi \in \left\{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \right\}.$$

$$R(G, K) = d_i(G, K), \quad \text{if agent } i \text{ is penalized due to other agents}$$

$$= \Phi[r_i(s_i, a_i)], \quad \text{otherwise.} \quad (4.8)$$

Trivially from (4.8) it can be inferred that

$$R(G, K) = \Phi[r_i(G, K)] \quad (4.9)$$

### 4.3.3 Scheme-I induced MAQL

The Q-value offered by scheme-I ( $\Omega \bar{Q}(G, K)$ ) is evaluated by obtaining the  $\Phi \in \left\{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \right\}$  of the summation of individual Q-value  $Q_i(s_i, a_i)$  and the immediate reward due to multi-agent coordination given by  $d_i(G, K)$ . For example, in multi-agent robotics,  $d_i(G, K)$  is the penalty because of collision among the agents. The  $\Omega \bar{Q}(G, K)$  is evaluated by the learning rule,

$$\Omega \bar{Q}(G, K) \leftarrow \Phi[Q_i(s_i, a_i) + d_i(G, K)] \quad (4.10)$$

at the joint state  $G$  because of joint action  $K$ . Assuming  $K^* = \langle a_1, a_2, \dots, a_m \rangle^*$  be the jointly optimized individual actions (joint action) at joint state  $G$ , at the end of learning phase agents evaluate the optimal pure strategy  $\Omega \bar{K}^*$  corresponding to the maximum of  $\Omega \bar{Q}(G, K)$ ,  $\Omega \in \{U, E, R, L\}$  and is given by (4.11).

$$\Omega \bar{K}^* = \arg \max_{\bar{K}} [\Omega \bar{Q}(G, \bar{K})] \quad (4.11)$$

**Note 4.1:** As the CQL and the proposed scheme-I based learning algorithms share common environment and agents, both have same joint action set given by (4.12).

$$\{\bar{K}\} = \{K\} \quad (4.12)$$

In  $\Omega QL$ -I, the optimal mixed strategy  $\Omega \bar{p}^*(K)$  is obtained by evaluating the maximum of the expected reward  $\sum_{\forall K} \Omega \bar{p}(K) \Omega \bar{Q}(G, K)$ ,  $\Omega \bar{p}: \{K\} \rightarrow [0, 1]$  and is given by (4.13).

$$\Omega \bar{p}^*(K) = \arg \max_{\Omega \bar{p}(K)} \left[ \sum_{\forall K} \Omega \bar{p}(K) [\Omega \bar{Q}(G, K)] \right] \quad (4.13)$$

On the other hand, Kok *et al.* observed that in most of the MAQL, agents required to coordinate their actions only in a few states, and in the remaining they act independently [44]. Motivated by their observations Note 4.2 and 4.3 are given below before proposing Theorem 4.1 and 4.2.

**Note 4.2:** Following the principle of [44], in CQL, Q-value of agent  $i$  at joint state  $G$  because of joint action  $K$  may be expressed by (4.14).

$$Q_i(G, K) = Q_i(s_i, a_i) + d_i(G, K), \quad (4.14)$$

where  $d_i(G, K)$  is explained in Section III.B, elements of  $G$  and  $K$  include  $s_i$  and  $a_i$  respectively.

**Note 4.3:** Again, following the principle of [44], in CQL, the Q-value of agent  $i$  at joint state  $G$  because of joint action  $K$  setting  $d_i(G, K) = 0$  may be expressed by (4.15).

$$Q_i'(G, K) = Q_i(s_i, a_i). \quad (4.15)$$

**Theorem 4.1:** *The optimal pure strategy,  $\Omega \bar{K}^*$  induced by scheme-I is an  $\Omega$ -equilibrium,  $\Omega K^*$  for  $\Omega \in \{U, E, R, L\}$  attained in CQL.*

*Proof.* Here,

$$\begin{aligned} \Omega \bar{K}^* &= \arg \max_{\bar{K}} [\Omega \bar{Q}(G, \bar{K})] && \text{[by (4.11)]} \\ &= \arg \max_K [\Omega \bar{Q}(G, K)] && \text{[by (4.12)]} \\ &= \arg \max_K [\Phi[Q_i(s_i, a_i) + d_i(G, K)]] && \text{[by (4.10) and } \Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \} \text{]} \\ &= \arg \max_K [\Phi[Q_i(G, K)]] && \text{[by (4.14)]} \\ &= \Omega K^* && \text{[by (4.5)]} \end{aligned} \quad (4.16)$$

Hence, the Theorem is proved.  $\square$

**Theorem 4.2:** *The optimal mixed strategy,  $\Omega \bar{p}^*(K)$  induced by scheme-I is an  $\Omega$ -equilibrium,  $\Omega p^*(K)$  for  $\Omega \in \{U, E, R, L\}$  attained in CQL.*

*Proof.* Here,

$$\begin{aligned} \Omega \bar{p}^*(K) &= \arg \max_{\Omega \bar{p}(K) \forall K} [\sum \Omega \bar{p}(K) [\Omega \bar{Q}(G, K)]] && \text{[by (4.13)]} \\ &= \arg \max_{\Omega \bar{p}(K) \forall K} [\sum \Omega \bar{p}(K) \Phi[Q_i(s_i, a_i) + d_i(G, K)]] && \text{[by (4.10) and } \Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \} \text{]} \\ &= \arg \max_{\Omega \bar{p}(K) \forall K} [\Phi(\sum \Omega \bar{p}(K) [Q_i(s_i, a_i) + d_i(G, K)])] && \text{[}\because \Phi \text{ is independent of } K\text{]} \\ &= \arg \max_{\Omega \bar{p}(K) \forall K} [\Phi(\sum \Omega \bar{p}(K) [Q_i(G, K)])] && \text{[by (4.14)]} \\ &= \arg \max_{\Omega p(K) \forall K} [\Phi(\sum \Omega p(K) [Q_i(G, K)])] && \text{[}\because \Omega \bar{p} : \{K\} \rightarrow [0, 1], \Omega p : \{K\} \rightarrow [0, 1]\text{]} \\ &= \Omega p^*(K). && \text{[by (4.6)]} \end{aligned} \quad (4.17)$$

Hence, the Theorem is proved.  $\square$

It may be noted that, if among  $m$  agents at least one is required to successfully transport an object, (e.g., small box-carrying) to a predefined goal state cooperatively for succeeding the team, then scheme-I is useful. In Scheme-I, while an agent searches for its goal state, at the same time remaining agents keep on static (current state becomes the next state) by maintaining the equilibrium (cooperating) with the active one at that joint state.

However, if in a task (e.g. long stick-carrying), all  $m$  agents need to reach their individual goal simultaneously to complete the assigned task, then scheme-II is employed instead of scheme-I. Details of scheme-II are given below:

### 4.3.4 Scheme-II induced MAQL

In the traditional single agent Q-learning, two types of rewards are used: immediate reward and optimal future reward. The principle of  $\Omega_{QL-II}$  does not differ much from single agent Q-learning except the consideration of  $R(G, K)$  at a given joint state  $G$  because of a joint action  $K$  and  $\Omega\hat{Q}^*(G')$  at next joint state  $G'$ . Naturally, to optimize total reward, we need to optimize both  $R(G, K)$  and  $\Omega\hat{Q}^*(G')$ . Definition of  $R(G, K)$  is given in Definition 4.2 and  $\Omega\hat{Q}^*(G')$  are formally defined in Definition 4.3 for convenience of the readers.

**Definition 4.3:** *Expected optimal  $\Omega \in \{U, E, R, L\}$  Q-value at next joint state  $G'$ ,  $\Omega\hat{Q}^*(G')$  is obtained by evaluating the  $\Phi \in \{\sum_{i=1}^m, Min, Max, \prod_{i=1}^m\}$  among the summation of expected optimal*

*Q-value of agent  $i$  at  $s'_i$ ,  $\hat{Q}_i^*(s'_i)$  and the expected optimal change in immediate reward due to multi-agent coordination at joint next state  $G'$ ,  $\Delta\hat{Q}_i^*(G')$ .  $\Omega\hat{Q}^*(G')$  is given by (4.18),*

$$\Omega\hat{Q}^*(G') = \Phi[\hat{Q}_i^*(s'_i) + \Delta\hat{Q}_i^*(G')] \quad (4.18)$$

where,  $\Omega\hat{Q}^*(G')$  and  $\Delta\hat{Q}_i^*(G')$  are given by (4.19) and (4.20) respectively.

$$\Omega\hat{Q}^*(G') = \sum_{\forall G'} p(G' | (G, K)) \sum_{\forall K'} \Omega\tilde{p}^*(K' | G') \Omega\tilde{Q}(G', K') \quad (4.19)$$

$$\Delta\hat{Q}_i^*(G') = \sum_{\forall G'} p(G' | (G, K)) \sum_{\forall K'} \Omega\tilde{p}^*(K' | G') d_i(G', K') \quad (4.20)$$

where,  $\Omega\tilde{p}^*(K' | G')$  be the probability of selecting joint action  $K'$  at joint next state  $G'$  in scheme-II and the  $\Omega$  Q-value by scheme-II ( $\Omega\tilde{Q}(G, K)$ ) following the principle of single agent Q-learning rule using Definition 4.2 and 4.3 is given in (4.21).

$$\Omega\tilde{Q}(G, K) \leftarrow (1-\alpha)\Omega\tilde{Q}(G, K) + \alpha[R(G, K) + \gamma\Omega\hat{Q}^*(G')] \quad (4.21)$$

At the end of learning phase agents evaluate the optimal pure strategy  $\Omega\tilde{K}^*$  corresponding to the maximum of  $\Omega\tilde{Q}(G, K)$ ,  $\Omega \in \{U, E, R, L\}$  and is given by (4.22).

$$\Omega\tilde{K}^* = \arg \max_{\tilde{K}} [\Omega\tilde{Q}(G, \tilde{K})] \quad (4.22)$$

**Note 4:** Similar to Note 1, the relation between joint action set in scheme-II,  $\{\tilde{K}\}$  and  $\{K\}$  at a given joint state is given by (4.23).

$$\{\tilde{K}\} = \{K\} \quad (4.23)$$

In  $\Omega_{QL-II}$ , the optimal mixed strategy  $\Omega\tilde{p}^*(K)$  is obtained by evaluating the maximum of the expected reward  $\sum_{\forall K} \Omega\tilde{p}(K) \Omega\tilde{Q}(G, K)$ ,  $\Omega\tilde{p} : \{K\} \rightarrow [0, 1]$  and is given by (4.24).

$$\Omega\tilde{p}^*(K) = \arg \max_{\Omega\tilde{p}(K)} [\sum_{\forall K} \Omega\tilde{p}(K) [\Omega\tilde{Q}(G, K)]] \quad (4.24)$$

Lemma 4.1 to 4.6 are required to prove Theorem 4.3 and 4.4.



**Lemma 4.1:** *The Inequality,  $\Psi(x_i) \pm \gamma \Psi(y_i) \leq \Psi(x_i \pm \gamma y_i)$ ,  $\Psi \in \{\text{Min}, \prod_{i=1}^m\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\gamma \in [0, 1)$ , where  $0 \in \{x_i\}$  and  $0 \in \{y_i\}$ .*

*Proof.* See Appendix [51].

**Lemma 4.2:** *The inequality,  $\Psi(x_i) \pm \gamma \Psi(y_i) \geq \Psi(x_i \pm \gamma y_i)$ ,  $\Psi \in \{\text{Max}, \sum_{i=1}^m\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\gamma \in [0, 1)$ .*

*Proof.* See Appendix [51].

**Lemma 4.3:** *The Inequality,  $(1-\alpha)\Psi(x_i) \pm \alpha \Psi(y_i) \leq \Psi[(1-\alpha)x_i \pm \alpha y_i]$ ,  $\Psi \in \{\text{Min}, \prod_{i=1}^m\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\alpha \in [0, 1)$ , where  $0 \in \{x_i\}$  and  $0 \in \{y_i\}$ .*

*Proof.* See Appendix [51].

**Lemma 4.4:** *The Inequality,  $(1-\alpha)\Psi(x_i) \pm \alpha \Psi(y_i) \geq \Psi[(1-\alpha)x_i \pm \alpha y_i]$ ,  $\Psi \in \{\text{Max}, \sum_{i=1}^m\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\alpha \in [0, 1)$ .*

*Proof.* See Appendix [51].

Now, if  $\Delta \hat{Q}_i^*(G') = 0$ , then let the expected Q-value at  $\Omega$ -equilibrium of agent  $i$  at next joint state  $G'$ ,  $\Omega \hat{Q}_i^*(G')$  is given in (4.25).

$$\Omega \hat{Q}_i^*(G') = \sum_{\forall G'} p(G' | (G, K)) \sum_{\forall K'} \Omega p^*(K' | G') Q_i^*(G', K') \quad (4.25)$$

Now, if  $\Delta \hat{Q}_i^*(G') \neq 0$ , (4.7) can be rewritten as in (4.26)

$$Q_i(G, K) \leftarrow (1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \cdot [\Omega \hat{Q}_i^*(G') + \Delta \hat{Q}_i^*(G')]] \quad (4.26)$$

Now, comparing (4.7) and (4.26) one can write,

$$\Omega \hat{Q}_i^*(G') = \Omega \hat{Q}_i^*(G') + \Delta \hat{Q}_i^*(G'). \quad (4.27)$$

**Lemma 4.5:** *If  $\Omega \hat{Q}_i^*(G') = \Omega \hat{Q}_i^*(G') + \Delta \hat{Q}_i^*(G')$ , then  $\Omega \hat{Q}_i^*(G') = \hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')$  at the next joint state  $G' = \langle s'_i \rangle_{i=1}^m$ .*

*Proof.* See Appendix [51].

**Lemma 4.6:**  $\Omega \tilde{Q}(G, K) = \Phi[Q_i(G, K)]$  for  $\Phi \in \{\sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m\}$ , holds for the equality cases

of Lemma 4.1-4.4.

*Proof.* See Appendix [51].

**Theorem 4.3:** *The optimal pure strategy,  $\Omega \tilde{K}^*$  induced by scheme-II is an  $\Omega$ -equilibrium,  $\Omega K^*$  for  $\Omega \in \{U, E, R, L\}$  attained in CQL, holds for the equality cases of Lemma 4.1-4.4.*

*Proof.* Here,

$$\Omega \tilde{K}^* = \arg \max_{\tilde{K}} [\Omega \tilde{Q}(G, \tilde{K})] \quad [\text{by (4.22)}]$$

$$= \arg \max_K [\Omega \tilde{Q}(G, K)] \quad [\text{by (4.23)}]$$

$$\begin{aligned}
&= \arg \max_K [\Phi[Q_i(G, K)]] && \text{[by Lemma 4.6]} \\
&= \Omega K^* && \text{[by (4.5)] (4.28)}
\end{aligned}$$

Hence, the Theorem is proved.  $\square$

**Theorem 4.4:** *The optimal mixed strategy,  $\Omega \tilde{p}^*(K)$  induced by scheme-II is an  $\Omega$ -equilibrium,  $\Omega p^*(K)$  for  $\Omega \in \{U, E, R, L\}$  attained in CQL, holds for the equality cases of Lemma 4.1-4.4.*

*Proof.* Here,

$$\begin{aligned}
\Omega \tilde{p}^*(K) &= \arg \max_{\Omega \tilde{p}(K)} [\sum \Omega \tilde{p}(K) [\Omega \tilde{Q}(G, K)]] && \text{[by (4.24)]} \\
&= \arg \max_{\Omega \tilde{p}(K)} [\sum \Omega \tilde{p}(K) [\Phi[Q_i(G, K)]]] && \text{[by Lemma 4.6]} \\
&= \arg \max_{\Omega \tilde{p}(K)} [\Phi[\sum \Omega \tilde{p}(K) [Q_i(G, K)]]] && [\cdot: \Phi \text{ is independent of } K] \\
&= \arg \max_{\Omega p(K)} [\Phi[\sum \Omega p(K) [Q_i(G, K)]]] && [\cdot: \Omega \tilde{p}: \{K\} \rightarrow [0, 1], \Omega p: \{K\} \rightarrow [0, 1]] \\
&= \Omega p^*(K) && \text{[by (4.6)] (4.29)}
\end{aligned}$$

Hence, the Theorem is proved.  $\square$

### 4.3.5 Algorithms for scheme-I and II

Scheme-I and -II induced  $\Omega$ QL algorithms are proposed in Algorithm 4.1. Now, for further improvement, constraint version of  $\Omega$ QL-I/ $\Omega$ QL-II (C $\Omega$ QL-I/C $\Omega$ QL-II) is given below.

Algorithm 4.1: Scheme-I and -II induced $\Omega$ QL ( $\Omega$ QL-I and $\Omega$ QL-II)
<p><b>Input:</b> Learning rate <math>\alpha \in [0, 1)</math> and discounting factor <math>\gamma \in [0, 1]</math>;</p> <p><b>Output:</b> Optimal joint Q-value <math>\Omega \bar{Q}^*(G, K), \forall G, \forall K</math>; <u>//for Scheme-I</u>  Optimal joint Q-value <math>\Omega \tilde{Q}^*(G, K), \forall G, \forall K</math>; <u>//for Scheme-II</u></p> <p><b>Begin</b></p> <p>Initialize: state <math>s_i, \forall i</math>, action set <math>A_i</math> at <math>s_i, \forall i</math>, <math>\Omega \bar{Q}(G, K) \leftarrow 0, \forall G, \forall K</math> for Scheme-I and <math>\Omega \tilde{Q}(G, K) \leftarrow 0, \forall G, \forall K</math> for Scheme-II;</p> <p><b>Repeat</b></p> <ol style="list-style-type: none"> <li>1. Select an action <math>a_i \in A_i, \forall i</math> by the Boltzmann strategy [48] and execute it for both the schemes;</li> <li>2. For both the schemes observe immediate rewards <math>r_i(s_i, a_i)</math> and <math>d_i(G, K), \forall i</math>, evaluate next state <math>s_i' \leftarrow \delta_i(s_i, a_i), \forall i</math> to obtain joint next state <math>G' = \langle s_i' \rangle_{i=1}^m</math> and individual Q-value <math>Q_i(s_i, a_i) \leftarrow (1 - \alpha)Q_i(s_i, a_i) + \alpha[r_i(s_i, a_i) + \gamma \cdot \hat{Q}_i^*(s_i')], \forall i</math>;</li> <li>3. <b>Update:</b> Joint Q-value <math>\Omega \bar{Q}(G, K)</math> by (4.10) for Scheme-I* and <math>\Omega \tilde{Q}(G, K)</math> by (4.21) for Scheme-II** and <math>G \leftarrow G'</math>; // <math>\Omega \in \{U, E, R, L\}, \Phi \in \{\sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m\}</math>.</li> </ol> <p><b>Until</b> <math>\Omega \bar{Q}(G, K), \forall G, \forall K</math> converges for Scheme-I and <math>\Omega \tilde{Q}(G, K), \forall G, \forall K</math> converges for Scheme-II;</p> <p><math>\Omega \bar{Q}^*(G, K) \leftarrow \Omega \bar{Q}(G, K), \forall G, \forall K</math> for Scheme-I and <math>\Omega \tilde{Q}^*(G, K) \leftarrow \Omega \tilde{Q}(G, K), \forall G, \forall K</math> for Scheme-II;</p> <p><b>End.</b></p>

\* In Scheme-I, suppose, there is a group of two robots, R1 and R2. R1 attempts to transport a box from one location to a fixed goal G. Once, R1 fails R2 takes in charge and continues the box transportation. If R2 reaches the goal state G, then it receives the maximum individual immediate reward, 100 (say). At the same time, R1 receives the minimum individual immediate reward, i.e., 0. In the above circumstance, we choose RE for group immediate reward evaluation, because the success of one agent is enough to make the successful. Group immediate reward =  $\max(100, 0) = 100$ .

\*\* In Scheme-II, for the stick-carrying problem [46] both R1 and R2 should receive 100 rewards to make the team successful. Here, EE is employed to evaluate the group immediate reward =  $\min(100, 100) = 100$ .

### 4.3.6 Constraint $\Omega\text{QL-I}/\Omega\text{QL-II}(\text{C}\Omega\text{QL-I}/\text{C}\Omega\text{QL-II})$

In constraint  $\Omega\text{QL}$  ( $\text{C}\Omega\text{QL}$ ) algorithms, agents have to satisfy one or more task constraints to determine next feasible joint state from a set of next joint states. For example, in the stick-carrying problem, two robots (agents) transport a stick from a given location to the other in their environment by holding it at two ends of the stick. During transportation, the stick should not collide with any obstacle. The transportation of a fixed-length stick without having encountered an obstacle by the stick acts as a task constraint that the agents have to maintain throughout their journey. Similarly, in triangle-carrying problem three robots carry a triangle satisfying the above mentioned constraint.

If joint next state ( $G'$ ) does not satisfy task constraint(s) after selecting a joint action ( $K$ ) from a feasible joint state ( $G_F$ ), then the next joint state ( $G'$ ) is removed from the feasible joint Q-table and also the joint action ( $K$ ) is dropped from the feasible joint state ( $G_F$ ), otherwise the joint action is executed for learning. It is apparent that  $G_F \subseteq G$  and  $K_F \subseteq K$ , where  $G$  and  $K$  are joint state and joint action of the agents respectively. The constraint  $\Omega\text{QL-I}/\Omega\text{QL-II}$  ( $\text{C}\Omega\text{QL-I}/\text{C}\Omega\text{QL-II}$ ) algorithm mentioning the changes as compared to  $\Omega\text{QL}$  (Algorithm 4.1) is given in Algorithm 4.2.

Algorithm 4.2: Constraint $\Omega\text{QL-I}/\Omega\text{QL-II}$ ( $\text{C}\Omega\text{QL-I}/\text{C}\Omega\text{QL-II}$ )
<p><b>Input:</b> As in Algorithm 4.1 plus task constraints;  <b>Output:</b> Optimal Q-values for feasible joint state-action space;  <b>Begin</b>  Initialize: Same as Algorithm 4.1;  <b>Repeat</b>  1. Select action as in Algorithm 4.1;  2. Receive immediate rewards <math>r_i(s_i, a_i), \forall i</math> and <math>d_i(G, K), \forall i</math> as in Algorithm 4.1 and evaluate joint next state <math>G' = \langle s_i' \rangle_{i=1}^m</math> and individual Q-value <math>Q_i(s_i, a_i), \forall i</math> as in Algorithm 4.1;  3. <b>Update:</b> <math>\Omega\bar{Q}(G_F, K_F)</math> for Scheme-I and <math>\Omega\tilde{Q}(G_F, K_F)</math> for Scheme-II; // <math>F</math> in the suffix indicates the feasible  4. If the feasibility checking fails, then delete the <math>G_F</math> from the joint Q-table and drop the <math>K_F</math> taken at the current joint state;  <b>Until</b> <math>\Omega\bar{Q}(G_F, K_F), \forall G_F, \forall K_F</math> converges for Scheme-I and <math>\Omega\tilde{Q}(G_F, K_F), \forall G_F, \forall K_F</math> converges for Scheme-II;  <math>\Omega\bar{Q}^*(G_F, K_F) \leftarrow \Omega\bar{Q}(G_F, K_F), \forall G_F, \forall K_F</math> for Scheme-I and <math>\Omega\tilde{Q}^*(G_F, K_F) \leftarrow \Omega\tilde{Q}(G_F, K_F), \forall G_F, \forall K_F</math> for Scheme-II;  <b>End.</b></p>

### 4.3.7 Convergence

In [44], Kok *et al.* mentioned that in most of the MAQL, agents require coordinating their actions only in a few states, while acting independently in the remaining states. Based on their observations, we jointly optimize a combination of i) individual Q-function and ii) the change in the individual Q-functions because of the multi-agent coordination. The function used to combine the above two, is determined based on the choice of type of equilibrium. In the present chapter, to consider the change in Q-values due to multi-agent coordination, we classify the immediate reward (Section 4.3.2) into two types, one defined in individual state-

action space, and the other in the joint state-action space. Theorems (4.1, 4.2) and (4.3, 4.4) respectively leads to the optimal global policy for deterministic and stochastic cases for both scheme-I and -II. Convergence proofs of the proposed algorithms are offered by Theorem 4.5 and 4.6 respectively for scheme-I and-II. To propose Theorem 4.5 and 4.6 Lemma 4.7-4.9 are the prerequisites.

**Lemma 4.7:**  $\alpha | [R_t(G, K) - R^*(G, K)] | = 0, \forall t$  hold for  $\alpha \in [0, 1)$ , where  $t$  is the learning epoch.

*Proof.* See Appendix [51].

**Lemma 4.8:**  $\gamma | [\Omega \hat{Q}_t^*(G^l) - \Omega \hat{Q}^*(G^l)] | = 0$ , holds for  $\gamma \in [0, 1)$  as learning epoch  $t \rightarrow \infty$ .

*Proof.* See Appendix [51].

**Lemma 4.9:** If  $|\Omega \tilde{Q}_{t-k}(G, K) - \Omega \tilde{Q}^*(G, K)| = \Delta \tilde{Q}_{t-k}(G, K)$ , then  $(1-\alpha) | [\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)] | = (1-\alpha)^k \Delta \tilde{Q}_{t-k}(G, K)$ , where,  $\alpha \in [0, 1)$ ,  $k \in \mathbb{R}^+$  and  $t$  is the learning epoch.

*Proof.* See Appendix [51].

**Theorem 4.5:** The proposed scheme-I induced  $\Omega$  Q-learning converges  $[\Omega \bar{Q}_t(G, K) \rightarrow \Omega \bar{Q}^*(G, K)]$  as learning epoch  $t \rightarrow \infty$ .

*Proof.* By (4.10)

$$\begin{aligned} & | \Omega \bar{Q}_t(G, K) - \Omega \bar{Q}^*(G, K) | \\ &= | \Phi[Q_i^t(s_i, a_i) + d_i^t(G, K)] - \Phi[Q_i^*(s_i, a_i) + d_i^*(G, K)] | \end{aligned} \quad (4.30)$$

Now, we have two cases:

Case I: For,  $\Phi \in \{ \text{Min}, \prod_{i=1}^m \}$  (4.30) becomes

$$\begin{aligned} & | \Omega \bar{Q}_t(G, K) - \Omega \bar{Q}^*(G, K) | && \text{[by Lemma 4.1 with } \gamma = 1 \text{]} \\ & \leq | \Phi[Q_i^t(s_i, a_i) + d_i^t(G, K)] - \Phi[Q_i^*(s_i, a_i) + d_i^*(G, K)] | \\ &= | \Phi[Q_i^t(s_i, a_i) - Q_i^*(s_i, a_i)] + [d_i^t(G, K) - d_i^*(G, K)] | \\ &= | \Phi[Q_i^t(s_i, a_i) - Q_i^*(s_i, a_i)] | && \text{[for any agent } i \text{ at a fixed } (G, K) \text{ } d_i(G, K) \text{ is constant]} \\ &= | \Phi[\Delta Q_i^t(s_i, a_i)] |. \end{aligned} \quad (4.31)$$

Here,  $\Delta Q_i^t(s_i, a_i)$  refers to the error in Q-value of agent  $i$  at  $t^{\text{th}}$  iteration.

Case II: For,  $\Phi \in \{ \text{Max}, \sum_{i=1}^m \}$  (4.30) becomes

$$\begin{aligned} & | \Omega \bar{Q}_t(G, K) - \Omega \bar{Q}^*(G, K) | && \text{[by Lemma 4.2 with } \gamma = 1 \text{]} \\ & \leq | \Phi[Q_i^t(s_i, a_i)] + \Phi[d_i^t(G, K)] - \Phi[Q_i^*(s_i, a_i)] - \Phi[d_i^*(G, K)] | \\ &= | \Phi[Q_i^t(s_i, a_i)] - \Phi[Q_i^*(s_i, a_i)] + \Phi[d_i^t(G, K)] - \Phi[d_i^*(G, K)] | \\ & \leq | \Phi[Q_i^t(s_i, a_i)] - \Phi[Q_i^*(s_i, a_i)] | + | \Phi[d_i^t(G, K)] - \Phi[d_i^*(G, K)] | && \text{[} \cdot \text{: } |a+b| \leq |a| + |b| \text{]} \\ & \leq \Phi | [Q_i^t(s_i, a_i) - Q_i^*(s_i, a_i)] | + \Phi | [d_i^t(G, K) - d_i^*(G, K)] | && \text{[} \cdot \text{: } |\Phi[a_i] - \Phi[b_i]| \leq \Phi | [a_i + b_i] | \text{]} \\ &= \Phi | [Q_i^t(s_i, a_i) - Q_i^*(s_i, a_i)] | && \text{[for any agent } i \text{ at a fixed } (G, K) \text{ } d_i(G, K) \text{ is constant]} \\ &= | \Phi[\Delta Q_i^t(s_i, a_i)] |. \end{aligned} \quad (4.32)$$

Now, by [17] as learning epoch  $t \rightarrow \infty$ ,  $\Delta Q_t^i(s_i, a_i) \rightarrow 0$ . Hence, the scheme-I induced  $\Omega$  Q-learning converges  $[\Omega \bar{Q}_t(G, K) \rightarrow \Omega \bar{Q}^*(G, K)]$  as  $t \rightarrow \infty$ .  $\square$

**Theorem 4.6:** *The proposed scheme-II induced  $\Omega$  Q-learning converges  $[\Omega \tilde{Q}_t(G, K) \rightarrow \Omega \tilde{Q}^*(G, K)]$  as learning epoch  $t \rightarrow \infty$ .*

*Proof.* By (4.21)

$$\begin{aligned}
& |[\Omega \tilde{Q}_t(G, K)] - [\Omega \tilde{Q}^*(G, K)]| \\
&= |(1-\alpha)\Omega \tilde{Q}_{t-1}(G, K) + \alpha[R_t(G, K) + \gamma.\Omega \hat{Q}_t^*(G')] - [(1-\alpha)\Omega \tilde{Q}^*(G, K) + \alpha[R^*(G, K) + \gamma.\Omega \hat{Q}^*(G')]| \\
&= |(1-\alpha)[\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)] + \alpha[R_t(G, K) - R^*(G, K)] + \gamma[\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')]| \\
&\leq |(1-\alpha)[\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)] + \alpha[R_t(G, K) - R^*(G, K)]| + |\gamma[\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')]| \\
&\hspace{15em} [:\cdot|a+b|\leq|a|+|b|] \\
&\leq (1-\alpha)|[\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)]| + |\alpha[R_t(G, K) - R^*(G, K)]| + |\gamma[\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')]| \\
&\hspace{15em} [:\cdot|a+b|\leq|a|+|b|] \\
&= (1-\alpha)|[\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)]| + \alpha|[R_t(G, K) - R^*(G, K)]| + \gamma|[\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')]| \\
&= (1-\alpha)^k \Delta \tilde{Q}_{t-k}(G, K). \hspace{15em} [\text{by Lemma 4.7-4.9}] \hspace{5em} (4.33)
\end{aligned}$$

Now,  $\because \alpha \in [0, 1)$ , with  $k \rightarrow \infty$ ,  $(1-\alpha)^k \Delta \tilde{Q}_{t-k}(G, K) \rightarrow 0$ , where  $k$  is the dummy variable indicates the learning epoch. Hence, the scheme-II induced  $\Omega$  Q-learning converges  $[\Omega \tilde{Q}_t(G, K) \rightarrow \Omega \tilde{Q}^*(G, K)]$  as learning epoch  $t \rightarrow \infty$ .  $\square$

### 4.3.8 Multi-agent planning

Multi-agent planning is followed by multi-agent learning. In Correlated-Q induced Planning (CQIP) algorithm [22], [39],  $m$  number of agents plan to reach a predefined joint goal state by determining CE using  $m$  joint Q-tables. In the proposed multi-agent planning algorithms, the (strategy or CE) joint action corresponding to the (maximum expected) maximum joint Q-value offered by the proposed multi-agent planning algorithms is selected. Two alternatives of multi-agent planning algorithms are proposed. The first one, called  $\Omega$  Multi-agent Planning ( $\Omega$ MP) does not require to satisfy task constraints as it has already been undertaken during the learning phase (by C $\Omega$ QL-I and C $\Omega$ QL-II algorithms). The  $\Omega$ MP algorithm is given in Algorithm 4.3. However, in case the task constraint (e.g., fixed stick length or triangle structure) is not undertaken during the learning phase (as happens to be in  $\Omega$ QL-I and  $\Omega$ QL-II), task constraint(s) are to be satisfied during the planning phase. Constraint  $\Omega$ MP (C $\Omega$ MP) algorithm takes into account the task constraint during the planning phase. To handle the task constraint (stick-length) in C $\Omega$ MP algorithm, the policy given in Algorithm 4.4 is adapted.

Let the space (grids) between the agents (robots) holding a stick called intermediate state. Since the stick lies over the intermediate space, it is difficult to ascertain whether the region containing the next (front/back) intermediate state is occupied by an obstacle or not. To handle this problem, an agent has to check whether joint Q-value of the next intermediate

space is nonzero (as Q-values are initialized as zero). A nonzero Q-value indicates that the next joint state is free from obstacles.

<b>Algorithm 4.3: <math>\Omega</math> Multi-agent Planning (<math>\Omega</math>MP)</b>
<p><b>Input:</b> Feasible joint state <math>G_F</math>, Goal state <math>G_L</math>, <math>\Omega\bar{Q}^*(G_F, K_F)</math> for scheme-I and <math>\tilde{\Omega}\bar{Q}^*(G_F, K_F)</math> for scheme-II;</p> <p><b>Output:</b> Optimal feasible joint action (or CE) <math>K_F^*</math> at <math>G_F</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>While</b> <math>G_F \neq G_L</math> <b>do Begin</b></p> <p style="padding-left: 40px;"><b>For</b> <math>K_F \in \{K_F\}</math></p> <p style="padding-left: 60px;"><b>If</b> <math>\Omega\bar{Q}^*(G_F, K_F^*) \geq \Omega\bar{Q}^*(G_F, K_F)</math> // for scheme-I</p> <p style="padding-left: 60px;"><math>\tilde{\Omega}\bar{Q}^*(G_F, K_F^*) \geq \tilde{\Omega}\bar{Q}^*(G_F, K_F)</math> // for scheme-II</p> <p style="padding-left: 40px;"><b>Then</b> <math>K_F^* \leftarrow K_F</math> and <math>G_F \leftarrow G_F^j</math>; // <math>G_F^j</math> is the joint next state</p> <p style="padding-left: 40px;"><b>End If;</b></p> <p style="padding-left: 20px;"><b>End for;</b></p> <p style="padding-left: 20px;"><b>End While;</b></p> <p><b>End.</b></p>

<b>Algorithm 4.4: Constraint <math>\Omega</math> Multi-agent Planning (C<math>\Omega</math>MP)</b>
<p><b>Input:</b> Feasible joint state <math>G_F</math>, Goal state <math>G_L</math>, <math>\Omega\bar{Q}^*(G, K)</math> for scheme-I and <math>\tilde{\Omega}\bar{Q}^*(G, K)</math> for scheme-II;</p> <p><b>Output:</b> Optimal feasible joint action (or CE) <math>K_F^*</math> at <math>G_F</math>;</p> <p><b>Begin</b></p> <p style="padding-left: 20px;"><b>While</b> <math>G_F \neq G_L</math> <b>do Begin</b></p> <p style="padding-left: 40px;"><b>For</b> <math>K \in \{K\}</math></p> <p style="padding-left: 60px;"><b>If</b> <math>\Omega\bar{Q}^*(G_F, K^*) \geq \Omega\bar{Q}^*(G_F, K)</math> // for scheme-I</p> <p style="padding-left: 60px;"><math>\tilde{\Omega}\bar{Q}^*(G_F, K^*) \geq \tilde{\Omega}\bar{Q}^*(G_F, K)</math> // for scheme-II</p> <p style="padding-left: 60px;">and feasible joint next state <math>G_F^j</math> satisfies task constraints;</p> <p style="padding-left: 40px;"><b>Then</b> <math>K_F^* \leftarrow K</math> and <math>G_F \leftarrow G_F^j</math>; // <math>G_F^j</math> is the joint next state</p> <p style="padding-left: 40px;"><b>End If;</b></p> <p style="padding-left: 20px;"><b>End for;</b></p> <p style="padding-left: 20px;"><b>End While;</b></p> <p><b>End.</b></p>

## 4.4 Complexity Analysis

In this section, evaluation of the space- and time-complexities of the proposed learning and planning algorithms in deterministic settings and comparing them respectively with the CQL and CQIP algorithms are done for  $\Omega \in \{E, R\}$ . The time-complexity for  $\Omega \in \{U, L\}$  are shown in the Appendix [51] and the space-complexity is not shown as it does not vary with the variation of  $\Omega$ . However, the run-time complexities of the proposed and existing algorithms for  $\Omega \in \{U, E, R, L\}$  are given in Section 4.5. For simplicities, let us assume that

$$|S_1| = |S_2| = \dots = |S_m| = |S| \quad (4.34)$$

$$\text{and } |A_1| = |A_2| = \dots = |A_m| = |A| \quad (4.35)$$

Now, the cardinality of joint state set

$$|G| = |S_1 \times S_2 \times \dots \times S_m|$$

$$\begin{aligned}
&= |S_1| \cdot |S_2| \cdot \dots \cdot |S_m| \\
&= |S|^m \quad \text{[by (4.34)]}
\end{aligned}$$

Further, the cardinality of joint action set

$$\begin{aligned}
|\{K\}| &= |A_1 \times A_2 \times \dots \times A_m| \\
&= |A_1| \cdot |A_2| \cdot \dots \cdot |A_m| \\
&= |A|^m \quad \text{[by (4.35)]}
\end{aligned}$$

Let  $t_{CQIP}$  and  $t_{C\Omega MP}$  be the number of steps required to satisfy task constraint in one epoch during the planning phase by CQIP and C $\Omega$ MP algorithms respectively. In CQIP algorithm, task constraint is satisfied for  $m$  joint Q-tables and in C $\Omega$ MP algorithm, task constraint is satisfied for one joint Q-table, so,  $t_{CQIP} > t_{C\Omega MP}$ . Also let  $t_{C\Omega QL-I}$  and  $t_{C\Omega QL-II}$  are the number of steps required to satisfy task constraint in one learning epoch by C $\Omega$ QL-I and C $\Omega$ QL-II algorithms respectively. Comparison of the CQL and CQIP algorithms with the proposed  $\Omega$ QL, C $\Omega$ QL and  $\Omega$  induced planning algorithms ( $\Omega$ MP and C $\Omega$ MP) are given below.

#### 4.4.1 Complexity of Correlated Q-Learning

Complexities of Correlated-Q induced learning and planning algorithms are given below.

##### 4.4.1.1 Space Complexity

**Learning:** In CQL algorithm, agents maintain their own Q-tables in joint state-action space. Hence, the space requirement for a joint Q-table is  $|S|^m \cdot |A|^m$ . Assuming there is no communication among the agents, during the learning phase, each agent has to maintain all agent's joint Q-tables individually by observing other agents state, action and rewards. So, in CQL algorithm, space complexity (SC) of one agent is

$$SC_{CQL} = m \cdot |S|^m \cdot |A|^m = O(m \cdot |S|^m \cdot |A|^m) \quad (4.36)$$

**Planning:** In the CQIP phase, agents need the same number of joint Q-tables as required during the learning phase. Hence, SC of CQIP algorithm is given by,

$$SC_{CQIP} = m \cdot |S|^m \cdot |A|^m = O(m \cdot |S|^m \cdot |A|^m) \quad (4.37)$$

##### 4.4.1.2 Time Complexity

**Learning:** In CQL, during the learning phase for  $\Omega \in \{E, R\}$  an agent has to find out CE among all the joint actions. Therefore, in CQL algorithm, time complexity (TC) in a single learning epoch is

$$TC_{CQL} = (m-1)|A|^m + (|A|^m - 1) = O(m|A|^m) \quad (4.38)$$

**Planning:** For  $\Omega \in \{E, R\}$  during the CQIP phase, except finding CE agents have to satisfy task constraint. Therefore, in CQIP algorithm, TC is

$$TC_{CQIP} = (m-1)|A|^m + (|A|^m - 1) + t_{CQIP} > O(m|A|^m) \quad (4.39)$$

## 4.4.2 Complexity of the proposed algorithms

Complexities of the proposed learning and planning algorithms are given below.

### 4.4.2.1 Space Complexity

**Learning:** In  $\Omega$ QL algorithms, an agent maintains  $m$  agents' Q-tables at individual state-action space and one joint Q-table. Hence, during the learning phase SC in  $\Omega$ QL algorithms is

$$SC_{\Omega QL} = m \cdot |S| \cdot |A| + |S|^m \cdot |A|^m > O(|S|^m \cdot |A|^m) \quad (4.40)$$

**Planning:** During the planning phase, in  $C\Omega$ MP, agents need only one joint Q-table to plan. Hence, SC in  $C\Omega$ MP algorithm is

$$SC_{C\Omega MP} = |S|^m \cdot |A|^m = O(|S|^m \cdot |A|^m) \quad (4.41)$$

### 4.4.2.2 Time Complexity

**Learning:** For  $\Omega \in \{E, R\}$  in  $\Omega$ QL algorithms, TC in a single learning epoch is the number of comparisons required to update joint Q-value for a joint state-action pair. In  $\Omega$ QL-I and  $C\Omega$ QL-I, during the learning phase an agent has to evaluate  $m$  individual Q-values by  $m(|A|-1)$  comparisons and evaluate the  $\Phi \in \{Min, Max\}$  among the summation of  $m$  individual Q-values and  $d_i(G, K)$  by  $(m-1)$  comparisons. Hence, TC during the learning phase in  $\Omega$ QL-I and  $C\Omega$ QL-I (satisfying task constraint) algorithms are respectively

$$TC_{\Omega QL-I} = m \cdot (|A|-1) + (m-1) \approx O(m \cdot |A|) \quad (4.42)$$

$$\text{and } TC_{C\Omega QL-I} = m \cdot (|A|-1) + (m-1) + t_{C\Omega QL-I} > O(m \cdot |A|) \quad (4.43)$$

In  $\Omega$ QL-II and  $C\Omega$ QL-II, during the learning phase, each agent has to find out the  $\Omega$ IR and  $\Omega\hat{Q}^*$ . For that, an agent requires  $2(m-1)$  number of comparisons. Also agents update  $m$  individual Q-values by  $m \cdot (|A|-1)$  comparisons. Hence, TC during the learning phases in  $\Omega$ QL-II and  $C\Omega$ QL-II (satisfying task constraint) algorithms are given respectively in (44) and (4.45).

$$TC_{\Omega QL-II} = m \cdot (|A|-1) + 2(m-1) \approx O(m \cdot |A|) \quad (4.44)$$

$$TC_{C\Omega QL-II} = m \cdot (|A|-1) + 2(m-1) + t_{C\Omega QL-II} > O(m \cdot |A|). \quad (4.45)$$

**Planning:** During the planning phase agents evaluate optimal joint action (CE) corresponding to the maximum joint Q-value at a given joint state. Hence, TC in  $\Omega$ MP and  $C\Omega$ MP (satisfying task constraint) algorithms are

$$TC_{\Omega MP} = (|A|^m - 1) = O(|A|^m) \quad (4.46)$$

$$\text{and } TC_{C\Omega MP} = (|A|^m - 1) + t_{C\Omega MP} > O(|A|^m) \quad (4.47)$$

**Space-complexity Analysis of Stick-Carrying and Triangle-Carrying Problem:** SC of  $C\Omega$ QL algorithms ( $SC_{C\Omega QL}$ ) and  $\Omega$ MP algorithm ( $SC_{\Omega MP}$ ) depend on the task to be solved. In the context of stick-carrying and triangle-carrying problems, a description is given below. In a grid map, usually there exist three types of cells, called Corner cell ( $c$ ), Wall cell ( $w$ ) and Other



Cell ( $oc$ ). From Fig. 4.1 it is observed, that in an  $n \times n$  grid map, number of Corner cells ( $C_c$ ), Wall cells ( $C_w$ ) and Other cells ( $C_{oc}$ ) are 4,  $4(n-2)$  and  $(n-2)^2$  respectively. An analysis of Fig. 4.2 indicates that in a two-agent system ( $m=2$ ), from a  $c$ ,  $w$  and  $oc$  there exist 3, 5 and 8 feasible joint states respectively for an agent. Therefore, assuming  $F_k$  be the number of feasible joint states from state  $k$  in an  $n \times n$  grid map, total number of feasible joint states by C $\Omega$ QL algorithm is

$$m \times \sum_{k \in \{c, w, oc\}} C_k \times F_k = 2[4 \times 3 + 4(n-2) \times 5 + (n-2)^2 \times 8] \quad (4.48)$$

which on simplification returns

$$m \times \sum_{k \in \{c, w, oc\}} C_k \times F_k = 16n^2 - 24n + 8, \text{ i.e., } O(n^2). \quad (4.49)$$

while that in  $\Omega$ QL algorithm, for  $m=2$  is

$$(n \times n)^m = (n \times n)^2 = (n^4), \text{ i.e., } O(n^4). \quad (4.50)$$

Similarly, for triangle-carrying problem  $m=3$  and total space complexity in C $\Omega$ QL algorithm is

$$m \times \sum_{k \in \{c, w, oc\}} C_k \times F_k = 24n^2 - 48n + 24, \text{ i.e., } O(n^2) \quad (4.51)$$

and space complexity in  $\Omega$ QL algorithm for  $m=3$  is

$$(n \times n)^m = (n \times n)^3 = (n^6), \text{ i.e., } O(n^6). \quad (4.52)$$

From (4.49), (4.50), (4.51) and (4.52) we conclude

$$SC_{\Omega QL} > SC_{C\Omega QL} \quad (4.53)$$

As only learned joint state-action pairs are utilized to plan

$$\therefore SC_{C\Omega MP} > SC_{\Omega MP}. \quad (4.54)$$

Corner Cell	Wall Cell	Corner Cell
Wall Cell	Other Cell	Wall Cell
Corner Cell	Wall Cell	Corner Cell

Fig. 4.1 Corner cell, boundary cell and other cell

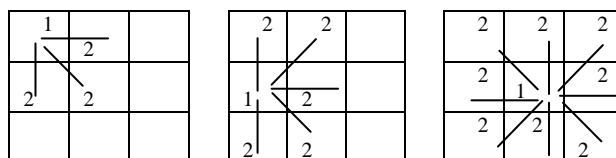


Fig. 4.2 Feasible joint states for two agent systems in stick-carrying problem

#### 4.4.3 Complexity comparison

Comparisons of complexities between CQL and proposed algorithms are given below.

#### 4.4.3.1 Space complexity

From (4.36) and (4.40),

$$\begin{aligned} \frac{SC_{\Omega QL}}{SC_{CQL}} &= \frac{m \cdot |S| \cdot |A| + |S|^m \cdot |A|^m}{m \cdot |S|^m \cdot |A|^m} \\ &= \frac{|S| \cdot |A|}{|S|^m \cdot |A|^m} + \frac{1}{m} \\ &\approx \frac{1}{|S|^{m-1} \cdot |A|^{m-1}} \quad \text{[after approximation]} \\ &< 1. \end{aligned} \tag{4.55}$$

From (4.37) and (4.41),

$$\begin{aligned} \frac{SC_{C\Omega MP}}{SC_{CQIP}} &= \frac{|S|^m \cdot |A|^m}{m \cdot |S|^m \cdot |A|^m} \\ &= \frac{1}{m}. \end{aligned} \tag{4.56}$$

$\therefore$  SC of  $C\Omega MP$  is  $1/m$  of the SC in  $CQIP$ .

We by (4.53) and (4.55) obtain,

$$SC_{CQL} > SC_{\Omega QL} > SC_{C\Omega MP}. \tag{4.57}$$

We by (4.54) and (4.56) obtain,

$$SC_{CQIP} > SC_{C\Omega MP} > SC_{\Omega MP}. \tag{4.58}$$

#### 4.4.3.2 Time complexity

From (4.38) and (4.43),

$$\begin{aligned} \frac{TC_{C\Omega QL-I}}{TC_{CQL}} &= \frac{m \cdot (|A| - 1) + (m - 1) + t_{C\Omega QL-I}}{(m - 1)|A|^m + (|A|^m - 1)} \\ &\approx \frac{1}{|A|^{m-1}} \quad \text{[after approximation]} \\ &< 1. \end{aligned} \tag{4.59}$$

From (4.42), (4.43) and (4.59),

$$TC_{CQL} > TC_{C\Omega QL-I} > TC_{\Omega QL-I}. \tag{4.60}$$

From (4.38) and (4.45),

$$\begin{aligned} \frac{TC_{C\Omega QL-II}}{TC_{CQL}} &= \frac{m \cdot (|A| - 1) + 2(m - 1) + t_{C\Omega QL-II}}{(m - 1)|A|^m + (|A|^m - 1)} \\ &\approx \frac{1}{|A|^{m-1}} \quad \text{[after approximation]} \\ &< 1. \end{aligned} \tag{4.61}$$

From (4.44), (4.45) and (4.61),

$$TC_{CQL} > TC_{C\Omega QL-II} > TC_{\Omega QL-II}. \tag{4.62}$$

Again from (4.39) and (4.47),

$$\frac{TC_{C\Omega MP}}{TC_{CQIP}} = \frac{(|A|^m - 1) + t_{C\Omega MP}}{(m - 1)|A|^m + (|A|^m - 1) + t_{CQIP}} \quad (t_{C\Omega MP} < t_{CQIP})$$

$$\approx \frac{1}{m}. \quad \text{[after approximation]} \quad (4.63)$$

From (4.46), (4.47) and (4.63) we obtain,

$$TC_{CQIP} > TC_{C\Omega MP} > TC_{\Omega MP}. \quad (4.64)$$

## 4.5 Simulation and experimental results

This section provides three experiments on multi-agent cooperation: one is the Experiment 4.1 where complexity analysis of the proposed and the existing algorithms are shown. Second one is Experiment 4.2 (for  $\Omega = R$ ) in which success of any one agent is enough to make the team successful. The last one is Experiment 4.3 (for  $\Omega = E$ ) where simultaneous success of all the agents is needed to make the team successful. Experiment 4.1 provides experiments on cooperation of multiple mobile robots (agents) during the learning phase in the framework of object-transportation (stick-carrying and triangle-carrying) problem. The stick-carrying (triangle-carrying) problem [38] refers to transposition of the stick (triangle) from a given location to a desired destination in the given workspace by two (three) robots, where the stick (triangle) is held at its end-points (vertices). In Experiment 4.2, two agents cooperatively transport an object (box) from a given location to a predefined destination. Agents cooperate among themselves by moving (passing) the box to another agent with an aim to achieve success. In Experiment 4.3, two/three agents transport a stick/triangle from a given location to a predefined destination. Obstacles are added to the workspace to add complexity for all the mentioned Experiments. These Experiments are undertaken to study the performance of the proposed learning and planning algorithms to compare their performance with equilibrium based MAQL (NQL, FQL, and CQL) algorithms. The performance metric used during learning phase is the number (or percentage) of converged Q-values in joint state-action pair required with learning epochs and run-time complexity per learning epoch, while that during the planning phase includes only the run-time required to completely execute the plan.

### 4.5.1 Experimental platform

Both computer simulations (for stick-carrying, triangle-carrying problem and box-carrying) and hardware testing (for stick-carrying and box-carrying problem) are performed to compare the relative performance of the proposed and existing algorithms.

#### 4.5.1.1 Simulation

The simulation is done employing MATLAB GUI (graphical user interface) R2015a version on an i7-3370 processor desktop computer with clock speed of 3.40GHz. The individual grid size in computer simulation is fixed at 20 pixels  $\times$  20 pixels. The total arenas in computer simulation are considered as of 5 unit  $\times$  5 unit grids size and 9 unit  $\times$  9 unit grids size.

In the present chapter, learning is conducted only in simulation, either by mimicking a real-world environment or by generating an arbitrary environment to avoid any damage of the real-robot. On the other hand, in simulation, planning is straight forward; where at the current joint state each agent evaluates the CE employing the identical Q-table maintained by each agent in joint state-action space, and moves to the next state by adding a fixed length in the direction of the action executed. In case of multiple equilibria, agents select the CE, which appears first.

#### **4.5.1.2 Hardware**

The hardware testing is done with two Khepera-II mobile robots [40]-[41], equipped with an onboard Microcontroller (Motorola 68331) having a flash memory of 512 kB and a clock speed of 25MHz. It has 8 inbuilt active infrared proximity sensors and they are the well-known semiconductor (GaAs) type proximity sensors [50]. In addition, both motor axes consist of an incremental encoder for position and speed measurement of the robot [40], [41]. Considering 2 unit and 5 unit speed (1 unit = 0.08mm/10ms), the stick-carrying problem is realized by connecting two robots through serial ports to two different Pentium IV desktop computers with clock speed of 2GHz.

As already discussed, to avoid damage of the real-robot, learning is conducted in simulation only and subsequently the planning is done in real-time. During the planning phase, the next positions of the robots are determined employing the proposed  $\Omega$  induced joint Q-table for  $C\Omega MP$ ,  $\Omega MP$  algorithms and the Correlated-Q induced joint Q-tables for CQIP algorithm stored in both the Pentium IV machines. As the joint Q-table maintained by each Pentium IV machine (or robot) is identical, hence, robots do not require communicating with fellow robots as well as with fellow robots' computers during coordination (i.e., evaluation of CE). In the perspective of implementation, robots identify the change in states (next state) by measuring the distance traversed, in the direction of the action executed using incremental encoders. The individual grid size in hardware testing is fixed at 80 mm  $\times$  80 mm. Total arena for hardware testing is 9 unit  $\times$  9 unit grids (720 mm  $\times$  720 mm).

Both in computer simulation, and hardware testing, each agent cooperates by selecting one set of actions among two sets. The first one consists of five actions: Left (L), Forward (F), Right (R), Back (B) and Pause (P) from a state. The second one includes nine actions, which are Left (L), Left-Forward (LF), Forward (F), Forward-back (FB), Right (R), Right-Back (RB), Back (B) and Pause (P) from a state.

#### **4.5.2 Experimental approach**

Experimental approaches for both learning and planning phases are given below.

#### 4.5.2.1 Learning phase

In the present chapter, it is assumed that agents can observe other agents' state, action and reward, i.e., the environment can be represented as a Multi-agent Markov Decision Process (MMDP) [3]. Hence, in the above perspective inter-agent communication is not required. The parameters required during the learning phase of the  $\Omega$ QL-I, C $\Omega$ QL-I,  $\Omega$ QL-II, C $\Omega$ QL-II, NQL, FQL and CQL algorithms are set as follows: discounting factor  $\gamma = 0.90$ , learning rate  $\alpha = 0.2$  and maximum immediate reward = 100. In NQL, FQL and CQL algorithms, a reward of -1 (penalty) is given to an agent, when it hits other agents besides when hitting an obstacle or boundary wall in the proposed Experiment 4.1, 4.2 and 4.3. However, in  $\Omega$ QL-I, C $\Omega$ QL-I,  $\Omega$ QL-II and C $\Omega$ QL-II algorithms, an agent receives the same reward (-1) due to penalty (hitting an obstacle or the boundary wall). In  $\Omega$ QL-I and  $\Omega$ QL-II, because of the collision among the agents and in C $\Omega$ QL-I and C $\Omega$ QL-II, because of collision among the agents as well as collision between the stick (or triangle) with an obstacle results in a reward of -1, however, individually agents are not penalized for the former cause. In simulation, the system joint state-transition probabilities are assigned as randomly generated constant values assuming the workspace to be slippery, satisfying the property of a Markovian matrix, subject to the sum of state-transition probabilities at each state is unity, and slippery workspace is the cause of uncertainty. In hardware, experiments are conducted only in the deterministic environment. To determine convergence performance of NQL, FQL, CQL,  $\Omega$ QL-I, C $\Omega$ QL-I,  $\Omega$ QL-II and C $\Omega$ QL-II algorithms, number (or percentage) of joint state-action pairs having converged Q-values after each learning epoch ( $N_{algo}$ ) is accumulated in an array to plot with respect to the learning epoch, where,  $algo \in \{NQL, FQL, CQL, \Omega QL-I, C\Omega QL-I, \Omega QL-II, C\Omega QL-II\}$ . The plots are shown under Experiment 4.1. In case of CQL, the average of  $N_{algo}$  among the four variants of CQL is done before comparison.

#### 4.5.2.2 Planning phase

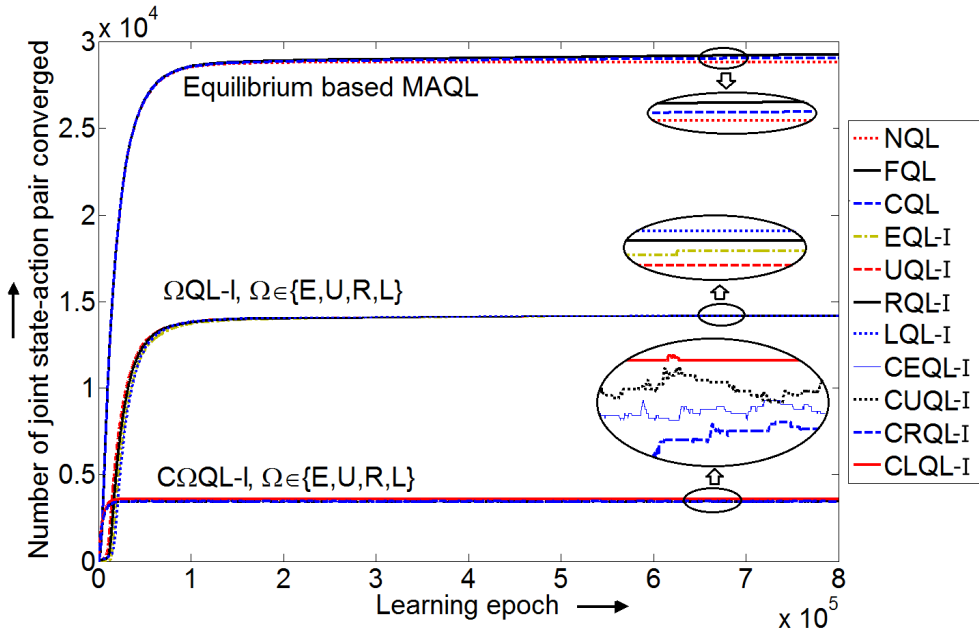
To study the total execution time in the planning phase (Experiment 4.2 and 4.3), the run-time complexity of  $\Omega$ MP, C $\Omega$ MP and CQIP algorithms are evaluated utilizing the 'run & time' button in MATLAB GUI. To study the run-time complexity during hardware test by Khepera-II mobile robot, a stop watch is used. Let  $T_{algo}$  be the run-time complexity of the  $algo \in \{CQIP, C\Omega MP, \Omega MP\}$ . For multiple solutions (joint action or equilibrium), the solution which appears first is selected by all the agents. Let  $S_i$  be the starting position of robot  $R_i$  and  $G_i$  is the goal position of the robot,  $i \in \{1, 2, 3\}$ . Real-time planning with Khepera-II mobile robots is followed after learning by computer simulation. Results for Experiment 4.1, 4.2 and 4.3 are given below.

### 4.5.3 Experimental results

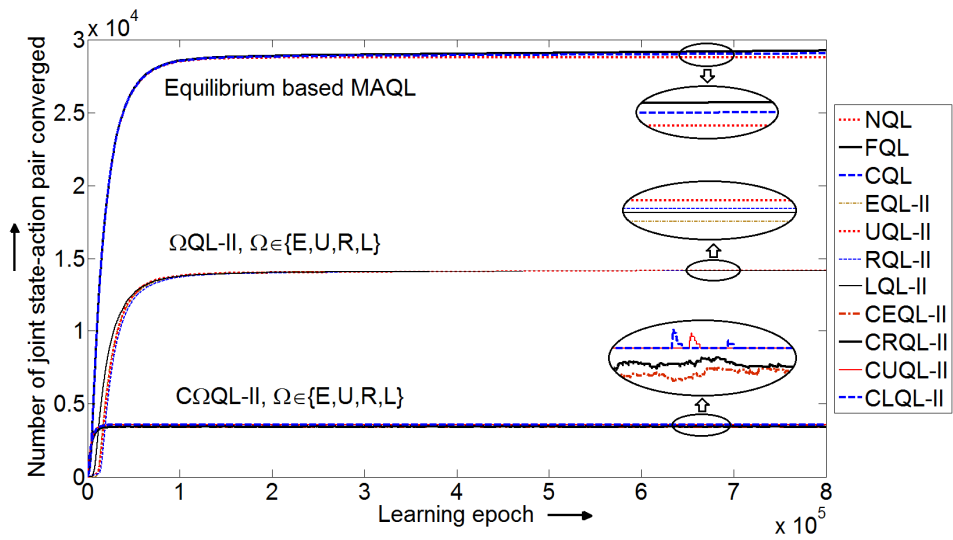
Experiment 4.1, 4.2 and 4.3 are given below in details.

**Experiment 4.1: Performance test of the proposed learning algorithms**

The motivation of this experiment is to examine the convergence of the  $\Omega$ QL-I, C $\Omega$ QL-I (Fig. 4.3.a, 4.4.a and 4.5.a) and  $\Omega$ QL-II, C $\Omega$ QL-II (Fig. 4.3.b, 4.4.b, and 4.5.b) algorithms with learning epoch for a two/three agent system. In Fig. 4.3, 4.4 and 4.5, all the algorithms are run for 20 times separately in an obstacle free  $5 \times 5$  grid map and the mean from the above 20 runs is evaluated. It is apparent from Fig. 4.3 that for both the schemes  $N_{CQL} > N_{\Omega QL} > N_{C\Omega QL}, \Omega \in \{E, U, R, L\}$ . This is supported by (4.57) and (4.58). It is also observed from Fig. 4.3 that  $N_{\Omega QL-I} \approx N_{\Omega QL-II}$ ,  $N_{C\Omega QL-I} \approx N_{C\Omega QL-II}$  and  $N_{NQL} \approx N_{FQL} \approx N_{CQL}$ . Further, the learning epochs required for convergence of NQL exceeds the same for  $\Omega$ QL algorithms, which in turn exceeds the same for C $\Omega$ QL algorithms. This is supported by (4.62) and (4.64). Hence, it is apparent from Fig. 4.3 that C $\Omega$ QL algorithms outperform NQL, FQL, CQL and  $\Omega$ QL algorithms in terms of speed of convergence. In Fig. 4.3, a fluctuation is observed in the curves of C $\Omega$ QL because of task-constraint checking (here, maintaining stick length in stick-carrying problem). Table-4.I offers the average of the percentage of joint state-action pair converged. It is apparent that Table-4.I supports (4.57), (4.58), (4.62) and (4.64). Similar inferences can be drawn from Fig. 4.4 and 4.5.

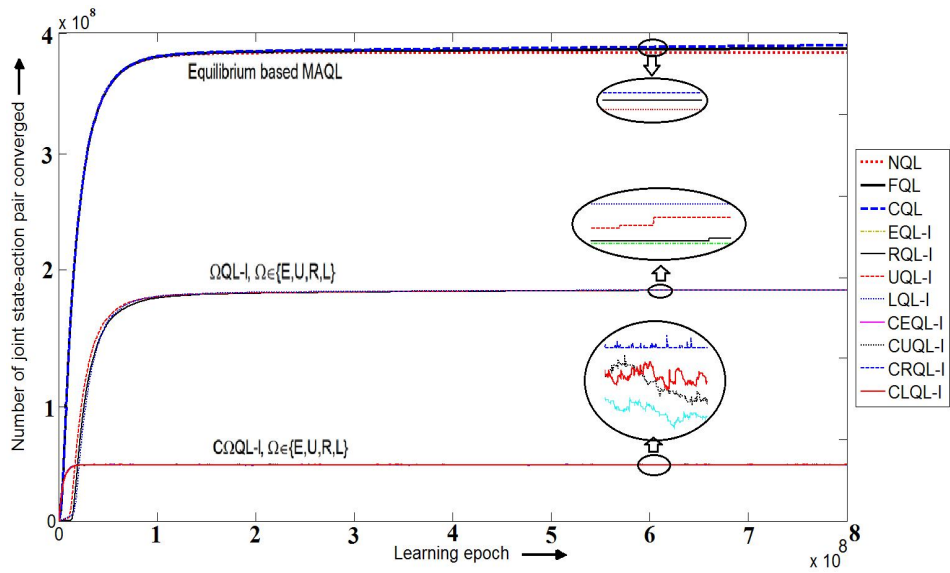


a. scheme-I

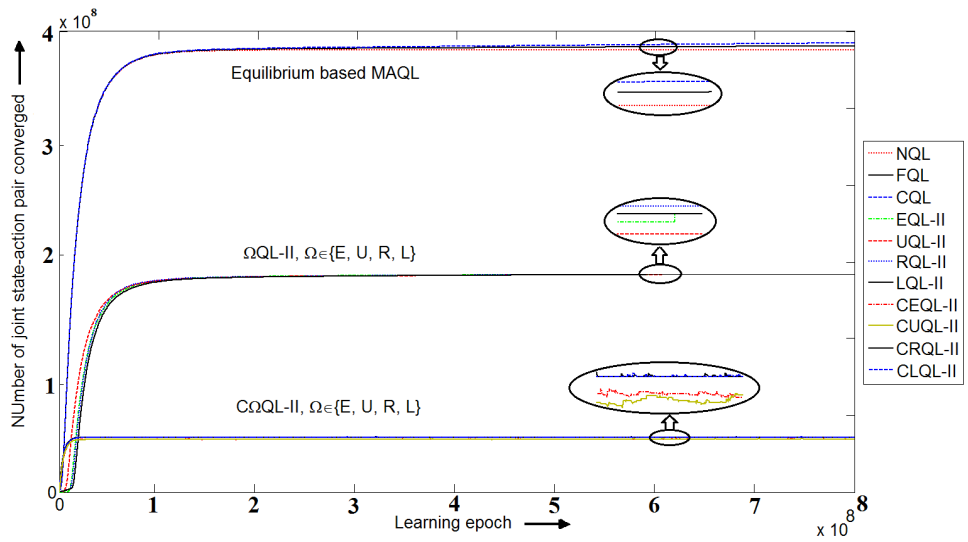


b. scheme-II

Fig. 4.3 Convergence comparison of  $\Omega\text{QL}$ ,  $C\Omega\text{QL}$ , NQL, FQL and CQL algorithms for two agents five actions, where  $\Omega \in \{U, E, R, L\}$

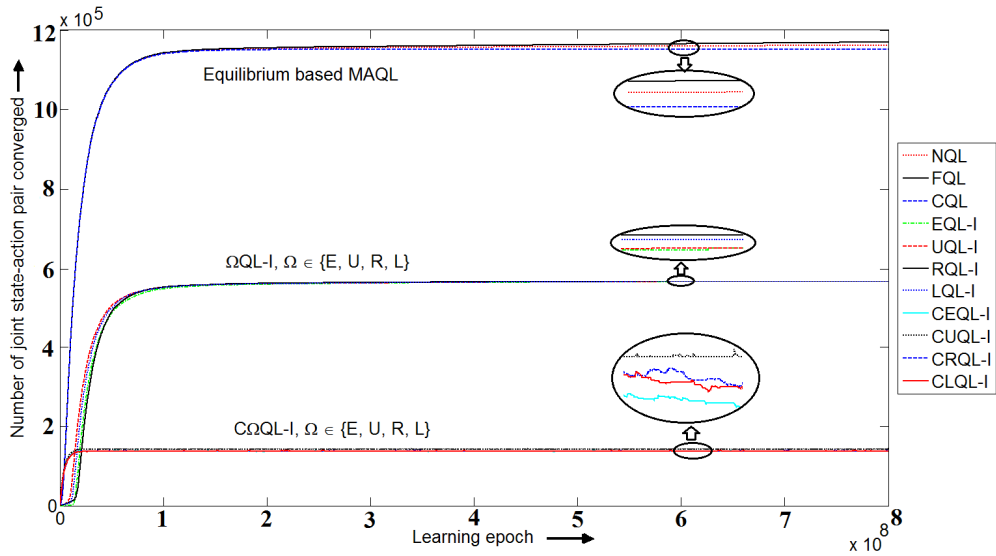


a. scheme-I



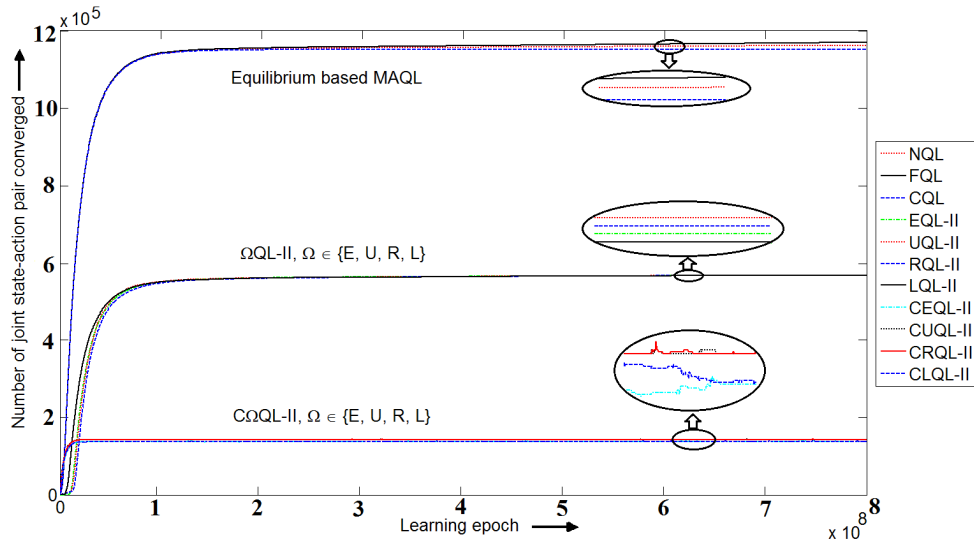
b. scheme-II

Fig. 4.4 Convergence comparison of  $\Omega$ QL, C $\Omega$ QL, NQL, FQL and CQL algorithms for three agents five actions, where  $\Omega \in \{U, E, R, L\}$



a. scheme-I





b. scheme-II

Fig. 4.5 Convergence comparison of  $\Omega$ QL,  $C\Omega$ QL, NQL, FQL and CQL algorithms for two agents nine actions, where  $\Omega \in \{U, E, R, L\}$

Table-4.I  
Average of the percentage (%) of joint state-action pair converged within  $1 \times 10^5$  learning epochs of different learning algorithms

Algorithm	unconstraint	constraint	Algorithm	unconstraint	constraint
EQL-I	99.76 %	100 %	LQL-II	99.96 %	100 %
UQL-I	99.57 %	100 %	EQL	89.76 %	X
RQL-I	99.46 %	100 %	UQL	90.75 %	X
LQL-I	99.78 %	100 %	RQL	90.17 %	X
EQL-II	99.96 %	100 %	LQL	89.98 %	X
UQL-II	99.84 %	100 %	NQL	89.67 %	X
RQL-II	99.84 %	100 %	FQL	90.61 %	X

Table-II shows the superiority of the proposed schemes in terms of the run-time complexity of single learning epoch over the equilibrium based MAQL including CQL for two agents.

Table-4.II  
Average run-time complexity of different learning algorithms (second)

Algorithm	unconstraint	constraint	Algorithm	unconstraint	constraint
EQL-I	0.008	0.014	LQL-II	1.011	1.017
UQL-I	0.019	0.023	EQL	12.019	X
RQL-I	0.008	0.015	UQL	22.018	X
LQL-I	1.008	1.015	RQL	12.020	X
EQL-II	0.010	0.016	LQL	32.801	X
UQL-II	0.021	0.025	NQL	14.201	X
RQL-II	0.012	0.016	FQL	16.206	X

**Experiment 4.2: Object (box)-carrying by scheme-I based proposed planning algorithms**

The motivation of this experiment is to transport a box by two agents from a given starting position to the desired destination utilizing the earlier experimental platform and approach for  $\Omega = R$ . Corresponding simulation and experimental results for the planning paths are shown in

Fig. 4.6 and 4.7. In Map 4.1 (Fig. 4.6 and 4.7), there are seven obstacles and two agents (robots). Each agent has a gripper to grip the box with an aim to transport it from one position to another. If two or more than two robots gather, then they exchange the box (indicated by thick arrow in Fig. 4.6 and by a circle in Fig. 4.7), motivated by the principle of market-based multi-robot coordination [42]. In Fig. 4.7, two Khepera-II mobile robots are employed in real-time, for box transportation task. In real-time, only robot 2 (R2) has a gripper to grip the box (Fig. 4.7) and robot 1 (R1) transports the same by carrying it on its top.

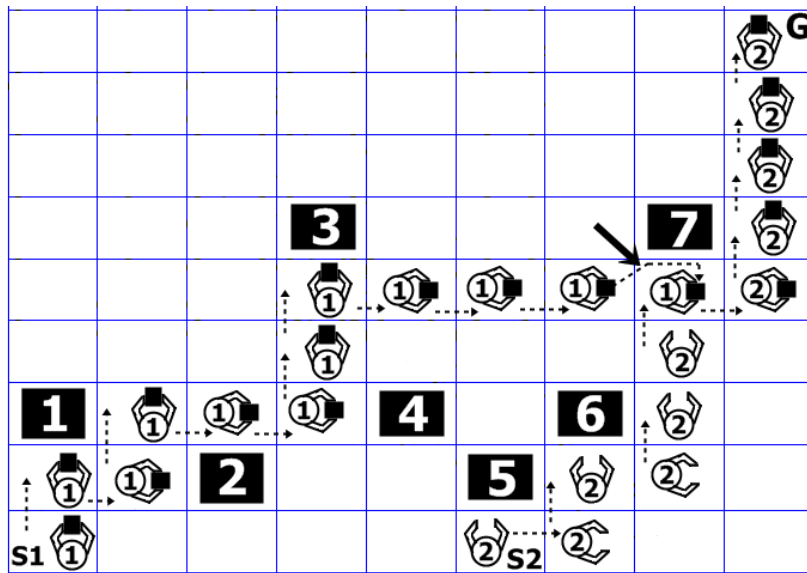


Fig. 6 (Map 1) Planning with box by CQIP, C $\Omega$ MP and  $\Omega$ MP algorithms

Table-III provides the mean of the run-time complexity of the 20 runs of the CQIP, C $\Omega$ MP and  $\Omega$ MP algorithms. It is apparent from Table-III that  $T_{CQIP} > T_{C\Omega MP} > T_{\Omega MP}$ , which is supported by (4.64). Hence, it is apparent from Table-III that  $\Omega$ MP algorithm outperforms the existing algorithms in terms of run-time complexity.

Table-4.III  
Average run-time complexity of different planning algorithms (second)

Map	CQIP algorithm	Proposed planning algorithms	
		C $\Omega$ MP	$\Omega$ MP
4.1 (Fig. 4.4)	15.93	10.92	8.90
4.1 (Fig. 4.5) with 2 unit speed	28.84	23.75	21.92
4.1 (Fig. 4.5) with 5 unit speed	18.65	13.65	11.97



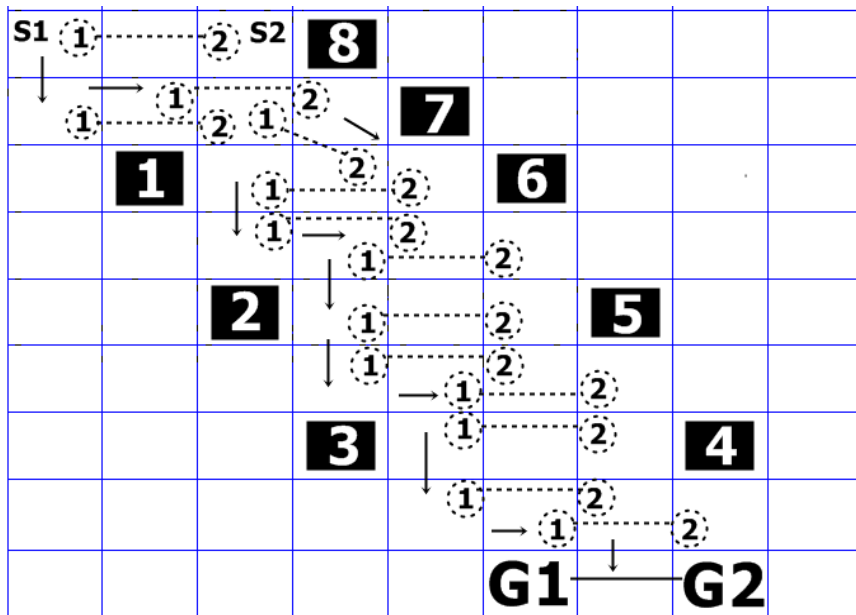


Fig. 4.8 (Map 4.2) Planning with stick by CQIP,  $\mathcal{C}\Omega\text{MP}$  and  $\Omega\text{MP}$  algorithms

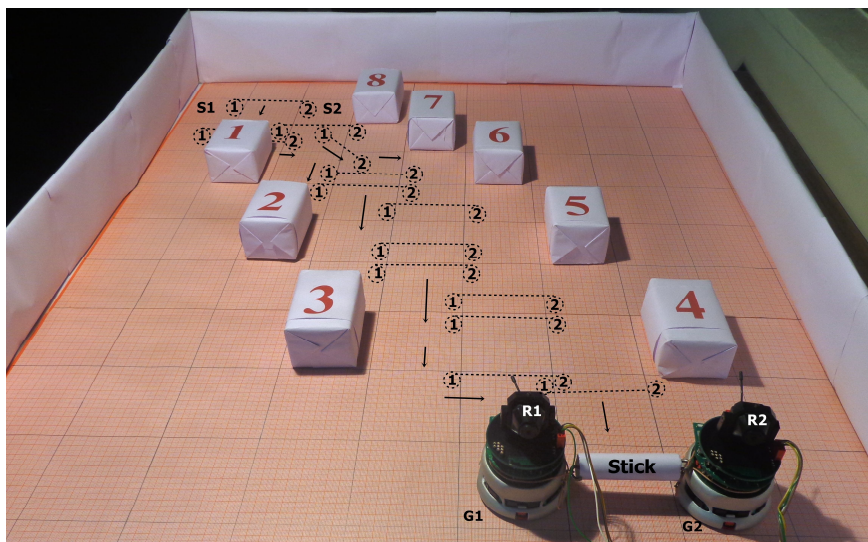


Fig. 4.9 (Map 4.2) Path planning using Khepera-II mobile robot by CQIP,  $\mathcal{C}\Omega\text{MP}$  and  $\Omega\text{MP}$  algorithms, the planned path of which is given in Fig. 4.6 also



proposed  $\Omega$ MP follows the  $C\Omega$ QL. An analysis reveals that both time- and space- complexities of proposed learning and planning algorithms are significantly less to those of the CQL. A further reduction in complexity is obtained by dropping the infeasible joint state-action pairs from the joint Q-table.

Simulation and practical experimental results are given to validate the superiority of the proposed algorithms over the reference algorithms considering space-, time- and run-time complexities as the performance metrics.

## **4.7 Summary**

Unlike correlated Q-learning, Chapter 4 proposes an attractive approach to adapt composite rewards of all the agents in one Q-table in joint state-action space during learning, and subsequently these rewards are employed to compute correlated equilibrium in the planning phase. Two separate schemes of multi-agent Q-learning have been proposed. If success of only one agent is enough to make the team successful, then scheme-I is employed. However, if an agent's success is contingent upon other agents and simultaneous success of the agents is required then scheme-II is employed. It is also shown that the correlated equilibrium obtained by the proposed algorithms and by the traditional correlated Q-learning are identical. In order to restrict the exploration within the feasible joint states, constraint versions of the said algorithms are also proposed. Complexity analysis and experiments have been undertaken to validate the performance of the proposed algorithms in multi-robot planning on both simulated and real platforms.

## 4.8 Appendix

The expected optimal Q-value of an agent  $i$  at next state  $s_i' \in S_i$  is denoted by  $\hat{Q}_i^*(s_i')$  and is expressed in (4A.1).

$$\hat{Q}_i^*(s_i') = \sum_{\forall s_i'} p_i(s_i' | (s_i, a_i)) \sum_{\forall a_i'} p_i^*(a_i' | s_i') Q_i(s_i', a_i') \quad (4A.1)$$

Here,  $a_i' \in A_i$  is the action at  $s_i'$ .  $Q_i(s_i', a_i')$  be the Q-value of agent  $i$  at next state  $s_i' \in S_i$  because of action  $a_i' \in A_i$ .  $p_i^* : A_i \rightarrow [0,1]$  is the optimal probability distribution over  $A_i$ .

<b>Algorithm 4A.1</b> Correlated Q-Learning (CQL)
<b>Input:</b> Learning rate $\alpha \in [0,1)$ and discounting factor $\gamma \in [0,1)$ ;
<b>Output:</b> Optimal joint Q-value $Q_i^*(G, K), \forall G, \forall K, \forall i$ ;
<b>Begin</b>
Initialize: Current state $s_i$ , action set $A_i$ at $s_i$ , joint Q-value $Q_i(G, K) \leftarrow 0, \forall G, \forall K, \forall i$ ;
<b>Repeat</b>
Randomly select an action $a_i \in A_i, \forall i$ and execute it;
Observe joint immediate reward $r_i(G, K), \forall i$ ;
Evaluate next state $s_i' \leftarrow \delta_i(s_i, a_i), \forall i$ to obtain joint next state $G' = \langle s_i' \rangle_{i=1}^m$ for $m$ agents;
$Q_i(G, K) \leftarrow (1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \cdot \Omega \hat{Q}_i^*(G')], \forall i$ and $G \leftarrow G'$ ;    // $\Omega \in \{U, E, R, L\}$
<b>Until</b> $Q_i(G, K), \forall G, \forall K, \forall i$ converges;
$Q_i^*(G, K) \leftarrow Q_i(G, K), \forall G, \forall K, \forall i$ ;
<b>End.</b>

**Lemma 4.1:** The Inequality,  $\Psi(x_i) \pm \gamma \Psi(y_i) \leq \Psi(x_i \pm \gamma y_i)$ ,  $\Psi \in \{\text{Min}, \prod\}_{i=1}^m$  holds for any real

values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\gamma \in [0,1)$ , where  $0 \in \{x_i\}$  and  $0 \in \{y_i\}$ .

*Proof.* Given  $x_i, y_i, \forall i, i \in [1, m]$  as real values and  $\gamma \in [0,1)$ ,

For any  $j \in [1, m]$  and  $\Psi = \text{Min}_{i=1}^m$ ,

$$x_j \geq \text{Min}_{i=1}^m(x_i) \text{ and } y_j \geq \text{Min}_{i=1}^m(y_i) \text{ always hold}$$

$$\therefore x_j \pm \gamma y_j \geq \text{Min}_{i=1}^m(x_i) \pm \gamma \text{Min}_{i=1}^m(y_i), \forall j, j \in [1, m]$$

$$\Rightarrow \text{Min}_{j=1}^m(x_j \pm \gamma y_j) \geq \text{Min}_{i=1}^m(x_i) \pm \gamma \text{Min}_{i=1}^m(y_i)$$

$$\Rightarrow \text{Min}_{i=1}^m(x_i \pm \gamma y_i) \geq \text{Min}_{i=1}^m(x_i) \pm \gamma \text{Min}_{i=1}^m(y_i) \quad [:\cdot i, j \in [1, m]]$$

$$\therefore \text{Min}_{i=1}^m(x_i) \pm \gamma \text{Min}_{i=1}^m(y_i) \leq \text{Min}_{i=1}^m(x_i \pm \gamma y_i) \quad (4A.1)$$

Similarly, if  $\Psi = \prod_{i=1}^m$ ,  $0 \in \{x_i\}$  and  $0 \in \{y_i\}$ , then

$$\prod_{i=1}^m (x_i) \pm \gamma \prod_{i=1}^m (y_i) \leq \prod_{i=1}^m (x_i \pm y_i), \quad (4A.2)$$

Thus, the desired inequality holds.  $\square$

**Lemma 4.2:** *The inequality,  $\Psi(x_i) \pm \gamma \Psi(y_i) \geq \Psi(x_i \pm \gamma y_i)$ ,  $\Psi \in \{\text{Max}, \sum\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\gamma \in [0, 1)$ .*

*Proof.* Given  $x_i, y_i, \forall i, i \in [1, m]$  as real values and  $\gamma \in [0, 1)$ ,

For any  $j \in [1, m]$  and  $\Psi = \text{Max}_{i=1}^m$ ,

$$x_j \leq \text{Max}_{i=1}^m(x_i) \text{ and } y_j \leq \text{Max}_{i=1}^m(y_i) \text{ always hold}$$

$$\therefore x_j \pm \gamma y_j \leq \text{Max}_{i=1}^m(x_i) \pm \gamma \text{Max}_{i=1}^m(y_i), \quad \forall j, j \in [1, m]$$

$$\Rightarrow \text{Max}_{j=1}^m(x_j \pm \gamma y_j) \leq \text{Max}_{i=1}^m(x_i) \pm \gamma \text{Max}_{i=1}^m(y_i)$$

$$\Rightarrow \text{Max}_{i=1}^m(x_i \pm \gamma y_i) \leq \text{Max}_{i=1}^m(x_i) \pm \gamma \text{Max}_{i=1}^m(y_i) \quad [ \because i, j \in [1, m] ]$$

$$\therefore \text{Max}_{i=1}^m(x_i) \pm \gamma \text{Max}_{i=1}^m(y_i) \geq \text{Max}_{i=1}^m(x_i \pm \gamma y_i) \quad (4A.3)$$

Similarly, if  $\Psi = \sum_{i=1}^m$ , then

$$\therefore \sum_{i=1}^m (x_i) \pm \gamma \sum_{i=1}^m (y_i) \geq \sum_{i=1}^m (x_i \pm \gamma y_i) \quad (4A.4)$$

Thus, the desired inequality holds.  $\square$

**Lemma 4.3:** *The Inequality,  $(1-\alpha)\Psi(x_i) \pm \alpha \Psi(y_i) \leq \Psi[(1-\alpha)x_i \pm \alpha y_i]$ ,  $\Psi \in \{\text{Min}, \prod\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\alpha \in [0, 1)$ , where  $0 \in \{x_i\}$  and  $0 \in \{y_i\}$ .*

*Proof.* Proof of Lemma 4.3 is similar to the proof of Lemma 4.1.  $\square$

**Lemma 4.4:** *The Inequality,  $(1-\alpha)\Psi(x_i) \pm \alpha \Psi(y_i) \geq \Psi[(1-\alpha)x_i \pm \alpha y_i]$ ,  $\Psi \in \{\text{Max}, \sum\}$  holds for any real values of  $x_i, y_i, \forall i, i \in [1, m]$  and  $\alpha \in [0, 1)$ .*

*Proof.* Proof of Lemma 4.4 is similar to the proof of Lemma 4.2.  $\square$

**Lemma 4.5:** *If  $\Omega \hat{Q}_i^*(G') = \Omega \hat{Q}_i^*(G') + \Delta \hat{Q}_i^*(G')$ , then  $\Omega \hat{Q}_i^*(G') = \hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')$  at the next joint state  $G' = \langle s'_i \rangle_{i=1}^m$ .*

*Proof.* By (4.26),

$$\begin{aligned} \Omega \hat{Q}_i^*(G') &= \sum_{\forall G'} p(G' | (G, K)) \sum_{\forall K'} \Omega p^*(K' | G') Q_i'(G', K') \\ &= \sum_{\forall G'} \prod_{j=1}^m p_j(s'_j | (s_j, a_j)) \sum_{\forall K'} \prod_{j=1}^m p_j^*(a'_j | s'_j) Q_i'(G', K') \quad [\text{by (4.3) and (4.5)}] \\ &= \prod_{j=1}^m \sum_{\forall s'_j} p_j(s'_j | (s_j, a_j)) \prod_{j=1}^m \sum_{\forall a'_j} p_j^*(a'_j | s'_j) Q_i'(s'_i, a'_i) \quad [\text{by (4.14), } Q_i'(G', K') = Q_i'(s'_i, a'_i)] \end{aligned}$$



$$\begin{aligned}
&= \sum_{\forall s'_i} p_i(s'_i | (s_i, a_i)) \sum_{\forall a'_i} p_i^*(a'_i | s'_i) Q_i(s'_i, a'_i) \times \prod_{\substack{j=1, \\ j \neq i}}^m \sum_{\forall s'_j} p_j(s'_j | (s_j, a_j)) \prod_{\substack{j=1, \\ j \neq i}}^m \sum_{\forall a'_j} p_j^*(a'_j | s'_j), \\
&\hspace{20em} [\because i, j \in [1, m]] \\
&= \sum_{\forall s'_i} p_i(s'_i | (s_i, a_i)) \sum_{\forall a'_i} p_i^*(a'_i | s'_i) Q_i(s'_i, a'_i) \\
&\hspace{10em} [\because \sum_{\forall s'_j} p_j(s'_j | (s_j, a_j)) = 1 \text{ and } \sum_{\forall a'_j} p_j^*(a'_j | s'_j) = 1, \forall j] \\
&= \hat{Q}_i^*(s'_i) \hspace{15em} \text{[by (4.1)]} \hspace{5em} (4A.5)
\end{aligned}$$

Given,  $\Omega \hat{Q}_i^*(G') = \Omega \hat{Q}_i^*(G) + \Delta \hat{Q}_i^*(G')$

$$\therefore \Omega \hat{Q}_i^*(G') = \hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G') \hspace{10em} \text{[by (4A.5)]}$$

Hence, the Lemma is proved.  $\square$

**Lemma 4.6:**  $\Omega \tilde{Q}(G, K) = \Phi[Q_i(G, K)]$  for  $\Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \}$ , holds for the equality cases

of Lemma 4.1-4.4.

*Proof.* Here,

$$\begin{aligned}
&\Omega \tilde{Q}(G, K) \leftarrow (1-\alpha)\Omega \tilde{Q}(G, K) + \alpha[R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] \hspace{5em} \text{[by (4.21)]} \\
&= (1-\alpha)^t \Omega \tilde{Q}(G, K) + (1-\alpha)^{t-1} \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + (1-\alpha)^{t-2} \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + \dots + \\
&\quad (1-\alpha) \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] \\
&\text{[by recursively substituting (4.21), where, learning epoch } t \rightarrow \infty] \\
&= \Phi[(1-\alpha)^t Q_i(G, K)] + (1-\alpha)^{t-1} \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + (1-\alpha)^{t-2} \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + \dots + \\
&\quad (1-\alpha) \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + \alpha [R(G, K) + \gamma \Omega \hat{Q}_i^*(G')] \\
&\hspace{10em} [\because t \rightarrow \infty \text{ and } \alpha \in [0, 1) \therefore (1-\alpha)^t \rightarrow 0, \text{ where } \Phi \in \{ \sum_{i=1}^m, \text{Min}, \text{Max}, \prod_{i=1}^m \}] \\
&= \Phi[(1-\alpha)^t Q_i(G, K)] + (1-\alpha)^{t-1} \alpha [\Phi[r_i(G, K)] + \gamma \Phi[\hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')]] + (1-\alpha)^{t-2} \alpha [\Phi[r_i(G, K)] + \\
&\quad \gamma \Phi[\hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')]] + \dots + (1-\alpha) \alpha [\Phi[r_i(G, K)] + \gamma \Phi[\hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')]] + \alpha [\Phi[r_i(G, K)] + \\
&\quad \gamma \Phi[\hat{Q}_i^*(s'_i) + \Delta \hat{Q}_i^*(G')]] \\
&\hspace{15em} \text{[by (4.9) and (4.18)]} \\
&= \Phi[(1-\alpha)^t Q_i(G, K)] + (1-\alpha)^{t-1} \alpha [\Phi[r_i(G, K)] + \gamma \Phi[\Omega \hat{Q}_i^*(G')]] + (1-\alpha)^{t-2} \alpha [\Phi[r_i(G, K)] + \\
&\quad \gamma \Phi[\Omega \hat{Q}_i^*(G')]] + \dots + (1-\alpha) \alpha [\Phi[r_i(G, K)] + \gamma \Phi[\Omega \hat{Q}_i^*(G')]] + \alpha [\Phi[r_i(G, K)] + \gamma \Phi[\Omega \hat{Q}_i^*(G')]] \\
&\hspace{15em} \text{[by Lemma 4.5]} \\
&= \Phi[(1-\alpha)^t Q_i(G, K)] + (1-\alpha)^{t-1} \alpha [\Phi[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + (1-\alpha)^{t-2} \alpha [\Phi[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \dots \\
&\quad + (1-\alpha) \alpha [\Phi[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \alpha [\Phi[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\hspace{10em} \text{[by statement considering only the equalities in Lemma 4.1 and 4.2]}
\end{aligned}$$

$$\begin{aligned}
&= (1-\alpha)^{t-1} \Phi[(1-\alpha)Q_i(G, K)] + (1-\alpha)^{t-1} \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')] + (1-\alpha)^{t-2} \Phi[\alpha[r_i(G, K) + \\
&\gamma \Omega \hat{Q}_i^*(G')]] + \dots + (1-\alpha) \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\quad [\because \Phi \text{ is independent of } t \text{ and } \alpha] \\
&= (1-\alpha)^{t-1} \Phi[(1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + (1-\alpha)^{t-2} \Phi[\alpha[r_i(G, K) + \\
&\gamma \Omega \hat{Q}_i^*(G')]] + \dots + (1-\alpha) \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\quad [\text{by statement considering only the equalities in Lemma 4.3 and 4.4}] \\
&= (1-\alpha)^{t-1} \Phi[Q_i(G, K)] + (1-\alpha)^{t-2} \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \dots + (1-\alpha) \Phi[\alpha[r_i(G, K) + \\
&\gamma \Omega \hat{Q}_i^*(G')]] + \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\quad [\text{by (4.7)}] \\
&= (1-\alpha)^{t-2} \Phi[(1-\alpha)Q_i(G, K)] + (1-\alpha)^{t-2} \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \dots + \\
&(1-\alpha) \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\quad [\because \Phi \text{ is independent of } t \text{ and } \alpha] \\
&= (1-\alpha)^{t-2} \Phi[(1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \dots + \\
&(1-\alpha) \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] + \Phi[\alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&\quad [\text{by statement considering only the equalities in Lemma 4.3 and 4.4}]
\end{aligned}$$

which on further simplification returns

$$\begin{aligned}
\Omega \tilde{Q}(G, K) &= \Phi[(1-\alpha)Q_i(G, K) + \alpha[r_i(G, K) + \gamma \Omega \hat{Q}_i^*(G')]] \\
&= \Phi[Q_i(G, K)] \quad [\text{by (4.7)}]
\end{aligned}$$

Hence, the Lemma is proved.  $\square$

**Lemma 4.7:**  $\alpha | [R_t(G, K) - R^*(G, K)] | = 0, \forall t$  holds for  $\alpha \in [0, 1)$ , where  $t$  is the learning epoch.

*Proof.* By (4.9),

$$\alpha | [R_t(G, K) - R^*(G, K)] | = \alpha | \Phi[r_i^t(G, K)] - \Phi[r_i^*(G, K)] | \quad (4A.6)$$

For  $\Phi \in \{ \text{Min}, \prod \}_{i=1}^m$  (4A.6) becomes

$$\begin{aligned}
\alpha | [R_t(G, K) - R^*(G, K)] | &\leq \alpha | \Phi[r_i^t(G, K)] - \Phi[r_i^*(G, K)] |, \quad [\text{by Lemma 4.1 with } \gamma = 1] \\
\alpha | [R_t(G, K) - R^*(G, K)] | &= 0. \quad [\because r_i^t(G, K) = r_i^*(G, K), \forall t, \forall i] \quad (4A.7)
\end{aligned}$$

for  $\Phi \in \{ \text{Max}, \sum \}_{i=1}^m$  (4A.6) becomes

$$\begin{aligned}
\alpha | [R_t(G, K) - R^*(G, K)] | &\leq \alpha | \Phi[r_i^t(G, K)] - \Phi[r_i^*(G, K)] |. \quad [\because | \Phi(a_i) - \Phi(b_i) | \leq \Phi | a_i - b_i |] \\
\alpha | [R_t(G, K) - R^*(G, K)] | &= 0. \quad [\because r_i^t(G, K) = r_i^*(G, K), \forall t, \forall i] \quad (4A.8)
\end{aligned}$$

Hence, by (4A.7) and (4A.8) the Lemma holds.  $\square$

**Lemma 4.8:**  $\gamma | [\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')] | = 0$ , holds for  $\gamma \in [0, 1)$  as learning epoch  $t \rightarrow \infty$ .

*Proof.* By (4.18),

$$\gamma | [\Omega \hat{Q}_t^*(G') - \Omega \hat{Q}^*(G')] |$$

$$= \gamma | [\Phi[\hat{Q}_i^{t*}(s_i^l) + \Delta\hat{Q}_i^{t*}(G^l)] - \Phi[\hat{Q}_i^*(s_i^l) + \Delta\hat{Q}_i^*(G^l)]] | \quad (4A.9)$$

Now, we have two cases:

Case I: For,  $\Phi \in \{ \underset{i=1}{\overset{m}{Min}}, \underset{i=1}{\overset{m}{\prod}} \}$  (4A.9) becomes

$$\begin{aligned} & \gamma | [\Omega\hat{Q}_i^*(G^l) - \Omega\hat{Q}^*(G^l)] | \\ & \leq \gamma | \Phi[\hat{Q}_i^{t*}(s_i^l) + \Delta\hat{Q}_i^{t*}(G^l) - \hat{Q}_i^*(s_i^l) - \Delta\hat{Q}_i^*(G^l)] | \quad [\text{by Lemma 4.1 with } \gamma = 1] \\ & = \gamma | \Phi[[\hat{Q}_i^{t*}(s_i^l) - \hat{Q}_i^*(s_i^l)] + [\Delta\hat{Q}_i^{t*}(G^l) - \Delta\hat{Q}_i^*(G^l)]] | \\ & = \gamma | \Phi[ \sum_{\forall s_i^l} p_i(s_i^l | (s_i, a_i)) \sum_{\forall a_i^l} p_i^*(a_i^l | s_i^l) [Q_i^t(s_i^l, a_i^l) - Q_i^*(s_i^l, a_i^l)] + \\ & \quad \sum_{\forall G^l} p(G^l | (G, K)) \sum_{\forall K^l} p^*(K^l | G^l) [d_i^t(G^l, K^l) - d_i^*(G^l, K^l)] | \quad [\text{by (4A.1) and (4.20)}] \\ & \quad [Q_i^*(s_i^l, a_i^l) \text{ is the maximum individual Q-value of agent } i \text{ at } (s_i^l, a_i^l) \\ & \quad \text{and } d_i^t(G^l, K^l) = d_i^*(G^l, K^l), \forall t, \forall i] \\ & = \gamma | \Phi[ \sum_{\forall s_i^l} p_i(s_i^l | (s_i, a_i)) \sum_{\forall a_i^l} p_i^*(a_i^l | s_i^l) [Q_i^t(s_i^l, a_i^l) - Q_i^*(s_i^l, a_i^l)] | \quad (4A.10) \end{aligned}$$

Case II: For,  $\Phi \in \{ \underset{i=1}{\overset{m}{Max}}, \underset{i=1}{\overset{m}{\sum}} \}$  (4A.9) becomes

$$\begin{aligned} & \gamma | [\Omega\hat{Q}_i^*(G^l) - \Omega\hat{Q}^*(G^l)] | \\ & \leq \gamma \Phi | [\hat{Q}_i^{t*}(s_i^l) + \Delta\hat{Q}_i^{t*}(G^l) - \hat{Q}_i^*(s_i^l) - \Delta\hat{Q}_i^*(G^l)] | \quad [ \cdot : \Phi(a_i) - \Phi(b_i) \leq \Phi | a_i - b_i ] \\ & = \gamma \Phi | [\hat{Q}_i^{t*}(s_i^l) - \hat{Q}_i^*(s_i^l)] + [\Delta\hat{Q}_i^{t*}(G^l) - \Delta\hat{Q}_i^*(G^l)] | \\ & = \gamma \Phi | \sum_{\forall s_i^l} p_i(s_i^l | (s_i, a_i)) \sum_{\forall a_i^l} p_i^*(a_i^l | s_i^l) [Q_i^t(s_i^l, a_i^l) - Q_i^*(s_i^l, a_i^l)] + \\ & \quad \sum_{\forall G^l} p(G^l | (G, K)) \sum_{\forall K^l} p^*(K^l | G^l) [d_i^t(G^l, K^l) - d_i^*(G^l, K^l)] | \quad [\text{by (4A.1) and (4.20)}] \\ & \quad [Q_i^*(s_i^l, a_i^l) \text{ is the maximum individual Q-value of agent } i \text{ at } (s_i^l, a_i^l) \\ & \quad \text{and } d_i^t(G^l, K^l) = d_i^*(G^l, K^l), \forall t, \forall i] \\ & = \gamma \Phi | \sum_{\forall s_i^l} p_i(s_i^l | (s_i, a_i)) \sum_{\forall a_i^l} p_i^*(a_i^l | s_i^l) [Q_i^t(s_i^l, a_i^l) - Q_i^*(s_i^l, a_i^l)] | \quad (4A.11) \end{aligned}$$

Now as  $t \rightarrow \infty$  by [17]  $Q_i^t(s_i^l, a_i^l) \rightarrow Q_i^*(s_i^l, a_i^l)$ . So, from (4A.10) and (4A.11)

$\gamma | [\Omega\hat{Q}_i^*(G^l) - \Omega\hat{Q}^*(G^l)] | = 0$ , as  $t \rightarrow \infty$ .

Hence, the Lemma holds.  $\square$

**Lemma 4.9:** If  $|\Omega\tilde{Q}_{t-k}(G, K) - \Omega\tilde{Q}^*(G, K)| = \Delta\tilde{Q}_{t-k}(G, K)$ , then  $(1-\alpha) | [\Omega\tilde{Q}_{t-1}(G, K) - \Omega\tilde{Q}^*(G, K)] | = (1-\alpha)^k \Delta\tilde{Q}_{t-k}(G, K)$ , where,  $\alpha \in [0, 1)$ ,  $k \in \mathbb{R}^+$  and  $t$  is the learning epoch.

*Proof.* By (4.21),

$$\begin{aligned}
& (1-\alpha) |\Omega \tilde{Q}_{t-1}(G, K) - \Omega \tilde{Q}^*(G, K)| \\
&= (1-\alpha) |[(1-\alpha)\Omega \tilde{Q}_{t-2}(G, K) + \alpha[R_{t-2}(G, K) + \gamma\Omega \hat{Q}_{t-2}^*(G')]] - \\
& \quad |[(1-\alpha)\Omega \tilde{Q}^*(G, K) + \alpha[R^*(G, K) + \gamma\Omega \hat{Q}^*(G')]]| \\
&= (1-\alpha) |(1-\alpha)[\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)] + \alpha[R_{t-2}(G, K) - R^*(G, K)] + \\
& \quad \gamma[\Omega \hat{Q}_{t-2}^*(G') - \gamma\Omega \hat{Q}^*(G')]| \\
&\leq (1-\alpha) |(1-\alpha)[\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)] + \alpha[R_{t-2}(G, K) - R^*(G, K)]| + \\
& \quad (1-\alpha) |\gamma[\Omega \hat{Q}_{t-2}^*(G') - \gamma\Omega \hat{Q}^*(G')]| \\
& \hspace{15em} (\because |a+b| \leq |a| + |b|) \\
&\leq (1-\alpha) |(1-\alpha)[\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)]| + (1-\alpha) |\alpha[R_{t-2}(G, K) - R^*(G, K)]| + \\
& \quad (1-\alpha) |\gamma[\Omega \hat{Q}_{t-2}^*(G') - \gamma\Omega \hat{Q}^*(G')]| \\
& \hspace{15em} (\because |a+b| \leq |a| + |b|) \\
&= (1-\alpha)^2 |[\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)]| + (1-\alpha)\alpha |[R_{t-2}(G, K) - R^*(G, K)]| + \\
& \quad (1-\alpha)\gamma |[\Omega \hat{Q}_{t-2}^*(G') - \gamma\Omega \hat{Q}^*(G')]| \\
&= (1-\alpha)^2 |[\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)]| + (1-\alpha)\gamma |[\Omega \hat{Q}_{t-2}^*(G') - \gamma\Omega \hat{Q}^*(G')]| \\
& \hspace{15em} [\text{by Lemma 4.7}] \\
&= (1-\alpha)^2 |\Omega \tilde{Q}_{t-2}(G, K) - \Omega \tilde{Q}^*(G, K)| \hspace{10em} [\text{by Lemma 4.8 with } t \rightarrow \infty.] \\
&= (1-\alpha)^k |\Omega \tilde{Q}_{t-k}(G, K) - \Omega \tilde{Q}^*(G, K)| \\
& \hspace{15em} [\text{For } k \in \mathbb{R}^+ \text{ and continuing recursively employing (4.21)}] \\
&= (1-\alpha)^k \Delta \tilde{Q}_{t-k}(G, K). \hspace{10em} [\text{by statement}]
\end{aligned}$$

Hence, the Lemma holds.  $\square$

The time-complexity for  $\Omega \in \{U, L\}$  are given in Table 4A.1 for the traditional as well as for the proposed algorithms. Let  $N$  be the maximum number of digits required to represent one Q-value all agents' Q-values.

Table 4A.1 Time-complexity analysis

	Algorithm	$\Omega = U$	$\Omega = L$
Learning	CQL	$m A ^m + ( A ^m - 1)$	$N^m  A ^m + ( A ^m - 1)$
	$\Omega$ QL-I	$m( A  - 1) + m$	$m( A  - 1) + N^m$
	C $\Omega$ QL-I	$m( A  - 1) + m + t_{C\Omega QL-I}$	$m( A  - 1) + N^m + t_{C\Omega QL-I}$
	$\Omega$ QL-II	$m( A  - 1) + 2m$	$m( A  - 1) + 2N^m$
	C $\Omega$ QL-II	$m( A  - 1) + 2m + t_{C\Omega QL-I}$	$m( A  - 1) + 2N^m + t_{C\Omega QL-I}$
Planning	CQIP	$m A ^m + ( A ^m - 1) + t_{CQIP}$	$N^m  A ^m + ( A ^m - 1) + t_{CQIP}$
	$\Omega$ MP	$( A ^m - 1)$	$( A ^m - 1)$
	C $\Omega$ MP	$( A ^m - 1) + t_{C\Omega MP}$	$( A ^m - 1) + t_{C\Omega MP}$

It is apparent from Table 4A.1 and earlier complexity analysis in Section 4.4 that the time-complexity of the proposed planning algorithms does not vary with the variation of  $\Omega$ .

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## **Chapter 5**

# **A Modified Imperialist Competitive Algorithm for Multi-Robot Stick-Carrying Application**

This chapter proposes a novel evolutionary optimization approach of solving a multi-robot stick-carrying problem. The problem refers to determine the time-optimal trajectory of a stick, being carried by two robots, from a given starting position to a predefined goal position amidst static obstacles in a robot world-map. The problem has been solved using a new hybrid evolutionary algorithm. Hybridization, in the context of evolutionary optimization framework, refers to developing new algorithms by synergistically combining the composite benefits of global exploration and local exploitation capabilities of different ancestor algorithms. The chapter proposes a novel approach to embed the motion dynamics of fireflies of the Firefly Algorithm (FA) into a socio-political evolution-based meta-heuristic search algorithm, known as the Imperialist Competitive Algorithm (ICA). The proposed algorithm also uses a modified random walk strategy based on the position of the candidate solutions in the search space to effectually balance the trade-off between exploration and exploitation. Thirteen other state-of-art techniques have been used here to study the relative performance of the proposed Imperialist Competitive Firefly Algorithm (ICFA) with respect to run-time and accuracy (offset in objective function from the theoretical optimum after termination of the algorithm). Computer simulations undertaken on a well-known set of 25 benchmark functions reveal that the incorporation of the proposed strategies into the traditional ICA makes it more efficient in both run-time and accuracy. The performance of the proposed algorithm has then finally been studied on the real-time multi-robot stick-carrying problem. Experimental results obtained for both simulation and real frameworks indicate that the proposed algorithm based stick-carrying scheme outperforms other state-of-art techniques with respect to two standard metrics defined in the literature. The application justifies the importance of the proposed hybridization and parameter adaptation strategies in practical systems.

## 5.1 Introduction

Multi-robot coordination has emerged as an important part of robotics research since late 1980s [1]. The problem of coordination of multiple robots arises in numerous applications, for example, in factory environment (to transfer materials and products between workstations), in patient-carrying systems in hospitals/airports and in defense and security systems. Coordination in multi-agent robotics aims at synchronizing and harmonizing the simultaneous actions of multiple robotic agents in pursuit of a specific goal. One of the crucial challenges for multi-agent coordination systems is to design appropriate coordination strategies between the robotic agents that enable them to perform effectively and time optimally in complex workspace. There exists extensive literature on multi-robot coordination employing different approaches including graphs [2], Voronoi diagrams [3], potential field [4], adaptive action selection [5], intention inference [6], cooperative conveyance [7], and perceptual cues [8]. The traditional mathematical model of a multi-robot coordination system can be recast in the settings of an optimization problem [9] with an aim to efficiently utilize the system resources. The objective of optimization here is to determine optimal robotic actions based on the sensory readings collected from the environment by the robots to meet one or more desired objectives of the problem. Thus optimization of the objective functions, characterizing the functionality of a multi-robot coordination system, provides the feasible solutions for the qualitative system performance. The chapter proposes a novel formulation of a multi-robot stick-carrying system as an optimization problem. The stick-carrying problem [10] includes two robots to jointly carry a stick from an assigned initial position to a specified final position in a given environment, without collision with the given obstacles near the robots and the stick, constrained by the constant distance (equal to the stick length) between the robots. The sensory data of the robots, offering the range measurements of the stick from the nearby obstacles and the workspace boundary, are the input variables of the optimization problem while the output variables being the necessary amount of rotation and translation of the stick (by the robots) to transfer it in small step towards the goal. The primary objective function of the stick-carrying optimization problem in this context is concerned with the minimization of the time consumed by the robots (i.e., the length of the path to be traversed by the robots) for complete traversal of the planned trajectory. In other words, we expect the robots to plan the local trajectory, so that the stick is shifted from a given position to the next position (sub-goal) in a time-optimal sense avoiding collision with the obstacles or the boundary of the world-map in the robots' workspace. The optimization algorithm is executed for each local planning step to carry the stick by a small distance. A sequence of local planning ultimately transports the stick to the desired goal position.

In the past decades, a plethora of computing algorithms has been proposed in the domain of numerical function optimization. Traditional derivative-based optimization techniques, such as Newton-Raphson method, quasi-Newton strategy, and steepest descent learning algorithm and the like completely rely on the derivative information of the objective function guiding the direction of search in the fitness landscape. These methods perform satisfactorily when the objective function to be optimized is globally concave over the search space. However, in real world scenario, the objective functions are sometimes found to be irregular and multimodal comprising multiple local optima, saddle points and discontinuities. Traditional gradient-based optimization algorithms are, therefore, ineffective to capture the global optima of these non-differentiable functions.

Since early 1990s, Evolutionary Algorithms (EAs) have emerged as a derivative-free stochastic global optimizer with capability of providing promising results to optimize the non-differentiable functions of the real world problems. EAs with the real-valued vector representation of the potential solutions of a complex physical system have earned wide popularity due to its flexibility and simple search strategy in the high-dimensional hyper-space and robust performance in the dynamic environment. They commence with a population of trial solutions, symbolizing the potential solutions of the problem. The relative integrity of a solution can be assessed by evaluating its associated objective function value (often called fitness). New solutions are then generated by population-based evolutionary procedure. Finally, a greedy selection step is employed being inspired by Darwinian principle of the survival of the fittest. The selection step is responsible for filtering and promoting better candidate solutions from the candidate pool to the next evolutionary generation.

The radical reduction in the computational time in the recent past coupled with the increasing demand to solve complex real world problems has enhanced the quest for more proficient nature-inspired metaheuristics. It is to be noted that two fundamental processes drive the evolution of an EA population—the diversification process, which enables exploring different regions of the search space and the intensification process, which ensures the exploitation of previous knowledge about the fitness landscape. The effects of such exploration and exploitation processes need to be competently balanced by an EA for its qualitative performance both w.r.t computational accuracy and run-time complexity over different fitness landscapes.

However, the superiority of an EA in optimizing different objective functions is subjected to the No Free Lunch Theorem (NFLT) [11]. According to NFLT the expected effectiveness of any two traditional EAs across all possible optimization problems is identical. A self-evident implication of NFLT is that the elevated performance of one EA, say A, over other EA, say B, for one class of optimization problems is counterbalanced by their respective

performances over another class. It is therefore practically difficult to devise a universal EA that would solve all the problems. This apparently paves the way for hybridization of EAs with other optimization strategies, machine learning techniques, and heuristics. In evolutionary computation paradigm, hybridization [12] refers to the process of integrating the attractive features of two or more EAs synergistically to develop a new hybrid EA. The hybrid EA is expected to outperform its ancestors w.r.t both accuracy and complexity over application-specific or general benchmark problems. The fusion of EAs through hybridization hence can be regarded as the key to overcome their individual limitations.

In this chapter, we propose a simple yet very powerful hybrid EA by collegially coalescing the attributes of two global optimizers— the traditional Imperialist Competitive Algorithm (ICA) [13], [14] and the traditional Firefly Algorithm (FA) [15]. ICA is a novel socio-politically motivated population-based meta-heuristic which has revealed remarkable performance in variant fields of optimization problems. The population individuals of ICA, resembling the countries in the world, are categorized as imperialist (best countries) and colonies (rest of the population) based on their associated objective function values. The entire population is subsequently divided into a number of sub-populations, known as empires, each consisting of an imperialist and a number of colonies (randomly selected based on the ruling power of the respective imperialist). The foundation of ICA is rooted in three elementary operations— 1) assimilation, which allows the possible movement of the colonies to their respective imperialist (strengthening exploitation), 2) revolution, which brings out sudden change in the countries' socio-political views (preventing premature convergence of ICA) and 3) the imperialistic competition, which reinforces the powerful empires with an attempt to collapse the weakest one. There exists a vast literature on the modification and application of ICA. Among these the following contributions need special mentioning.

A new EA has been proposed in [16] by combining ICA, Differential Evolution (DE) [17] and K-means clustering algorithm. ICA is also successfully hybridized with EA [18] and Genetic Algorithm (GA) [19]. ICA is treated as a local search strategy to develop a new memetic algorithm in [20]. A new variant of ICA has been introduced in [21] by strengthening the interaction among the imperialists of all the empires. A modified version of ICA is proposed in [22] based on the attraction and repulsion profiles between countries in an empire and is applied to solve a brushless direct current wheel motor design problem. A hybrid ICA, along with an artificial neural network, is implemented in [23] for oil flow rate prediction combining the local search facility of back-propagation and the global search ability of ICA. Seven different chaotic maps are utilized in [24] to improve the convergence characteristics of traditional ICA. Chaos has also been employed in [25] to adapt the angle of movement of colonies in ICA. In [26], an adaptive colony-radius selection strategy is

proposed for improvement in the assimilation policy of ICA. ICA has also been extended to solve constrained optimization policy in [27] equipped with a classical penalty technique.

In [28], ICA is used to solve the dynamic economic dispatch problem while ICA is utilized for parameter identification of a reduced detailed R-C-L-M model of transformer in [29]. The capability of ICA to efficiently control the traffic of a metropolis is studied in [30]. ICA has also shown its potential to design a novel rotor configuration [31]. In [32], promising results are obtained for optimal design of a brushless doubly fed induction generator using ICA. ICA has also been successfully applied for clustering [33] and optimizing epoxy adhesive layer in fiberglass [34]. ICA has also been utilized to optimize the skeletal structures [35] and for solving the integrated product mix-outsourcing optimization problem [36]. The efficiency of ICA has also been validated in the field of template matching [37], IIR filter design [38], non-linear multiple response [39], graph coloring problem [40], PID controller tuning [41] and scheduling in a hybrid flexible flow-shop [42]. A comparative study carried out in [43] exposes superiority of ICA over Particle Swarm Optimization (PSO) in solving the inverse problem in eddy current non-destructive evaluation.

On the other hand, FA is a population based metaheuristic search algorithm for numerical function optimization that draws inspiration from the collective behavior and biochemical properties of fireflies. The motion dynamic of fireflies is derived from four properties of the social interaction between a group of mobile agents—following, dispersion, aggregation and homing. In [44], a fuzzy controller is employed to adaptively tune FA parameters for its better performance. In contrast, a novel strategy employing optimal deviation based FA tuned fuzzy membership function is introduced in [45] for multi-objective unit commitment problem. Another multi-objective variant of FA, adaptively tuning its control parameters using beta distribution, is proposed in [46]. A high convergence speed is obtained in [47] by using a Gaussian probability distribution based position renewal of fireflies. FA has been extensively applied to many optimization fields, including annual crop planning problem [48], complex and nonlinear problem [49], data mining [50], digital image processing [51], structural size and shape optimization [52], hybrid flow shop scheduling problems [53], QAP problem [54], queuing system optimization [55], economic emissions load dispatch problem [56], object tracking [57], traveling salesman problem [58], and so on. According to [59] FA outperforms PSO in finding optimal solutions of noisy non-linear continuous mathematical models in presence of higher level of noise. A hybridized version of FA is found in [60] for forecasting day-ahead electricity price. FA has also been successfully hybridized with cellular learning automata in [61]. A speciation-based FA has been implemented in [62] to solve dynamic optimization problem. A detailed description of hybridization aspects of FA with learning automata, GA and directed direction based search is provided in [63].

In our proposed hybrid stratagem, the foraging behavioral dynamic of fireflies in FA is used to assimilate the colonies of ICA. The proposed assimilation dynamic enforces the colonies within an empire to follow the socio-political aspects of all relatively better countries, even including the imperialist, of the same dominion empire. It in turn increases the explorative revelation of the colonies in the empire. For improving the performance of the hybrid algorithm further, we modulate the step-size for random movement of each firefly according to its relative position in the search space, such that an inferior solution is driven by the explorative force while a qualitative solution should be confined in its local neighborhood in the search space. The chapter also recommends a novel approach of evaluating the threshold value for uniting empires, accelerating the convergence speed. Combinations of the FA-type motion dynamic along with the adaptive step-size and search range based threshold computation do not impose any serious computational overhead on the traditional ICA as evident from the simulation results.

The complicated real world fitness landscape, induced by uncertain and imprecise environment, encompasses multimodality, deception and isolation. Explorative and exploitative capabilities are two cornerstones of EA that determines its efficacy in tracking the global optimum in such ill-conditioned/diverse fitness landscape. Hence the performance of EA is constrained by a trade-off between two antagonist processes: exploration and exploitation. Exploitation favors good convergence speed by orienting the search towards the desired global optimum through local refinement, whereas exploration aids in searching new promising regions in a large search space without getting stuck at the local basins of attraction. In population-based EAs the explorative power is manipulated by the population diversity. A population consisting of almost identical candidates has a low exploration power. Generally the search space is vigorously explored by the trial solutions (using the search operators) in the earlier generations of EAs with a high population variance. The population gradually loses its diversity during the convergence towards the global optimum (via greedy selection) through evolutionary generations. In the earlier explorative phases of optimization problems, low population diversity could induce premature convergence towards a sub-optimal solution. In this article we analyze the evolution of the population variance of ICFA and its two parent algorithms including the traditional ICA and FA over generations delineating its impact on their explorative power. The simulation results reveal that the proposed hybrid algorithm realized with the traditional ICA and the traditional FA (hereafter referred to as Imperialist Competitive Firefly Algorithm –ICFA) enjoys a greater potential of balancing the explorative and exploitative powers as compared to individual balancing propensity of the original ICA and FA.

Experiments have been undertaken to test the expertise of the proposed hybrid algorithm on a test-suite of 25 benchmark functions [64]. The performance of the proposed ICFA is compared with ICA with DE (ICA-DE) [16], Interaction Enhanced ICA using Artificial Imperialists (ICAAI) [21], Memetic ICA (Memetic-ICA) [20], ICA with Adaptive Radius of Colonies Movement (ICAR) [26], Social-Based Algorithm (SBA) [65], Hybrid Evolutionary ICA (HEICA) [18], Chaotic ICA (CICA) [24], Modify ICA with K-means (K-MICA) [33], Recursive ICA with GA (R-ICA-GA) [19], the traditional Artificial Bee Colony (ABC) [66], traditional FA [15], the traditional ICA [13] and the traditional global best PSO [67]. Experiments reveal that the proposed realization outperforms other competitor algorithms both by computational accuracy and run-time complexity.

Lastly, the efficacy of the proposed hybrid evolutionary algorithm is validated in the present context of the proposed multi-robot stick-carrying problem. Here too, the performance of the proposed algorithm is compared with the state-of-art techniques for the same application and the results are in favor of the proposed algorithm. Experiments undertaken further to compare the relative performance of the ICFA based path-planner with other swarm/evolutionary algorithm based design reveal that the proposed ICFA based planner outperforms other realizations. We have arrived at this conclusion by performing comparative analysis of the contender algorithms, used to plan the time-optimal trajectory of the stick, by using two performance metrics which have been previously used in the existing literature [37].

The chapter is divided into five sections. Section 2.2 provides the formulation of the multi-robot stick-carrying problem. Section 2.3 overviews the traditional ICA and FA. The section then explores the proposed hybridization mechanism along with the experimental settings for the benchmarks and simulation strategies. Computer simulation of multi-robot stick-carrying problem in conjunction with the experiments with Khepera-II mobile robots is given in section 2.4. Section 2.5 concludes the chapter with future research direction.

## **5.2 Problem Formulation for Multi-Robot Stick-Carrying**

The problem is demonstrated by considering two homogeneous robots, capable of jointly carrying a stick (by transporting it through a desired angle and distance) from a given starting position to a given goal position avoiding collision with static obstacles in the workspace. There exist two different planning approaches to address the stick-carrying problem: i) local planning and ii) global planning. The global planning is concerned with the planning of the entire trajectory of the robots with the stick from the given initial position to the final position. Contrarily in the local planning, the local movement of the system (the robots with the stick) is executed optimally in small steps towards the goal. The local planning has more flexibility than the global counterpart for the following reasons. First, it can take care of dynamic obstacles. Second, local planning requires small time to determine the next position of the stick

only, rather than deriving the entire trajectory of motion for the robots carrying the stick. Here, the local planning is used for its time-efficiency.

The stick-carrying problem undertaken here is aimed at minimizing the time required by the robots for the execution of each local plan of transportation of the stick. In the present context, this is realized by minimizing the distance between the next positions of the robots with respect to their goal position. It ensures that the robots will follow the shortest path, in turn reducing time required to execute the plan. In order to take care that the next position of the stick is not in the close vicinity of obstacles, a penalty is introduced. It offers a large (or a small) penalty when the next position is close enough to (or far away from) any obstacle.

The mathematical model of the stick-carrying problem is configured with the distances of the stick as well as the robots,  $R_1$  and  $R_2$ , from the sidewall of the workspace as input variables (Fig. 4.1) and the next position of the robots (carrying the stick) as output (estimator) variables. The mathematical model is recast as minimizing an objective function, concerned with the optimal selection of the next position of the system (i.e., the robots with the stick) avoiding collision with obstacles for execution of each local plan. The hybrid evolutionary/swarm algorithm to be proposed is used to determine the next local position of the stick to satisfy the objective. Fig. 4.1 provides the distance measures and ((4.1)) combines these distances into a single entity [69]. Here

$$d = \min(d_{w1}, d_{w2}) + \min(d_{l1}, d_{l2}) + \min(d_{w3}, d_{w4}) \quad (5.1)$$

where the parameters used on the right side of ((4.1)) represent range-measures (indicated in Fig. 4.1) and  $R_1$  and  $R_2$  represent the centers of gravity of two robots carrying the stick.

The following principles are used for formulating the problem.

- 1) The robots first determine their next positions in order to align themselves with the goal and thus plan for a local motion at that current position.
- 2) This alignment may result in a possible collision with static obstacles, if the determined next position of either of the robots or the stick has already been occupied by a static obstacle. Under this circumstance, the robots should turn left or right by certain angle and hence new next positions are to be determined.
- 3) While planning locally, the most important issue to be taken care of is the distance between two robots. If the distance becomes greater than the stick length, the stick will fall.



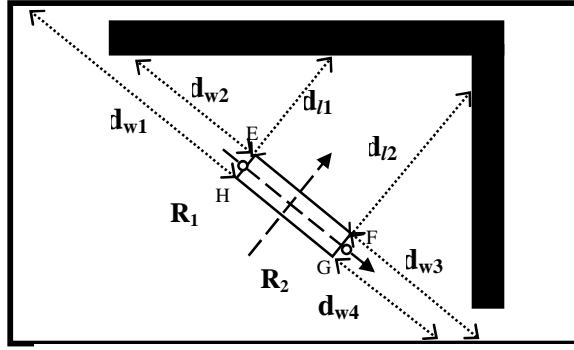


Fig. 5.1 Diagram illustrating the calculation of  $d$

- 4) If the robots can align themselves towards the goal position of the stick without any collision satisfying the distance constraint between them, the local motion will be executed.

Let,  $(x_i, y_i)$  and  $(x'_i, y'_i)$  be the current and the next positions of the robot  $R_i$  with  $\theta_i$  representing its angle of rotation for  $i= [1, 2]$ . Furthermore,  $(x_{i-goal}, y_{i-goal})$  is regarded as the goal position of  $R_i$  for  $i= [1, 2]$ . So, for unit time interval, we have

$$\begin{aligned} x'_i &= x_i + v_i \times \cos \theta_i \\ y'_i &= y_i + v_i \times \sin \theta_i \end{aligned} \quad (5.2)$$

To ensure that the robots should follow the shortest path we need to minimize i) the total Euclidean distance traversed by robots from current position  $(x_i, y_i)$  to next position  $(x'_i, y'_i)$  and ii) the expected Euclidian distance to be covered from the next position  $(x'_i, y'_i)$  to the goal position  $(x_{i-goal}, y_{i-goal})$ , which is given by (5.3).

$$f = \sum_{i=1}^2 \sqrt{(x_i - x'_i)^2 + (y_i - y'_i)^2} + \sqrt{(x'_i - x_{i-goal})^2 + (y'_i - y_{i-goal})^2} \quad (5.3)$$

Combining (5.2) and (5.3), we have the primary objective function to be minimized as given in (5.4).

$$f = \sum_{i=1}^2 v_i + \sqrt{(x_i + v_i \cos \theta_i - x_{i-goal})^2 + (y_i + v_i \sin \theta_i - y_{i-goal})^2} \quad (5.4)$$

Simultaneously, the robots need to satisfy the equality constraint of (5.5) for successfully carrying the stick to execute each local step of the entire task.

$$d_{1,2} = l \quad (5.5)$$

Here,  $d_{1,2}$  is the distance between the robots and  $l$  represents the length of the stick.

In a nutshell, the optimization problem here includes an objective function  $f$ , concerning minimization of the Euclidean distance between the current positions of the robots with their respective goal positions, avoiding collision with obstacles and subjected to the equality constraint as in (5.5). Hence, the objective function for the proposed optimization problem is given by

$$f = \sum_{i=1}^2 v_i + \sqrt{(x_i + v_i \cos \theta_i - x_{i-goal})^2 + (y_i + v_i \sin \theta_i - y_{i-goal})^2} - \lambda(d_{1,2} - l) + 2^{-d} \times K \quad (5.6)$$

Here  $\lambda$  is the Lagrangian multiplier, which needs to be evaluated to satisfy (5.5). The last term in (5.6) is the penalty where  $K$  is a constant. The last term offers a large (or a small) value when the next position is close enough to (or far away from) any obstacle in the workspace.

## 5.3 Proposed Hybrid Algorithm

In this section, firstly the traditional Imperialist Competitive Algorithm (ICA) and then the Firefly Algorithm (FA) are overviewed. It then proposes their hybridization methodology following the simulation strategies to substantiate the merit of the proposed algorithm over their traditional counterparts.

### 5.3.1 An Overview of Imperialist Competitive Algorithm (ICA)

Imperialist Competitive Algorithm (ICA) is a population-based stochastic algorithm, which is inspired by the socio-political evolution and the imperialistic competitive policy of a government to extend its power beyond its boundaries. It has earned wide popularity because of its noticeable performance in computational optimization with respect to the quality of solutions [13]. Like any other evolutionary algorithms, ICA starts with an initial population of solutions, called countries. The countries are classified into two groups—imperialists and colonies, based on their power (which is inversely proportional to their objective function values). The colonies (weaker countries) with their relevant imperialist (stronger country) form some empires. In each empire, the imperialist pursues an assimilation policy to improve the economy, culture, and political situation of its colonies, thus winning their loyalty. Moreover, the empires take part in the imperialistic competition in an attempt to gain more colonies. In ICA, the assimilation of colonies towards their respective imperialists along with the competition among empires eventually results in just one empire in the world with all the other countries as colonies of that unique empire. An overview of the main steps of the ICA is presented next.

#### 5.3.1.1 Initialization

ICA starts with a population  $P_i$  of  $NP$ ,  $D$ -dimensional countries,  $\bar{X}_i(t) = \{x_{i,1}(t), x_{i,2}(t), x_{i,3}(t), \dots, x_{i,D}(t)\}$  for  $i = [1, NP]$  representing the candidate solutions, at the current generation  $t=0$  by randomly initializing in the range  $[\bar{X}^{\min}, \bar{X}^{\max}]$  where  $\bar{X}^{\min} = \{x_1^{\min}, x_2^{\min}, \dots, x_D^{\min}\}$  and  $\bar{X}^{\max} = \{x_1^{\max}, x_2^{\max}, \dots, x_D^{\max}\}$ . Thus the  $d$ -th component (socio-political feature) of the  $i$ -th country at  $t=0$  is given by

$$x_{i,d}(0) = x_d^{\min} + \text{rand}(0,1) \times (x_d^{\max} - x_d^{\min}) \quad (5.7)$$

where  $rand(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $d = [1, D]$ . The objective function value  $f(\vec{X}_i(0))$  of the country  $\vec{X}_i(0)$  is evaluated for  $i = [1, NP]$ .

### 5.3.1.2 Selection of Imperialists and Colonies

The population  $P_0$  is sorted in ascending order of  $f(\vec{X}_i(0))$  for minimization problem with  $i = [1, NP]$ . The first  $N$  countries with less cost function values are selected as imperialists while the remaining  $M = NP - N$  countries are declared as colonies. Hence the population individuals are categorized into two groups of countries— imperialists and colonies.

### 5.3.1.3 Formation of Empires

The empire under the  $j$ -th imperialist is constructed based on its ruling power. To accomplish this, first the normalized power of the  $j$ -th imperialist country,  $p_j$ , is evaluated by (5.8) with  $f(\vec{X}_{NP}(0))$  representing the objective function value of the weakest country in the current sorted population  $P_0$ .

$$p_j = \frac{f(\vec{X}_{NP}(0)) - f(\vec{X}_j(0))}{\sum_{l=1}^N f(\vec{X}_{NP}(0)) - f(\vec{X}_l(0))} \quad (5.8)$$

It is evident from (5.8) that better the  $j$ -th imperialist (i.e., less objective function value  $f(\vec{X}_j(0))$  for minimization problem), higher is the difference  $f(\vec{X}_{NP}(0)) - f(\vec{X}_j(0))$  leading to the enhancement of its corresponding ruling power,  $p_j$ . Now the initial number of colonies under in the  $j$ -th empire, denoted by  $n_j$  is computed by (5.9)

$$n_j = \lfloor M \times p_j \rfloor \quad (5.9)$$

$$\text{such that } \sum_{j=1}^N n_j = M \quad (5.10)$$

Here  $\lfloor \cdot \rfloor$  represents the floor function. According to (5.9) the stronger imperialists with higher ruling power now possess larger empires. Hence  $p_j$  symbolizes the fraction of the colonies occupied by the  $j$ -th imperialist. Subsequently the  $j$ -th empire is formed by randomly selecting  $n_j$  countries from  $M$  colonies provided that there will be no common colony between two different empires. Hence the number of countries within the  $j$ -th empire including its imperialist is  $n_j + 1$ . Let the  $k$ -th country belonging to the  $j$ -th empire is denoted by  $\vec{X}_k^j(t)$  (at generation  $t=0$ ) for  $k = [1, n_j + 1]$ . The countries within the  $j$ -th empire are now sorted in ascending order of their objective function values such that the imperialist  $\vec{X}_1^j(t)$  in the  $j$ -th empire attains the first rank. This step is repeated for  $j = [1, N]$ .

### 5.3.1.4 Assimilation of Colonies

Each imperialist country now attempts to improve its empire by enhancing the socio-political influences of its colonies. To accomplish this, each country  $\vec{X}_k^j(t)$  in the  $j$ -th empire now

moves towards its corresponding imperialist  $\bar{X}_1^j(t)$  by changing its characteristic features following (5.11) for  $k= [2, n_j+1]$ .

$$\bar{X}_k^j(t+1) = \bar{X}_k^j(t) + \beta \times rand(0,1) \times (\bar{X}_1^j(t) - \bar{X}_k^j(t)) \quad (5.11)$$

Here  $rand(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $\beta$  is the assimilation coefficient. The objective function value of the modified colony  $f(\bar{X}_k^j(t+1))$  is evaluated for  $k= [2, n_j+1]$ . After assimilation, all the countries in the  $j$ -th empire are sorted in ascending order of the objective function values and the first ranked country is declared as the imperialist  $\bar{X}_1^j(t+1)$  of the same empire for the next generation (i.e.,  $t = t+1$ ). The step is repeated for  $j = [1, N]$ .

### 5.3.1.5 Revolution

Revolution creates sudden fluctuation in the economic, cultural and political aspects of countries in an empire. The colonies in an empire are now equipped with the power of randomly changing their socio-political attributes instead of being assimilated by their corresponding imperialist. It resembles the mutation of trial solutions in the traditional EA. The revolution rate  $\eta$  in the algorithm indicates the percentage of colonies in each empire which will undergo the revolution process. A high value of revolution rate therefore fortifies the explorative power at a cost of poor exploitation capability. Hence a moderate value of revolution rate is favored. Revolution is implemented by randomly selecting  $\eta \times n_j$  countries (including the imperialist) in the  $j$ -th empire (for  $j = [1, N]$ ) and then they are replaced by randomly initialized countries characterized by new socio-political nature. After revolution, as in case of assimilation, all the countries in each empire are sorted in ascending order of the objective function values so that its imperialist is at the first position. The step is repeated for all empires.

### 5.3.1.6 Imperialistic Competition

All the  $N$  empires now participate in an imperialistic competition to take possession of colonies of other weaker empires based on their ruling power. The colonies of the weaker empires will be gradually eluded from the ruling power of their corresponding imperialists and will be thereafter controlled by some other stronger empires. Consequently, the weaker empires will be losing their power and ultimately may be eradicated from the competition. The imperialistic competition along with the collapse mechanism will progressively result in an increment in the power of more dominant empires and diminish the power of weaker ones. The imperialistic competition encompasses the following steps.

### 5.3.1.6.1 Total Empire Power Evaluation

Once an empire is constructed under the dominance of the  $j$ -th imperialist  $\bar{X}_1^j(t+1)$ , the power of the respective empire is compositely influenced by the objective function value of  $\bar{X}_1^j(t+1)$  as well as the constituent colonies  $\bar{X}_k^j(t+1)$  (after assimilation) under the respective  $j$ -th empire for  $k= [2, n_j+1]$ . The total objective function value of the  $j$ -th empire is evaluated as follows.

$$tc_j = f(\bar{X}_1^j(t+1)) + \xi \cdot \frac{1}{n_j} \sum_{k=2}^{n_j+1} \bar{X}_k^j(t+1) \quad (5.12)$$

Here  $\xi < 1$  is a positive number which regulates the influence of the constituent colonies to control the ruling power of the empire. A minuscule value of  $\xi$  causes the total power of the  $j$ -th empire to be determined by its imperialist  $\bar{X}_1^j(t+1)$  only, while increasing the value of  $\xi$  accentuates the importance of the colonies in deciding the total power of the respective empire. The  $N$  empires now are sorted in ascending order of  $tc_j$  for  $j = [1, N]$ . Then the normalized possession power of the  $j$ -th empire,  $pp_j$ , is evaluated by (5.13) with  $tc_N$  representing the total objective function value of the weakest empire in the current population  $P_t$ .

$$pp_j = \frac{tc_N - tc_j}{\sum_{l=1}^N tc_N - tc_l} \quad (5.13)$$

It is evident from (5.13) that stronger the  $j$ -th empire (i.e., less the total objective function value  $tc_j$  for minimization problem), higher is the possession power,  $pp_j$ , which consecutively increases its probability of seizing colonies from weaker empires. This step is repeated for  $j = [1, N]$ .

### 5.3.1.6.2 Reassignment of Colonies and Removal of Empire

The empire with least possession power is interpreted as being defeated in the competition. Let the weakest colony of this weakest empire be denoted as  $\bar{X}_{worst}$ , which is now removed from the dominance of its currently ruling imperialist and reassigned as a new colony to one of the stronger empires based on their possession probabilities. It is noteworthy that  $\bar{X}_{worst}$  will not be possessed by the most powerful empires, but stronger the empire, more likely to possess  $\bar{X}_{worst}$ . To accomplish this, the possession probability of the  $j$ -th empire is computed as follows for  $j= [1, N]$ .

$$prob_j = pp_j - rand(0,1) \quad (5.14)$$

Now  $\bar{X}_{worst}$  is assigned as a new colony to the  $j$ -th empire for which the possession probability  $prob_j$  is maximum. However, if the worst colony consists of only its imperial

before exclusion operation, (i.e.,  $\vec{X}_{worst}$  is the imperialist of the weakest empire), the removal of  $\vec{X}_{worst}$  will result in the collapse of the weakest empire.

### 5.3.1.6.3 Union of Empires

The disagreement between two empires may be assessed by the difference in their respective socio-political features. This dissimilarity between any two empires,  $j$  and  $l$  is evaluated by taking the Euclidean distance between the respective imperialists  $\vec{X}_1^j(t+1)$  and  $\vec{X}_1^l(t+1)$  as in (5.15) for  $j, l = [1, N]$ .

$$Dist_{j,l} = \left\| \vec{X}_1^j(t+1) - \vec{X}_1^l(t+1) \right\| \quad (5.15)$$

If  $Dist_{j,l}$  is less than a predefined threshold,  $Th$ , the two empires are merged into one empire. The stronger country among  $\vec{X}_1^j(t+1)$  and  $\vec{X}_1^l(t+1)$  is declared as the imperialist of the newly formed empire.

After each evolution, we repeat from step 5.3.1.4 until one of the following conditions for convergence is satisfied. Stop criteria include a bound by the number of iterations, achieving a sufficiently low error or aggregations thereof.

## 5.4 An Overview of Firefly Algorithm (FA)

In Firefly Algorithm (FA) [15], a potential solution to an optimization problem is encoded by the position of a firefly in the search space and the light intensity at the position of the firefly corresponds to the fitness of the associated solution. Each firefly changes its position iteratively by flying towards brighter fireflies at more attractive location in the fitness landscape to obtain optimal solutions.

### 5.4.1 Initialization

FA commences with a population  $P_t$  of  $NP$ ,  $D$ -dimensional firefly positions,  $\vec{X}_i(t) = \{x_{i,1}(t), x_{i,2}(t), x_{i,3}(t), \dots, x_{i,D}(t)\}$  for  $i = [1, NP]$  by randomly initializing in the search range  $[\vec{X}^{\min}, \vec{X}^{\max}]$  where  $\vec{X}^{\min} = \{x_1^{\min}, x_2^{\min}, \dots, x_D^{\min}\}$  and  $\vec{X}^{\max} = \{x_1^{\max}, x_2^{\max}, \dots, x_D^{\max}\}$  at the current generation  $t = 0$ . Thus the  $d$ -th component (socio-political feature) of the  $i$ -th firefly at  $t=0$  is given by

$$x_{i,d}(0) = x_d^{\min} + rand(0,1) \times (x_d^{\max} - x_d^{\min}) \quad (5.16)$$

where  $rand(0,1)$  is a uniformly distributed random number lying between 0 and 1 and  $d = [1, D]$ . The objective function value  $f(\vec{X}_i(0))$  (which is inversely proportional to the light intensity for minimization problem) of the  $i$ -th firefly is evaluated for  $i = [1, NP]$ .

### 5.4.2 Attraction to Brighter Fireflies

Now the firefly  $\vec{X}_i(t)$  is attracted towards the positions of the brighter fireflies  $\vec{X}_j(t)$  for  $i, j = [1, NP]$  but  $i \neq j$  such that  $f(\vec{X}_j(t)) < f(\vec{X}_i(t))$  for minimization problem. Now the attractiveness

$\beta_{i,j}$  of  $\vec{X}_i(t)$  towards  $\vec{X}_j(t)$  is proportional to the light intensity seen by adjacent fireflies. However attractiveness  $\beta_{i,j}$  decreases exponentially with the distance between them, denoted by  $r_{i,j}$  as given in (5.17).

$$\beta_{i,j} = \beta_0 \exp(-\gamma \times r_{i,j}^m), \quad m \geq 1 \quad (5.17)$$

where  $\beta_0$  denotes the maximum attractiveness experienced by the  $i$ -th firefly at its own position (i.e., at  $r_{i,i} = r_{i,i} = 0$ ) and  $\gamma$  is the light absorption coefficient, which controls the variation of  $\beta_{i,j}$  with  $r_{i,j}$ . This parameter is responsible for the convergence speed of FA. A setting of  $\gamma=0$  leads to constant attractiveness while  $\gamma$  approaching infinity is equivalent to the complete random search [15]. In (5.17)  $m$  is a positive constant representing a non-linear modulation index. The distance between  $\vec{X}_i(t)$  and  $\vec{X}_j(t)$  is computed using the Euclidean norm as follows.

$$r_{i,j} = \|\vec{X}_i(t) - \vec{X}_j(t)\| \quad (5.18)$$

This step is repeated for  $i, j = [1, N]$ .

### 5.4.3 Movement of Fireflies

The firefly at position  $\vec{X}_i(t)$  moves towards a more attractive position  $\vec{X}_j(t)$  occupied by a brighter firefly (i.e.,  $f(\vec{X}_j(t)) < f(\vec{X}_i(t))$ ) for  $j = [1, N]$  but  $i \neq j$  following the dynamic given in (5.19).

$$\vec{X}_i(t) = \vec{X}_i(t) + \beta_{i,j} \times (\vec{X}_j(t) - \vec{X}_i(t)) + \alpha \times (\text{rand}(0,1) - 0.5) \quad (5.19)$$

The first term in the position updating formula (5.19) represents the  $i$ -th firefly's current position. The second term in (5.19) denotes the change in the position of the firefly at  $\vec{X}_i(t)$  due to the attraction towards the brighter firefly at  $\vec{X}_j(t)$ . Hence it is apparent that the brightest firefly with no more attractive firefly in the current sorted population  $P_t$  will have no motion due to the second term and may get stuck at the local optima. To circumvent the problem, the last term is introduced in (5.19) for the random movement of the fireflies with a step-size of  $\alpha \in (0, 1)$ . Here  $\text{rand}(0,1)$  is a random number generator uniformly distributed in the range (0, 1). This step is repeated for  $i = [1, NP]$ . After completion of its journey mediated by the brighter ones, the updated position of the  $i$ -th firefly is represented by  $\vec{X}_i(t+1)$  for  $i = [1, NP]$ .

After each evolution, the steps 5.4.2 and 5.4.3 are repeated until one of the following conditions for convergence is satisfied. The conditions include restraining the number of iterations, maintaining error limits, or the both, whichever occurs earlier.

## 5.5 Proposed Imperialist Competitive Firefly Algorithm

In our proposed hybridization stratagem, the light-intensity based attraction driven movement of fireflies is embedded into the modified version of ICA to utilize the composite benefits of the explorative and exploitative capabilities of both the ancestor algorithms. The fitness profile based colonizing behavior of the countries in ICA provides them the local exploitation

capability surrounding the local optima (as discovered by their imperialists). In addition, the imperialistic competition eventually helps the algorithm to converge towards the desired global optimum. On the other hand, FA draws inspiration from the self-organizing behavior of fireflies, which offers it potential for global exploration. The information of a better position in the search space, as acquired by a brighter firefly, is distributed among others through the motion dynamic as evident from (5.19). These facts have motivated us to propose a new hybrid algorithm, named Imperialist Competitive Firefly Algorithm (ICFA). In ICFA, the intensification process is controlled by the formation and revolution of empires (clusters) in the search space by ICA, while the diversification is influenced by the foraging behavior of fireflies.

In the modified ICA, each colony tries to contribute to the improvement of its governing empire by improving its socio-political attributes following a new assimilation policy. This is different from the traditional ICA where the revolution of a colony is instigated by the features of its respective imperialist only. Hence the evolving colony is not guided by the experience of more powerful colonies within the same empire. This issue is resolved here being inspired by the self-organizing dynamics (5.19) of the fireflies in the traditional FA. In the present context, the socio-political features of the assimilating colony are stimulated by that of all other powerful colonies within the same empire including its imperialist. This is implemented here by the assimilation dynamic in (5.20) employed by the  $k$ -th colony  $\bar{X}_k^j(t)$  within the  $j$ -th empire for  $k = [2, n_j+1]$ . Here it is presumed that the countries in the  $j$ -th empire are sorted in ascending order of their respective objective function values such that the imperialist  $\bar{X}_1^j(t)$  occupies the first position.

$$\bar{X}_k^j(t) = \bar{X}_k^j(t) + \beta_{k,l}^j \times (\bar{X}_l^j(t) - \bar{X}_k^j(t)) + \alpha \times (\text{rand}(0,1) - 0.5) \text{ if } f(\bar{X}_l^j(t)) < f(\bar{X}_k^j(t)) \quad (5.20)$$

Expression (5.20) indicates that the colony  $\bar{X}_k^j(t)$  follows the nature of a stronger colony  $\bar{X}_l^j(t)$  (including the imperialist  $\bar{X}_1^j(t)$ ) with  $f(\bar{X}_l^j(t)) < f(\bar{X}_k^j(t))$  in the  $j$ -th empire. The  $k$ -th modernized country is now represented by  $\bar{X}_k^j(t+1)$ .

Again, it is noteworthy that the random movement of a firefly (or a colony) with step-size  $\alpha$  in (5.19) (or in (5.20)) in traditional FA helps the population individuals to avoid local optima by their expedition proficiency. Particularly, the convergence of fireflies towards the global optimum greatly relies on the step-size ( $\alpha$ ) profile. However, in the traditional FA,  $\alpha$  is taken to be constant for all fireflies in the current population. It indicates that  $\alpha$  assists in the exploration of the fireflies in the fitness landscape irrespective of their fitness. Consequently, fireflies in vicinity of the global optimum may be deviated away (with  $\alpha$  value greater than the requirement) and may get trapped at local optima. Contrarily, fireflies far away from the global optimum in the fitness landscape (with  $\alpha$  smaller than necessity), may not be given any



opportunity to be attracted towards the global optimum. To overcome this problem,  $\alpha$ , used for the random movement of a firefly, needs to be modulated with its relative position with respect to the current best firefly position.

It in turn ensures that the best candidate solution should search in the local neighborhood with a small step-size to prevent the omission of the global optimum whereas a poor performing member should participate in the global search to explore promising regions in the search space. Under this proposed scheme, the step-size value  $\alpha_{i,d}$  assigned to the  $d$ -th positional component of the  $i$ -th firefly at location  $\vec{X}_i(t)$  is varied based on its spatial distance from the best firefly rather than being constant as outlined in (5.21) for  $d = [1, D]$ . It is apparent that in the sorted population  $\vec{X}_1(t)$  corresponds to the position of the brightest firefly.

$$\alpha_{i,d} = \alpha_{\min} + (1 - \alpha_{\min}) \times rand(0,1) \times \frac{|X_{1,d}(t) - X_{i,d}(t)|}{X_d^{\max} - X_d^{\min}} \quad (5.21)$$

Here  $|\cdot|$  represents the absolute value and  $rand(0,1)$  is a uniformly distributed random number lying in  $(0, 1)$ . It is apparent from (5.21) that if  $X_{i,d}(t)$  is close to  $X_{1,d}(t)$ ,  $\alpha_{i,d}$  reduces to its small minimum value  $\alpha_{\min}$  confining  $X_{i,d}(t)$  in its small neighborhood. Again if the difference  $|X_{i,d}(t) - X_{1,d}(t)|$  increases and approaches to  $X_d^{\max}(t) - X_d^{\min}(t)$ ,  $\alpha_{i,d}$  also approaches to unity offering  $X_{i,d}(t)$  a large magnitude of perturbation. Apparently, the step-size is now treated as a  $D$ -dimensional vector as symbolized by  $\vec{\alpha}_i = \{\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,D}\}$  with its  $d$ -th component  $\alpha_{i,d} \in (\alpha_{\min}, 1)$ .

Moreover, in the traditional ICA, the dissimilarity threshold,  $Th$ , used for uniting two empires is kept as a predefined constant disregarding the search space dimension. It is obvious that the selection of threshold, being responsible for the union of empires, determines the performance of ICA. In the proposed work, a new empirical formula is recommended to calculate the threshold as in (5.22) with the search range  $[\vec{X}^{\min}, \vec{X}^{\max}]$  and  $D$  and  $N$  as the search space dimension and number of empires respectively.

$$Th = \frac{\|\vec{X}^{\max} - \vec{X}^{\min}\|}{N \times D} \quad (5.22)$$

Motivated by these observations, we extend the traditional ICA with the proposed strategies of hybridization with  $\alpha$  modulated FA and uniting threshold  $Th$  selection. The extended ICA, called Imperialist Competitive Firefly Algorithm (ICFA) is similar to the traditional ICA except for the assimilation of the colonies and the union of the empires which are given below.

### 5.5.1 Assimilation of Colonies

The assimilation of the  $k$ -th country  $\vec{X}_k^j(t)$  under the sorted  $j$ -th empire (led by the imperialist  $\vec{X}_1^j(t)$ ) is performed by the following two steps for  $k = [1, n_j+1]$  and  $j = [1, N]$ .

### 5.5.1.1 Attraction to Powerful Colonies

The distance between the  $k$ -th and  $l$ -th countries,  $\bar{X}_k^j(t)$  and  $\bar{X}_l^j(t)$ , under the  $j$ -th empire is computed using the Euclidean norm as follows.

$$r_{k,l}^j = \|\bar{X}_k^j(t) - \bar{X}_l^j(t)\| \quad (5.23)$$

Now, as in case of FA, the country  $\bar{X}_k^j(t)$  is attracted towards a more powerful one  $\bar{X}_l^j(t)$  (including the respective imperialist) with an attractiveness  $\beta_{k,l}^j$  for  $k, l = [1, n_j+1]$  but  $k \neq l$  such that  $f(\bar{X}_l^j(t)) < f(\bar{X}_k^j(t))$ . The associated attractiveness  $\beta_{k,l}^j$  is evaluated by (5.24).

$$\beta_{k,l}^j = \beta_0 \exp(-\gamma \times (r_{k,l}^j)^m), \quad m \geq 1 \quad (5.24)$$

where  $\beta_0$  and  $\gamma$  are same as defined in (5.17).

### 5.5.1.2 Modification of Empire Behavior

The country  $\bar{X}_k^j(t)$  updates its socio-political features inspired by the stronger one  $\bar{X}_l^j(t)$  in the same empire following the dynamic given in (2.25) for  $k, l = [1, n_j+1]$  but  $k \neq l$  provided  $f(\bar{X}_l^j(t)) < f(\bar{X}_k^j(t))$ .

$$\bar{X}_k^j(t+1) = \bar{X}_k^j(t) + \beta_{k,l}^j \times (\bar{X}_l^j(t) - \bar{X}_k^j(t)) + \bar{\alpha}_k^j \times (\text{rand}(0,1) - 0.5) \quad (5.25)$$

The  $D$ -dimensional step-size vector  $\bar{\alpha}_k^j$  for random movement in (5.25) is now modulated based on the relative socio-political aspects of the country  $\bar{X}_k^j(t)$  with respect to that of its respective imperialist  $\bar{X}_1^j(t)$  (with least objective function value in the  $j$ -th empire) as given in (5.26) for  $d = [1, D]$ .

$$\alpha_{k,d}^j = \alpha_{\min} + (1 - \alpha_{\min}) \times \text{rand}(0,1) \times \frac{|\bar{X}_{1,d}^j(t) - \bar{X}_{k,d}^j(t)|}{X_d^{\max} - X_d^{\min}} \quad (5.26)$$

Here  $\text{rand}(0, 1)$  is a uniformly distributed random number lying in  $(0, 1)$ . It is apparent from (5.26) that the imperialist  $\bar{X}_1^j(t)$  is assigned with a step size  $\alpha_{\min}$  for all its components. After each movement of a country, the  $j$ -th empire is sorted in ascending order of the objective function values of its constituent countries so that the first rank is always occupied by the imperialist  $\bar{X}_1^j(t)$ . The step is repeated for  $k = [1, n_j+1]$ . At the end of this step the modified country is demoted by  $\bar{X}_k^j(t+1)$  for  $k = [1, n_j+1]$ .

### 5.5.1.3 Union of Empires

The dissimilarity  $Dist_{j,l}$  between any two empires,  $j$  and  $l$  is evaluated by taking the Euclidean distance between the respective imperialists  $\bar{X}_1^j(t+1)$  and  $\bar{X}_1^l(t+1)$  as in (5.15) for  $j, l = [1, N]$ . The two empires are merged into one empire if  $Dist_{j,l}$  is less than the threshold,  $Th$  as computed using (5.22). The stronger country among  $\bar{X}_1^j(t+1)$  and  $\bar{X}_1^l(t+1)$  is declared as the

imperialist of the newly formed empire and the rest is treated as a colony under the former one. The algorithm for ICFA is given in next page.

<b>Algorithm 5.1 Imperialist Competitive Firefly Algorithm (ICFA)</b>
<p><b>Begin</b></p> <ol style="list-style-type: none"> <li>1. Initialize a population of <math>NP</math>, <math>D</math>-dimensional countries <math>\vec{X}_i(t)</math> at generation <math>t=0</math> using (5.7) and evaluate <math>f(\vec{X}_i(t))</math> for <math>i= [1, NP]</math>;</li> <li>2. Sort <math>P_i</math> in ascending order of cost function values and select the top <math>N</math> countries as the imperialists with the remaining <math>M=NP-N</math> countries as colonies;</li> <li>3. Assign randomly selected <math>n_j</math> colonies under the <math>j</math>-th imperialist, based on its power <math>p_j</math> as computed using (5.9) and (5.8) respectively for <math>j= [1, N]</math>;</li> <li>4. <b>While</b> termination condition is not reached <b>do</b>  <b>Begin</b> <ol style="list-style-type: none"> <li>4.1. <b>For</b> each empire <math>j= 1</math> to <math>N</math> <b>do begin</b>  Sort the countries in the <math>j</math>-th empire in ascending order of their respective cost function values with the imperialist <math>\vec{X}_1^j(t)</math> at the first position;  <b>For</b> each country under the <math>j</math>-th empire <math>k= 1</math> to <math>n_j+1</math> <b>do begin</b>  <b>For</b> each country under the <math>j</math>-th empire <math>k= 1</math> to <math>n_j+1</math> <b>do begin</b>  <b>If</b> <math>f(\vec{X}_l^j(t)) &lt; f(\vec{X}_k^j(t))</math> <b>then do</b>  <ol style="list-style-type: none"> <li>a. Evaluate the distance <math>r_{k,l}^j</math> and hence the attraction <math>\beta_{k,l}^j</math> between <math>\vec{X}_k^j(t)</math> and <math>\vec{X}_l^j(t)</math> using (5.23) and (5.24) respectively;</li> <li>b. Compute the step-size of random movement of <math>\vec{X}_k^j(t)</math> using (5.26);</li> <li>c. Update the socio-political nature of <math>\vec{X}_k^j(t)</math> using (5.25);</li> <li>d. Sort the countries in the <math>j</math>-th empire in ascending order of their respective cost function values with the imperialist <math>\vec{X}_1^j(t)</math> at the first position;</li> </ol> <b>End If.</b>  <b>End For.</b>  Denote the updated country with <math>\vec{X}_k^j(t+1)</math>;  <b>End For.</b>  <b>End For.</b> </li> <li>4.2. <b>For</b> each empire <math>j= 1</math> to <math>N</math> <b>do begin</b>  <ol style="list-style-type: none"> <li>i) Randomly select <math>\eta \times n_j</math> countries from the <math>j</math>-th empire and perform revolution by re-initialization</li> <li>ii) Sort the countries in the <math>j</math>-th empire in ascending order of their respective cost function values with its imperialist at the first position;</li> </ol> <b>End For.</b> </li> <li>4.3. Evaluate the possession power <math>pp_j</math> and the possession probability <math>prob_j</math> of the <math>j</math>-th empire using (5.13) and (5.14) respectively for <math>j= [1, N]</math>;</li> <li>4.4. Sort the empires in descending order of the possession power <math>pp_j</math> for <math>j= [1, N]</math>. Identify the worst colony <math>\vec{X}_{worst}</math> of the weakest empire <math>\vec{X}_N(t+1)</math> and reassign <math>\vec{X}_{worst}</math> as a colony of new empire, <math>j</math>, with the largest possession probability <math>prob_j</math> for <math>j \in [1, N]</math>;</li> <li>4.5. <b>If</b> <math>\vec{X}_{worst}</math> was the only country in the <math>N</math>-th empire <b>then</b> the <math>N</math>-th empire collapses with <math>N \leftarrow N-1</math>  <b>End If.</b> </li> <li>4.6. Evaluate the disagreement threshold <math>Th</math> and the dissimilarity <math>Dist_{j,l}</math> between any two imperialists <math>\vec{X}_j(t+1)</math> and <math>\vec{X}_l(t+1)</math> using (5.22) and (5.15) respectively. Combine the <math>j</math>-th and <math>l</math>-th empires if <math>D_{j,l} &lt; Th</math>. The stronger among <math>\vec{X}_j(t+1)</math> and <math>\vec{X}_l(t+1)</math> will be considered as the imperialist of the combined empire while the remaining will be treated as its colony. Decrement the number of colonies by setting <math>N \leftarrow N-1</math>. This is done for <math>j, l= [1, N]</math>;</li> <li>4.7. Set <math>t \leftarrow t+1</math>;</li> </ol> <b>End While.</b>  <b>End.</b> </li> </ol>

## 5.6 Simulation Results

The performance of the proposed ICFA algorithm is examined here with respect to the minimization of 25 benchmark functions recommended in [64].

### 5.6.1 Comparative Framework

The comparative framework includes ICA with DE (ICA-DE) [16], Interaction Enhanced ICA using Artificial Imperialists (ICAAI) [21], Memetic ICA (Memetic-ICA) [20], ICA with Adaptive Radius of Colonies Movement (ICAR) [26], Social-Based Algorithm (SBA) [65], Hybrid Evolutionary ICA (HEICA) [18], Chaotic ICA (CICA) [24], Modify ICA with K-means (K-MICA) [33], Recursive ICA with GA (R-ICA-GA) [19], the traditional Artificial Bee Colony (ABC) [66], the traditional FA [15], the traditional ICA [67] and the traditional PSO [54]. The above the traditional evolutionary/swarm optimization algorithms are chosen because of their wide popularity in solving numerical single objective optimization problems.

### 5.6.2 Parameter Settings

To make the comparison fair, the populations for all the algorithms (over all problems tested) are initialized using the same random seeds and the population size is kept at 50. We employ the best parametric set-up for all these algorithms as prescribed in their respective sources. In our proposed ICFA, the initial number of empires  $N$  is taken to be  $\sqrt{NP}$  which is equals to 7 for a population size  $NP$  of 50 and  $\alpha_{\min}$  is taken to be 0.3 so that maximum permissible value of step size is 1. The maximum attractiveness  $\beta_0$  and the light absorption coefficient  $\gamma$  both are taken to be 1 with the non-linear modulation index  $m$  determined by  $\sqrt{D \times \max_{j=1}^D (X_j^{\max} - X_j^{\min})}$  as described in [68].

### 5.6.3 Analysis on Explorative Power of ICFA

The explorative and exploitative capabilities of an algorithm can be assessed by the population variance. Let  $\vec{X}_i(t)$  be the  $D$ -dimensional  $i$ -th solution of the population  $P_t$  at generation  $t$ . The variance of the population  $P_t$  considering all its  $NP$  solutions is given by (5.27).

$$V(P_t) = \frac{1}{D} \sum_{j=1}^D \left( \frac{1}{NP} \sum_{i=1}^{NP} x_{i,j}^2 - \left( \frac{1}{NP} \sum_{i=1}^{NP} x_{i,j} \right)^2 \right) \quad (5.27)$$

4.2shows the evolution of the population variance over FEs for f15 in case of 50- $D$  problem. The plots of other functions are omitted for the sake of space economy. The plot indicates that ICA offers a good level of exploitation capability from the initiation of the algorithm whereas FA performs better in balancing the diversification at the earlier exploration stage and gradually converges to the optimal point with a relatively smaller population variance.

However, it is evident that ICFA outperforms the rest two ancestors in providing a well trade-off between high explorative and exploitative powers which prevail during the earlier and later phases of the search respectively. Hence it can be concluded that the hybridization of ICA and FA has empowered ICFA with better exploration and exploitation capabilities.

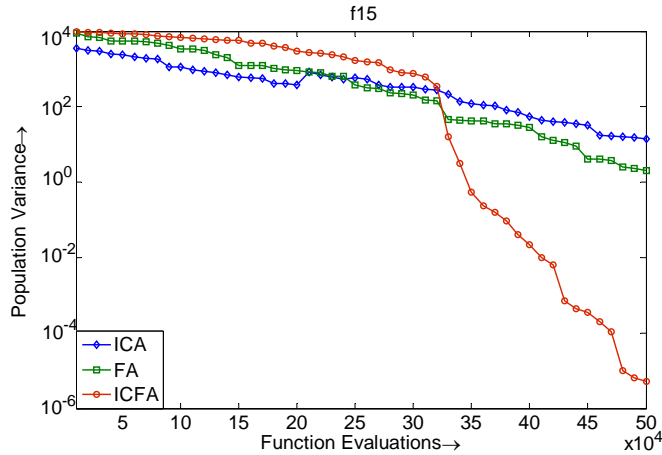


Fig. 5.2 Evolution of the expected population variance

### 5.6.4 Comparison of Quality of the Final Solution

Here, we test the relative performance of our algorithm with other competitor algorithms using 25 benchmark functions [64] of 10, 30 and 50 dimensions. The experiments are conducted for 50 independent runs. For lack of space, the mean and standard deviation (within parenthesis) of the benchmarks function values of 25 independent runs for each of the fourteen algorithms are presented in Table 5.1 for 30-*D* problems only. Please note that the results excluded follow a similar trend like those reported in Table 5.1. Maximum number of function evaluations (Max\_FEs) is set at 300,000 for 30-*D*.

In the last column of Table 5.1, the statistical significance level of the difference of the means (of the final accuracies) of the best two algorithms, obtained using t-test with 25 samples, is reported. Here ‘+’ indicates that the t value of 49 degrees of freedom is significant at a 0.05 level of significance by two-tailed test, while ‘-’ means the difference of the means is not statistically significant and ‘NA’ stands for Not Applicable, referring to the cases for which two or more algorithms achieve the same best accuracy results.

### 5.6.5 Performance Analysis

A close scrutiny of Table 5.1 indicates that the performance of the proposed ICFA has remained effectually and consistently superior to that of the other algorithms. It is noteworthy that out of 25 benchmark instances, in 21 cases ICFA outperforms its nearest neighbor competitor in a statistically significant fashion. In three cases, (f10 and f19) ICA-DE, which

remains the second best algorithm, has achieved the best average accuracy surpassing ICFA. It is evident from Table 5.1 that ICFA performs consistently better than all algorithms over most of the 25 benchmark instances, and the advantage of ICFA is very prominent as well.

Table 5.1 COMPARATIVE ANALYSIS OF PERFORMANCE OF THE PROPOSED ICFA WITH OTHER ALGORITHMS BASED ON SOLUTION QUALITY FOR f01 to f25

Functions	ICFA	Memetic											Stat. Sig.		
		ICA-DE	ICAAI	ICA	ICAR	SBA	HEICA	CICA	K-MICA	R-ICA-GA	ABC	FA		ICA	PSO
f01	<b>0.00e+00</b> (0.00e+00)	2.88e-29 (1.99e-27)	1.10e-26 (2.74e-26)	1.94e-23 (2.95e-25)	3.39e-22 (4.59e-24)	3.82e-21 (4.75e-23)	4.42e-21 (5.20e-20)	5.70e-20 (5.30e-18)	6.34e-17 (5.54e-17)	6.39e-16 (5.60e-17)	6.70e-14 (5.94e-16)	6.70e-14 (6.41e-15)	7.75e-10 (6.53e-11)	7.79e-10 (6.71e-10)	+
f02	<b>1.22e-04</b> ( <b>8.31e-04</b> )	3.24e-04 (1.13e-03)	2.32e-03 (1.30e-03)	2.67e-03 (1.93e-02)	1.19e-02 (3.11e-01)	1.93e-01 (3.42e-01)	2.21e-01 (3.48e+00)	2.67e-01 (4.52e+02)	3.07e-01 (4.58e+02)	4.58e-01 (4.75e+02)	4.86e-01 (4.96e+02)	4.94e-01 (5.28e+02)	5.76e-01 (5.35e+03)	7.65e-01 (5.56e+03)	+
f03	<b>1.97e-01</b> ( <b>1.37e+00</b> )	1.04e+00 (1.50e+00)	1.56e+00 (1.75e+00)	1.78e+01 (1.77e+00)	1.80e+01 (2.44e+01)	2.38e+01 (2.46e+01)	3.54e+01 (3.31e+01)	3.83e+02 (3.84e+01)	4.09e+03 (4.09e+01)	4.89e+03 (4.31e+02)	5.25e+03 (5.69e+02)	6.23e+03 (5.81e+02)	6.71e+04 (5.50e+02)	6.71e+03 (5.88e+02)	+
f04	<b>1.37e-05</b> ( <b>0.00e+00</b> )	1.53e-04 (4.58e-26)	1.90e-04 (1.13e-25)	2.00e-04 (1.15e-23)	2.66e-04 (1.84e-22)	3.28e-03 (2.17e-20)	3.71e-03 (2.35e-20)	3.97e-02 (3.15e-16)	3.98e-02 (3.69e-16)	5.27e-02 (4.21e-14)	5.30e-02 (4.57e-10)	5.45e-01 (4.82e-03)	6.53e+01 (5.55e-01)	6.42e-01 (5.23e-02)	+
f05	<b>1.03e-11</b> ( <b>4.59e-09</b> )	2.54e-10 (6.95e-08)	3.74e-10 (1.01e-08)	1.06e-09 (1.01e-08)	1.60e-09 (1.27e-07)	3.09e-08 (1.81e-07)	3.76e-08 (1.84e-06)	5.42e-07 (2.79e-05)	5.72e-07 (3.01e-04)	5.78e-04 (3.84e-04)	6.08e-03 (4.05e-04)	6.39e-03 (5.60e-03)	6.97e-03 (7.37e-03)	6.73e-03 (6.08e-02)	+
f06	<b>7.34e-01</b> ( <b>3.67e-05</b> )	9.53e-01 (5.77e-05)	1.06e+00 (1.69e-04)	1.28e+00 (1.92e-04)	1.67e+00 (2.36e-03)	1.67e+00 (2.58e-03)	2.45e+00 (2.72e-02)	2.81e+00 (2.82e-01)	2.92e+00 (3.42e-01)	3.59e+00 (3.43e+00)	4.35e+00 (5.46e+00)	5.97e+00 (6.30e+00)	6.61e+00 (6.69e+01)	6.31e+00 (6.59e+01)	+
f07	<b>1.10e-05</b> ( <b>8.56e-01</b> )	1.30e-05 (1.28e+00)	1.41e-05 (1.32e+00)	1.18e-05 (2.14e+00)	1.64e-04 (2.57e+00)	2.07e-04 (3.05e+00)	2.47e-04 (3.12e+00)	3.15e-03 (3.40e+00)	3.82e-03 (4.37e+00)	4.02e-03 (4.80e+00)	4.53e-02 (5.21e+00)	4.54e-02 (5.42e+00)	5.74e-01 (6.50e+00)	5.12e-01 (5.46e+00)	+
f08	<b>2.45e+01</b> ( <b>1.19e-02</b> )	2.65e+01 (1.29e-02)	3.55e+01 (1.36e-02)	3.57e+01 (1.45e-02)	3.72e+01 (1.58e-01)	3.85e+01 (1.59e-01)	4.10e+01 (1.61e-01)	4.35e+01 (2.10e-01)	4.51e+01 (2.17e-01)	5.56e+01 (3.01e+00)	5.68e+01 (3.04e+00)	5.72e+01 (3.29e+01)	7.57e+01 (5.96e+01)	6.13e+01 (5.91e+01)	+
f09	<b>0.00e+00</b> ( <b>2.20e-28</b> )	4.82e-24 (3.59e-26)	1.55e-23 (1.66e-25)	1.80e-23 (1.83e-22)	1.83e-22 (2.96e-22)	2.07e-22 (3.21e-18)	2.23e-21 (3.42e-18)	2.86e-19 (3.55e-17)	3.07e-19 (3.82e-16)	4.16e-19 (4.04e-13)	4.21e-10 (5.11e-12)	4.97e-10 (5.60e-10)	8.85e-09 (7.74e-09)	6.33e-09 (6.50e-09)	+
f10	1.59e+01 (5.95e-06)	<b>1.26e+01</b> ( <b>2.74e-06</b> )	1.62e+01 (1.38e-05)	1.83e+01 (3.33e-04)	2.57e+01 (4.34e-04)	2.76e+01 (4.45e-03)	3.42e+01 (5.57e-03)	3.64e+01 (5.89e-02)	4.36e+01 (6.00e-01)	4.75e+01 (6.04e+00)	5.57e+01 (6.45e+00)	6.19e+01 (7.23e+00)	6.39e+01 (8.32e+00)	7.91e+01 (8.32e+00)	-
f11	<b>2.07e-03</b> ( <b>1.29e-13</b> )	1.21e-02 (3.41e-12)	1.28e-02 (4.50e-12)	1.67e-02 (2.67e-11)	3.35e-02 (1.17e-10)	3.50e-01 (3.29e-10)	4.03e-01 (3.42e-09)	4.26e-01 (3.50e-09)	4.32e+00 (3.65e-09)	5.20e+00 (4.77e-09)	5.63e+00 (4.98e-08)	6.01e+00 (5.72e-08)	6.20e+00 (5.72e-08)	6.33e+00 (5.85e-08)	+
f12	<b>2.58e-04</b> ( <b>6.10e-05</b> )	7.93e-04 (1.42e-04)	1.04e-03 (1.74e-04)	1.21e-03 (1.87e-04)	2.73e-03 (2.04e-04)	3.02e-03 (2.79e-04)	3.17e-03 (2.91e-03)	3.63e-02 (3.02e-03)	4.54e-02 (3.68e-03)	4.61e-02 (4.39e-02)	5.05e-02 (4.59e-02)	5.60e-02 (5.62e-02)	5.77e-02 (5.81e-02)	6.81e-02 (5.88e-02)	+
f13	<b>1.26e-01</b> ( <b>0.00e+00</b> )	1.58e-01 (1.89e-19)	1.88e-01 (1.19e-19)	2.37e-01 (1.24e-18)	2.60e-01 (2.11e-15)	2.92e-01 (3.77e-14)	2.95e-01 (3.92e-14)	3.42e-01 (4.66e-13)	3.83e-01 (4.66e-11)	5.16e-01 (4.68e-10)	6.44e-01 (4.88e-09)	6.51e-01 (4.90e-09)	6.66e-01 (6.17e-08)	6.88e-01 (7.99e-07)	+
f14	<b>1.09e+00</b> ( <b>1.76e-01</b> )	<b>1.09e+00</b> ( <b>1.76e-01</b> )	1.33e+00 (1.85e-01)	1.33e+00 (2.03e-01)	1.58e+00 (2.40e-01)	2.58e+00 (2.69e-01)	2.63e+00 (4.08e-01)	2.99e+00 (4.08e-01)	3.22e+00 (4.31e-01)	3.37e+00 (5.11e-01)	4.12e+00 (5.77e-01)	4.51e+00 (6.15e-01)	5.98e+00 (6.34e-01)	7.87e+00 (8.87e-01)	NA
f15	<b>9.15e-02</b> ( <b>1.23e-03</b> )	1.11e-01 (3.48e-03)	1.13e-01 (6.73e-03)	1.25e-01 (2.23e-02)	1.82e-01 (2.27e-02)	2.18e-01 (2.85e-02)	2.96e+00 (3.71e-02)	2.97e+00 (3.71e-02)	3.29e+00 (4.27e-01)	4.16e+00 (4.46e-01)	4.18e+00 (4.58e-01)	4.87e+00 (5.02e-01)	4.89e+00 (5.73e-01)	5.72e+00 (5.78e-01)	+
f16	<b>4.63e+01</b> ( <b>1.04e-04</b> )	1.07e+02 (1.47e-04)	1.86e+02 (1.56e-04)	1.96e+02 (1.68e-04)	2.96e+02 (1.78e-03)	3.08e+02 (2.02e-03)	3.20e+02 (2.41e-02)	3.62e+02 (4.67e-02)	3.68e+02 (4.70e-02)	4.46e+02 (4.72e-01)	5.45e+02 (4.73e-01)	6.12e+02 (4.86e-01)	6.60e+02 (5.46e+00)	6.70e+02 (5.91e+00)	+
f17	<b>0.00e+00</b> ( <b>7.63e-08</b> )	0.25e+02 (1.06e-07)	1.23e+02 (1.32e-07)	2.25e+02 (1.34e-07)	2.70e+02 (1.69e-06)	2.97e+02 (1.88e-05)	3.22e+02 (2.01e-04)	3.23e+02 (2.38e-04)	3.29e+02 (3.31e-04)	4.21e+02 (4.03e-03)	5.05e+02 (4.25e-02)	5.39e+02 (5.16e-02)	5.49e+02 (5.35e+01)	6.41e+02 (6.42e+01)	+
f18	<b>1.27e+02</b> ( <b>1.79e-05</b> )	1.46e+02 (1.85e-05)	1.65e+02 (2.45e-05)	2.98e+02 (2.91e-05)	3.15e+02 (3.21e-05)	3.82e+02 (3.78e-04)	4.25e+02 (4.07e-04)	4.45e+02 (4.29e-04)	4.51e+02 (4.63e-04)	4.53e+02 (4.63e-04)	4.75e+02 (5.39e-03)	4.78e+02 (5.83e-03)	4.96e+02 (5.89e-03)	5.61e+02 (6.08e-03)	+
f19	1.33e+02 (1.08e-06)	<b>1.44e+02</b> ( <b>1.15e-06</b> )	1.53e+02 (1.27e-06)	2.22e+02 (1.27e-06)	3.35e+02 (1.34e-06)	3.65e+02 (2.56e-05)	3.80e+03 (2.83e-05)	3.81e+03 (3.13e-05)	4.47e+03 (4.39e-05)	4.51e+03 (4.87e-04)	4.53e+03 (5.34e-04)	5.04e+03 (5.40e-04)	6.57e+03 (6.52e-04)	6.95e+03 (7.08e-04)	-
f20	<b>3.31e-03</b> ( <b>1.13e-01</b> )	1.04e-02 (1.37e-01)	1.83e-02 (1.69e-01)	2.43e-01 (1.89e-01)	2.75e+00 (2.31e-01)	3.39e+00 (2.51e-01)	3.64e+00 (2.76e-01)	3.67e+01 (2.97e-01)	3.71e+01 (3.09e+00)	4.10e+02 (3.09e+00)	4.70e+03 (4.78e+00)	5.18e+03 (4.81e+00)	5.28e+03 (4.92e+01)	5.82e+03 (5.15e+01)	+
f21	<b>2.51e+02</b> ( <b>1.08e+00</b> )	3.29e+02 (1.17e+00)	3.64e+02 (1.39e+00)	4.73e+02 (2.84e+00)	4.77e+02 (3.07e+00)	4.00e+02 (3.60e+00)	5.69e+02 (4.11e+00)	5.28e+03 (5.24e+00)	6.38e+03 (5.38e+01)	7.53e+03 (5.77e+01)	7.65e+03 (5.83e+01)	7.75e+03 (5.97e+01)	8.21e+03 (6.18e+01)	8.64e+03 (6.92e+01)	+

Functions	ICFA	ICA-DE	ICAAI	Memetic ICA	ICAR	SBA	HEICA	CICA	K-MICA	R-ICA-GA	ABC	FA	ICA	PSO	Stat. Sig.
f22	<b>1.08e+02</b> ( <b>0.00e+00</b> )	1.52e+02 (1.59e+00)	1.78e+02 (3.35e+00)	1.95e+02 (3.69e+00)	2.03e+02 (4.07e+00)	2.32e+02 (4.28e+00)	3.22e+02 (4.87e+00)	3.46e+02 (5.60e+00)	3.46e+02 (5.70e+00)	3.73e+02 (6.05e+00)	3.92e+02 (6.15e+00)	4.75e+02 (6.50e+00)	5.52e+02 (6.92e+00)	5.95e+02 (6.92e+00)	+
f23	<b>3.38e+02</b> ( <b>0.00e+00</b> )	4.72e+02 (5.22e-18)	4.48e+02 (1.84e-17)	5.02e+02 (1.46e-14)	5.10e+02 (3.39e-13)	5.38e+02 (3.86e-09)	5.62e+03 (4.06e-08)	5.66e+03 (4.07e-06)	6.10e+03 (4.30e-05)	6.17e+03 (4.40e-04)	6.91e+03 (5.38e-04)	7.23 e+03 (5.91e-03)	7.30 e+03 (6.03e-02)	7.87e+03 (6.49e-02)	+
f24	<b>1.29e+02</b> ( <b>3.57e-17</b> )	<b>1.29e+02</b> ( <b>3.57e-17</b> )	1.46e+02 (1.44e-16)	1.62e+02 (1.45e-15)	1.78e+02 (1.89e-14)	1.84e+03 (2.91e-12)	1.93e+03 (3.11e-10)	1.97e+03 (3.43e-10)	2.53e+03 (3.75e-09)	3.42e+03 (3.95e-07)	3.49e+03 (4.48e-06)	3.77e+03 (5.97e-05)	4.44e+03 (6.11e-05)	4.86e+03 (6.63e+01)	NA
f25	<b>1.36e+02</b> ( <b>1.17e+00</b> )	2.44e+02 (1.21e+00)	2.73e+02 (1.93e+00)	2.99e+02 (1.94e+00)	3.16e+02 (2.06e+00)	4.01e+02 (2.10e+00)	4.34e+02 (2.33e+00)	5.10e+02 (2.79e+00)	5.16e+02 (3.26e+00)	6.01e+02 (3.59e+01)	6.02e+02 (4.53e+01)	6.51e+02 (5.49e+01)	6.54e+02 (5.89e+01)	6.89e+02 (6.57e+01)	+

Table 5.2 COMPARATIVE ANALYSIS OF PERFORMANCE OF THE PROPOSED ICFA WITH OTHER ALGORITHMS BASED ON CONVERGENCE TIME IN SECONDS FOR f01 TO f25

Functions	Tolerance	ICFA	ICA-DE	ICAAI	Memetic ICA	ICAR	SBA	HEICA	CICA	K-MICA	R-ICA-GA	ABC	FA	ICA	PSO
f01	1.00e-18	24.742	27.809	108.375	137.003	213.684	278.096	412.032	468.265	524.497	565.394	566.416	577.663	627.761	645.142
f02	2.00e-14	31.957	42.212	190.793	204.307	228.951	256.775	276.649	298.114	330.707	356.942	377.611	437.234	441.208	454.723
f03	4.00e-02	35.716	54.877	85.719	86.560	86.560	151.270	223.543	232.788	241.192	289.094	414.312	426.918	1554.722	489.107
f04	2.00e-04	44.991	46.230	222.895	280.683	359.109	404.514	584.069	693.453	864.752	899.838	1190.840	1197.032	1203.223	1271.330
f05	1.00e-10	56.435	66.432	354.738	570.807	738.502	964.245	980.369	1206.112	1241.586	1354.457	1699.522	1886.566	2438.024	2028.461
f06	4.00e-02	65.785	74.136	368.645	509.179	606.941	729.145	800.430	975.587	1120.194	1242.397	1317.756	1323.866	1751.577	1344.233
f07	3.00e-05	71.749	72.299	224.046	259.784	384.8657	412.356	426.101	501.700	531.939	607.538	716.125	801.345	874.195	1213.701
f08	2.00e-01	80.550	85.743	195.016	225.752	289.344	393.212	463.163	474.822	518.277	519.337	526.756	549.013	594.587	552.193
f09	2.00e-02	82.757	108.455	452.986	879.839	1520.119	1568.031	2060.219	2068.930	2121.198	2386.892	2700.498	2748.410	2883.435	2974.904
f10	3.00e-03	83.044	87.470	299.377	416.525	609.168	632.598	991.851	1239.163	1473.458	1496.888	3358.236	5987.554	12183.370	6430.112
f11	1.00e-10	87.984	136.390	335.992	410.024	586.563	879.845	936.792	1059.230	1104.789	1261.396	1312.649	1383.833	1765.384	2007.413
f12	8.00e-02	103.247	116.073	1154.319	1346.705	1513.440	2103.425	2218.857	2622.869	3078.184	3283.396	3341.112	3347.525	3629.692	3719.472
f13	9.00e-02	103.916	104.565	184.451	223.420	326.037	357.212	462.427	467.623	474.118	494.901	567.643	618.302	758.589	775.475
f14	7.00e-04	113.784	146.048	431.003	679.753	684.679	756.102	758.565	770.880	844.766	898.949	943.281	1098.442	1155.088	1293.009
f15	7.00e-01	117.032	144.925	312.086	374.504	2145.597	2360.156	3881.580	6807.394	8406.839	8874.969	9577.165	11157.105	11332.653	12795.561
f16	9.00e-03	120.138	177.510	81.890	87.284	622.757	2049.704	2093.837	2407.667	2505.739	2540.064	2785.244	2795.051	2834.280	3442.327
f17	1.00e-01	120.310	131.810	293.698	302.544	337.929	360.930	375.084	461.778	470.624	485.663	5051.256	5316.646	6015.507	6227.819
f18	2.00e-01	123.131	123.131	574.288	622.956	958.769	1065.840	1241.046	1620.661	1990.541	2710.835	3022.314	3178.053	3650.137	4263.361
f19	3.00e-02	128.413	168.148	327.428	185.883	325.722	3871.156	4280.442	5184.280	5798.208	6633.832	8339.188	8782.580	9925.168	11801.059
f20	2.00e+00	152.039	164.528	323.989	360.189	427.159	428.969	477.839	765.629	948.439	1035.319	1076.949	1136.679	1174.689	1230.799
f21	4.00e+01	152.635	201.814	250.264	324.213	743.142	863.356	1366.070	1653.855	1690.283	1850.569	2032.712	2112.854	2549.997	3398.782
f22	2.00e+00	156.405	184.566	1869.495	2129.448	2166.275	2751.169	4180.911	7603.627	9639.926	9661.589	10983.017	25995.306	34443.781	60439.088
f23	7.00e+01	165.270	207.414	224.767	243.773	269.390	329.714	428.049	437.966	444.576	454.493	472.672	545.391	2247.674	4016.065
f24	2.00e+02	180.090	266.919	543.129	571.715	618.167	1032.661	1386.410	1486.460	1650.828	1915.2471	19581.26	6860.587	7432.303	8254.144
f25	2.00e+02	182.976	365.211	1437.141	1659.379	2266.831	2718.715	3296.535	4178.080	4259.568	11852.711	12000.870	12371.267	20445.927	28965.063

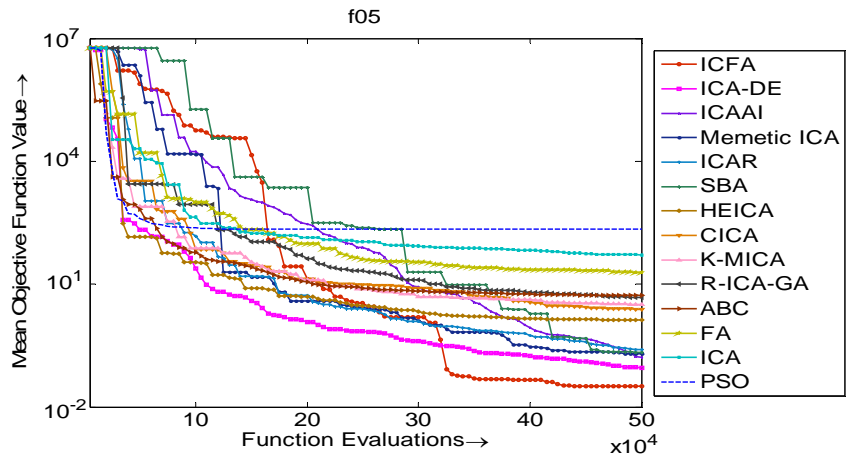


Fig. 5.3 Relative performance in mean best objective function versus function evaluation for f05 with

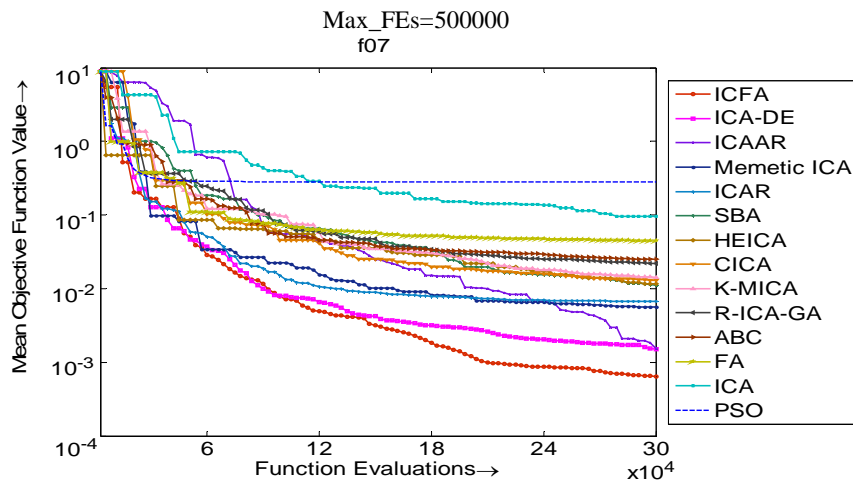


Fig. 5.4 Relative performance in mean best objective function versus function evaluation for f17 with

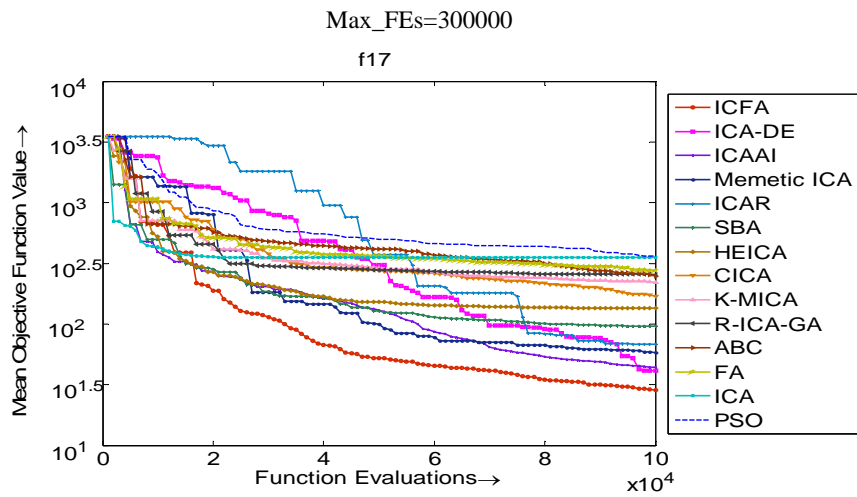


Fig. 5.5 Relative performance in mean best objective function versus function evaluation for f07 with

Max\_FEs=100000



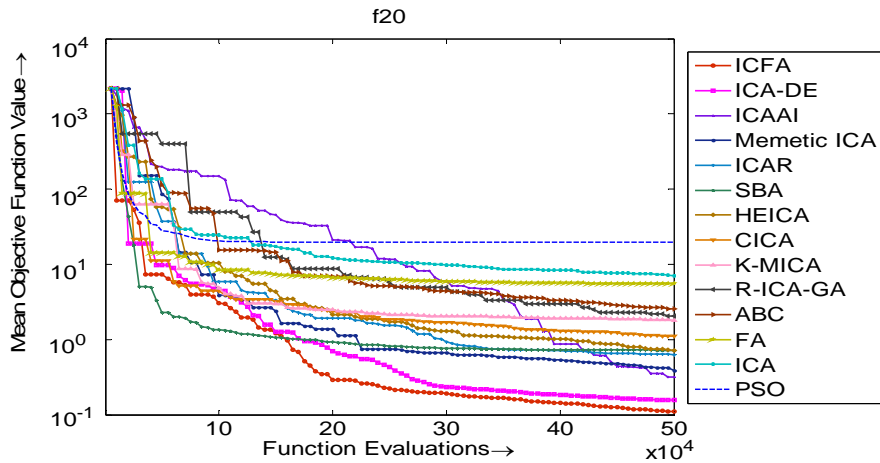


Fig. 5.6 Relative performance in mean best objective function versus function evaluation for f20 with Max\_FEs=500000

In order to compare the speeds of different algorithms, we present the time taken by each algorithm to converge to the prescribed threshold value of the objective function for the minimization problem of 100-D search space in Table 5.2. The simulation results in Table 5.2 apparently substantiate the highest convergence speed of the proposed ICFA as compared to its other thirteen contestants.

To compare the relative speed of convergence and quality of solution (accuracy) of ICFA with its thirteen competitors, we plot the mean objective function values taken over 25 runs versus FEs over four representative benchmark instances (f05, f07, f17 and f20) for different settings of problem dimensions in Fig. 5.3 -5.6. The plots for all the functions are omitted for space economy. In Fig. 5.3 -5.6, the maximum number of function evaluations Max\_FEs is set to be 100000, 300000 and 500000 for 10-D, 30-D and 50-D problems respectively. It is observed from Fig. 5.3 -5.6 that ICFA outperforms all other algorithms in terms of convergence speed and solution quality.

In Fig. 5.4 plot of accuracy versus run-time complexity for all fourteen algorithms over the benchmark function f25 is presented with Max\_FEs of 500000 for 50-D problem. It is to be noted that the accuracy corresponds to absolute difference between the best-of-the-run value  $f(\vec{X}_{best})$  (obtained after the termination of the algorithm) and the theoretical optimum  $f^*$  of a particular objective function, i.e.,  $|f(\vec{X}_{best}) - f^*|$ . This provides a visual means of illustrating the performance of the algorithms with respect to both accuracy and FEs/run-time. In order to have uniformity in order of magnitude, the x- and the y-coordinates are scaled appropriately and then we refer to the distance of a point from the origin as a measure of its performance. The smaller the measure, the better is the performance of the algorithm. The relative performance of two algorithms is symbolized by ‘ $\geq$ ’. Using this convention, Fig. 4.4 reveals that the performance of the fourteen algorithms respectively is  $ICFA \geq ICA-DE \geq ICAAI$

$\geq$ Memetic ICA  $\geq$  ICAR  $\geq$  SBA  $\geq$  HEICA  $\geq$  CICA  $\geq$  K-MICA  $\geq$  R-ICA-GA  $\geq$  ABC  $\geq$  FA  $\geq$  ICA  $\geq$  PSO.

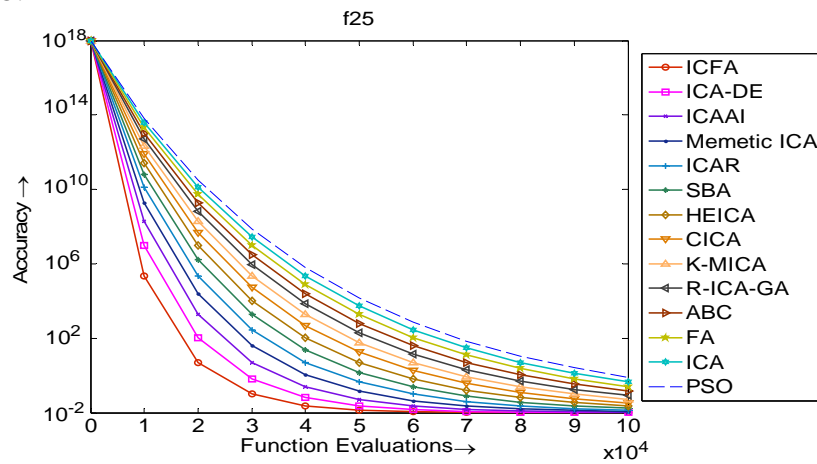
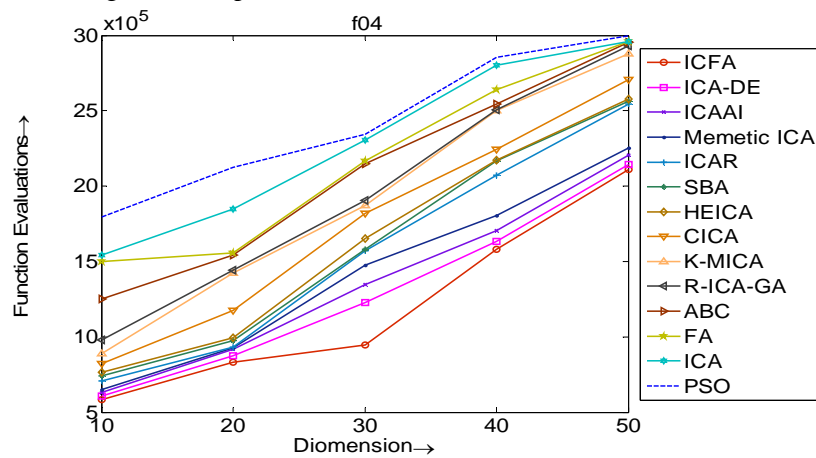


Fig. 5.7 Relative performance in accuracy versus function evaluation for ICFA over other competitive algorithms for f25 with Max\_FEs= $5 \times 10^6$

The scalability of an algorithm signifies the consistency in its qualitative performance with the growth of dimensionality of the search space. Increase of dimensions indicates a rapid growth of the hyper-volume of the search space and this in turn decelerates the convergence speed of most of the global optimizers. Fig. 5.7 illustrates the scalability of the fourteen algorithms over two benchmark functions delineating the variation of the average computational cost (measured in number of FEs required to yield a predefined accuracy threshold) to capture the global optimum with the enhancement in the dimensionality of the search space. It can be observed that ICFA requires smaller number of FEs to achieve the threshold value irrespective of the problem dimensions. Hence the superiority of the proposed ICFA over other algorithms is prominent as well.



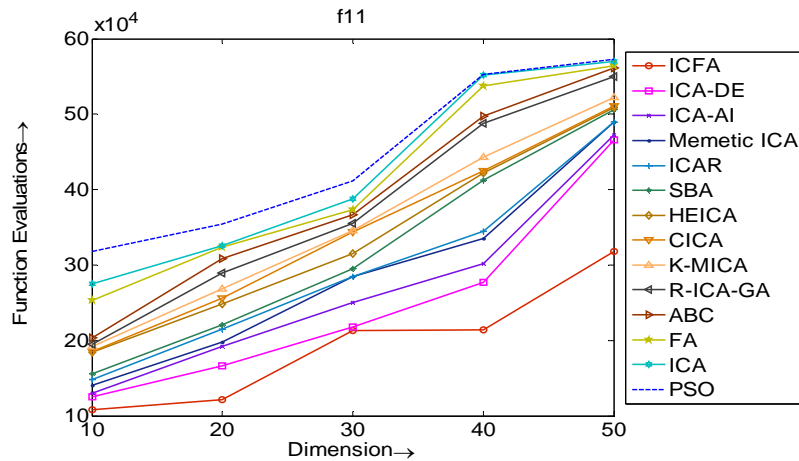


Fig. 5.8 Variation of FEs required for convergence to predefined threshold accuracy ( $1.00e-08$ ) with increase in search space dimensionality for f04 and f11

A non-parametrical statistical test, known as Friedman two-way analysis of variances by ranks [51], is also performed on the mean of the objective function values for 50 independent runs of each of the fourteen algorithms, for 50- $D$  problems. Additionally, we use Iman-Davenport test as a variant of Friedman test that provides better statistics [52]. Table 5.3 summarizes the rankings obtained by Friedman procedure. The results emphasize ICFA as the best algorithm, so the post-hoc analysis [40] is applied with ICFA as the control method. With the level of significance  $\alpha=0.05$ , both the Friedman and the Iman-Davenport statistics explain significant differences in operators with test values of 275.7054 and 20591.88, respectively, and  $p<0.001$  (the estimated probability of rejecting the null hypothesis ( $H_0$ ) of a study question when that hypothesis is true).

In the post-hoc analysis the Bonferroni-Dunn test [53] is employed over the results of Friedman procedure. The outcome of the analysis provides a measure of the level of significance of the superiority of the control algorithm over each of the remaining algorithms (i.e., when the null hypothesis is rejected). For the Bonferroni-Dunn test, a critical difference (CD) [53] is calculated which for these data appears as 2.639. It elucidates that the performance of two algorithms is significantly different, only if their corresponding average Friedman ranks differ by at least a critical difference. It is pictorially depicted in Fig. 5.9 . It can be perceived that only ICA-DE and ICAAI, the null hypothesis cannot be rejected with any of the tests for  $\alpha=0.05$ . The performance of other eleven algorithms, however, may be regarded as significantly poor than the ICFA in the present context.

Table 5.3 AVERAGE RANKINGS OBTAINED THROUGH FRIEDMAN’S TEST

Algorithm	Ranking
ICFA	1.117
ICA-DE	1.883
ICAAI	3.000
Memetic ICA	4.000
ICAR	5.000
SBA	6.000
HEICA	7.000
CICA	8.000
K-MICA	9.000
R-ICA-GA	10.000
ABC	11.000
FA	12.000
ICA	13.000
PSO	14.000
Critical Difference $\alpha=0.05$	2.639

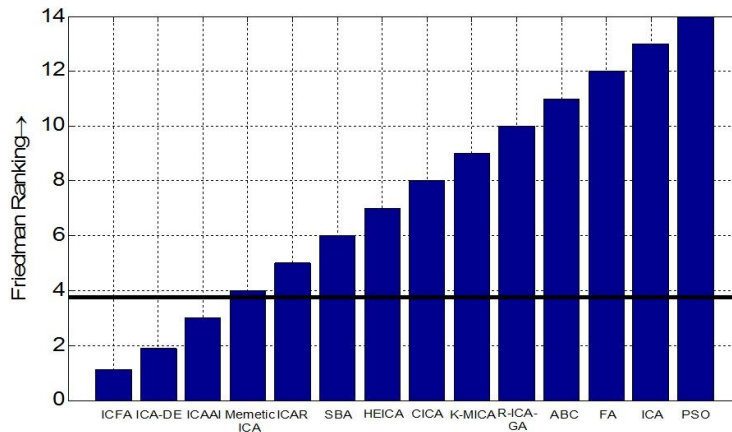


Fig. 5.9 Graphical representation of Bonferroni-Dunn’s procedure considering ICFA as control method

## 5.7 Computer Simulation and Experiment

Here the proposed algorithm is employed to carry a fixed length stick from a starting position to a fixed destination by twin robots (multi-robot stick-carrying problem) both in computer simulation and real-time environment. To analyze the performance of the proposed multi-robot stick-carrying problem, the following performance metrics [69] are used.

### 5.7.1 Average total path deviation (ATPD)

Let  $P_{ik}$  be a path from the starting point  $S_i$  to the goal point  $G_i$  generated by the  $i$ -th robot in the  $k$ -th run of the algorithm. If  $P_{i,1}, P_{i,2}, \dots, P_{i,k}$  are the paths generated over  $k$  runs then the

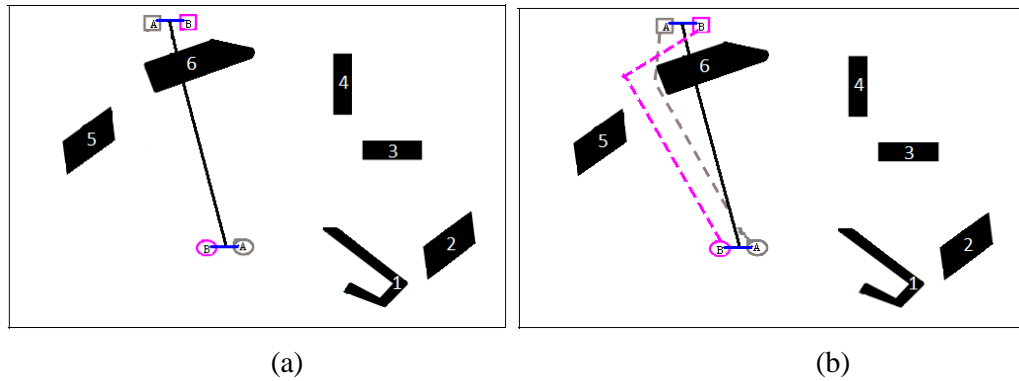
average total path traversed (*ATPT*) by the  $i$ -th robot is given by  $\sum_{j=1}^k P_{i,j} / k$  and the average path deviation (*APD*) is evaluated by measuring the difference between *ATPT* and the ideal shortest path between  $S_i$  to  $G_i$ . If the geometrically ideal path in a particular workspace is  $P_{i-ideal}$ , then the average path deviation is given by  $P_{i-ideal} - \sum_{j=1}^k P_{i,j} / k$ . Therefore for two robots in the workspace the average total path deviation (*ATPD*) is  $\sum_{i=1}^2 (P_{i-ideal} - \sum_{j=1}^k P_{i,j} / k)$ .

### 5.7.2 Average Uncovered Target Distance (AUTD)

Given a goal position  $G_i$  and the current position  $C_i$  of a robot on a 2-dimensional workspace, where  $G_i$  and  $C_i$  are 2-dimensional vectors, the uncovered distance of robot  $i$  is  $\|G_i - C_i\|$ , where  $\|\cdot\|$  denotes the Euclidean norm. For two robots, uncovered target distance  $UTD = \sum_{i=1}^2 \|G_i - C_i\|$ . Now, for  $k$  runs of the program, we evaluate the average of *UTDs* and call it the average uncovered target distance (*AUTD*). For all experiments conducted in this study, we have considered  $k=10$ .

### 5.7.3 Experimental Setup in Simulation Environment

The multi-robot stick-carrying problem is implemented in C on a Pentium processor. The experiment is performed with two similar soft-bots of circular cross-sections of radius 6 pixels and 10 differently shaped obstacles. While performing the experiments, old obstacles are retained and new obstacles are added in the workspace. The experiments are accomplished with equal velocities for two robots in a given run of the program; however, the velocities are regulated over different runs of the same program. Some instances of the workspace of the robots, employing different evolutionary optimization algorithms, are given in Fig. 5.10. It reveals that the robots successfully follow the shortest path with minimum path deviation in case of ICFA-based realization of the multi-robot stick carrying problem.



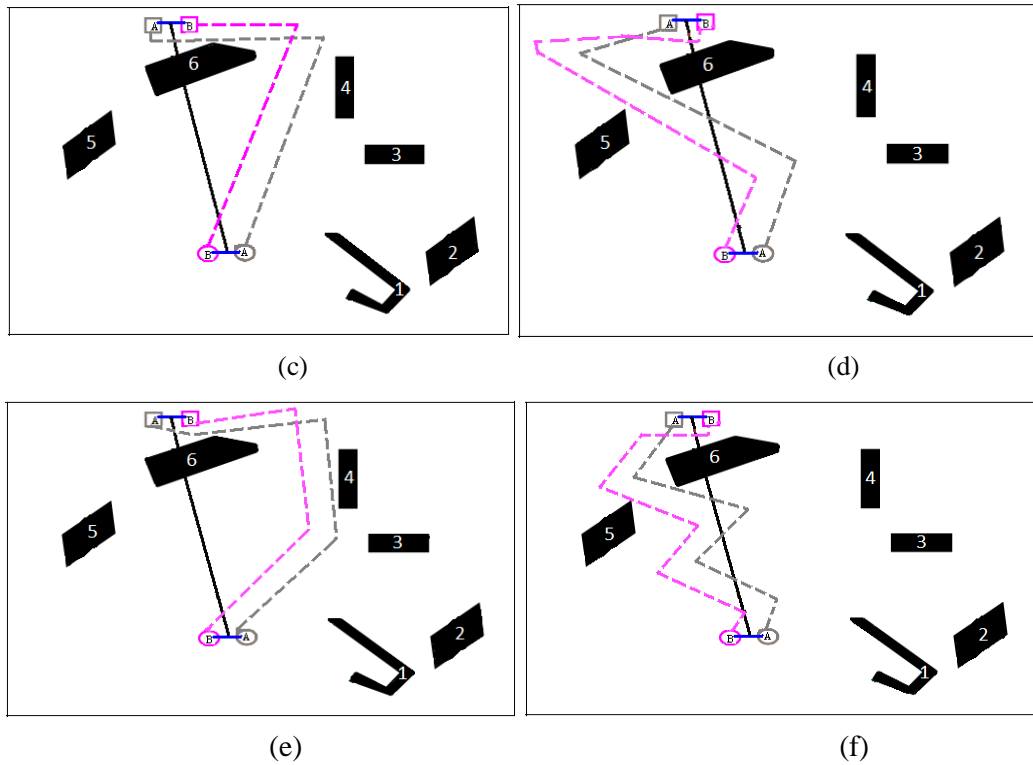


Fig. 5.10 Initial (a) and final configuration of the world-map after execution of the (b) ICFA- (c) ICA-DE- (d) ICAAI- (e) FA- and (f) ICA-based simulations with 5 obstacles requiring 23, 29, 32 and 34 steps respectively

### 5.7.4 Experimental Results in Simulation Environment

First, we plot *ATPT* by varying number of obstacles from 2 to 10 by generating paths using five different algorithms, including ICFA, ICA-DE, ICAAI, FA and ICA. It is worth mentioning from Fig. 5.11, that ICFA has the least *ATPT* in comparison to other algorithms irrespective to the number of obstacles.

The second study on performance analysis is undertaken by plotting *ATPD* by generating paths by five different evolutionary/swarm algorithms (as used in *ATPT*). Fig. 5.11 provides the results of *ATPD* computation when the number of obstacles varies from 2 to 10. Here too, we observe that ICFA outperforms the remaining four algorithms as *ATPD* remains the smallest for ICFA based simulation irrespective to the number of obstacles.

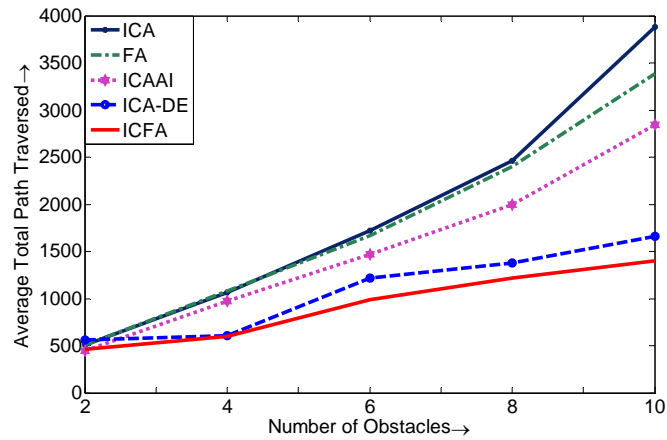


Fig. 5.11 Average total path traversed versus number of obstacles

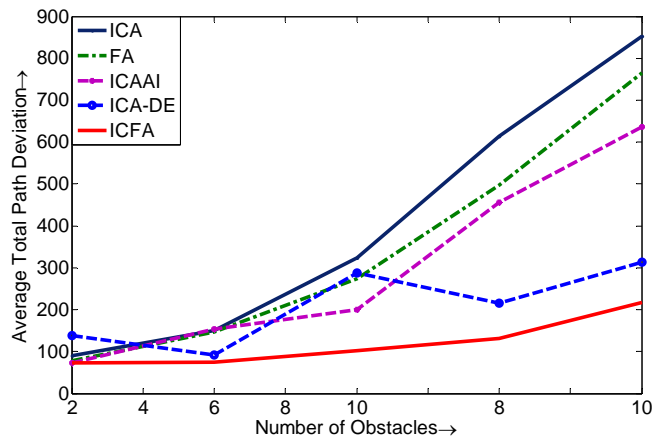


Fig. 5.12 Average total path deviation versus number of obstacles

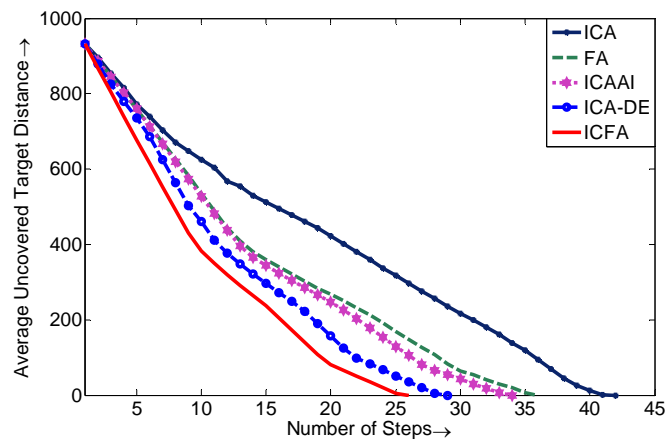


Fig. 5.13 Average uncovered target distance vs. number of steps with number of obstacles= 5 (constant)

The last analysis on performance is undertaken by comparing *AUTD* over the number of planning steps. Fig. 5.13 provides a plot of *AUTD* when the paths are planned using the five

algorithms referred to above with number of obstacles=5. It is apparent from Fig. 5.13, that *AUTD* returns the smallest value for ICFA irrespective of the number of planning steps. In brief, the proposed ICFA-based stick-carrying methodology outperforms the remaining four algorithms with respect to all three popular metrics.

### 5.7.5 Experimental Setup with Khepera Robots

The experiment is also undertaken in real environment with two homogeneous Khepera-II mobile robots [70, 71] (diameter of 7 cm) in a world-map of  $8 \times 6$  grids of equal size. Each robot is equipped with 8 infrared sensors, two motor driven side wheels and one caster wheel with a flash memory of 512 KB, and a Motorola 68331, 25MHz processor. The range sensors placed at fixed angles have limited range detection capabilities. The robot represents measured range data in the scale: [0, 1023]. When the distance of the obstacle from the sensor exceeds 5cm, it is represented by zero. When an obstacle is approximately 2 cm away from the sensor, it corresponds to 1023.

The robots are controlled by two Pentium-IV personal computers (PCs) through wired connections. An optimization algorithm-based control program determining the next position of each robot from its current position is run on the attached Pentium machine with the range data obtained from sensory measurements of the robots. The necessary commands for controlling motor movements are transferred to the robots from their connected computers. One sample run of the stick-carrying problem on the Khepera environment is given in Fig. 5.14. The experiment is performed on 20 different world-maps of different grid counts, each with five different obstacle-maps, and in all the 100 environments, the robots could successfully follow the shortest paths avoiding collision with the obstacles.

### 5.7.6 Experimental Results with Khepera Robots

Results of the experiments performed are summarized in Table 5.4. Three performance metrics, namely 1) total number of steps taken to reach the goal, 2) *ATPT*, and 3) *ATPD* have

Table 5.4 COMPARISON OF NUMBER OF STEPS, AVERAGE PATH TRAVERSED AND AVERAGE TOTAL PATH DEVIATION BY THE KHEPERA ROBOTS

Algorithms	Total Number of Steps	ATPT (inch.)	ATPD (inch.)
ICFA	<b>10</b>	<b>41.2</b>	<b>7.5</b>
ICA-DE	13	44.2	9.5
ICAAI	16	45.7	12.1
FA	18	48.3	13.7
ICA	22	52.0	16.4



been used here too to determine the relative merits of ICFA over other algorithms. Table 5.4 substantiates that ICFA outperforms the remaining four algorithms with respect to all the three metrics.

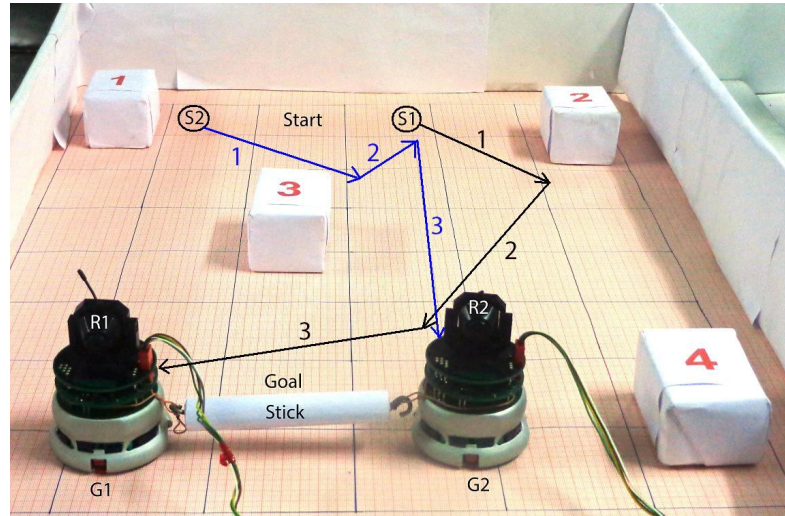


Fig. 5.14 Final configuration of the world-map after experiment using Khepera-II mobile robots

## 5.8 Conclusion

The most intriguing issue of the present chapter is the unique formulation of a multi-robot stick-carrying problem in the framework of an evolutionary optimization problem. The integrity of the work lies in online optimization of the trajectory of the stick (carried by the robots) in a sequence of local steps from the predefined starting position to a given goal position without collision with obstacles in the workspace. The stick-carrying optimization problem is then solved using the proposed hybrid evolutionary optimization algorithm.

The chapter also introduced a novel approach for efficiently employing both ICA and FA to develop a hybrid algorithm with an aim to utilize the composite benefits of the explorative and exploitative capabilities of both ancestor algorithms. The potential of local exploitation is captured by the colonizing behavior of countries (representing candidate solutions) surrounding the respective imperialist (representing the local optima in the search space) of the traditional ICA. Alternatively, the global explorative proficiency of FA is signified by the self-organizing behavior of fireflies (representing candidate solutions) based on their light intensity (symbolizing the fitness) profile. The merit of the proposed hybridization policy lies in devising two interesting stratagems to realize the communal benefits of the two ancestor algorithms: 1) integration of the light intensity (fitness) induced motion dynamics of fireflies in the traditional ICA, and 2) modulation of step size for random motion of fireflies based on the

best position in the search space (discovered so far). The chapter also recommends a new policy of evaluating the threshold value for union of colonies (set of candidate solutions) based on search space dimensions.

The incorporation of firefly motion dynamics of the traditional FA in the traditional ICA is significant due to its efficacy of distributing the information of the promising regions in the search space among the fireflies (candidate solutions) through the brighter ones (quality solutions with better fitness). The strategy thus allows each country (candidate solution) of ICA to improve its socio-political characteristics by following the more powerful countries (quality solutions with better fitness) in the colony, not only being guided by the respective imperialist country (local optima). Hence, it evades the possibility of the search strategy to get stuck at local optima.

The second strategy provides a unique estimate of step size for random movement of fireflies based on their position in the search space relative to the so far best position. It differs from the conventional approach, which considers equal step size for random movement of all fireflies irrespective of their fitness, thereby offering a poor convergence rate in most of the real world applications. In the proposed alternative approach, the step size for random motion is decreased for a brighter firefly (quality solution with better fitness) to confine its search process in close proximity of the best position discovered so far. It thus assists in the local exploitation of quality solutions. Contrarily, the fireflies far away from the best location in the search space are assigned with a large value of step size to inspire them in global exploration.

The experimental study undertaken reveals the effectiveness of the proposed hybrid algorithm in counterbalancing the trade-off between the global exploration and the local exploitation. It is capable of maintaining large population variance to ensure population diversity at earlier explorative phase of ICFA, while confining the search process in the local neighborhoods at later exploitative phase by preserving low population diversity. We have then undertaken a comparative study of the proposed ICFA algorithm with thirteen state-of-the-art hybrid/traditional evolutionary/swarm algorithms. The efficacy of all the fourteen contender algorithms is scrutinized with respect to the test suit of 25 CEC 2005 benchmark functions. The relative performance of all the algorithms has been compared based on the solution quality and the convergence time. The quality performance of ICFA is substantiated by the reported simulation results. The experimental study clearly reveals that ICFA outperforms its competitor algorithms with respect to the computational accuracy and the run-time complexity required for convergence, irrespective of settings of problem dimension.

Three non-parametric tests including the Friedman test, the Iman-Davenport statistic, and the Bonferroni–Dunn post hoc analysis, are used to validate the statistical significance of the

results. The results of both the Friedman and the Iman-Davenport tests affirm the rejection of the null hypothesis, concerned with the equivalent performance of all the contender algorithms. Moreover, ICFA emerges as the winner achieving the highest average Friedman rank. The outcome of the Bonferroni-Dunn test further reveals that, apart from ICA-DE and ICAAI, the remaining eleven algorithms are outperformed by ICFA in a statistically significant manner.

Finally, the proposed ICFA is employed to handle the multi-robot stick-carrying problem. The experiments undertaken reveal that the ICFA-based program here too outperforms all its competitors with respect to two parameters AUTD and ATPD. The experiments performed with Khepera-II mobile robots also indicate that ICFA outperforms other realizations in real environment, thereby justifying the merit of the proposed algorithm.

A potential extension of the proposed multi-robot stick-carrying problem is to consider noise contaminating the sensory measurements of the robots. In real world problems, the sensory data of the robots are often found to be contaminated with noise due to sensor aging or noisy ambience or faulty measurement procedure. The application of traditional/hybrid evolutionary/swarm optimization algorithms may fail to solve such practical multi-robot coordination problems. The existing algorithms are biased towards the selection of candidate solutions with better fitness measures over evolutionary generations. However, this conventional fitness based selection of candidate solutions may lead the search process towards sub-optimal or deceptive regions in the search space in the presence of noise. Hence, new robust selection strategy needs to be incorporated in the traditional/hybrid evolutionary/swarm algorithms to cope with the uncertainty involved in the noisy sensory data of the robots. Although the quality performance of ICFA is evident on different complicated fitness landscapes, there is still scope to further amend its effectiveness to capture the global optima in the real world multi-modal objective surface. It can be accomplished by the online tuning of the algorithm control parameters to learn the objective space characteristics using, for instance, machine-learning methods.

## **5.9 Summary**

Chapter 5 hybridizes the Firefly Algorithm and the Imperialist Competitive Algorithm. The above explained hybridization results in the Imperialist Competitive Firefly Algorithm, which is employed to determine the time-optimal trajectory of a stick, being carried by two robots, from a given starting position to a predefined goal position amidst static obstacles in a robot world-map. The motion dynamics of fireflies of the Firefly Algorithm is embedded into the socio-political evolution-based meta-heuristic Imperialist Competitive Algorithm. Also the trade-off between the exploration and exploitation is balanced by modifying the random walk strategy based on the position of the candidate solutions in the search space. The superiority

of the proposed Imperialist Competitive Firefly Algorithm is studied considering run-time and accuracy as the performance metrics. Finally, the proposed algorithm has been verified in real-time multi-robot stick-carrying problem.

## 5.10 Appendix 5.1

In order to compare the speeds of different competitive algorithms listed under section 5.6.1, we record the number of FEs the algorithm takes to reach below a predefined cut-off value of the objective function for the minimization problem. A lower number of FEs corresponds to a faster algorithm. Table 5.A.1 details the number of runs (out of 25) that successfully locate the optimum solution (within the given tolerance) as well as the success performance attained by the algorithms to converge within the prescribed threshold value. Table 5.A.1. A designates that the number of runs that converge below a pre-specified threshold value is also greater for ICFA over most of the benchmark problems considered here. This indicates the higher robustness of the algorithm as compared to its other thirteen contestants.

Table 5.A.1 NO OF SUCCESSFUL RUNS OUT OF 25 RUNS AND SUCCESS PERFORMANCE IN PARENTHESIS (SUCCESS PERFORMANCE=MEAN (FES FOR SUCCESSFUL RUNS)\*(# OF TOTAL RUNS) / (# OF SUCCESSFUL RUNS)) FOR f01 TO f25

Functions	Tolerance	ICFA	ICA-DE	ICAAI	Memetic ICA	ICAR	SBA	HEICA	CICA	K-MICA	R-ICA-GA	ABC	FA	ICA	PSO
f01	1.00e-18	25	25	25	24	24	24	23	23	22	21	21	21	21	20
		(2.42e+03)	(2.72e+03)	(1.06e+04)	(1.34e+04)	(2.09e+04)	(2.72e+04)	(4.03e+04)	(4.58e+04)	(5.13e+04)	(5.53e+04)	(5.54e+04)	(5.65e+04)	(6.14e+04)	(6.31e+04)
f02	2.00e-14	24	24	23	23	22	22	22	21	21	21	21	20	19	18
		(4.02e+03)	(5.31e+03)	(2.40e+04)	(2.57e+04)	(2.88e+04)	(3.23e+04)	(3.48e+04)	(3.75e+04)	(4.16e+04)	(4.49e+04)	(4.75e+04)	(5.50e+04)	(5.55e+04)	(5.72e+04)
f03	4.00e-02	25	24	24	24	24	24	23	23	22	21	21	21	21	21
		(4.25e+03)	(6.53e+03)	(1.02e+04)	(1.03e+04)	(1.03e+04)	(1.80e+04)	(2.66e+04)	(2.77e+04)	(2.87e+04)	(3.44e+04)	(4.93e+04)	(5.08e+04)	(1.85e+05)	(5.82e+04)
f04	2.00e-04	25	24	24	24	24	23	23	23	21	21	20	19	19	18
		(2.18e+03)	(2.24e+03)	(1.08e+04)	(1.36e+04)	(1.74e+04)	(1.96e+04)	(2.83e+04)	(3.36e+04)	(4.19e+04)	(4.36e+04)	(5.77e+04)	(5.80e+04)	(5.83e+04)	(6.16e+04)
f05	1.00e-10	25	25	24	24	23	23	23	23	22	22	21	21	21	21
		(1.75e+02)	(2.06e+02)	(1.10e+03)	(1.77e+03)	(2.29e+03)	(2.99e+03)	(3.04e+03)	(3.74e+03)	(3.85e+03)	(4.20e+03)	(5.27e+03)	(5.85e+03)	(7.56e+03)	(6.29e+03)
f06	4.00e-02	25	24	23	23	22	22	22	21	21	21	20	20	19	19
		(3.23e+03)	(3.64e+03)	(1.81e+04)	(2.50e+04)	(2.98e+04)	(3.58e+04)	(3.93e+04)	(4.79e+04)	(5.50e+04)	(6.10e+04)	(6.47e+04)	(6.50e+04)	(8.60e+04)	(6.60e+04)
f07	3.00e-05	25	24	24	24	23	23	23	22	22	21	21	20	19	18
		(5.22e+02)	(5.26e+02)	(1.63e+03)	(1.89e+03)	(2.80e+03)	(3.00e+03)	(3.10e+03)	(3.65e+03)	(3.87e+03)	(4.42e+03)	(5.21e+03)	(5.83e+03)	(6.36e+03)	(8.83e+03)
f08	2.00e-01	25	25	25	25	25	23	23	22	22	21	20	20	19	19
		(7.60e+03)	(8.09e+03)	(1.84e+04)	(2.13e+04)	(2.73e+04)	(3.71e+04)	(4.37e+04)	(4.48e+04)	(4.89e+04)	(4.90e+04)	(4.97e+04)	(5.18e+04)	(5.61e+04)	(5.21e+04)
f09	2.00e-02	24	24	23	23	22	22	21	21	20	20	20	19	19	19
		(1.90e+02)	(2.49e+02)	(1.04e+03)	(2.02e+03)	(3.49e+03)	(3.60e+03)	(4.73e+03)	(4.75e+03)	(4.87e+03)	(5.48e+03)	(6.20e+03)	(6.31e+03)	(6.62e+03)	(6.83e+03)
f10	3.00e-03	24	24	24	24	23	23	22	21	21	20	19	19	18	19
		(3.19e+02)	(3.36e+02)	(1.15e+03)	(1.60e+03)	(2.34e+03)	(2.43e+03)	(3.81e+03)	(4.76e+03)	(5.66e+03)	(5.75e+03)	(1.29e+04)	(2.30e+04)	(4.68e+04)	(2.47e+04)
f11	1.00e-10	25	25	25	25	24	24	23	23	23	21	20	20	20	20
		(3.09e+03)	(4.79e+03)	(1.18e+04)	(1.44e+04)	(2.06e+04)	(3.09e+04)	(3.29e+04)	(3.72e+04)	(3.88e+04)	(4.43e+04)	(4.61e+04)	(4.86e+04)	(6.20e+04)	(7.05e+04)
f12	8.00e-02	25	25	25	24	24	23	22	22	22	21	21	21	20	20
		(1.61e+03)	(1.81e+03)	(1.80e+04)	(2.10e+04)	(2.36e+04)	(3.28e+04)	(3.46e+04)	(4.09e+04)	(4.80e+04)	(5.12e+04)	(5.21e+04)	(5.22e+04)	(5.66e+04)	(5.80e+04)
f13	9.00e-02	25	25	23	22	21	21	20	20	20	19	19	19	19	19
		(8.00e+03)	(8.05e+03)	(1.42e+04)	(1.72e+04)	(2.51e+04)	(2.75e+04)	(3.56e+04)	(3.60e+04)	(3.65e+04)	(3.81e+04)	(4.37e+04)	(4.76e+04)	(5.84e+04)	(5.97e+04)

Functions	Tolerance	ICFA	ICA-DE	ICAAI	Memetic ICA	ICAR	SBA	HEICA	CICA	K-MICA	R-ICA-GA	ABC	FA	ICA	PSO
f14	7.00e-04	<b>25</b> (4.62e+02)	25 (5.93e+02)	24 (1.75e+03)	24 (2.76e+03)	24 (2.78e+03)	23 (3.07e+03)	23 (3.08e+03)	23 (3.13e+03)	22 (3.43e+03)	22 (3.65e+03)	21 (3.83e+03)	20 (4.46e+03)	20 (4.69e+03)	18 (5.25e+03)
f15	7.00e-01	<b>25</b> (6.00e+02)	25 (7.43e+02)	25 (1.60e+03)	25 (1.92e+03)	24 (1.10e+04)	24 (1.21e+04)	24 (1.99e+04)	23 (3.49e+04)	22 (4.31e+04)	22 (4.55e+04)	22 (4.91e+04)	21 (5.72e+04)	21 (5.81e+04)	21 (6.56e+04)
f16	9.00e-03	<b>24</b> (2.45e+03)	24 (3.62e+03)	23 (1.67e+03)	23 (1.78e+03)	22 (1.27e+04)	22 (4.18e+04)	21 (4.27e+04)	21 (4.91e+04)	21 (5.11e+04)	20 (5.18e+04)	20 (5.68e+04)	20 (5.70e+04)	19 (5.78e+04)	19 (7.02e+04)
f17	1.00e-01	<b>25</b> (1.36e+03)	25 (1.49e+03)	23 (3.32e+03)	23 (3.42e+03)	22 (3.82e+03)	21 (4.08e+03)	21 (4.24e+03)	21 (5.22e+03)	21 (5.32e+03)	20 (5.49e+03)	20 (5.71e+04)	19 (6.01e+04)	19 (6.80e+04)	19 (7.04e+04)
f18	2.00e-01	<b>25</b> (2.53e+03)	25 (2.53e+03)	24 (1.18e+04)	24 (1.28e+04)	23 (1.97e+04)	23 (2.19e+04)	21 (2.55e+04)	21 (3.33e+04)	20 (4.09e+04)	20 (5.57e+04)	20 (6.21e+04)	19 (6.53e+04)	19 (7.50e+04)	17 (8.76e+04)
f19	3.00e-02	<b>24</b> (7.53e+03)	24 (9.86e+03)	24 (1.92e+04)	23 (1.09e+04)	23 (1.91e+04)	23 (2.27e+05)	22 (2.51e+05)	21 (3.04e+05)	20 (3.40e+05)	20 (3.89e+05)	20 (4.89e+05)	19 (5.15e+05)	19 (5.82e+05)	18 (6.92e+05)
f20	2.00e+00	<b>25</b> (8.40e+03)	24 (9.09e+03)	23 (1.79e+04)	22 (1.99e+04)	22 (2.36e+04)	22 (2.37e+04)	21 (2.64e+04)	21 (4.23e+04)	21 (5.24e+04)	19 (5.72e+04)	19 (5.95e+04)	19 (6.28e+04)	19 (6.49e+04)	18 (6.80e+04)
f21	4.00e+01	<b>25</b> (4.19e+03)	25 (5.54e+03)	24 (6.87e+03)	24 (8.90e+03)	24 (2.04e+04)	23 (2.37e+04)	22 (3.75e+04)	22 (4.54e+04)	21 (4.64e+04)	21 (5.08e+04)	21 (5.58e+04)	21 (5.80e+04)	21 (7.00e+04)	20 (9.33e+04)
f22	2.00e+00	<b>24</b> (7.22e+02)	24 (8.52e+02)	24 (8.63e+03)	23 (9.83e+03)	23 (1.00e+04)	22 (1.27e+04)	22 (1.93e+04)	21 (3.51e+04)	21 (4.45e+04)	19 (4.46e+04)	19 (5.07e+04)	19 (1.20e+05)	18 (1.59e+05)	18 (2.79e+05)
f23	7.00e+01	<b>24</b> (2.00e+02)	24 (2.51e+02)	24 (2.72e+02)	23 (2.95e+02)	23 (3.26e+02)	22 (3.99e+02)	22 (5.18e+02)	21 (5.30e+02)	20 (5.38e+02)	20 (5.50e+02)	20 (5.72e+02)	20 (6.60e+02)	19 (2.72e+03)	18 (4.86e+03)
f24	2.00e+02	<b>24</b> (5.04e+02)	24 (7.47e+02)	24 (1.52e+03)	23 (1.60e+03)	23 (1.73e+03)	22 (2.89e+03)	22 (3.88e+03)	20 (4.16e+03)	20 (4.62e+03)	20 (5.36e+03)	20 (5.48e+04)	20 (1.92e+04)	20 (2.08e+04)	19 (2.31e+04)
f25	2.00e+02	<b>24</b> (2.47e+02)	23 (4.93e+02)	23 (1.94e+03)	23 (2.24e+03)	22 (3.06e+03)	22 (3.67e+03)	22 (4.45e+03)	22 (5.64e+03)	21 (5.75e+03)	20 (1.60e+04)	20 (1.62e+04)	19 (1.67e+04)	17 (2.76e+04)	17 (3.91e+04)

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## **Chapter 6**

# **Conclusions and Future Directions**

This chapter concludes the thesis. Here novelties, originality of thesis are reclaimed and the future research directions are indicated.

## 6.1 Conclusions

The thesis identifies a few fundamental problems in multi-robot coordination and proposes solutions to handle these problems by extending the traditional evolutionary algorithm (EA) and multi-agent Q-learning (MAQL). Chapter 1 provides the preliminaries of Reinforcement Learning (RL) and EA in view of the multi-robot coordination. Chapter 2 proposes two useful characteristic properties for exploration of the team-goal and joint action selection in multi-agent system. The incorporation of the first property with traditional MAQL (TMAQL) ensures exploration of the team-goal by multi-phased transitions of the agents asynchronously or synchronously to finally reach the team-goal, and thereby offer high reward values to such pre-goal state to the goal state transitions. The second property helps in identifying common preferred joint actions for the entire team, thus avoiding same joint actions at the same states and thereby enhancing the learning speed of the agents. The Q-table obtained in joint state-action space using the proposed fast cooperative multi-agent Q-learning (FCMQL) algorithms have been employed in the multi-agent planning algorithm to autonomously select goal state-transitions (team-goal) from the pre-goal states based on their high reward values stored in the Q-table. TMAQL induced planners occasionally fail to reach the team-goal as such state-transitions which might result in due to follow-up actions of Property 1 in FCMQL are missing from the Q-table obtained by TMAQL. It is shown in a Theorem, that the expected convergence time of the proposed FCMQL algorithms is less than the same of TMAQL algorithms. The complexity analysis reveals the superiority of the proposed FCMQL algorithms over the TMAQL algorithms.

Chapter 3 proposes a novel Consensus Q-learning (CoQL) algorithm for multi-robot cooperative planning. The proposed CoQL algorithm addresses the problem of equilibrium selection among multiple equilibria, by evaluating the consensus (joint action) at the current joint state. An analysis reveals that a consensus at a joint state is a pure strategy Nash equilibrium (NE) as well as pure strategy correlated equilibrium (CE). The novelty of the CoQL lies in the adaption of the joint Q-values at consensus. The superiority of the proposed CoQL algorithm is verified over the reference algorithms in terms of the average of the average rewards (AAR) earned by the agents against the learning epoch. In addition, consensus-based multi-robot cooperative planning algorithm is proposed and its superiority is verified over reference algorithms, considering path length and torque requirement as the performance metrics.

Chapter 4 introduces a novel approach to correlated Q-learning (CQL) and subsequent multi-robot planning. Two models are proposed in chapter 4. The principles adapted in the proposed models yield a single Q-table in joint state-action space, which contains sufficient information to plan by employing the proposed multi-agent planning algorithms. The Q-table

obtained from model-I and II have less computational cost than the traditional CQL. An analysis reveals that both time- and space- complexities of proposed learning and planning algorithms are significantly less to those of the CQL. A further reduction in complexity is obtained by dropping the infeasible joint state-action pairs from the joint Q-table. Unlike traditional CQL, in the proposed models computation of the CE is done partly in the learning and partly in the planning phases, thereby requiring CE computation once. It has been proved in a Theorem, that the CE obtained by the proposed models is same as that obtained by the traditional CQL algorithms.

Chapter 5 introduces a novel approach for efficiently employing both imperlistic competitive algorithm (ICA) and firefly algorithm (FA) to develop a hybrid algorithm with an aim to utilize the composite benefits of the explorative and exploitative capabilities of both ancestor algorithms. The potential of local exploitation is captured by the colonizing behavior of countries (representing candidate solutions) surrounding the respective imperialist (representing the local optima in the search space) of the traditional ICA. Alternatively, the global explorative proficiency of FA is signified by the self-organizing behavior of fireflies (representing candidate solutions) based on their light intensity (symbolizing the fitness) profile. The merit of the proposed hybridization policy lies in devising two interesting stratagems to realize the communal benefits of the two ancestor algorithms: 1) integration of the light intensity (fitness) induced motion dynamics of fireflies in the traditional ICA, and 2) modulation of step size for random motion of fireflies based on the best position in the search space (discovered so far). The chapter also recommends a new policy of evaluating the threshold value for union of colonies (set of candidate solutions) based on search space dimensions.

Finally, all the proposed learning and planning algorithms are verified first in simulation and then are implemented for real-time planning using Khepera mobile robots. The proposed learning-based planning and ICFA is employed to handle the multi-robot object-transportation tasks. The experiments performed with Khepera mobile robots also indicate that the proposed algorithms outperform other realizations in real environment, thereby justifying the merit of the proposed algorithms.

## **6.2 Future directions**

Cooperative robots have wide applications in flexible manufacturing systems (FMS) and factory automation, where the servicing robots picks up items from the conveyer and places the items again once the operation on the item by the servicing robot is over. In defense sector, cooperative robots would find applications in landmine/water mine clearing. In building construction/repair, robot team is a good choice as the skyscrapers invite high risks

for the masons or their assistants. We hope for the best, when a pair of robots might serve as a surgeon and nurse, where the latter may assist the former in surgery.

The above dreams will be realized in near future by extending the thesis in the following dimension: i) multi-agent Fuzzy-Q learning, ii) multi-agent reinforcement learning employing function approximation techniques, iii) multi-agent reinforcement learning for distributed Q-learning with the flavor of partially observable Markov decision process, iv) optimal trajectory generation in presence of dynamic obstacles employing PrEference Appraisal Reinforcement Learning, v) efficient strategies for mixed coordination and vi) deep reinforcement learning.