Ref. No.: Ex/PHY/TG/102/20/2019

MASTER OF SCIENCE EXAMINATION, 2019

(1st Year, 1st Semester)

PHYSICS

Mathematical Methods - I

Paper - PHY/TG/102

Time : Two hours **Full Marks : 40**

Answer ***Q.no. 1*** and any ***two*** from the rest.

1(a). Evaluate $\oint_\Gamma \bar{z}\, dz$ and $\oint_\Gamma z^{\frac{1}{2}}\, dz$ (consider the Principal Branch of $z^{\frac{1}{2}}$) in the two cases, when: (i) Γ is the circle $|z| = 1$, and (ii) Γ is the circle $|z-1| = 1$.

1(b). Let f have an isolated singularity at the point $z = z_0$. In other words, f is analytic in a punctured neighbourhood of the point $z = z_0$. Then show that the residue of $f'(z)$ at $z = z_0$ is *zero*.

1(c). Define an *analytic* (holomorphic) function. Examine whether the following functions are analytic: (i) Im z , (ii) $|z|^2$, (iii) $\ln|z|$.

[6 + 4 + 6]

2(a). Evaluate the integral:

$$\int_0^\infty \frac{\sin^3 x}{x^3}\, dx \qquad OR \qquad \int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}}$$

2(b). Let Γ be a smooth, simply-connected closed contour, and let $f(z)$ be analytic inside and and on Γ, with the exception of (may be) a finite number of points inside of Γ . Then show that

$$\oint_{\Gamma} \frac{dz}{2\pi i} \left(\frac{f'(z)}{f(z)} \right) = \# \text{ Zeros of } f - \# \text{ Poles of } f, \text{ (counted with multiplicity)}$$

[6 + 6]

3(a). What are the Cauchy-Riemann conditions and what are they useful for? Find all $v(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, such that for $z = x + iy$, the function: $f(z) = (x^3 - 3xy^2) + iv(x, y)$ is analytic.

3(b). Identify (with proper mathematical justification) the type of singularity of the function $z^2 e^{1/z}$ at $z = 0$.

3(c). Compute the residue(s) of the following function at all its poles, including infinity.

$$f(z) = \left(\frac{z^{n-1}}{z^n - 1} \right)$$

[4 + 2 + 6]

4(a). Evaluate the integral (any ONE):

$$\int_0^\infty \frac{dx}{(1 + x^n)} \qquad OR \qquad \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2} dx \qquad (a > 0).$$

4(b). Let $f(z) = \frac{z^5}{|z|^4}$ $(z \neq 0)$, and $f(0) = 0$. Show that the real and imaginary parts of f does satisfy Cauchy-Riemann equations at $z = 0$, yet f is *not* analytic at $z = 0$! Why?

4(c). Let $\mathcal{T}$ be the linear operator on $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by

$$\mathcal{T}(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is $\mathcal{T}$ invertible? If so, find a rule for the action (analogous to the above formulation) of $\mathcal{T}^{-1}$. Hence or otherwise, show that $\mathcal{T}$ also satisfies the equation:

$$(\mathcal{T}^2 - \mathbb{I})(\mathcal{T} - 3\mathbb{I}) = 0$$

[3 + 4 + 5]

5(a). Of which function is the Laplace transform given by ? .

$$\frac{1}{(s^2+1)(s-1)}$$

5(b). Given two functions $f_1(x)$ and $f_2(x)$, their *convolution* $*$ is defined by:

$$g(x) := (f_1 * f_2)(x) = \int_{-\infty}^{\infty} dy\, f_1(y)\, f_2(x-y)$$

Show that the Fourier transform $\mathcal{F}$ for the convolution satisfies the simple product rule:

$$\mathcal{F}[g(x)] = \mathcal{F}[f_1(x)] \cdot \mathcal{F}[f_2(x)] \times \text{constant}$$

where the "constant" factor depends on the normalization conventions chosen.

5(c). Let $F(s)$ denote the Laplace transform of a real-valued function $f(x)$. Then show that the formula for the Laplace transform ($\mathcal{L}$) of $f''(x)$ can be written as:

$$\mathcal{L}[f''(x)] = s^2F(s) - sf(0) - f'(0).$$

[4 + 4 + 4]