Master of Science Examination, 2019

(2nd Year, 2nd Semester)

Mathematics

Unit - 4.1

Advanced Functional Analysis

Time : Two hours | Full Marks : 50

Answer question No. 1 and ***any three*** from the rest.

1. Is the topological vector space boundedness equivalent to metric boundedness for a metrizable topological vector space ? Justify your answer. 2

2. a) In a topological vector space X, for any $A \subset X$ prove that $\overline{A} = \bigcap (A+V)$, where V runs over all utds of θ. 3

   b) Define a Balanced subset of a topological vector space X. If B is balanced then show that $\vec{B}$ is also so and if also $\theta \in B^{o}$ then B^{o} is also balanced. 5

   c) Let X be a complex topological vector space, Y be a subspace of X, n be a positive integer and dim Y = n. Then prove that (i) every isomorphism of C^n onto Y is a homeomorphism and (ii) Y is closed. 8

3. a) Prove that every locally compact topological vector space has finite dimension. 7

[Turn over

b) If X is a topological vector space with a countable local base then show that there is a metric d on X such that d is compatible with the topology of X, the open balls centered at are balanced and d is invariant. 9

4. I. When a linear map $\Delta : X \to Y$, where X and Y are topological vector spaces, called bounded ? For a linear map Δ prove that (a) $\to$ (b) $\to$ (c), where

a) Δ is continuous

b) Δ is bounded

c) If $x_n \to \theta_x$, then $\{\Delta x_n : n = 1, 2, \cdots\}$ is bounded. 5

II. If X is metrizable then show that (c) $\to$ (a). 3

III. Let A be a convex absorbing set in a topological vector space X. Then show that

i) $\mu_A(x+y) \le \mu_A(x) + \mu_A(y)$

ii) $\mu_A(tx) = t\mu_A(x)$ if $t \ge 0$

iii) μ_A is a seminorm is A is balanced.

iv) If $B = \{x : \mu_A(x) < 1\}$ and $C = \{x : \mu_A(x) \le 1)$ then $B \subset A \subset C$ and $\mu_B = \mu_A = \mu_C$. 8

5. a) Suppose Y is a subspace of a topological vector space X and Y is an F-space in the topology inherited from X. Then prove that Y is a closed subspace of X. 7

b) Prove that a topological vector space X is normable iff its origin has a convex bounded neighborhood. 4

c) Let X and Y be topological vector spaces, $\{\Delta_n\}$ is a sequence of continuous linear maps of X into Y. If C is the set of all $x \in X$ for which $\{\Delta_n x\}$ is a Cauchy sequence in Y and C is of 2nd category then prove that C = X. 5

6. a) Let A and B be disjoint nonemply convex sets in a locally convex topological vector space X where A is compact and B is closed. Then prove that there is a $\Delta \in X^*$, $r_1, r_2 \in R$ such that

$$\operatorname{Re}\Delta(x) < r_1 < r_2 < \operatorname{Re}\Delta y$$

$\forall x \in A$ and $\forall y \in B$. 8

b) If f is a continuous linear functional on a subspace M of a locally convex space X then prove that there exists $\Delta \in X^*$ such that $\Delta = f$ on M. 8