Ex./M.Sc./M/A-1.1/35/2019

**MASTER OF SCIENCE EXAMINATION, 2019**

**(2nd Year, 1st Semester)**

**MATHEMATICS**

**Advanced Algebra**

**Paper : - 3.3, (A - 1.1)**

Time : Two hours | Full Marks : 50

(Unexplained Notations, Symbols and terms have their usual meaning)

Answer ***Q.no.1*** and any ***four*** from the rest.

1. Answer any ***five*** : 2x5=10

(a) $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module but its submodule {0, 2, 4} is not free – Explain. What is illustrated in this statement ?

(b) Give an example (with explanation) to illustrate that $\sqrt{Q}$ is a prime ideal but Q is not a primary ideal.

(c) $\mathbb{Q}$ (the field of rational numbers) is not an integral extension of $\mathbb{Z}$ – Justify.

(d) Suppose A is a torsion abelian group and Q is the additive group of rationals. Then $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ – Explain.

(Turn over)

(e) For a prime number p, $\mathbb{Z}_{(p)}$ is the valuation ring of the p-adic valuation of $\otimes$ – Justify.

(f) Injective $\mathbb{Z}$-modules are precisely the divisible abelian groups – Explain.

(g) In the ring $\mathbb{R}$[x,y],
$(x^2,xy) = (x) \cap (y,x)^2$ and $(x^2, xy) = (x) \cap (x^2,y)$ are two minimal primary decompositions of the ideal $(x^2, xy)$. Does it violate the Primary Decomposition theorem ?

2. (a) Let R be a commutative ring with identity. Let M be a maximal ideal of R and Q be an ideal of R such that $M^n \subseteq Q \subseteq M$ for same n $\geq 1$. Prove that Q is a primary ideal and $\sqrt{Q} = M$ .

(b) Let R be a commutative ring with identity and N be a primary submodule of a unitary R-module M. Prove that the annihilator Q(say) of the quotient R-module M/N is a primary ideal of R. Also determine the associated prime ideal P(say). How do we express the relation between N and P ? 4+(3+2+1)



(ii) the canonical map from R to $R_P$ induces an injection from the integral domain $R/P$ to the field $R_P/P^e$ .

(iii) $R_P/P^e$ is isomorphic to the field of fractions of $R/P$ .

(b) What is the relation between $\mathbb{Z}_p$ and $\mathbb{Z}_{(p)}$, for any prime p ?

(c) What is meant by a local property of a commutative ring with identify ?

(d) Prove that a discrete valuation ring is a local ring. (2+1+3)+1+1+2

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5. (a) i) State Noether Normalization lemma.

ii) State and prove the Weak form of Hilbert Nullstellensatz.

(b) Prove that an integral extension is preserved by localization. (2+5)+3

6. (a) (i) State the Lying over theorem.

(ii) State and prove the Going-up theorem.

(b) Prove that any UFD is integrally closed

(c) Lt $\{\nu_1, \nu_2\}$ be a basis of the vector space $\mathbb{R}^2$. Show that the element $\nu_1 \otimes \nu_2 + \nu_2 \otimes \nu_1$ of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ cannot be written as a simple tensor $\nu \otimes \omega$ for any $\upsilon, \omega, \in \mathbb{R}^2$. (1+5)+2+2

7. (a) Let R be a commutative ring with identity. Let $R_P$ be the localization of R at the prime ideal P of R and $P^e$ be extension of P to $R_P$. Prove that

(i) $R_P$ is a local ring with $P^e$ as the unique maximal ideal.



3. (a) Suppose R is a ring with identity, M is a right R-module and N is a left R-module. Define the tensor product of M and N

(i) using the universal mapping properly,

(ii) using the constructive method

(b) Write three equivalent conditions (for each) for a module to be

(i) projective, (ii) injective, (iii) flat.

(c) Determine (specifying the relevant results) if the $\mathbb{Z}$-module $\not\otimes \oplus \mathbb{Z}$ is

(i) projective, (ii) injective, (iii) flat. (2+2)+3+3

4. (a) Let R be a commutative ring with identity and J be its Jacobson radical. Prove that $x \in J$ if and only if $1 - rx$ is a unit for all $r \in R$.

(b) Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at (p). Find the nil radical and the Jacobson radical of $\pi_{(p)}$.

(c) Suppose R is a commutative Noetherian ring with identity. Prove that the Zariski topology on R is discrete if and only if R is Artinian. 3+2+5

(Turn over)