Ex/UNIT4.3-A2.1/2019

MASTER OF SCIENCE EXAMINATION, 2019

(2nd Year, 2nd Semester)

UNIT – 4.3 (A-2.1)

ADVANCED ALGEBRA - II

Time : Two hours Full Marks : 50

Answer **questions No. 1** and any ***four*** from the rest.

[Unexplained symbols and notations have their usual meanings unless otherwise mentioned R-module means left R-module]

1. Answer ***any five*** questions from the following : 2×5=10

   a) Suppose M is a simple R-module. Then the ring $R/_{\mathrm{Ann}_R(M)}$ is primitive – Explain !

   b) The Jacobson radical J(R) of a ring R contains all nil right or left ideals of R – Explain !

   c) Let V be an infinite dimensional vector space over a field F. Then the ring L(V) of all linear operators on V cannot be left Artinian – Justify !

   d) Let V be a vector space over a field F and T be a linear operator on V. Then V is an F[x]-module via T. Suppose V is a semisimple F[x] - module. What does it mean in terms of V as a vector space over F ?

   e) i) The character table of a finite group is a square matrix – Justify !

[Turn over

ii) The group $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ does not have any faithful irreducible representations over the field $\mathbb{C}$ of complex numbers. Justify.

f) Every primitive ring is Jacobson semisimple. – Justify ! What can you say about the converse ? Answer with justification.

g) What is meant by a tensor of type (p, q) where p, q are nonnegative integers ?

Tensors of type (p, q) can also be considered as suitable multilinear functionals – Explain !

2. Let R be a ring having at least one faithful simple left module and at least one faithful simple right module. Then any one of the following is defined to be the Jacobson radical of R.

i) $\bigcap_{M \in X} \text{Ann}_R(M)$, where X is the set of all simple left R-modules.

ii) $\bigcap_{M \in Y} \text{Ann}_R(M)$, where Y is the set of all simple right R-modules.

Prove that the sets in (i) and (ii) are equal i.e., the Jacobson radical of any ring is left-right symmetric. 10

d) Suppose G is a group of order 12. What are the possible degrees of all the irreducible representations of G ?

2+4+2+2

7. a) Suppose V is a finite dimensional vector space over a field F and V* denotes the dual of V.

i) A covariant tensor space over V* is a contravariant tensor space over V- Justify !

ii) What is meant by a symmetric tensor algebra of V ?

iii) What is meant by an exterior algebra of V ?

b) Let V and U be two vector spaces over the same field. Let U $\otimes$ V denote the tensor product of U and V (defined using universal mapping property for bilinearity measured by linearity).

Suppose $(B = \{u_i : i \in I\})$ is a basis for U and $C = \{v_j : j \in J\}$ is a basis for V. Prove that $\{u_i \otimes v_j : i \in I, j \in J\}$ is a basis for $U \otimes V$. (1+2+2)+5 to the statement that 'the group algebra $\mathbb{C}G$ - is Jacobson semi simple'.

b) Suppose G is a finite group.

i) Prove that g, h $\in$ G are conjugate to each other if and only if $\chi(g) = \chi(h)$ for all characters χ of G.

i) Prove that g $\in$ G is conjugate to g^{-1} if and only it $\chi(g)$ is real for all characters χ of G.

c) Prove that the column orthogonality and the row orthogonality for the character table of any finite group are equivalent. 2+(3+2)+3

6. a) Suppose χ is a character of a finite group G. Let g $\in$ G be such that $\chi(g)$ is a rational number.

Then x(g) is an integer i.e., an element of Z. – Justify.

b) Using the theory of group representations (the relevant lemma may be assumed) prove Burnside's $p^a q^b$ Theorem : "Let p and q be prime numbers and let a, b be nonnegative integers with $a + b \geq 2$. If G is a group of order $p^a q^b$ then G is not simple".

c) The regular representation of a finite Abelian group G $(|G| \geq 2)$ over the field of complex numbers cannot be irreducible – Justify !

3. a) Let M be a simple left R-module. Let V be a finite dimensional vector subspace of the D = $Hom_R(M,M)$ vector space M. Let $v \in M$ be such that $v \notin V$. Prove that there exists $r \in R$ such that $rv \neq 0$ and $rv = 0$.

b) State and prove Jacobson Density theorem for primitive rings. 5+5

4. a) Prove that for a left Artinian ring R the following are equivalent :

i) R is simple,

ii) R is primitive,

iii) R is isomorphic to the endomorphism ring of a non zero finite dimensional vector space over a division ring D.

iv) For some positive integer n, R is isomorphic to the ring of all $n \times n$ matrics over a division ring.

b) The above result is usually known as Wedderburn-Artin theorem for simple left Artinian rings. State (proof is not required) Wedderburn-Artin Theorem for Jacobson semisimple left Artinian rings. 8+2

5. a) State (proof is not required) Maschke's theorem for representations of a finite group G over the field $\mathbb{C}$ of complex numbers. Explain how is the theorem equivalent