M. Sc. Mathematics Examination, 2019

(1st Year, 2nd Semester)

Mathematics

Functional Analysis

Unit - 2.3

Time : Two hours Full Marks : 50

(25 marks for each part)

Use a separate Answer-Script for each part

PART - II

(Answer ***any five*** questions)

1. a) If $\{x_n\}_n$ and $\{y_n\}_n$ are two sequences in an inner product space X converging to x and y respectively, then prove that $< x_n, y_n > \to < x, y > a, n \to \infty$.

   b) Prove that in an innerproduct space X,

$$\| x - y \|^2 + \| x - z \|^2 = 2(\| x - \frac{1}{2}(y+z) \|^2 + \| \frac{1}{2}(y-z) \|^2),$$

   $\forall$ x, y, z $\in$ x. 2+3

2. a) Let X denotes the inner product space of real valued continuous function on $[0, 2\pi]$ with inner product defined by $< x, y > = \int_0^{2\pi} x(t)y(t)dt.$

If $e_o(t) = \frac{1}{\sqrt{2\pi}}$ and $e_n(t) = \frac{\cos nt}{\sqrt{\pi}}, n = 1, 2, \cdots$

then prove that $\{e_n\}_n$ is an orthonormal sequence in X.

3+2

b) Prove that an orthonormal set in an inner product space is linearly independent.

3. a) If M and N are closed subspaces of a Hilbert spae H such that $M \perp N$, then prove that the subspace $M + N$ is closed.

b) What happens when $N = M^{\perp}$? 4+1

4. State and prove Bessel's inequality regarding orthonormal sequence in a Hilbert space. 1+4

5. Define isometric operator for a Hilbert space H. Prove that a unitary operator is isometric. Is the converse true ? Justify.

1+2+2

6. a) For an adjoint operator A on a Hilbert space, prove that $(\lambda A)^* = \bar{\lambda} A^*$, where λ is a scalar.

b) Prove that if T is any continuous linear operator on a Hilbert space H, then T can be expressed uniquely in the form T = A+iB, where A and B are self adjoint.

7. a) If T is a normal operator on a Hilbert space H, then prove that $Tx = \lambda x$ if and only if $T^*x = \bar{\lambda}x$ for $\forall x \in H$ and λ is a scalar.

b) Give an example of a projection operator on a Hilbert space.

Notations and symbols have their usual meaning.