Ex./M.SC/M/1.3/32/2019

MASTER OF SCIENCE EXAMINATION, 2019

(1st Year, 1st Semester)

MATHEMATICS

Complex Analysis

Paper : 1.3

Time : Two hours Full Marks : 50

Answer any ***five*** questions.

All questions carry equal marks

Symbols/Notations used have their usual meaning.

1. (a) Define a curve in a complex plane.

   (b) Let f(z) = u(x,y) + iv(x,y) be defined in an open region R. If f(z) is holomorphic in R then prove that $u_x = \nu_y$, $u_y = -\nu_x$ hold in R. Hence prove that if f(z) is holomorphic in R and $f'(z) = 0$ in R then f(z) is constant in R. 2+8

2. (a) Evaluate $\int_\Gamma z^3\, dz$, where $\Gamma = L(0,1) + L(1,2+i) + L(2+i, i)$. Here L(a,b) is a line joining the points z = a and z = b.

(Turn over)

(b) If f(z) is holomorphic in a simply connected open region R and Γ is a rectifiable closed curve contained in R then prove that $\oint_{\Gamma} f(z)dz = 0$. 4+6

3. (a) Define winding $W(\Gamma, z_0)$. Show that if Γ is a rectifiable closed Jordon curve, then either $W(\Gamma, z_0) = 1$ for every $z_0 \in I(\Gamma)$, or $W(\Gamma, z_0) = -1$ for every $z_0 \in (\Gamma)$.

(b) Let R be an open region and Γ be a rectifiable curve such that R and Γ are disjoint point sets. If f(z) is continuous on Γ, then prove that the function

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z} d\varsigma, z \in R$$

is holomorphic and

$$F^{/}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{(\varsigma - z)^2} d\varsigma, z \in R$$ 2+3+5

7. (a) Let f(z) and g(z) be both holomorphic in an open region R and Γ be a rectifiable closed Jordon curve such that $I(\Gamma) \cup \Gamma \subseteq R$. If on Γ, $f(z) \neq 0$ and $|f(z)| > |g(z)|$, then f(z) and f(z) + g(z) have same number of zeros in $I(\Gamma)$.

(b) Show that a polynomial of degree n has exactly n zeros in the complex plane. 5+5

8. Use the method of contour integration to prove the following results (any ***two***)

(a) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)} dx = \frac{\pi}{8\, a^3}, \mathrm{Re}\{a\} > 0.$

(b) $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$

(c) $\int_{0}^{\infty} \frac{\cos x}{\sqrt{x}} dx = \frac{\sqrt{\pi}}{\sqrt{2}}.$ 5+5

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5. (a) Let $\omega = f(z) = u(x,y) + iv(x,y)$ is defined in an open region R such that the partial derivatives of u and v are continuous in R and the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ in R. If the mapping $\omega = f(z)$ is conformal in R then prove that f(z), is holomorphic in R and $f'(z) \neq 0$ in R.

(b) Find the general bilinear transformation which maps the circle $|z| = r$ onto the circle $|\omega| = r'$. 7+3

6. (a) If z_0 is an isolated essential singularity of f(z), then prove that given any complex number c and any two positive numbers $\in, \delta$ however small they may be, there exists $z \in N'(z_0, \delta)$ such that $|f(z) - c| < \in$.

(b) Let R be an open region and $z_0 \in R$. Suppose f(z) be holomorphic and not identically zero in $R - \{z_0\}$. If E be the set of all zeros of f(z) in $R - \{z_0\}$ and z_0 be a limit point of E, then z_0 is an essential singularity of f(z). 6+4

4. (a) Let f(z) be holomorphic in an annulus $A(z_0, r_1, r_2)$ and

$$F(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0) n,\ z \in A(z_0, r_1, r_2).$$

If $K \equiv K(z_0, r)$ is a circle such that $r_1 < r < r_2$, and $|f(z)| < M$ on $K(z_0, r)$ then prove that

$|a_n| \leq \frac{M}{r_n}$, $n = 0, \pm 1, \pm 2 \ldots$ where M is a constant.

(b) If f(z) is holomorphic and bounded in some $N'(z_0)$, then z_0 is not a singularity of f(z).

(c) Let R be a bounded open region and B its boundary and $R^* = R \cup B$. If f(z) is nonconstant and holomorphic in R and continuous in R^*, then prove that $|f(z)| < M$ for every $z \in R$ where $M = \max_{z \in R^*} |f(z)|$. 3+2+5