7. a) i) Find all possible Jordan canonical forms of a real square matrix of order 3 having eigen values 5, 5, 5.

ii) Determine all possible Jordan canonical forms for a 5×5 matrix whose minimal polynomial is $(x-2)^2$.

3+2

b) Find the Jordan canonical form of the real matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 7 & -4 \\ 0 & 9 & -5 \end{pmatrix}.$$

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M. Sc. Mathematics Examination, 2019

(1st Year, 2nd Semester)

Mathematics

Algebra

Unit - 2.1

Time : Two hours | Full Marks : 50

Answer ***any five*** questions. 10×5=50

1. a) What do you mean by field extension ? Show that every field F is either an extension of $\mathbb{Q}$ or an extension of $\mathbb{Z}$ for some prime p. 1+4

b) i) Let F be a field and a, b be two elements of a field containing F. Suppose that a and b are algebraicover F of degree m and n respectively such that ged (m, n) = 1. Show that [F(a,b) : F] = mn.

ii) With proper justification give an example to show that the above result in (i) need not be true if m and n are not relatively prime. 3+2

2. a) i) Find the degree of the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2})$ over $\mathbb{Q}$. Is it a simple extension ? Justify your answer.

ii) Find the degree of the splitting field of $x^p - 1$ over $\mathbb{Q}$, where p is prime. (2+1)+2

[Turn over

b) Let K be a field and f(x) be a non constant polynomial in K[x]. Show that there is a field extension F of K such that f(x) has a root α in F. 5

3. a) i) Let p(x) be an irreducible polynomial over a field F and be the formal derivative of p(x). Show that p(x) is seperable if and only if $p'(x) \neq 0$.

ii) Show that $x^4 + x^2 + [1]$ is seperable over $\mathbb{Z}_2$. 3+2

b) Let GF(p^n) be a field of order p^n. Show that the mapping f : GF(p^n) $\rightarrow$ GF(p^n) defined by f(a) = a^p is an antomorphism. Hence conclude that every finite field is a perfect field. 3+2

4. a) Prove that every element in a finite field can be written as the sum of two squares. 5

b) i) Let $K \subseteq L \subseteq F$ be a chain of fields. Suppose that F/K is a normal extension, show that F/L is a normal extension. Is L/K a normal extension ? Justify your answer.

ii) Show that $\mathbb{Q}(\sqrt{2})$ is a normal extension of $\mathbb{Q}$ but $\mathbb{Q}(\sqrt{3})$ is not a normal extension of $\mathbb{Q}$. 3+2

5. a) Show that

i) the regular hexagon is constructible by straightedge and compassonly.

ii) it is impossible to trisect an angle of 60^o by straightedge and compassonly. 2+3

b) i) Find the Galosis group of the field extension $\mathbb{C}/\mathbb{R}$.

ii) Show that the Galosis group of the polynomial $f(x) = x^5 - 25x + 5$ over $\mathbb{Q}$ is S_5. Hence conclude that the equation f(x) = 0 is not solvable by radicals. 2+3

6. a) i) Let v be a finite dimensional vector space over a field F and T : V $\rightarrow$ V be a linear operator such that $T^3 = T$. Is P triangularizable ? Is T diagonalizable ? Justify your answer.

ii) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator. Show that if T is not trangularizable over $\mathbb{R}$ then T is diagnalizable over $\mathbb{C}$. 2+3

b) Let $A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}$.

Are A and B simultaneously diagonalizable over $\mathbb{R}$? Justify your answer. 5