Ex./M.SC/M/1.1/32/2019

MASTER OF SCIENCE EXAMINATION, 2019

(1st Year, 1st Semester)

MATHEMATICS

Algebra - I

Paper : 1.1

Time : Two hours Full Marks : 50

Use a separate Answer-Script for each part.

PART - I (25 marks)

Unexplained symbols and notations have their usual meanings.

Answer ***q.no. 1*** and any ***three*** from the rest.

1. Answer any ***five*** questions : 5x2=10

(a) Suppose p be a prime integer and $n>1$ be any integer. Then no group of order p^n is simple – Explain.

(b) For $n \geq 5$, S_n is not solvable – Explain.

(c) Suppose G be non commutative group of order p^3, p be a prime. Find $|Z(G)|$.

(d) There is no simple group of order 56 – Justify.

(e) Any group of order 35 is solvable – Explain.

(Turn over)

(f) Suppose G be a group and $f : G \to G$ be an isomorphism defined by $f(a) = a^n$, $\forall\ a \in G$ and n be integer. Then show that a^{n-1} Z(G), $\forall\ a \in G$.

2. Let G be a finite commutative group of order n. If m is a positive divisor of n then show that G has a subgroup of order m.
Is the above result true for any finite group ? Justify your answer. 3+2

3. Define group action. Let G be a finite group. Let H be a subgroup of G of index p, smallest prime dividing the order of G. Show that H is normal subgroup of G. 2+3

4. Define solvable group. Suppose G be a group. Prove that G is solvable iff there is a positive integer m such that $G^{(m)} = \{e\}$. 2+3

5. State Sylow's Third theorem. If there exists an epimorphism of a finite group G onto the group $\mathbb{Z}_8$, then show that G has normal subgroups of index 4 and 2.

13. Define quotient module. Let R be a ring with identity and A, B be two submodules of an R-module M such that $A \subseteq B$. Show that $\frac{M/A}{B/A} \cong M/B$. 2+3

14. Define free module. Give an example to show that (i) submodule of a free module need not be a free module. (ii) a torsion free module need not be free module. 1+2+2

—— X ——

6. Suppose G is a group of order pqr, where p, q, r are primes with $p > q > r$. Show that G is solvable. 5

PART - II (25 marks)

GROUP - A (15 marks)

Answer any ***three*** questions. 3x5=15

7. Let R be a commutative ring with identity and I be a prime ideal of R. Show that I[x] is a prime ideal of R[x]. Is I[x] a maximal ideal of R[x] if I is a maximal ideal of R ? Justify your answer. 3+2

8. Let R be a PID and $P \in R$. Show that p is an irreducible element of R if and only if p is a prime element of R. Is the element 5 irreducible in $\mathbb{Z}[i]$? Justify your answer. 4+1

9. Prove that $\mathbb{Z}\left[\sqrt{-6}\right]$ is not a unique factorization domain (UFD). Find examples of each of the following in $\mathbb{Z}\left[\sqrt{-6}\right]$.

   (i) an irreducible element that is not prime

(Turn over)

   (ii) non zero elements a and b for which gcd(a,b) = 1 but no α, β exist in $\mathbb{Z}\left[\sqrt{-6}\right]$ such that $a\alpha + b\beta = 1$.

1+2+2

10. Show that x^2+1 is irreducible in $\mathbb{R}[x]$ and $\mathbb{R}[x] / \langle x^2+1 \rangle \cong \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. 1+4

11. Show that an Artinian integral domain is a field. Let R be a commutative ring with identity such that R is Artinian. Is every prime ideal of R a maximal ideal of R ? Justify your answer. 3+2

GROUP - B (10 marks)

Answer any ***two*** questions.

12. (i) Let R be a commutative ring with identity. Let M be an R module and L, L^1, N, N^1 be submodules of M such that $M = L \oplus L^1 = N \oplus N^1$. Give an example to show that $L = N$ but $L^1 \neq N^1$.

    (ii) Let R be a ring with identity 1 and e be a central idempotent in R. If M is an R-module then show that $M = eM \oplus (1-e)M$. 3+2