13. (a) By using the formula

$$N = \frac{4\pi Vd}{h^3} \int_0^\infty \frac{1}{e^{\beta(\in-\mu)}+1} p^2\, dp,$$

find the total number of electron occupied by Fermi sphere of radius p_F at temperature, T, which is much smaller than the Fermi temperature, T_F, where d=2, is the spin degeneracy of electron.

(b) Similarly, by using the formula for pressure,

$$P = \frac{4\pi d}{3h^3} \int_0^\infty \frac{1}{e^{\beta(\in-\mu)}+1} \left(p\frac{de}{dp} \right) p^2\, dp\,;$$

Show that pressure exerted by relativistic electrons is

$$P = \frac{8\pi}{3h^3} m^4 c^5 A(x),\ where\ A(x) = \int_0^{\theta_F} \sin h^4 \theta d\theta,$$

and $x = \sin h\, \theta_F = \frac{pF}{mc}$.

(c) Considering the equilibrium of the white dwarf star having N/2 helium nuclei, mass M, radius R and gravitational self energy $E_g = -\alpha GM^2/R$, derive the mass-radius relationship of it. Symbols have their usual meaning. 1+2+2

Ex:B.SC/PHY/32/H12/GR-A/2019

FINAL B.Sc. EXAMINATION, 2019

(3rd Year, 2nd Semester)

PHYSICS

Paper : HO-12

Time : Two hours Full Marks : 50

Use separate answer-script for each group.

GROUP - A

(Quantum Mechanics - II)

Answer ***q.no. 1*** and any ***two*** from the rest.

1. (a) A particle in $\mathbb{R}^3$ has a wave-function given by $\psi(\vec{r}) = Ne^{-\alpha r}$, where N and α are real parameters/constants, (and $\alpha > 0$). Calculate the uncertainty $(\Delta r)^2$.

(b) Consider the very well known one-dimensional problem of a particle in a box (or equivalently in an infinitely square well), i.e., a particle placed in a potential :

$$V(x) = 0,\ for\ 0 < x < L$$
$$= \infty,\ otherwise$$

Evaluate the $x-p$ uncertainty product $(\Delta x)^2\,(\Delta p)^2$, for the ground state as well as the first excited states. 4+5

2. (a) A particle in the (one-dimensional) infinite square well has the initial wave function given by :

$$\Psi(x,0) = Ax\,(L-x), \quad (0 \le x \le L),$$

where A and L are constants. Outside the well, of course $\Psi(x > L, t) = 0$. Find the value of the wave-function $\Psi(x,t)$ for all subsequent times.

(b) Check whether $-i\frac{d}{dx}$ *and* $\frac{d^2}{dx^2}$ are Hermitian operators in the space of functions (on the real line) which vanish at the points $x = \pm \propto$. (Why is this vanishing condition necessary?) 5+3

3. Consider a particle of mass 'm' moving/confined inside an infinite square well of width 2L. Initially $(t=0)$, the particle is in its lowest energy state. Assume now that at $t=0^+$, both the walls of the infinite well suddenly move apart/outwards instantaneously and symmetrically, so that now the width of the well doubles to 4L. Find the expression for the wave-function of the particle at subsequent times, ie, for $t > 0$. From the physical point of view (role of adiabatic hypothesis), explain how the outcome of this action of moving the wall depends on whether you move the walls apart – slowly or fast? 8

8. Consider a system with N particles, each of which has two non-degenerate energy levels having energies $\in$ and $-\in$. The system is in contact with a heat bath at temperature T.

(a) Derive the expressions of specific heat of the system.

(b) Obtain the form of specific heat at the (i) very high and (ii) very low temperature limits (iii) Plot the variation of specific heat with respect to temperatures. 2+(1+1+1)

9. (a) Find the average energy of an electromagnetic radiation of frequency ν at temperature T, when the quanta of energy comes in multiples of $h\nu$, i. e. $E_\nu = mh\nu$, $m = 0,1,2,....$

(b) Obtain the expression of density of states of the electromagnetic radiation of frequency ν stored in a volume V by counting the number of quantum cells in the phase space. $2^1/_2+2^1/_2$

10. (a) Using the expression of pressure, $P = \frac{k_B T}{Va}\sum_{\in} \ln\left(1 + aze^{-\beta\in}\right)$ and converting the

(Turn Over)

4. Starting from the commutation relations of the angular momentum operator **L**, (or equivalently, starting with the generatrs of SO(3), which you assume to be given), obtain the complete solution for the eigenvalue problem for $|\mathbf{L}^2, \mathbf{L}_z>$. How is the operator $\mathbf{L}^2$ related to the angular part of the Laplacian in 3d ? 8

5. (a) Establish the general inequality (for any stationary state of a quantum system in 3d under the action of a central force), $L_X^2 + L_Y^2 \geq 0$.

   (b) The hydrogen atom wave-function is known to be the form $\psi(\vec{r}) = Ne^{-\alpha r}$. If p_r denotes the radial momentum operator, then find the value of $< p_r >$ for this ground state wave function ? 4+4

GROUP - B (25 marks)

(Statistical Mechanics - II)

Answer any ***five*** questions.

6. Consider the simple harmonic oscillator (SHO) with energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where n = 0,1,2,3,...,

(Turn Over)

   (a) Write down the partition function, Z_{SHO}.

   (b) Find the density of states, g(E) for SHO from Z_{SHO}.

   (c) Derive the expressions of internal energy, E and specific heat, C_V.

   One may use Laplace Transform : $\alpha^{-1}(e - \beta\omega) = \delta(E - \omega)$

   1+1+3

7. (a) Let the pure state density matrix, ρ, of a pure state vector, $|\psi\rangle$, $be\, \rho\, |\psi\rangle\langle\psi|$, and the expectation value of an observable, A, be $\langle A\rangle = \langle\psi|A|\psi\rangle$. Now, show that $Tr(\rho A) = \langle A\rangle$, where $Tr(A) = \sum_n \langle n|A|n\rangle$ and $\{|n\rangle\}$ is a complete set of ortho-normalised state vectors.

   (b) Let the mixed state density matrix, ρ_{mix}, of a mixed state be defined as a weighted sum of ρ_i, like $\rho_{mix} = \sum_i pi\, \rho i$ where ρ_i be a pure state density matrix of a pure state vector, $|\psi_i\rangle$ and $\sum_i pi = 1$. Hence, show that $Tr\left(\rho_{mix}^2\right) \leq 1$. 2+3 summation into integration in the phase-space show that

$$P = \frac{4\pi}{3h^3}\int_0^\infty \frac{1}{z^{-1}e^{\beta\epsilon}+a}\left(p\frac{d\epsilon}{dp}\right)p^2\,dp$$

(b) Similarly, using the expression of mean energy, $E = \sum_\epsilon \frac{\epsilon}{z^{-1}\,e^{\beta\epsilon}+a}$ and converting the summation into integration in the phase-space show that E=3PV/2, when $\epsilon = p^2/2m$. Symbols have their usual meaning. 2+3

11. By using the following relationships valid for the ideal Fermi gas

$$\frac{PV}{k_BT} = -\sum_\epsilon \ln\left(1+z\,e^{-\beta\epsilon}\right) \text{ and } N = \sum_\epsilon \frac{1}{z^{-1}\,e^{\beta\epsilon}+1},$$

where ϵ is the single particle energy, $z = e^{\mu/k_BT}$ and $\beta^{-1} = k_BT$, show that when the volume V is very large, the above relationships can be expressed as,

$$\frac{P}{k_BT} = \frac{f_{\frac{5}{2}}(z)}{\lambda^3} \text{ and } \frac{N}{V} = \frac{f_{\frac{3}{2}}(z)}{\lambda^3}$$

where $\lambda = h/\sqrt{2\pi m k_BT}$ and the Fermi-Dirac function, $f_\nu(z)$ is defined as

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}\,dx}{z^{-1}e^x+1}$$

other symbols have the usual meanings. 3+2

12. (a) Show that the total partition function of a single hydrogen atom is $Z^H(1) = \frac{V}{\lambda_H^3}\,e^{\beta R}$, where R is the Rydberg constant.

(b) Consider the ionized hydrogen gas at high enough temperature where ionization and recombination lead to the equilibrium governed by $H \rightleftharpoons p^+ + e^-$. Derive the Saha equation in the equilibrium,

$$\frac{n_p n_e}{n_H} = \frac{e^{-\beta R}}{\lambda_e^3}$$

Symbols have their usual meanings. 2+3