13. Let f(x) be Riemann integrable, 2π periodic function and

$f(x) \sim \sum_{n=\infty}^{\infty} c_n e^{inx}$. Then prove that

(i) $\lim_{N\to 0} \frac{1}{2\pi} \int_{\pi}^{\pi} \left|f(x) - S_N(f;x)\right|^2 dx = 0$

(ii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(x)\right|^2 dx = \sum_{-\infty}^{\infty} \left|C_n\right|^2$ 3+2

14. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined as follows :

$$f(x) = \begin{cases} Cos\,x\ , 0 \le x \le \pi \\ -Cos\,x\ , -\pi \le x \le 0 \end{cases}$$

obtain the Fourier series for the function f(x). Hence find the sum of the series

$$\frac{2}{1.3} - \frac{6}{5.7} + ...$$

what is the value of the Fourier series at $x = 0$? $2^1/_2$+$1^1/_2$+1

—— X ——

Ex./FM/5.4/39/2019

BACHELOR OF SCIENCE EXAMINATION, 2019

(3rd Year, 1st Semester)

MATHEMATICS (HONOURS)

Analysis - III

Paper - 5.4

Time : Two hours Full Marks : 50

Symbols/Notations have their usual meaning.

PART - I (25 marks)

Answer any ***five*** questions.

1. Let X denote the set of all sequences of real numbers. Define $d : X \times X \to R$ by

$$d(\tilde{x}, \tilde{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for all $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$ in X. Show that d is a metric on X. 5

2. (a) Define distance between two sets and diameter of a set in a metric space (X,d).

(b) Prove that dist (x, A) = 0 if and only if $x \in \overline{A}$. 2+3

3. (a) Define nowhere dense subset in a metric space. Give an example of it.

(b) Prove that in a metric space (X, d), a subset A of X is nowhere dense iff every non empty open set U in X contains a non empty open set V such that $V \cap A = \phi$. 2+3

4. State and prove Baire's Category theorem. 5

5. Show that a mapping $f : (X, d) \to (Y, \rho)$ is continuous on X if and only if $\overline{f}^{1}(G)$ is open in X, for all open subset G of Y. 5

6. State Banach contraction principle theorem.
Let X = {x is real : x ≥ 1} and let a mapping $T : X \to X$ be defined by $T(x) = \frac{1}{2}x + \frac{1}{x}, \forall\, x \in X$.
Then show that T is a contraction mapping and find the unique fixed point of T in X, with respect to usual metric. 2+3

7. Prove that every closed interval of reals is a compact set. 5

PART - II (25 marks)
Answer any ***five*** questions.

8. Let $E \subset \mathbb{R}$. Show that a sequence of functions $\{f_n(x)\}$; $f_n : E \to \mathbb{R}$ is uniformly convergent on E if and only if for every $\in > 0$, $\exists$ a positive integer 'p' such that $|f_n(x) - f_m(x)| < \in$, $\forall\, x \in E$ and $\forall\, m, n \geq p$. 5

9. Let a series of functions $\sum_{k=1}^{\infty} U_k(x)$ converge uniformly to a sum function f(x) in [a,b] and each $U_k(x)$ is continuous in [a,b]. For each $x \in [a,b]$ define

$$g_n(x) = \sum_{k=1}^{n} \int_a^x U_k(t)\,dt$$

Show that $g_n(x)$ converges iniformly to g(x) in [a,b]. 5

10. Prove that a power series $\sum an\, x^n$ with radius of convergence 'ρ', converges absolutely and uniformly on $[a,b] \subset (-\rho, \rho)$.5

11. (a) Let $\{f_n\}$ be a sequence of continuous real-valued functions converging uniformly to 'f' on a set $E(\subset \mathbb{R})$. If $x_n \to x$ then show that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$ 3

(b) Examine uniform convergence of the following infinite series :

$$\sum_{n=1}^{\infty} \frac{2^n\, x^{2n-1}}{1 + x^{2n}}$$ 2

12. Let 'f' be a bounded improper integrable function on $[0, 2\pi]$ and let 'a' be a point of Lipschitz continuity of f. Then show that the Fourier series of 'f' is convergent at x = a with sum f(a). State the sumfunction at a point of discontinuity. 4+1

(Turn over)