13. Define unique factorization domain (UFD). Note that $5 = (2+i)(2-i) = (1+2i)(1-2i) \in \mathbb{Z}[i]$. Does this contradicts that $\mathbb{Z}[i]$ is a UFD ? Justify your answer, Hence conclude that 5 is not prime in $\mathbb{Z}[i]$. 2+2+1

14. (i) Let $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3+1$ be two polynomials in Q[x]. Find gcd (f(x), g(x)) and lcm(f(x), g(x)).

    (ii) Show that the polynomial $x^3 + 8ix^2 - 6x - 1 + 3i$ is irreducible in $(\mathbb{Z}[i])[x]$. 2+3

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Ex:MATH/H/32/6.3/86/2019

BACHELOR OF SCIENCE EXAMINATION, 2019

(3rd Year, 2nd Semester)

MATHEMATICS (HONOURS)

Algebra - IV

Paper : 6.3

Time : Two hours Full Marks : 50

Use a separate Answer-Script for each part.

PART - I (25 marks)

Answer any ***five*** questions.

1. Define the group of automorphisms of a group G. Find the group of automorphisms of a finite cyclic group of order n. 5

2. Show that in a group G of order 49, any normal subgroup of order 7 must lie in the center of G. 5

3. Let G be a finite group and $T : G \to G$ be a fixed point free automorphism ($T(x) = x \Rightarrow x = e_G$). Show that if T^2 is the identity map on G, then G is abelian. 5

4. Define the conjugacy relation and conjugacy classes $C_x (x \in G)$ of a finite group G. Prove that the number of elements of C_x is same as the index of the normalizer of x in G. 5

(Turn Over)

5. Let G be a finite group of order n and p be a prime number such that p^m divides n, where m is a natural number. Then show that G has a subgroup of order p^m. 5

6. Define elementary divisors of a finite abelian group. Find all elementary divisors of the group $\mathbb{Z}_{20} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{50}$, where $\mathbb{Z}_n$ denotes the group of integers moduls n. 5

7. Let G be an abelian group. Prove that G has a finite basis if and only if G is isomorphic to a direct sum of finite copies of the group of integers. 5

PART - II (25 marks)

Answer any ***five*** questions.

8. Let R be a commutative ring with identity and N be the set of all nilpotent elements of R. Show that N is an ideal of R and the quotient ring R/N has no non zero nilpotent elements. Is commutativity of R essential ? Justify your answer. 2+2+1

9. Let R and R^1 be two commutative rings with identity $|_R$ and $|_{R^1}$ respectively. If $f : R \to R^1$ be a non zero ring homomorphism and R^1 is an integral domain then show that $f(|_R) = |_{R^1}$. Give an example to show that the above result does not hold if R^1 has divisor of zero. 3+2

10. (i) Show that a polynomial in $\mathbb{Z}_2[x]$ has a factor (x–[1]) if and only if it has even number of non zero coefficients.
(ii) Let F be a field. Is F[x] a field ? Justify your answer.
(iii) What is the quotient field of a finite integral domain ? 2+2+1

11. (a) Define maximal ideal and prime ideal of a commutative ring with identity.
(b) Let R be a commutative ring with identity such that for every $x(\in R)$ satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal of R is a maximal ideal of R. 2+3

12. (a) Give an example to show that in a Euchidean Domain (ED), the quotient and remainder are not unique.
(b) Define Principal Ideal Domain (PID). Let R be a PID and P be a prime ideal of R. Is R/P a PID ? Justify your answer. 2+(1+2)

(Turn Over)