(b) Define a quadratic form on a vector space V over a field F. Prove that every real inner product defines a quadratic form. Also prove that such a quadratic form is positive definite.

(c) Let A be an n x n matrix over F(= $\mathbb{R}$ or $\mathbb{C}$) such that its columns are orthonormal with respect to the standard inner product of F^n. Prove that its rows are also orthonormal. Hence prove that $A^* A = I_n$ implies that $AA^* = I_n$.

2+4+4

7. (a) Let A and B be nxn matrices over a field F. Prove that if $I - AB$ is invertible then $I - BA$ is invertible and $(I - BA)^{-1} = I + B(I - AB)^{-1} A$. Hence or ohterwise prove that λ is an eigenvalue of AB iff it is an eigenvalue of BA.

(b) Let V, W be two finite dimensional vector spaces over a field F and $T_1 : V \to V$, $T_2 : W \to W$ be two linear operators with minimal polynomials $x^2 - 5x + 6$ and $x^3 - 2x^2 + x - 2$ respectively. Let $T : V \oplus W \to V \oplus W$ be defined by $T(\nu + \omega) = T_1(\nu) + T_2(\omega)$ for all $\nu \in V$, for all $\omega \in W$. Prove that V and W are invariant under T. Also find the minimal polynomial of T. 6+4

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Ex./FM/5.3/39/2019

BACHELOR OF SCIENCE EXAMINATION, 2019

(3rd Year, 1st Semester)

MATHEMATICS (Honours)

Paper - 5.3

Algebra - III

Time : Two hours Full Marks : 50

Answer ***Q.No. 1*** and any ***four*** from the rest.

1. Answer any ***five*** questions : 2x5=10

(a) Suppose A is a 3x3 real matrix which is not similar to a triangular matrix over $\mathbb{R}$. Then the degree of the minimal polynomial for A is of degree 3 – Justify.

(b) Suppose T is a diagonalizable linear operator on an n-dimensional vector space V such that T has a cyclic vector. Then all the eigenvalues of T are distinct – Justify.

(c) Geometric multiplicity of the eigenvalue of a nonzero nilpotent operator T on an n-dimensional vector space V cannot be n – Justify.

(d) Suppose $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $\langle (x_1, x_2), (y_1, y_2) \rangle := 3x_1y_1 + 2x_1y_2 + 4x_2y_1 + 6x_2y_2$. Which condition(s) for $\langle\ ,\ \rangle$ to be an inner product is (are) lacking ?

(Turn over)

(e) Let F be a field and m,n be two positive integers. Let T : $F^n \to F^m$ be a linear oeprator. If T is onto then $m \geq n$ implies m = n. – Justify.

(f) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2⊗$ where a,b,c,d are integers. If A has eigenvalues then they are integers. – Justify.

(g) Determine the sign of the quadratic form

$q(x, y, z) = x^2 + 4xy + 4yz + 5y^2 + 6z^2$ on $\mathbb{R}^3$.

2. (a) State (proof is not required) three necessary and sufficient conditions for a linear operator on a finite dimensional vector space to be diagonalizable.

(b) Prove that a linear transformation is (i) 1 – 1 iff it maps any linearly independent set to a linearly independent set, (ii) onto iff it maps any spanning set to a spanning set.

(c) Prove that two finite dimensional vector spaces over the same field are isomorphic iff they are of same dimension.

(d) Prove that every one dimensional T-invariant subspace (T is a linear transformation) is an eigenspace of T.

2+5+2+1

3. (a) Let $A \in M_{mxn}(F)$ (F is a field and m, n are integers). Prove that there exists a linear transformation from F^n to F^m whose matrix with respect to the standard ordered bases is A.

(b) Let V be a vector space over a field F. What is the canonical mapping from V to V^{**}. Is it a linear transformation? Is it 1–1 ? Justify your answer.

(c) Trace of an $n \times n$ square matrix over a field F is a linear functional on F^n. – Justify.

3+5+2

4. (a) There exist nonsimilar matrices having the same minimal polynomial m(x) and the same characteristic polynomial f(x). – Justify.

(b) If two 3 x 3 matrices over the same field have the same minimal polynomial m(x) and the same characteristic polynomial f(x) then they cannot be nonsimilar. – Justify.

(c) Find the matrix of the differentiation operator D on the vector space V of all real polynomials of degree less then or equal to 4 with respect to the canonical basis $\{1, x, x^2, x^3, x^4\}$ Find also the Jordon form of D.

3+3+4

5. (a) Give example of two nonsimilar 4 x 4 real matrices having two eigenvalues ± 1 over $\mathbb{R}$ and four eigenvalues $\pm 1, \pm i$ over $\mathbb{C}$.

(b) Suppose T is a linear operator on a finite dimensional vector space V over a field F. Suppose rank T^2 and rank T are the same. Prove that $\mathrm{Im}T \cap \mathrm{Ker}\, T = \{0\}$.

5+5

6. (a) 0 and 1 are the only possible eigenvalues of an idempotent operator. Give an example to illustrate that the converse is not true.

(Turn over)