Ex/UG/Sc./Core/Math/TH/10/2019

B. Sc. Mathematics Examination, 2019

(2nd Year, 2nd Semester)

Mathematics (Honours)

Core - 10

Ring Theory and Linear Algebra-II

Time : Two hours Full Marks : 50

Use a separate Answer-Script for each part

(25 marks for each part)

PART - I

(Answer ***any five*** questions) 5×5=25

1. Let R be a commutative ring with identify. Show that $R[x]/\langle x\rangle \cong R$. Hence conclude that $\langle x\rangle$ is a prime ideal of $\mathbb{Z}[x]$ but $\langle x\rangle$ is not a maximal ideal of $\mathbb{Z}[x]$. 3+2

2. Show that in an integral domain every prime element is irreducible. Is integral domain necessary in the above result ? Justify your answer. 3+2

3. Define Euclidean domain. Show that every Euclidean domain is a principal ideal domain. 2+3

4. i) Let F be a field and $f(x), g(x) \in F[x]$ such that $f(a) = g(a)$ for all $a \in F$. Is $f(x) = g(x)$? Justify your answer.

ii) With proper justification given an example to show that subring of a principal ideal domain (PID) need not be a principal ideal domain (PID)

iii) Is $\mathbb{Z}_n[x]$ a PID for every n ? Justify your answer.

2+2+1

5. Define unique factorization domain (UFD). Let $\mathbb{R}$ be a UFD and P be a prime ideal of $\mathbb{R}$. Is $\mathbb{R}/P$ a UFD ? Justify your answer. 2+3

6. Let F be a field and $f(x) \in F[x]$ such that degree of f(x) be 3. Show that f(x) is irreducible over F if and only if f(x) has no root in F. Does the above result hold for polynomials of degree more than 3 ? Justify your answer. 3+1

7. State Eisenstein criterion for irreducibility of a polynomial. Is Eisenstein criterion necessary for irreducibility of a polynomial ? Is the polynomial $x^3 - 3|23|2x + |23|23$ irreducible in f[x] ? Justify your answer. 2+2+1

PART - II

(Attempt question No. ***1*** and ***any three*** from the rest.)

1. State whether the following statement is true or false. $1\times7=7$

a) For any linear operator Ton a finite dimensional inner product space $(r, <\cdot, \cdot>)$, we have $(T^*)^* = T$

b) Every non-null finite dimensional inner product space has an orthogonal basis.

c) Every linear transformation is a linear

d) Every finite dimensional vector space V functional isomorphic to V^{**}, its doulde dual.

e) Every square matrix is similar to a Jordan Block matrix.

f) The Gram⁻ Schmidt orthogonalisation process allows us to construct an orthonormal set from an arbitrary set of vectors.

g) If a linear operator is diagnolisable, every generalised eigen vector must necessarily be an eigenvector.

2. Let M be a subspace of a finite dimensional vector space v(f). Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_n\}$ be a basis for v such that M = span $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_m\} m < n$ Let $M^o ; \{f : v \to \Im \mid f(\underset{\sim}{x}) = O \ \forall \underset{\sim}{x} \ \varepsilon \ M$ and f is linear$\}$ be the annihilator of M. Prove that,

M^O = span $\{f_{m+1}, \dots, f_n)$ where $\{f_1, \dots, f_n\}$ is the dual basis corresponding to the basis In other words, prove that

$$\dim(v) = \dim(M) + \dim(M^O) = \dim(v'),$$

where v' is the dual of r(f). 6

3. a) $\begin{bmatrix} 3 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$

b) $\begin{bmatrix} -7 & 8 & 2 \\ -4 & 5 & 1 \\ -23 & +21 & 7 \end{bmatrix}$

For both of the above matrices, find the Jordan form. Is the Jordan form of these matrices unique ?

In each of the above cases, if A is the given matrix and J is its Jordan matrix, find an invertible matrix P such that

$P^{-1} AP = J$ 6

4. Fir each of the following vector spaces v, linear transformations T and vectors Z. Find an ordered basis for the T-cyclic subspace generated by the vector Z.

a) $V = P_3(\mathbb{R}),\ T(f(x) = f''(x),\ \forall f \varepsilon P_3(\mathbb{R}) z = x^3.$

$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 6

b) $V = M_{2\times x}(\mathbb{R}),\ T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A,\ \forall A \varepsilon M_{2\times 2}(\mathbb{R})$ 6

5. Let r = and define <A, B> = true (B*.A) Prove that < . , . > is an inner product on and calculate ||A||, ||B|| and $M_{m\times n}(\mathfrak{I})$, and $<A, B>$ if $A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix},\ B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}$

6. Let T be a linear operator on a finite dimensional inner product space. Suppose the characteristic polynimoal of T splits. Prove that three exists an orthonormal basis β of r such that the matrix $[T]_\beta$ is upper triangular. 6

7. In each of the following, find the orthogonal projection of the vector u on the given subspace w of the inner product space r.

a) $r = \mathbb{R}^2,\ u = (2,6),\ W = \{(x,y) \mid y = 4\infty\}$

b) $r = P(\mathbb{R})$ with the inner product

$$< f, g > = \int_0^1 f(t)\, g(t)\, dt.$$

$u(t) = 4 + 3t - 2t^2$ and $W = P_1(\mathbb{R})$.