(b) Let V be a vector space over a field F. Let f and g be two non-zero functionals defined on V such that $f(x) = 0$ implies that $g(x) = 0$ for all $x \in V$. Show that $g = \lambda f$ for some $\lambda \in F$. 2+3

13 (a) Let A be a 2x2 real matrix with trace 5 and determinant value 6. Find the eigen values of the matrix $B = A^2 - 2A + I_2$.

(b) Show that two similar matrices have same eigenvalue. Is the converse true ? Justify your answer. 2+2+1

14. (a) Show that the eigenvalues of a real symmetric matrix are all real.

(b) Let V be a finite dimensional real vector space and $T : V \to V$ be a linear operator such that $T^2 = I$. Show that the sum of the eigenvalues of T is an integer. 3+2

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Ex./UG/M/CORE6/29/2019

INTER BACHELOR OF SCIENCE EXAMINATION, 2019

(2nd Year, 1st Semester)

MATHEMATICS (HONOURS)

Ring Theory and Linear Algebra - I

Paper : CORE - 06

Time : Two hours Full Marks : 50

GROUP - A (25 marks)

(Ring Theory - I)

Answer any ***five*** questions.

1. Define subring of a ring. With proper justification, give an example of a ring with identity 1_R and a subring S with identity 1_S such that $1_R \neq 1_S$. If R is an integral domain then show that $1_R = I_S$. 2+2+1

2. Define field. Show that every finite integral domain is a field. Is the converse true ? Justify your answer. 1+3+1

3. Show that any ring of order 6 is commutative. Does there exist an integral domain of order 6 ? Justify your answer. 3+2

4. Let R be a ring with identity. With proper justification give an example of a left ideal of R which is not a right ideal of R.

(Turn over)

Let R be an integral domain such that $A \cap B = AB$ for all ideals A and B of R. Show that R is a field. 2+3

5. Define quotient ring. Is the quotient ring of an integral domain always an integral domain ? Justify your answer. What is the quetient field of a finite integral domain ? 2+2+1

6. State first isomorphism theorem of ring. Show that any epimosophism from the ring $(\mathbb{Z}, +;)$ onto itself is an isomorphism. 2+3

7. Let R be a commutative ring with identity. Define prime ideal of R. If every ideal of R is a prime ideal of R then show that R is a field. If R is a finite commutative ring with identity then show that every prime ideal of R is a maximal ideal of R. 1+2+2

GROUP - B (25 marks)

Answer any ***five*** questions.

8. Define vector space. Let $S_1 = \{A \in M_n(\mathbb{R}) : A^T = A\}$ and $S_2 = \{B \in M_n(\mathbb{R}) : B^T = -B\}$. Show that S_1 and S_2 are two subspaces of the vector space $M_n(\mathbb{R})$ such that $M_n(\mathbb{R}) = S_1 \oplus S_2$. 2+3

9. Define basis of a vector space. State a necessary and sufficient condition for a finite dimensional vector space Vover a field F has precisely one basis. Extend the linearly independent set $\{(1,-1,1,-1), (1,1,-1,1)$ to a basis of $\mathbb{R}^4$. 2+1+2

10. (a) State rank-nullity theorem. Let V be a real vector space of dimension d and $T : V \to V$ be a linear mapping with rank r and nullity n. Show that $rn \le \frac{1}{4}d^2$.

    (b) Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be a linear mapping such that $\text{Ker} t = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$. Is T surjective ? Justify your answer. 2+1+2

11. Let V be a vector space over a field F and let W_1, W_2 be two subspaces of V. Show that

$$W_1 / W_1 \cap W_2 \cong (W_1 + W_2) / W_2 .$$

Hence conclude that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$, if W_1 and W_2 are finite dimensional subspaces of V. 3+2

12. (a) Give an example of a linear operator T on $\mathbb{R}^3$ such that $T \neq 0$, $T^2 \neq 0$ but $T^3 = 0$.