4. i) Apply Descarte's rule of signs to find the nature of the roots of the equation $x^6 + x^4 + x^2 + x + 3 = 0$ 2

ii) Solve the equation $x^3 - 12x + 65 = 0$ by Cardan's method. 3

Ex/IM/III/12/19(Old)

First B. Sc. Examination, 2019

(1st year, 1st Semester, Old Syllabus)

Mathematics

Paper : 1.3 (Algebra - I)

Time : Two hours Full Marks : 50

(25 marks for each part)

Use a separate Answer-Script for each part

PART - I

Answer ***any five*** questions.

Let $\mathbb{R}$ denotes the field of all real numbers.

1. Define a vector space.

Let $V = \mathbb{R}^2$. For $(a, b), (c, d) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, define

$(a, b) + (c, d) = (a + 2c, b + 3d)$ and $\alpha(a, b) = (\alpha a, \alpha b)$.

Is V a vector space over $\mathbb{R}$ with these operations ? Justify your answer. 5

2. Define a subspace of a vector space. Show that $\mathbb{R}^n$ is the direct sum of subspaces

$$w_1 = \{(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \mid a_n = 0\}$$

and $w_2 = \{(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n \mid a_1 = a_2 = \cdots = a_{n-1} = 0\}$.

5

[Turn over

3. Solve the following system of linear equations :

$$a_1 + 2a_2 + a_3 - a_4 = 0$$

$$3a_1 + 2a_2 - 3a_3 = 0$$

$$-4a_1 - 4a_2 + 2a_3 + a_4 = 0$$

$$2a_1 \quad -4a_3 \quad = 0$$

5

4. Define a linearly independent subset of a vector space. Let S be a linearly independent subset of a vector space V and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in$ span (S). 5

5. Define a basis of a vector space. Prove that the set of solutions to the system of linear equations :

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$. Find a basis for this subspace. 5

6. Define a linear transformation of vector spaces.

Let V and W be vector spaces of equal (finite) dimension and let $T : V \to W$ be a linear transformation. Prove that the following are equivalent :

a) T is one-to-one

ii) If the system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has a non-zero solution then show that $a^2 + b^2 + c^2 + 2abc = 1$. 2

11. i) Show that the eigenvalues of a real symmetric matrix are all real. 2

ii) Let λ be eigen value of a real skew symmetric matrix A. Show that $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$ 3

12. i) If for given A and B, the matrix equation $AX = B$ has more than one solution then show that it has infinitely many solutions. 2

ii) Let A be a real square matrix of order 2 with trace 5 and determinant value 6. Find the eigen values of the matrix A^{-1}. 3

13. i) What is the multiplicity of the root $x = 1$ of the equation $x^n - nx + ((n-1) = 0(n > 1)$? Justify your answer. 2

ii) If α, β, γ are the roots of $f(x) = x^3 - px^2 + qx - r = 0$, where p, q, r are positive integers and α is the harmonic mean of the roots of $f'(x) = 0$ then show that $\alpha^2 = \beta\gamma$. 3

PART - I

Answer ***any five*** questions. 5×5=25

8. i) Let A be a real skew-symmetric matrix of order n such that $A^2 + I_n = 0$. Show that A is an orthogonal matrix. 2

ii) Let A and B be two square matrices of the same order n such that $AB = I_n$. Show that $BA = I_n$. 2

iii) Let $A = (a_{ij})_{n\times n}$ be a nipotent matrix of index k. Show that $I_n - A$ is invertible, where I_n be the identity matrix of order n. 1

9. i) Let A be a square matrix of order n. Show that $A(\text{adj}A) = (\det A)I_n$. Hence show that $\det(\text{adj}A) = (\det A)^{n-1}$ if $\det A \neq 0$. 2

ii) By using Laplace's expansion, show that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2$$ 3

10. i) Let A be a square matrix of order 3 such that A is not a symmetric matrix. Show that rank of the matrix $A - A^T$ is 2. 3

b) T is onto

c) rank (T) = dim(V). 5

7. Let A and B be two $n \times n$ real matrices such that det $B \neq 0$. Show that A and BAB^{-1} have some eigen values. 5