c) If $u_n > 0 \; \forall n \in \mathbb{N}$ and $\underset{n\to\infty}{\text{lt}}\, n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = l$,

show that $\sum u_n$ is convergent if $l > 1$. 4+3+3

11. a) Find the region of Convergence of the series

$$x + \frac{2^2 x^2}{\lfloor 2} + \frac{3^3 x^3}{\lfloor 3} + \frac{4^4 x^4}{\lfloor 4} + \ldots \; (x > 0).$$

b) If $\{x_n\}$ is a sequence of non-zero real numbers and $\underset{n\to\infty}{\text{lt}}\, x_n = x_* \; (\neq 0)$ then prove that $\underset{n\to\infty}{\text{lt}}\, \dfrac{1}{x_n} = \dfrac{1}{x_*}$.

c) Define limit of a sequence.

d) Show that $\underset{n\to\infty}{\text{lt}} \left(\dfrac{\sin n}{n}\right) = 0$. 5+3+1+1

Ex/UG/Sc./Core/Math/TH/03/2019

First B. Sc. Examination, 2019

(1st Year, 2nd Semester)

Mathematics (Honours)

Core - 3

Real Analysis

Time : Two hours Full Marks : 50

Use a separate Answer-Script for each part

(25 marks for each part)

PART - I

(Answer ***any five*** questions)

1. i) Let X be any set. Then show that Card X < Card P(X).

   ii) Show that (a, b] and [c, d) have the same cardinality, where a, b, c d are real numbers and a < b, c < d. 3+2

2. Let F be an Archimedean ordered field. Show that if F satisfies Cantor's nested interval property then F satisfies lub property. 5

3. Find the derived set of the following sets in R :

   i) $\left\{\dfrac{1}{2^n} + \dfrac{1}{3^m} + \dfrac{1}{5^p} : m, n, p \in N\right\}$;

   ii) $\left\{m + n\sqrt{2} : m, n \in Z\right\}$ 2+3

4. Let $A, B \subset R$. Then prove that $\overline{A \cap B} \subset \bar{A} \subset \bar{B}$. Give example to show that $\overline{A \cap B} \neq \bar{A} \subset \bar{B}$. 3+2

5. Prove that in R finite intersection of open sets is open. Give examples to show that arbitrary intersection of open sets may not be open. 3+2

6. Let S be a non-empty subset of R. If S is a clopen set then show that $S = R$. 5

7. Show that if a subset F of R is compact then every sequence in F has a convergent subsequence in F. Using this justify that the set Q of rationals is not compact. 3+2

PART - II (25 Marks)

Answer ***Q. No. 8*** and any ***two*** from the rest

8. a) Show that the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ is divergent.

b) Give an example of an unbounded above, bounded below sequence but does not diverge to $+\infty$. Justify your answer. 3+2

9. a) State and prove Cauchy's general principle of convergence of a sequence.

b) If $u_n > 0 \; \forall \; n \in \mathbb{N}$ and $\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = l$, then show that $\sum u_n$ is divergent if $l < 1$.

c) Test the Convergence of the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ 5+3+2

10. a) If $\{u_n\}$ is a decreasing sequence of positive real numbers with $\lim_{n\to\infty} u_n = 0$, then show that $\sum (-1)^{n+1} u_n$ is convergent.

b) A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ $\forall n \geq 1$ and $0 < u_1 < u_2$. Show that the sequence $\{u_n\}$ converges to $\frac{u_1 + 2u_2}{3}$.