Ex./UG/Sc/CORE-2/TH/50/2019

BACHELOR OF SCIENCE EXAMINATION, 2019

(1st Year, 1st Semester)

MATHEMATICS(Honours)

Group Theory - I

Paper : CORE - II

Time : Two hours Full Marks : 50

Symbols have their usual meanings.

Answer any ***five*** questions.

1. (a) Define a group. Let G be a group and $a, b \in G$ such that

   $ab^3 a^{-1} = b^2$ and $b^{-1} a^2 b = a^3$

   Show that $a = b = e$, where e is the identity of the group. 5

   (b) Let G be a group and $x \in G$ such that $o(x) = mn$ where m, n are two relatively prime positive integers. Show that there exist $y, z \in G$ such that $x = yz = zy$ and $o(y) = m$, $o(z) = n$. 5

2. (a) Define even and odd permutations in a permutations group. Prove that in a finite permutation group the number of even permutations is same as the number of odd permutations. 5

(Turn over)

(b) Prove that the permutations group S_n is generated by the set $\{(1\ 2), (1\ 3), (1\ 4), \ldots (1\ n)\}$ where $n \geq 2$. 5

3. (a) Define a subgroup of a group. Let G be a group and H be a subgroup of G. Show that for any $g \in G, K = gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G and $|K| = |H|$. 5

(b) Define a cyclic group. Let G be a finite cyclic group of order n. Then show that for every positive divisor d of n, there exists a unique subgroup of G of order d. 5

4. (a) Show that any finite subgroup of the group of nonzero couplex numbers under multiplication is a cyclic group. 5

(b) Let G be a group and H be a subgroup of G. Show that the following conditions are equivalent :

(1) $gH = Hg$ for all $g \in G$

(2) $gHg^{-1} = H$ for all $g \in G$

(3) $gHg^{-1} \subseteq H$ for all $g \in G$

(4) $g\,h\,g{-}1 \in H$ for all $g \in G$, for all $h \in H$. 5

5. (a) Define the index of a subgroup in a group. Let G be a finite group and H be the only subgroup of G with index n. Show that H is a normal subgroup of G. 5

(b) State and prove the first isomorphism theorem for groups. 5

6. (a) Define homomorphisms of groups. Let $f : G \to G_1$, be an onto homomorphism of G. If H is a normal subgroup of G, then show that f(H) is a normal subgroup of G_1. 5

(b) Show that

$$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong (\mathbb{R}^*, .),$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. 5

7. (a) Let m, n be two positive integers greater than 1. Prove that

$$\mathbb{Z}_m \times \mathbb{Z}n \cong \mathbb{Z}_{mn}$$

if and only if gcd (m,n) = 1. 5

(b) Define the dihedral group D_4. Determine the center of D_4. 5

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