Failure of Laminated Composite Shell Roofs

Thesis Submitted

by

Arghya Ghosh

Doctor of Philosophy

(Engineering)

Civil Engineering Department
Faculty Council of Engineering & Technology
Jadavpur University
Kolkata, India
2019

JADAVPUR UNIVERSITY

KOLKATA – 700 032, INDIA

INDEX NO. 271/14/E

1. **Title of the Thesis:** Failure of Laminated Composite Shell Roofs.

2. **Name, Designation & Institution of the Supervisor/s:** Dr. Dipankar Chakravorty, Professor, Jadavpur University.

3. **List of Publications:**

   International Journal Papers:

a) A. Ghosh, D. Chakravorty, First Ply Failure Analysis of Laminated Composite Thin Hypar Shells Using Nonlinear Finite Element Approach, *Thin Walled Structures*, 2018, Vol. 131, pp. 736-745.
b) A. Ghosh, D. Chakravorty, Nonlinear First Ply Failure Analysis of Composite Skewed Hypar Shells Using FEM, *Structural Engineering and Mechanics,* 2018, Vol. 68, No. 1, pp. 81-94.
c) A. Ghosh, D. Chakravorty, First-Ply-Failure Performance of Composite Clamped Spherical Shells, *Mechanics of Composite Materials,* 2018, Vol. 54, No. 2, pp. 191-206.

   International Conference Publications:

a) A. Ghosh, D. Chakravorty, Nonlinear First Ply Failure of Composite Hypars – A Finite Element Application, *Structural Engineering Convention 2018 (11th International Conference), Jadavpur University*, December 19-21, 2018.
b) A. Ghosh, D. Chakravorty, Failure Analysis of Civil Engineering Composite Shell Roofs, *11th International Symposium on Plasticity and Impact Mechanics, IIT Delhi,* December 11-14, 2016.
c) A. Ghosh, D. Chakravorty, Review on Failure Study of Laminated Composite Plates and Shells, *International Conference on Frontiers of Mechanical Engineering, Materials and Energy, Beijing, China*, December 20-21, 2012.

4. **List of Patents**: No.

5. **List of Presentations in International:**

a) A. Ghosh, D. Chakravorty, Nonlinear First Ply Failure of Composite Hypars – A Finite Element Application, *Structural Engineering Convention 2018 (11th International Conference), Jadavpur University*, December 19-21, 2018.

b) A. Ghosh, D. Chakravorty, Failure Analysis of Civil Engineering Composite Shell Roofs, *11th International Symposium on Plasticity and Impact Mechanics*, *IIT Delhi*, December 11-14, 2016.

CERTIFICATE FROM THE SUPERVISOR

*This is to certify that the thesis entitled "***Failure of Laminated Composite Shell Roofs***" submitted by Shri Arghya Ghosh, who got his name registered on 18.02.2014 for the award of Ph.D. (Engineering.) degree of Jadavpur University is absolutely based upon his own work under the supervision of Dr. Dipankar Chakravorty and that neither his thesis nor any part of the thesis has been submitted for any degree/diploma or any other academic award anywhere before.*

......……………………………………

Dr. Dipankar Chakravorty

Professor, Civil Engineering Department,

Jadavpur University, Kolkata.

CERTIFICATE

The foregoing thesis is hereby approved as a creditable study of an engineering subject carried out and presented in a manner satisfactory to warrant its acceptance as a prerequisite to the Ph.D. degree (Engineering) for which it has been submitted. It is understood that by this approval the undersigned do not necessarily endorse or approve any statement made, opinion expressed or conclusion drawn therein, but approve the thesis only for the purpose for which it is submitted.

Final Examination for

Evaluation of Thesis

1. ______________________________

2. ______________________________

3. ______________________________

(Signatures of Examiners)

ACKNOWLEDGEMENT

The author owes his most sincere thanks and profound gratitude to his supervisor, Prof. Dr. Dipankar Chakravorty for the indispensable advice and inspiration rendered at each phase of the research work. His valuable suggestions, constructive criticism and critical evaluation throughout the research work are thankfully acknowledged.

The author is grateful to the Head of the Department and to all the faculty members of Civil Engineering Department for their cooperation. The author also expresses his gratitude to staff members of the departmental library and the central library of Jadavpur University.

The excellent cooperation and support of the co-researchers are thankfully acknowledged.

It would not have been possible for the author to pursue the research work without the constant effort, sacrifices and encouragement of his parents Mr. Amarnath Ghosh and Mrs. Mita Ghosh, from the very first day of his student life till date. The author also like to mention the name of his younger sister Ankita who always kept him happy and energetic. Her positive thinking, encouragement and mental support are always there from the beginning of the work till date.

The author is thankful to all those, whose efforts either directly or indirectly have contributed substantially during the course of this thesis work.

Signature of the Candidate:

Date:

Place: Kolkata

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOTATIONS

The following notations are used in the text of the thesis. The symbols which are not listed in the following are explained where they appear for the first time.

1. VARIABLES IN MATRIX FORM

$[B]$	Linear part of strain-displacement matrix
$[B']$	Nonlinear part of strain-displacement matrix
$[E]$	Laminate stiffness matrix
$[J]$	The Jacobian matrix
$[K_T]$	Tangent stiffness matrix
$[K_S]$	Secant stiffness matrix

2. VARIABLES AS VECTORS

$\{d_e\}$	Element nodal displacement vector
$\{F\}$	Generalized stress resultant vector
$\{P\}$	External generalized forces
$\{\varphi\}$	Resultant of internal forces and external generalized forces

3. OTHER VARIABLES

X, Y, Z	Global coordinate axes.
1, 2, 3	Local coordinate axes.
a,b	Length and width of shell in plan
c	Rise of skewed hypar shell
h	Shell thickness
A	Area of shell
q	Intensity of uniformly distributed load
R_{xy}	Radius of cross curvature of hypar shell
$\varepsilon_x, \varepsilon_y$	In-plane strains along X and Y axes of the shell.

γ_{xy} , γ_{xz}, γ_{yz}	In-plane and transverse shear strains, respectively.
$\kappa_x, \kappa_y, \kappa_{xy}$	Curvatures of the shell due to loading.
N_x, N_y, N_{xy}	In-plane normal and shear force resultants.
M_x, M_y, M_{xy}	Bending moment and torsional moment resultants.
Q_x, Q_y	Transverse shear resultants.
np	Total number of ply.
z_k , z_{k-1}	Top and bottom distances of the k^{th} ply from mid-plane of a laminate.
N_1 to N_8	Shape functions for first to eight nodes of the element
A_0-A_7	Constant terms of displacement polynomial
$\sigma_1, \sigma_2, \tau_{12}$	Normal stresses along 1 and 2 axes and shear stress acting on 1-2 surface of a lamina, respectively.
$\varepsilon_1, \varepsilon_2, \gamma_{12}$	In-plane normal strains along 1 and 2 axes and shear strain acting on 1-2 surface of a lamina, respectively.
E_{11}, E_{22}, E_{33}	Modulus of elasticity along the directions 1, 2 and 3.
G_{12}, G_{13}, G_{23}	Shear modulus of a lamina in 1-2, 1-3, and 2-3 planes corresponding to the local axes of that lamina, respectively.
ν_{ij}	Poisson's ratio.
$\sigma_{1T}^u, \sigma_{2T}^u$	Ultimate normal tensile stresses along 1 and 2 direction, respectively.
$\sigma_{1C}^u, \sigma_{2C}^u$	Ultimate normal compressive stresses along 1 and 2 direction, respectively.
$\varepsilon_{1T}^u, \varepsilon_{2T}^u$	Ultimate normal tensile strains along 1 and 2 direction, respectively.
$\varepsilon_{1C}^u, \varepsilon_{2C}^u$	Ultimate normal compressive strains along 1 and 2 direction, respectively.
$\tau_{12}^u, \tau_{13}^u, \tau_{23}^u$	Ultimate shear stress values in 1-2, 1-3, and 2-3 planes corresponding to the local axes of that lamina, respectively.
$\gamma_{12}^u, \gamma_{13}^u, \gamma_{23}^u$	Ultimate shear strain values in 1-2, 1-3, and 2-3 planes corresponding to the local axes of that lamina, respectively.
θ	Orientation of fibers in a lamina with respect to the global x-axis of shell

ABSTRACT

The aesthetically appealing shell roofs have gained wide popularity in civil engineering applications due to their ability to cover large column free unsupported areas with lesser thicknesses compared to those required for plates. In several civil engineering projects, apart from regular shell geometries (cylindrical and spherical), complicated shell forms like elliptic paraboloids, skewed hypars and conoids have been preferred by the practicing engineers. In recent years, the laminated composite shell structures have achieved immense popularity in aerospace, offshore, mechanical and especially in civil engineering applications. Due to their superior mechanical properties such as high strength to weight ratio and high stiffness to weight ratio, excellent corrosion and thermal resistances, ease of fabrication, these shell structures have drawn special attention from the practicing engineers. It is observed from the detailed review of literature that the first ply failure strengths of industrially preferred and aesthetically appealing doubly curved synclastic spherical and doubly ruled anticlastic skewed hypar shell forms have not been studied in depth by the researchers. Thus, it is felt that failure characteristics of these forms with different laminations need to be studied in details to apply them efficiently as roofing units and to popularize their use further in the industry.

In the present work, the finite element method is employed to study the first ply failure behaviour of laminated composite shell roofs considering geometrically linear and nonlinear strains. An eight noded curved quadratic Serendipity element having five degrees of freedom at each node is used to model the shell surface. The nonlinear static equilibrium equation is satisfied through the Newton – Raphson iteration process. Different well-established failure criteria like maximum stress, maximum strain, Hoffman, Tsai-Hill, Tsai- Wu, Hashin and Puck failure criteria are used to obtain the linear and nonlinear first ply failure load values.

The finite element code developed for assessing the first ply failure load is validated through solutions of benchmark problems before applying it to generate new results.

Furthermore, a set of author's own problems are taken up with different parametric variations to arrive at some meaningful conclusions regarding the first ply failure behaviour of laminated composite spherical and skewed hypar shell roofs. Apart from failure criteria governed by strength, the author also studies the failure characteristics of composite shells employing the serviceability criterion of deflection. Failure modes or tendencies, failed plies and failure zones are furnished in different chapters of this thesis. The results obtained are analyzed and post-processed from different engineering standpoints to extract meaningful conclusions regarding failure behaviour of the composite shells and to arrive at important design guidelines which are expected to be beneficial for practicing engineers.

The literature review, mathematical formulation, details of investigations are presented systematically as different chapters in the thesis. Scope of future research work are also included in the end of this thesis.

ORGANISATION OF THE THESIS

The thesis is divided into nine chapters and one appendix. Chapter 1 contains the general introduction and some of the salient features of the present investigation. In Chapter 2, the review of the existing literature is reported meticulously. The available literature is thoroughly analyzed and critically discussed to identify the lacunae present therein. Based on the elaborate review exercise, the actual scope of the present work is outlined and presented in Chapter 3. Having defined the scope, Chapter 4 contains the mathematical formulation employed in the present analysis. Results are obtained for some specific benchmark problems solved by earlier investigators to establish the validity of the present formulation. A wide spectrum of author's own problems are taken up and solved with different practical parametric variations and are discussed in details in Chapters 5 to 8. Chapter 5 deals with first ply failure problems of clamped laminated composite spherical shells using geometrically linear strains. Chapter 6 contains first ply failure problems of clamped skewed hypar shells using geometrically linear and nonlinear strains with varying planform and thickness. In Chapter 7 linear and nonlinear first ply failure of clamped hypar shells with varying curvature including guidelines to non-destructive test monitoring are presented systematically. Chapter 8 furnishes the nonlinear first ply failure characteristics of simply supported hypar shell roofs. The future scope of research is indicated in Chapter 9. The references are presented in the Appendix.

KEYWORDS

- Shell
- Skewed Hypar
- Spherical
- Laminated Composite
- Finite Element Method
- Nonlinear Analysis
- First Ply Failure
- Failure Modes
- Failure Zones
- Design Guidelines

Chapter 1

INTRODUCTION

1.1 GENERAL

An arbitrarily curved structural surface which is capable of resisting superimposed loads by combined in-plane thrust and bending of the surface is a shell. Eggs, nuts, animals skulls are natural examples of shell units and man came to understand the load carrying potentialities of shell surfaces from very early days. Naturally, many manmade constructions use the shell surfaces starting from the earliest examples of Chou vases and Greek urns followed later by the submarine in 1620, the earliest pressure cooker in 1688 and thin walled pressure vessels and pneumatic tyres in the eighteenth and nineteenth centuries. As the course of civilization went on flowing, the civil engineers use shells as parts of roof covers, chimneys, water tanks, cooling towers etc. getting the idea from very old structures like the Pantheon of Rome built in AD 125, the Mosque of Santa Sophia, Istanbul, built in AD 538.

The hunt in the search of advanced materials resulted in the introduction of laminated composite in the different weight sensitive disciplines of engineering and the civil engineers too started using these advanced materials for shell fabrication. The plastic domes at Benghazi and at Dubai Airport are earliest examples of composite shells built in the second half of the last century. In order to successfully apply these materials in the construction industry the characteristics of the composite are to be very clear to the practicing engineers. A look at the volume of literature on shells clearly shows that researchers on around the planet are engaged in characterization of the laminated composite.

1.2 PRESENT PROBLEM – ITS IMPORTANCE IN RESEARCH

The laminated composites have a unique property of getting tailored by different prefixed fiber orientation which suits to the different specific requirements in different application areas. Despite having very high specific strength and stiffness properties, these materials are weak in transverse shear and fabrication defects may lead to failure at service which may also happen otherwise due to overloading. In laminated composites the first ply failure indicates initiation of failure no doubt but the structural units still retain sufficient reserve strength and so the first ply failure phenomenon may remain unnoticed. The failure may initiate at the fiber level or at the matrix and different researchers have proposed different failure theories which are stress based or strain based or interactive in nature. Failures of laminated composite plates have received some attention from researchers but similar studies on composite shells, particularly using the nonlinear strains, are very few in the published research reports. The linear first ply failure approach is easier to implement and proper choice of factor of safety may form a basis of design recommendation while on the other hand an analysis using the nonlinear strains is a more appropriate approach and is expected to yield improve results. The nonlinear approach requires greater involvement from implementation point of view. Most of the research reports that have been published on the analysis of composite shells utilize the finite element approach which are now become relatively easier to implement with the advent of high speed computers. Keeping the above mention development in view, it is felt that failure analysis of civil engineering composite shells is an area that needs greater attention and this defines the motivation behind the present research.

Chapter 2

LITERATURE REVIEW

2.1 GENERAL

Research related to shell structures have a long history and the literature that has accumulated in this area are vast. If one closely examines the published papers then it is found that the recent focus is on laminated composite shells. The journey that began with static analysis of isotropic shell surfaces gradually encompassed free and forced vibration studies using closed form solutions and numerical approaches like the finite element method. Numerical studies on laminated composite shells are published in different technical journals and failure characteristics is one of the areas which is focused on recently. The review of literature that is presented in this chapter discusses the historical review of shell research in Section 2.2, mentions the latest literature on shells in Section 2.3 and furnishes the papers on failure of laminated composites separately in Section 2.4. Section 2.5 discusses critically the entire body of literature to bring out the broad scope of research that is yet to be covered.

2.2 HISTORICAL REVIEW OF SHELL RESEARCH

Construction, design and research of shell structures have long history and research reports covering various parametric studies are vast. Sechler (1974) in his book presented the historical course of shell research and design. In this section the gradual course of shell research is presented. For clear understanding of the systematic and chronological development of the different aspects of shell research, this section is further divided into distinct parts.

2.2.1 Review on Thin Shell Theory

Three dimensional structure, produced by two curved surfaces, thickness being small compared to plan dimensions, may be classified as shell. One of the most common ways of classifying shell theories is on the basis of thickness of shell. A thin shell is defined as a shell with a thickness which is small compared to its other dimensions and its radii of curvatures. A number of researchers were interested to analysis the thin shells from the first half of nineteenth century. Lame and Clapeyron (1833) analyzed shells of revolution under symmetric loading. Aron (1874) was the first to analyze the shell problem using the general theory of elasticity and obtained the components of bending and twist in a form that were independent of the tangential displacements, which was based on the Kirchhoff's hypothesis (1876) for plates. However a new vista opened when Love (1888) formulated his theory of shells and observed that the Kirchhoff's hypothesis, where normal to the middle surface before deformation was assumed to remain normal to the middle surface even after deformation was not strictly correct. Love was the first investigator to present a good approximation of thin shell bending behaviour, based on the classical theory of elasticity.

2.2.2 Review on Membrane Theory

The membrane theory proposed by Lame and Clapeyron (1833) neglected all moment expressions and the flexural strains were assumed to be zero or negligible compared to the axial strains. The general form of equations of this theory was proposed by Beltrami (1881).

2.2.3 Review on Membrane and Bending Theory

The membrane theory used by earlier researchers had certain limitations that motivated Marguerre (1938), Vlasov (1947) and Reissner (1955) to develop the general theory of shell bending. Then onwards, both the membrane and bending theories were found to be applicable to analyze different problems depending upon the nature of the shell forms. Besides static analysis, vibration characteristics of shallow spherical shells were studied by Reissner (1946, 1955) and other authors [Johnson and Reissner (1958), Kalnins and Naghdi (1960), Hoppmann (1961)] using analytical methods. Nazarov (1949) and Wang (1953) concentrated on the bending theories of shells, while Vlasov (1964) continued to work on the membrane theory. Flügge and Conrad (1956) proposed more suitable expressions for the set of differential equations and improved the bending theory of Marguerre (1938). The membrane theory of hyperbolic and elliptic paraboloid was extended by Parme (1956) to obtain the stresses expressing the shell surface in Cartesian co-ordinates. With the progress of shell research, different theories were proposed by Sanders (1959), Koiter (1960), Naghdi (1963) and Goldenveizer (1968). The theories due to Koiter (1960) and Budiansky and Sanders (1963) appeared to be more popular because they were consistent with the basic Love-Kirchhoff hypotheses. Among the other theories proposed, membrane theories of Fischer (1960), Flügge and Geyling (1957), Soare (1966), Ramaswamy and Rao (1961) and bending theories of Bongard (1959) and Chetty and Tottenham (1964) are some of the important mile stones. Investigations into the bending theory of shells were carried out by Iyengar and Srinivasan (1968) employing equations of Bongard (1959) by expressing the displacement functions satisfying the boundary conditions and by Ramaswamy (1971) expressing the problem in terms of three coupled differential equations.

2.2.4 Review on Shallow Shell Theory

The development of shallow shell theory is principally credited to Marguerre (1938), Reissner (1946), Donnell (1933) and Vlasov (1964). Vlasov (1964) considered the shell as shallow if its rise is not more than one-fifth times of its smallest plan dimension. Munro (1961) derived a general heterogeneous equation for the linear analysis of thin shallow second order shells of positive, zero or negative Gaussian curvature. The bending analysis of translational shells of rectangular planforms and hypars was reported by Apeland and Popov (1961). Russel and Gerstle (1967, 1968) further extended the theoretical and experimental investigations of Brebbia (1966) related to umbrella-like hyperbolic paraboloid shells. Forced vibrations of axisymmetric shallow spherical shells were reported by Reisman and Culowski (1968). The performance of hypars in elastic and ultimate ranges was studied experimentally by Bandyopadhyay and Ray (1971, 1972). The buckling criterion of different shell forms was studied by Abel and Billington (1972), while Gergely (1972) used the energy principle to estimate the buckling load. Vibration and stability of laminated composite cylindrical and other shallow shell forms were reported by Dong (1968) and Dulaska (1969). Dynamic characteristics of doubly curved shells with triangular finite elements were studied by Olson and Lindberg (1968), while Brebbia and Hadid (1971) solved conoidal shell problems using rectangular finite element. Leissa and his colleagues [Leissa et al. (1981, 1983), Narita and Leissa (1984) and Qatu and Leissa (1991a)] employed the Ritz method to determine the free vibration characteristics of shallow shells. Natural frequencies and mode shapes were presented for saddle, cylindrical and spherical shells of rectangular planform with different parametric variations. Vibrations of simply supported shells of elliptic planform were reported by Coleby and Majumdar (1982). Xiang-Sheng (1985) studied forced vibrations of elastic shallow shells under moving loads. Ghosh and Bandyopadhyay (1989) worked on bending analysis of isotropic conoidal shells while Qatu (1989) investigated bending and free vibrations of laminated shallow shells and Chao and Tung (1989) studied response of clamped orthotropic hemispherical shells subjected to step pressure and blast loads. Chandrashekhara (1989) used a nine noded isoparametric finite element for free vibration analysis of composite cylindrical and spherical shells by extending Sanders' theory.

2.2.5 Review on Laminated Composite Shells

While the theory on shell structures was being simplified and improved from time to time by many researchers, another group of investigators started developing exotic materials with high strength and stiffness properties. As a result, laminated composites were introduced in engineering application from the second half of the last century. The popularity of composite laminates is mainly attributed to their high stiffness/strength to weight ratio, superior fatigue resistance, lesser susceptibility to be damaged by weathering actions and, most significantly, the flexibility to tailor its stiffness and strength properties by varying fiber orientations and lamina stacking sequences. This resulted in the use of laminated composite materials to fabricate shell forms. As a result, bending analysis of laminated composite shells emerged as a new field of research. Hildebrand et al. (1949) first applied shell equations to an anisotropic material. The first analysis that combined bending-stretching coupling was due to Ambartsumyan (1953). He looked into the behaviour of laminated orthotropic shells rather than laminated anisotropic shells. Subsequently, Dong et al. (1962) introduced a theory of thin shells laminated with anisotropic material and Stavsky (1963) also proposed this theory for plates of anisotropic material. A discussion on orthotropic shell theory was reported by Vlasov (1964). Widera and his colleagues (1970, 1980) derived a first-approximation theory for the unsymmetrical deformation of non-homogeneous, anisotropic, elastic cylindrical shells by integrating the elastic equations asymptotically. For a homogeneous, isotropic material the theory reduced to that due to Donnell. Gulati and Essenberg (1967) and Zukas and Vinson (1971) incorporated the transverse shear deformation effects in composites. Whitney and Sun (1973, 1974) introduced a shear deformation theory which includes both transverse shear deformation and transverse normal strain together with the expansional strains.

A number of researchers like Reddy and Chao (1981), Bert and Reddy (1982), Bert and Kumar (1982) and Reddy (1984) reported closed form solutions for laminated composite shells. Bert and Reddy (1982) and Bert and Kumar (1982) worked on simply supported cylindrical and spherical shells subjected to sinusoidal distribution of transverse load while Reddy (1984) reported exact bending and vibration solutions for moderately thick laminated composite cylindrical and spherical shells. The author extended Sanders' shell theory and showed, unlike plates, antisymmetric angle ply laminates with simply supported boundaries do not admit exact solutions. Reddy (1984) and Vinson and Sierakowski (1986) first worked on anisotropic laminated composite shell structures including transverse shear deformation theories. Noor and Burton (1989, 1990), Kapania and Raciti (1989a, 1989b) and Kapania (1989) reported comprehensive review articles pertaining to composite laminated plates and shells.

Vibration characteristics of thick cylindrical shallow shells of rectangular planform for various boundary conditions and shell geometries were analyzed using higher order shear deformation theory in the article of Lim et al. (1995). A simple higher order shear deformation theory of laminated composite shells was developed by Xiao-ping (1996). Using Love's first order geometric approximation and Donnell's simplification, the governing equations of shallow shells were established by him. A vibration and damping analysis based on individual layer deformation was presented by Khatri and Asnani (1996) for axisymmetric vibrations of laminated composite and fiber reinforced composite conical shells. In their study the shell forms which were discussed were mostly of non-civil engineering applications. Free vibration analysis of laminated anisotropic shallow shells based on first order transverse shear deformation theory was done by Wang and Schweizerhof (1997). They used boundary domain element method. The effects of both in-plane and rotary inertia on the natural frequencies were included. Gendy et al. (1997) examined the effect of large spatial rotations on the geometric stiffness for stability analysis and inertia operators for vibration of laminated plates and shells, in conjunction with the recently developed mixed finite element formulation with lower order displacement/strain interpolation. At the same time Aksu (1997) introduced a finite element formulation for the free, undamped vibration analysis of a shell of general shape with a curved isoparametric trapezoidal element, including thickness shear deformations and without neglecting *z/R* in comparison with unity. He considered the rotary inertia effects in consistent mass matrix. The shell element with eight nodes and forty degrees of freedom is applicable for both thin and moderately thick shell. Large amplitude free vibration behaviour of doubly curved shallow open shells with simply supported edges was presented by Shin (1997). The free vibration characteristics of isotropic and laminated orthotropic spherical caps were carefully examined by Gautham and Ganesan (1997). At the same time Sivasubramonian et al. (1997) reported free vibration of curved panels with cutouts and the free vibration of thin shallow shells with slits on rectangular planform was studied by Crossland and Dickinsion (1997). Chakravorty et al. (1995a, 1995b, 1996) studied free vibration of laminated composite conoidal, hyperbolic paraboloid and elliptic paraboloidal shells for varying boundary conditions, aspect ratios, thicknesses and lamination values.

2.2.6 Review on Different Methods of Shell Analysis Including Finite Element Method

To solve shell problems, different solution techniques including variational methods like Ritz, Collocation, subdomain (Biezeno-Koch), least squares, Galerkin, Kantorovich, Trefftz method and finite element methods were proposed by shell researchers. The area of shell research received a new dimension when Greene et al. (1961) suggested the use of finite element method utilizing the flat elements. These elements were found to be good for a limited class of problems, but when the membrane-flexural stiffnesses were strongly coupled, the element failed completely. Singly curved element was first developed by Grafton and Strome (1963). Since then the finite element method also acquired a prominent position as an efficient analytical tool in the area of shell research.

The finite element method, due to its ability to cater to any arbitrary geometry, loading and boundary conditions started being used for the analysis of laminated composite shells. Earlier works in this aspect were due to Lakshminarayana and Viswanath (1976) and Rao (1978) who developed a high precision triangular cylindrical and a rectangular shallow finite element respectively. Since then the finite element method began to be used widely and improved elements started being developed and used from time to time. It is interesting to note that researchers concentrated on parallel developments of refined theories as well as better finite elements. Greene et al. (1968) used a doubly curved finite element for the dynamic analysis of shells. A four noded rectangular shallow shell element having three radii of curvature and five degrees of freedom per node was used by Connor and Brebbia (1967). The concept of isoparametric curved elements, where the displacement and co-ordinates are interpolated from the nodal values by the same set of shape functions, was first introduced by Ergatoudis et al. (1968) and applications of finite elements to the analysis of plates and shells were reported by Gallagher (1969). Ahmad et al. (1970) introduced the concept of treating shell element as a special case of three-dimensional analysis. Dhatt (1970) solved thin shell problems with curved triangular element using discrete Kirchhoff's hypothesis and translational shells were analyzed by Choi and Schnobrich (1970) using the finite element method. Reddy (1981) gave an elaborate presentation of the evolution of different elements used in shell analysis dealing with the finite element modeling of composite plates and shells.

Simultaneous works were continued applying different analysis methods. Qatu and Leissa (1991a) obtained natural frequencies of cantilevered doubly curved laminated composite shells using Ritz method. The effects of various parameters including material, number of layers, fiber orientation and curvature on natural frequencies were studied. They also studied the vibrations of twisted composite cantilever plates having the radius of cross curvature [Qatu and Leissa (1991b)]. Here studies of first six mode shapes, each for seven fiber orientation angles and four angles of twist showed that all the mode shapes were symmetric or anti-symmetric about the x-axis (axis perpendicular to the clamped edge) at fiber orientation angle 0^o and 90^o. For other angles of fiber orientations the symmetry or anti-symmetry of the mode shapes was lost due to coupling between the modes. The work further showed that maximum fundamental frequencies were observed when the fibers were perpendicular to the clamped edge. The work acted as a strong foundation for further studies of mode shapes. Tsai and Palazotto (1991) reported the nonlinear free and forced vibrations of isotropic and composite shells using a curved quadrilateral finite element. Chia and Chia (1992) solved problems of nonlinear vibration of moderately thick composite spherical shells by the method of harmonic balance. Dey et al. (1994) investigated the bending analysis of laminated composite elliptic paraboloid and conoid. Sheinman and Reichman (1992) studied the buckling and vibration characteristics of shallow composite shell panels by employing Ritz method. Touratier (1992) developed a refined theory of composite shallow shells in which a simple shear deformation theory considered cosine distribution of transverse shear strains and tangential stress free boundaries. Bending and vibration of some isotropic and composite shells were studied. An excellent review work was done by Qatu (1992) discussing the developments of shallow shell vibration research. Hwang and Foster (1992) presented free vibration results of axisymmetric shallow spherical shell with a circular hole. Both analytical and finite element solutions were obtained and mutually compared. A finite strip analysis of diaphragm supported composite shells was conducted by Mohd and Dawe (1993). Vibrations of isotropic shallow shells with two adjacent edges clamped and others free were investigated by Qatu and Leissa (1993), using Ritz method. Bhaskar and Varadan (1993) studied about inter-laminar stresses in composite shells under transient loading using Navier approach together with Laplace transform technique, while Ritz minimization procedure was used by Liew and Lim (1994) to analyze vibrations of perforated isotropic shallow shells. Kant et al. (1994) investigated shell dynamics with three dimensional degenerated finite elements. Cylindrical and spherical cap problems were solved. Sathyamoorthy (1994) reported vibrations of moderately thick spherical shells with large amplitudes using Galerkin method. Behaviour of spherical shells subjected to periodic load using a shear flexible element was investigated by Ganapathi et al. (1994). Different aspects of laminated composite shell behaviour were taken up by several investigators. Jing et al. (1995) studied bending behaviour while Ye and Soldatos (1995) investigated buckling characteristics. Mizusawa and Kito (1995), Sathyamoorthy (1995), Piskunov et al. (1994) and Tessler et al. (1995) put their emphasis on vibrations of laminated composite shells. Ye and Han (1995) used semi-analytical method to investigate a nonlinear bending and buckling behaviour of polar orthotropic shells of revolution. Noor and Burton (1989, 1990) and Burton and Noor (1995) presented excellent review work where they focused on the assessment of computational model used by different authors to analyze multilayered composite plates and shells and sandwich panels. The aspect of shear deformation was specially highlighted in one of their papers. The Ritz method with algebraic polynomials was used by Qatu (1996) and he obtained the natural frequencies of cantilevered shallow shells having triangular and trapezoidal planforms. Free vibrations of laminated composite, noncircular thick cylindrical shells by exact solution were analyzed by Suzuki, Shikanai and

Leissa (1996). Schokker et al. (1996) concentrated on dynamic buckling behaviour of composite shells.

2.3 LITERATURES ON LATEST SHELL RESEARCH

For last few years in the field of shell research, we see applications of different shell theories, different analytical methods applicable to different types of shell structures. Researchers like Chun and Lam (1998), Tan (1998), Korjakin et al. (1998), Lim et al. (1998), Laura et al. (1999), Sivakumar et al. (1999), Sivasubramonian et al. (1999), Reddy and Palaninathan (1999), Hu and Tsai (1999a), Yadav and Verma (2001), Anlas and Goker (2001), Korjakin et al. (2001), Lee and Han (2001) investigated different dynamic aspects of isotropic and composite plates and shells. Free and forced vibration analyses of laminated singly curved and doubly curved shells with concentric and eccentric cutouts were analyzed by Chakravorty et al. (1998). Free vibration problems of corner supported, simply supported and clamped cylindrical, saddle, hyperbolic paraboloid bounded by straight edges (commonly called skewed hypar), conoidal and spherical shells were carried out with the effects of concentric and eccentric cutouts. Forced vibration problems are solved for three different load time histories for corner supported hypar and conoidal shells.

Kistler and Waas (1999) used static response characteristics as a tool for understanding transverse impact response of thin fiber reinforced composite cylindrical panels. The effect of impact velocity, panel curvature, thickness, both in-plane and out of plane boundary conditions and the validity of linear and nonlinear plate theory on the resulting impact force and panel displacement were investigated. Qatu (1999) derived accurate equations of elastic deformation for laminated composite deep, thick shells. The equations included shells with a pretwist and accurate force and moment resultants were derived, which were considerably different than those used for plates. Günay (1999) worked on geometrically nonlinear static stiffened cylindrical shallow shell problems.

Toorani and Lakis (2000) worked on generalization of geometrically linear shear deformation theory for multilayered anisotropic plates and shells including transverse shear deformations, rotary inertia and initial curvature effects. These equations were applied to different geometries, such as revolution cylindrical, spherical and conoidal shells and as well as to rectangular and circular plates. Chang and Chang (2000) carried out geometrically nonlinear vibration analysis of shell structures adopting Green's strains. Newton – Raphson method and Newmark's method were adopted to solve the problems. Nonlinear free vibration characteristics of first and second vibration modes of laminated shallow shells with rigidly clamped edges were examined by Abe et al. (2000). Nonlinear equations of motion for the shells based on the first order shear deformation and classical shell theories were derived by means of Hamilton's principle. The study of tapered laminated composite, formed by dropping of some of the plies was done by He, Hoa and Ganesan (2000). An excellent survey of recent shell finite elements was done by Yang et al. (2000).

Zhang (2001) analyzed a clamped composite cylindrical shell for free vibration using wave propagation approach. The first eight frequencies obtained by wave propagation approach were compared with that obtained through finite element method and the results were in good agreement. A geometrically nonlinear bending analysis for functionally graded plates and shallow shells was carried out by Woo and Meguid (2001). The nonlinear transverse deflections, stresses and bending moments of thin plates and spherical shells were reported by the authors.

Ram and Babu (2002) studied fundamental frequencies of laminated composite spherical shell cap with and without cutout. The authors adopted higher order shear deformation theory and isoparametric finite element formulation to report frequencies for simply supported and clamped shells of varying thickness. Application of linear shallow shell theory to frequency response of thin cylindrical panels with arbitrary lamination was done in the paper of Kabir (2002). An exhaustive review of the research advances in the dynamic behaviour of laminated shells during the years 1989-2000 was presented by Qatu (2002a, 2002b). Turkmen (2002) in his paper presented the results from a theoretical and experimental investigation of the dynamic response of cylindrically curved laminated composite shells subjected to normal blast loading. The dynamic equations of motion for cylindrical laminated shells were derived using the assumptions of Love's theory of thin shells. Ganapathi et al. (2002) did the dynamic analysis of laminated cross ply noncircular thick cylindrical shells subjected to thermal and mechanical loadings based on higher order theory. Ganapathi et al. (2003) continued research work on free and forced vibration subjected to the same type of transient loadings but with a different type of shell. The analysis, here, was concerned with the free vibration characteristics and transient dynamic responses of simply supported oval crossply cylindrical shells. The finite element technique for studying the dynamic behaviour of laminated noncircular thin and moderately thick elliptic cylindrical shells was also used by Ganapathi et al. (2004). The shells were supported with simply supported boundary condition and the analysis was based on first order shear deformation theory. Nayak and Bandyopadhyay (2002) studied free vibration of laminated composite clamped stiffened conoidal shells using eight noded doubly curved finite elements. The authors (2005) later used nine noded elements to report fundamental frequencies and mode shapes of simply supported stiffened shells. The nine noded elements showed better accuracy than the eight noded elements, and thus they (2006) used the nine noded ones to work on forced vibration of isotropic stiffened conoids.

A review on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid-structure interaction was reported by Amabili and Païdoussis (2003). Djoudi and Bahai (2003) studied nonlinear displacements and natural frequencies of shallow cylindrical shells. An advanced geometrically nonlinear shell theory for doubly curved shell panels with transversely compressible core was reported by Hohe and Librescu (2003). The proposed formulation was based on Kirchhoff's hypothesis for face sheets and higher order power series for the core displacements. The model was developed by using von – Kármán nonlinearity and was capable to account for dynamic effects and initial geometric imperfections. According to Sze et al. (2004), most of the geometrically nonlinear finite element shell problems reported load – deflection curves in figure format until 2004. The authors realized that while solving benchmark problems, reconstructing those data points from a graphical measurement was a tedious and difficult job. Thus, they solved cantilever beam, isotropic and laminated plate, full cylindrical shell and cylindrical shell panel problems using Newton – Raphson method and furnished the results in tabular and graphical forms. Reddy (2004) explained the shear locking phenomenon. It generally occurs in thin plate and shell finite elements formulated using elements degenerated from three dimensional formulation and first order shear deformation theory. Since, the transverse displacement and rotation were treated as independent degrees of freedom in first order shear deformation theory, the transverse shear does not vanish when the element is subjected to pure bending. Consequently, in case of thin plates and shells, strain energy due to bending became negligibly small compared to the shear strain energy and the structure appeared to be infinitely rigid showing very small displacement values. Zienkiewicz et al. (1971) introduced selective or reduced integration as a remedy to this numerical problem for those elements formulated based on first order shear deformation theory. The transverse shear deformation was better represented using the non-exact i.e. reduced numerical integration technique since, many incorrect shear terms present in the element formulation are not integrated [Parisch (1979)].

Sahoo and Chakravorty (2004) investigated the static bending characteristics i.e. transverse displacement, in-plane tensile and bending stress resultants of laminated composite skewed hypar shells. Numerical results were obtained for uniformly loaded laminated composite hypar shells with six practical boundary conditions, where the edges were differently restrained. Eight different types of lamination including single layered and multilayered, cross ply and angle ply, symmetric and anti-symmetric stacking orders were chosen. Sahoo and Chakravorty (2005, 2006a) continued the research on composite skewed hypar shell roofs with and without stiffeners for its free vibration characteristics i.e. fundamental frequency and mode shapes. A detailed study on dynamic stiffness of hypar shells were studied with various practical combinations of support constraints and laminations. The bending behaviour of laminated composite stiffened hypar shell was also studied by the authors (2006b, 2008a) using geometric linear strain terms.

Liew et al. (2006) studied stability analysis of composite laminated cylindrical shells. Kandasamy and Singh (2006) presented a numerical study for the free vibration of skewed open circular cylindrical deep shells using first order shear deformation theory of shells. This formulation includes rotary inertia and shear deformation so that thin to moderately thick shells can be analyzed. The Vlasov-Reddy higher order shear deformation theory of laminated orthotropic elastic shells was implemented by Ferreira et al. (2006) through a multi-quadrics discretization of governing equations and boundary conditions. Forced vibration response study for composite cylindrical and spherical shells were carried out by Lee et al. (2006). Both the thin and thick shells were considered in the analysis. The central displacement of shells subjected to uniformly distributed load was reported for varying radius, boundary condition and thickness values. The finite element formulation was developed using higher order shear deformation theory and Sanders' nonlinear shell kinematics. The authors solved the time dependent governing equation combining Newmark's direct time integration method and Newton – Raphson method.

Civalek (2007) presented a numerical study on the free vibration analysis of laminated conical and cylindrical shell using Love's first approximation thin shell theory and discrete singular convolution (DSC) method. Free vibration analysis of laminated cylindrical shells with clamped-clamped, clamped-simply supported, clamped-free and simply supported-simply supported boundary conditions by using Fourier series expansion method were carried out by Shao and Ma (2007). Das and Chakravorty (2007) reported a research paper on static bending characteristics of laminated conoidal shell using linear finite element method. The boundary condition for the conoidal shell consist of different combinations of clamped, simply supported and free edges which was not reported in any of the earlier research papers. Sahoo and Chakravorty (2007) reported static bending behaviour of composite hypar shells under the action of uniformly distributed load with three different arrangements of point supports. Both the static displacement and in-plane force and moment resultants were studied in the research paper. This report was based on geometrically linear finite element formulation. Nanda and Bandyopadhyay (2007, 2008) used the von-Kármán type geometric nonlinearity along with the first order shear deformation theory to study the nonlinear to linear fundamental frequency values of cylindrical panels with cutouts and backbone curves of spherical panels with cutouts considering simply supported boundary conditions. This approach was extended by the authors (2009) to study the geometrically nonlinear transient responses of laminated composite cylindrical, spherical and hyperbolic paraboloid shells with and without square cutouts and initial geometric imperfection. Kundu and Sinha (2007) reported geometrically nonlinear behaviour of laminated composite doubly curved shells and used arc-length method to solve the governing equation. Rougui et al. (2007) studied free and forced vibrations of simply supported circular cylindrical shells using Donnell's nonlinear shell theory and transverse deformations. Arciniega and Reddy (2007) worked on tensor based geometrically nonlinear finite element formulation considering transverse stretching, thus, establishing a three dimensional constitutive relationship.

Shaoo and Chakravorty (2008b) continue their research work on point supported composite hypar shell roofs with its free vibration characteristics i.e. fundamental frequency and mode shapes. A detailed study on dynamic stiffness of hypar shells is studied with various practical combinations of laminations. Das and Chakravorty (2008) again reported about the vibration aspects and mode shapes of laminated conoidal shells with the same boundary conditions as in their previous paper reported in this section. In both the papers the authors carried out a detailed research work on the shell stiffness by varying the lamination angles and stacking sequences along with the boundary conditions. Free and forced vibration study of a clamped isotropic cylindrical shallow shell considering first order shear deformation theory was carried out by Ribeiro (2008). The author studied natural frequencies of cylindrical shell of various thicknesses and mode shapes were also plotted. Poore et al. (2008) reported the natural frequencies and mode shapes of laminated cylindrical shells containing a circular cutout and also investigated the effects of varying cutout size, shell radius and laminate layup as well as the effects of two types of boundary conditions on the shell vibration response. Pradyumna and Bandyopadhyay (2008) analyzed static bending behaviour of hypar shells using higher order shear deformation theory with simply supported and clamped boundary condition. The shell structure was subjected to uniform and sinusoidal loading. Free vibration analysis of hyperbolic paraboloid, hypar and conoidal shell was carried out with simply supported, corner supported and clamped boundary conditions. Fundamental frequencies were studied with various lamination schemes. A nonlinear zig – zag theory for highly shear deformable laminated anisotropic shells were reported by Chaudhuri (2008). The nonlinear finite element formulation was established using virtual work principle and total Lagrangian approach. The author considered the shells as transversely inextensible and fully nonlinear strain –

displacement relationship was applied for the remaining five degrees of freedom. Han et al. (2008) worked on geometrically nonlinear analysis of laminated composite thin shells using a modified first order shear deformation theory and nine noded Lagrangian elements.

A free vibration analysis of a simply supported shallow cylindrical shell was performed by Dogan and Arslan (2009). Authors used Hamilton's principle to obtain the governing differential equations for a general curved shell. The authors used two different theories to perform the analysis and in first of them the shear deformation is not considered and in the second theory it was considered. The values of the fundamental frequencies using these theories which were obtained by varying the elasticity ratio, thickness to width ratio and width to radius ratio were compared with the results obtained through finite element method. Kishimoto et al. (2009) studied forced vibration response of isotropic conical shell with simply supported boundary condition at the opposite edges. Hamilton's principle with the Rayleigh–Ritz method was employed to derive the equation of motion of the conical shell. Dynamic displacement in two tangential directions and in transverse direction was studied under the action of unit amplitude sinusoidal time dependent load. The dynamic displacement in three orthogonal directions was plotted with the loading frequency and some peaks values of displacement were observed in frequency – response curve. They were identified as resonant responses of the conical shell under the external dynamic loads. Das and Chakravorty (2009) solved free vibration problems of full conoidal shells using a finite element code which was developed using an eight noded curved quadratic isoparametric element. Numerical problems were solved for six different practical boundary conditions and for eight laminations of graphite-epoxy conoidal shells on square planform to study the performances of different shell options in terms of fundamental frequencies. Effects of number of boundary constraints and arrangement of boundary conditions along the four edges on fundamental frequencies were studied meticulously and also performances of symmetric and anti-symmetric laminations were examined in details. Study of comparative performance of cross and angle ply laminations was also made. Zhao and Liew (2009) studied geometrically nonlinear behaviour of isotropic and functionally graded cylindrical shells. Sanders' nonlinear strain theory and arc-length method was adopted by the authors to report displacements and stresses of shells subjected to concentrated load at the center.

Das and Chakravorty (2010) again put their contribution to the research work on laminated shells by this paper. In this paper the authors studied bending behaviour of composite conoidal shells with three different combinations point supported boundary conditions and four different laminations. Relative performance of different shell options were studied in details. Suitable approaches were proposed to choose the best shell option among many in a practical situation. A new nonlinear theory was reported by Amabili and Reddy (2010) for doubly curved closed and open shells. The proposed theory was derived considering shear deformation and fully nonlinear strain – displacement relations for in-plane and transverse degree of freedom. The authors studied large amplitude forced vibration of deep and moderately thick laminated composite circular cylindrical shells with initial geometric imperfections and the proposed theory yielded superior results than the existing nonlinear theories.

Kumari and Chakravorty (2010, 2011) worked on damaged conoidal shells and reported static deflections and deflection profiles of delaminated graphite-epoxy shells. Neogi et al. (2011) studied impact response of simply supported skewed hypar shell roofs using geometrically linear finite element method. Nanda and Pradyumna (2011) worked on free and forced vibrations of laminated composite cylindrical and spherical shells with imperfections subjected to hygrothermal environments. Eight noded Serendipity elements, first order shear deformation theory, von – Kármán kinematics and modified Newton – Raphson method were adopted by the authors to solve vibration problems of shells. The free vibration response was solved using direct integration method for the eigenvalue problem and the forced vibration response was solved using Newmark's average acceleration method. The authors (2013) later studied transient responses of functionally graded cylindrical, spherical and hyperbolic paraboloid shell panels. Amabili (2011) compared the Amabili – Reddy theory with von – Kármán nonlinear theory by studying forced vibration of simply supported thick circular cylindrical shells and the proposed theory established its superiority. The accuracy of the proposed formulation was established for deep and thick cylindrical panels also by Alijani and Amabili (2013). Amabili (2013) later refined the Amabili – Reddy theory by adding a uniform normal transverse strain in the thickness direction and the effects of geometric imperfections. A full three dimensional model of the laminated shell was possible to derive using shell theory considering transverse stress and strain. As a result, the refined theory proved to be efficient than the Amabili – Reddy theory in predicting natural frequencies and large amplitude forced responses of laminated composite circular cylinders. The theory was again modified by Amabili (2014) by adding a third order transverse normal strain. Alijani and Amabili (2014) reported a detailed review on nonlinear vibrations of shell structures.

The nonlinear vibration modes of simply supported cylindrical shells were reported by Avramov (2012) applying Donnell's shell theory and Galerkin method. The author also worked on stability analysis of periodic motions. Bich and Nguyen (2012) studied nonlinear vibration of simply supported and functionally graded cylindrical shells. Governing equations were derived using improved Donnell's shell theory and kinematic nonlinearity. The Galerkin method, the Volmir's assumption and fourth order Runge – Kutta method were used as solution algorithms. Natural frequencies and dynamic displacements of axially and transversely loaded shells were studied by the authors. An extension of Kármán – Donnell's theory for non-shallow, long cylindrical shells undergoing large deflection was studied by Xue et al. (2013). The governing equation was derived by considering the influence of initial curvature of the shell and the authors indicated that ignoring that influence overestimated the buckling pressure conspicuously. The static responses including the deflections and stresses of laminated composite skew shells considering different geometry, boundary conditions, ply orientation, loadings and skew angles were reported by Kumar et al. (2013). Shell forms considered in this study include spherical, conical, cylindrical and hypar shells.

The von – Kármán nonlinear theory and total Lagrangian approach were combined with first order shear deformation theory to study displacements of simply supported and clamped square and circular laminated composite plates and also clamped and hinged composite cylindrical shells by Van et al. (2014). Ahmed and Sluys (2014) worked on geometrically nonlinear dynamic responses of thick and thin, isotropic and anisotropic plates and cylindrical shells using solid-like shell elements. By using the eight noded solid elements, the authors were able to conduct a three dimensional dynamic analysis of the laminated composites. Shear locking of these elements were avoided by using the Assumed Natural Strain (ANS) method.

2.4 REVIEW ON FAILURE OF COMPOSITE LAMINATES

Failure of a composite laminates or laminated composite is progressive in nature which initiates with failure of the weakest ply in the laminate. The initiation of failure is termed as first ply failure. After failure of the weakest ply, redistribution of stresses takes place and remaining laminae continue to carry the load until the total laminate fails. The maximum stress and maximum strain theories applicable to metals are also useful to predict failure of the fiber reinforced composites. These theories also indicate failure modes of the composites which are frequently observed in practice, and hence, maximum stress and strain criterion are physically realistic. A laminated composite may be subjected to biaxial stress state in its service life. It may happen that none of the stresses exceed their respective permissible values individually, but their combination indicates failure. The interactive type failure theories were reported by

Hill (1948), Azzi and Tsai (1965), Hoffman (1967) and Tsai and Wu (1971). Hill (1948) suggested anisotropic generalization of the von-Mises criterion for the shear yield behaviour of anisotropic ductile metals. Azzi and Tsai (1965) modified Hill's theory to the fracture of an orthotropic lamina and proposed the Tsai-Hill failure theory. Hoffman (1967) proposed a fracture criterion to predict brittle strength of orthotropic materials. The proposed criterion was based on Mises-Schleicher isotropic yield condition and Hill's orthotropic yield condition (1948). Tsai and Wu (1971) proposed a tensor polynomial type interactive strength criterion, which was an improvement over earlier proposed theories in predicting combined stress-space i.e. in the space away from the coordinate axes of the failure surface. They observed that failures of composites are further complicated by a multitude of independent and interacting mechanisms which includes filament breaks and micro-buckling, delamination, dewetting, matrix cavitation and crack propagation. They opined that operationally simple strength criteria cannot possibly explain the actual mechanisms of failures. They intended to evolve a useful tool for materials characterization, which determines how many independent strength components exist and how they are measured. Hashin (1980) proposed a separate failure criterion for fiber and matrix collapse by introducing fracture plane-dependent stress components. Then, Puck and Schürmann (1998) enhanced Hashin's failure criterion by implementing the angle of fracture plane and proposed three kinds of fracture modes: inter-fiber failures under tensile stress on the plane perpendicular to the fiber direction and in-plane shear stress (mode A), failure due to a small compressive stress on the plane perpendicular to the fiber direction and large in-plane shear stress (mode B), and failure due to a large compressive stress on the plane perpendicular to the fiber direction and small in-plane shear stress (mode C). A number of researchers used these failure criteria in their research works for the study of failures of composite laminates.

Turvey (1980) studied an initial flexural failure of simply supported GFRP and CFRP plates subjected to three lateral pressure distributions. The author restricted his study to symmetric, cross ply layups. The author presented the results graphically as a set of initial failure load and corresponding plate central deflection curves to facilitate design application.

Reddy and Pandey (1987) reviewed a number of existing failure criteria, including the maximum stress, the maximum strain, Tsai – Hill's, Tsai – Wu and Hoffman's, to analyze the first ply failure of laminated composite plates subjected to in-plane and transverse loads using geometrically linear finite element analysis procedure for the static analysis. The finite element program formulated by the authors allows the user to specify the desired failure criteria and element type, and yields the first ply failure results (i.e. failure load, ply and element in which failure occurs).

The first order shear deformation theory and the finite element method to compute the linear and nonlinear first ply failure loads for three different types of loads on the plates and four different types of boundary conditions were proposed by Reddy and Reddy (1992). The failure loads and locations predicted by different well-established failure criteria can differ from each other significantly. The authors found large difference between the linear and the nonlinear failure loads in the case of transverse loading, considerably less in the case of in-plane (tensile) loading with hinged ends, and almost zero in the case of in-plane (tensile) loading with clamped ends. The difference was large for thin (three or four layer) laminates and simply supported boundary conditions and much less for thick (16 layer) laminates and clamped end boundary conditions. The displacements and first ply failure loads obtained by finite element analysis were compared for a typical laminate and it is concluded that the failure loads depend on the nature of the displacements and their derivatives throughout the laminate rather than the central deflections.

A progressive failure algorithm was proposed by Reddy et al. (1995) to report the first and ultimate ply failure of composite laminates in bending. The authors studied the effect of geometric nonlinearity, span to depth ratio, lamination sequence and the boundary condition on the proposed algorithm. The von – Kármán type geometric nonlinearity was considered and the nonlinear governing equation was solved using Newton-Raphson method. The authors adopted Tsai-Wu failure theory to report the failure loads and stress based independent failure theory to identify the modes of failure. They found that the transverse normal stress had considerable influence on the failure of composite plates and concluded that a three dimensional plate theory should be preferred over equivalent single layer theories to study failure behaviour of composite laminates accurately.

Kam and his co-researchers conducted experiments to study first ply failure of laminated plates subjected to concentrated load at the center. The authors used layer wise linear displacement theory [Kam and Jan (1995)], Ritz method [Kam and Sher (1995)] and finite element method [Kam et al. (1996)] to obtain theoretical failure loads of those plates. Maximum stress, Tsai-Hill, Tsai-Wu and Hoffman failure theories were applied to obtain the failure loads.

Kam and Jan (1995) used layer wise linear displacement theory and geometrically linear strain – displacement relationship to formulate bending stiffness of the plate. The authors compared the failure loads obtained using the proposed theory and Mindlin theory with experimental results. It was found that the layer wise linear displacement theory showed better agreement with the experimental results than the Mindlin theory.

Kam and Sher (1995) conducted experiment on centrally loaded, clamped, laminated composite plates and studied the load – deflection relationship up to total ply failure of the laminate. Acoustic emission sensors were used to identify first ply failure of those plates. Von – Kármán – Mindlin nonlinear plate theory, Ritz method and the failure theories reported by

Reddy and Pandey (1987) were used by the authors to obtain the theoretical failure loads. Theoretical results were compared with the experimental values. Along with the failure loads, the failed ply numbers were also reported. The authors also proposed a stiffness reduction model where the laminate stiffness was reduced progressively.

A nonlinear finite element method for the prediction of nonlinear load – deflection curves and first ply failure loads of centrally loaded and partially clamped laminated composite plates using several well-established failure criteria was developed by Kam et al. (1996). The authors adopted a hypothetical stiffness reduction model to improve the prediction of load – displacement curves of damaged laminated composite plates. Experimental results on load – displacement curve and first ply failure load of laminated plates with four different lamination arrangements were used to verify the accuracy of the solutions. The results shown that the method, especially when used with nine noded Lagrangian elements, predicted fairly good load – deflection curves and ultimate strengths for plates with damage. The capabilities of the well-established failure criteria in predicting first ply failure load was investigated and discussed. The authors found fairly accurate first ply failure loads of laminated composite plates based on maximum stress, Hoffman's and Tsai-Hill's failure criteria.

Kam and Chang (1997) reported reliability analysis of laminated composite plates based on the concept of first ply failure and structural reliability theory. Maximum stress and Tsai-Wu failure criteria were used to identify failure of those plates. Stress analysis was carried out using nine noded Lagrangian elements and 2 × 2 Gauss rule based on first order shear deformation theory. The lamina strength parameters were treated as baseline random variables. The feasibility and accuracy of the proposed method was validated by comparing its results with the experimental results and findings from normal, lognormal and Weibull distributions. The Weibull distribution yielded conservative results. These probability density functions were also used by Lin et al. (1998) to study first ply failure and buckling probabilities of laminated composite plates. The material properties, fiber angles and layer thicknesses of the laminates were treated as random variables in the reliability analysis. Stress analysis was carried out using eight noded quadratic elements and 2×2 Gauss rule. The theoretical results were validated by comparison with experimental data.

Sciuva et al. (1998) studied the influence of nonlinearities and the stress distribution. Author implemented geometrically nonlinear first ply failure procedure in a home-developed finite element code and investigated the behaviour of simply supported and clamped laminated plates under a uniformly distributed transverse load. The authors tested the procedure for thin and thick plates and in the case of shear stresses (obtained by means of the constitutive equations and by integration of the local equilibrium equations). Singh and Kumar (1998) studied post buckling failure behaviour and progressive failure of thin simply supported symmetric rectangular laminates subjected to in-plane shear loads. The buckling load, first and ultimate ply failure loads and maximum transverse deflection associated with failure loads were reported for variation of boundary condition, lamination, plate aspect ratio, fiber orientation and lamina material properties of those laminates. The authors used three dimensional Tsai – Hill failure criterion to predict the failure of lamina and maximum stress criterion to predict the onset of delamination. First order shear deformation theory and von – Kármán type geometric nonlinearity were combined with nine noded Lagrangian elements to formulate the governing equation of plate bending and the nonlinear equation was solved using the Newton – Raphson approach. The authors found that the buckling and failure loads were strongly dependent on boundary condition and lamination of plates. Moreover, the ultimate failure loads were found to be significantly greater than the first ply failure loads.

The first ply failure study was extended by Kam and Lai (1999) and failure strengths of laminated composite plates were determined using experimental techniques. Acoustic emission sensors were used to detect the failure sources in the laminated plates and the first ply failure loads. The authors proposed a finite element formulation based on layer wise linear displacement theory and nine noded Lagrangian elements. Theoretical failure loads were determined using the Tsai-Wu failure criterion. Comparisons between the experimental and theoretical first ply failure strengths showed reasonable accurately.

Buckling of laminated cylindrical shells was reported by Ferreira and Barbosa (2000) using nine noded Lagrangian elements and von – Kármán nonlinearity. The Newton – Raphson method and arc length method was used to solve the differential equation.

First ply failure analysis of composite laminated stiffened plate and shell panels using finite element method was made by Prusty et al. (2001a, 2001b). Modified approach of shell analysis was adopted, which takes care of shallow or deep shell elegantly. Stiffener was modeled suitably so as to accommodate arbitrary oriented stiffener anywhere inside the shell element. Various failure theories such as Maximum stress, Maximum strain, Hoffman's, Tsai-Wu's and Yeh-Stratton's theories were implemented for prediction of first ply failure loads using iterative procedure. A few laminated composite unstiffened and stiffened panels with various loading cases were solved for first ply failure analysis.

Kumar and Srivastava (2003) presented a new laminated stiffened plate element for first ply failure analysis. The element was able to accommodate any number of arbitrarily oriented stiffeners and did not need a mesh line along the stiffener. The mesh division was no longer constrained by the stiffener disposition. The first ply failure analysis of laminated stiffened plates was carried out and the results were in good agreement with published literature. A few parametric studies of laminated stiffened plates with various stiffener sections such as blade, I–section and hat section and with varying fiber angles ranging from 15˚to 75˚were carried out. The plates stiffened with blade stiffeners are found to have higher failure loads than plates stiffened with I-sections at fiber angles less than 45˚. The negative cross coupling terms of hat stiffener have ensured higher failure loads of hat stiffened plates.

A three dimensional layer wise mixed finite element approach was considered by Ramtekkar et al. (2004) for analysis of first ply failure of laminated composite plates and it was observed that the three dimensional failure theories yielded approximately the same failure loads as their two dimensional counterparts. Garai and Ray (2005) and Ray and Dey (2008) presented the first ply failure analysis of laminated composite plates exposed to moist condition and linearly varying temperature through the thickness of the laminate respectively in their research reports.

Kelly and Hallström (2005) studied behaviour of laminated composites subjected to transverse load by experimentally and numerically. The authors reported the load – displacement relations of those composites. The initial damage was identified by sound of cracking of the composite. The first ply failure load was found at 20 – 30% of the ultimate load carrying capacity of the specimen. However, the initial damage was not noted on the load – displacement curve. Moreover, the authors did not found any noticeable loss of stiffness of the specimen at the initial failure load. The authors concluded that accumulation of damage at such low load levels may affect fatigue strength of the laminate. It was further reported that the initial failure mode was predominantly intra-laminar matrix shear failure and ultimately the laminate failed through inter-laminar shear failure. The authors proposed a three dimensional finite element formulation to predict the first ply failure numerically. The nonlinear load – displacement relationship of the composite was accurately predicted by the proposed finite element formulation. The numerically predicted failure loads and locations were also in reasonable agreement with the experimental results.

Zahari and Zaffrany (2006) reviewed progressive damage analysis of composite layered plates using a mesh reduction method giving a progressive damage methodology and algorithm for composite laminates was successfully developed for the new finite strip methods using stress-based failure criteria.

Maimí et al. (2007) proposed a new constitutive model for the prediction of the onset and growth of intra-laminar failure mechanisms in composite laminates under plane stress using a simplification of the LaRC04 failure criteria.

Progressive failure of tapered laminated composite plates subjected to uniaxial compression was reported by Ganesan and Liu (2008). The authors used von – Kármán type geometric nonlinearity and nine noded Lagrangian elements to solve the plate bending problems. Tensor polynomial form of maximum stress criterion was used by the authors to obtain the first and ultimate ply failure loads. Along with the failure loads, maximum transverse deflections at first and ultimate ply failures and buckling loads were also reported. It was found that the buckling loads of all the plates were smaller than their first ply failure loads and the ultimate ply failure load was significantly greater than the first ply failure load. The authors also worked on failure modes of those plates and reported that the first ply failure was initiated through matrix cracking of a lamina and ultimately the laminate fails through delamination. Orifici et al. (2008) conducted a nonlinear finite element analysis by introducing a stress-based adhesive degradation model in order to predict the failure mechanisms introduced by the debonding behaviour in blade stiffened composite panels under compressive loads. The influence of the shape of optimised cutouts on the buckling behaviour of tensile stressed thin walled composite plates was reported by Kremer and Schürmann (2008).

Wagner and Balzani (2010) presented a numerical model to predict the post buckling response of stringer-stiffened curved composite airframe panels under axial compression including fiber fracture, matrix cracking and fiber-matrix debonding by means of extended Hashin's criteria considering both geometric and material nonlinearities. Liu and Zheng (2010) reviewed the developments on damage modelling and finite element analysis of composite laminates. The authors reviewed different damage models based on continuum damage mechanics, popular failure theories like maximum stress, Hashin's, Hoffman's, Yamada-Sun's, Tsai-Hill's and Tsai-Wu's criteria and finally the finite element implementation of progressive failure analysis. The authors found that among all failure criteria, the most suitable failure criterion for composite laminates is the polynomial stress tensor criterion proposed by Tsai and Wu. The damage model based on continuum damage mechanics was yet to describe specific features of the damage evolution of composite laminates. The finite element technique provided good approximation of progressive failure with continuous stiffness degradation, specifically when combined with advanced techniques like cohesive theory used to predict damage evolution properties of laminated composites. Chang and Chiang (2010) studied first and ultimate ply failures of anti-symmetrically laminated composite plates using experimental and theoretical techniques. A number of parameters like length to depth ratio, aspect ratio and fiber orientation were varied by the authors to study failure of centrally loaded plates. They used eight noded elements and first order shear deformation theory to develop an isoparametric finite element formulation which was used to solve the plate bending problem. Maximum stress, maximum strain, Hoffman's, Tsai – Hill's and Tsai – Wu's failure theories were used to predict the theoretical failure loads. The experimental results were used to verify the theoretical predictions. The authors found that the ultimate ply failure loads were significantly greater than the first ply failure loads for a given laminate. They recommended adopting the first ply failure load as a criterion to design the laminated composite plates. Falzon and Apruzzese (2010) proposed a progressive intra-laminar failure methodology to simulate damage growth of laminated composite materials and some structural applications of this progressive failure model on composite plates were implemented later (2011)**.** The authors (2011) also highlighted the importance of considering the nonlinear behaviour in shear by comparing the model proposed by them with the Hashin's model. In particular, the focus was on the nonlinear response of the shear failure mode and its interaction with the other failure modes.

Maimí et al. (2011) studied the first ply failure and the onset of delamination in laminated composite under multi axial loads, and proposed a continuum bulk model for the representation of the nonlinear constitutive response before matrix cracking. A composite finite element to predict failure progress in composite laminates accounting for nonlinear material properties was developed by Abu-Farsakh and Almasri (2011). The material nonlinearity has a great effect on initial and final failures of angle plied laminated composite shells (cylindrical and spherical) for some laminae configuration where shear failure in the matrix is expected. In some cases, material nonlinearity would decrease the failure load by around 30%. In addition, it was clear that failure mechanism of composite shells highly depends on load type and geometry of the shell. For example, in cylindrical shells subjected to internal when failure starts in a layer it propagates in the whole layer before it starts in another layer, while for spherical shells under transverse load the failure sequence propagates interchangeably between the layers. First ply failure and buckling of simply supported and clamped cylindrical shell panels were reported by Adali and Cagdas (2011). The effect of fiber orientation, aspect ratio and panel thickness on failure and buckling loads were studied by the authors. They conducted stability and stress analysis on symmetrically laminated angle ply laminates subjected to uniaxial compression using eight node finite elements and shear deformation theories. The dominant failure mode was defined as the minimum of buckling and first ply failure loads. The authors found that the dominant failure mode for thick panels was first ply failure and that for the thin panels was buckling.

Oterkus et al. (2012) proposed a combination of the finite element method and the peri-dynamic theory to predict the initial and final failure loads of a stiffened composite cylindrical panel with a central slot under combined internal pressure and axial tension. Different case studies of laminated composite plates were introduced by Nali and Carrera (2012) in order to compare the failure results corresponding to the different two dimensional popular failure criteria including Hashin's criterion subjected to mono-axial and bi-axial loadings. Gohari et al. (2012) studied failure of a circular cylindrical thin walled shell made of GFRP composite subjected to static internal and external pressures. Deformation, delamination, shear deformation and micro buckling failure were investigated.

Gupta et al. (2012, 2013) investigated the effect of evolving damage on static response characteristics of laminated composite cylindrical / conical panels (Gupta et al., 2012) and plates (Gupta et al., 2013) subjected to uniformly distributed transverse loading with inclusion of geometric nonlinearity based on the first order shear deformation theory. The detailed parametric study was carried out later by the same authors (2015) to investigate the effects of evolving damage, span-to-thickness ratio, lamination scheme, boundary conditions and semi-cone angle on the post buckling response and failure load of laminated cylindrical / conical panels under meridional compression considering geometric nonlinearity and evolving material damage.

A finite element model was proposed by Chen et al. (2012, 2014) to study progressive failure of laminated composites. The proposed model was an elasto-plastic damage model which considered plasticity effects exhibited by composite materials. The authors first developed that model considering in-ply damage (2012) only and later modified it to account for delamination damage also (2014). Eight noded three dimensional cohesive elements were implemented in the finite element model to simulate in-plane and out of plane failure in the composite and interface layers, respectively. The authors applied that model to study progressive failure of punctured laminates subjected to in-plane tensile load and transverse low velocity impact load. A progressive failure study of laminated composite plates was reported by Ellul et al. (2014). The authors adopted shell elements, first order shear deformation theory and geometrically nonlinear theory to predict the load deformation behaviour and Tsai-Wu failure theory to obtain the progressive and ultimate ply failures of plates with different boundary conditions, ranging from fully clamped to simply supported ones. The authors adopted different progressive failure models and suggested to decrease the elastic properties of laminae depending on the failure mode. Romanowicz (2014) conducted a study to determine the in situ first ply failure strength of lamina in a symmetric cross ply laminate subjected to uniaxial tension as well as to investigate the effect of matrix ductility on the transverse failure behaviour.

Bakshi and Chakravorty (2012, 2013, 2014a, 2015) studied the first ply failure analysis of laminated composite conoidal shells considering geometric linear finite element formulation. The authors investigated the failure loads for clamped and simply supported boundary conditions. Different parametric variations including lamina sequences (cross and angle ply), angle of laminations and number of repetitions of symmetric and anti-symmetric units were done in these studies. The author (2014b) also investigated the first ply failure behaviour of composite cylindrical shells considering geometric nonlinearity and proposed some practical guidelines to the practicing engineers.

Lal et al. (2012) evaluated nonlinear stochastic first ply failure response of composite plate under compressive loading using classical failure criteria such as Tsai-Wu's and Hoffman's for random input material and strength properties. This stochastic first ply failure study using material nonlinearity under hygro thermal environment and different biaxial loadings with clamped, simply supported and hinged end conditions using Puck failure criterion was done by Gadade et al. (2016a, 2016b).

Coelho et al. (2015) worked on punctured laminated composites under in-plane tension load and predicted damage initiation using the Hashin's failure criterion. The authors used brick elements for whom the kinematic and constitutive behaviours were like shell elements. The nonlinear governing equation was solved using Newton – Raphson method. The Hashin's failure theory was adopted by the authors to obtain the failure of the composite laminates.

Finally, the authors validated the results with case studies. First ply failure prediction of an internally pressurized shell was carried out by Gohari et al. (2015). They studied theoretical failure loads of unsymmetrically laminated ellipsoidal woven GFRP composite shell. The Tsai-Wu failure criterion and linear interpolation technique were adopted to obtain the failure. The influence of different parameters like thickness, aspect ratio and stacking sequence on the first ply failure of laminated ellipsoidal shell were investigated and the authors found that the analytical results showed good agreement with experimental values. Initial and progressive failures of glass / carbon fiber reinforced laminated composite plates were studied by Lee et al. (2015) using the Puck failure criterion and damage mechanics respectively. Dong et al. (2014) endeavoured to ameliorate the applicability of Puck's failure theory by proposing simplest methods for evaluating various parameters required and Matthias and Kröplin (2012) extended the theory up to three dimensional stress analysis.

The first ply failure stresses of laminated composite plates and cylindrical shell panels with modified Tsai-Wu's and Hashin's failure criteria using nonlinear finite element method were discussed by Chróścielewski et al. (2016) and these modifications were also reported in this paper. Reinoso and Blázquez (2016) reported the post buckling failure responses of composite cylindrical stiffened panel subjected to uniform pressure load employing geometric nonlinearity.

A three dimensional consistent anisotropic damage model for laminated fiber reinforced composites relying on the Puck failure criterion was established by Reinoso et al. (2017). The damage model was modified with a transverse shear and trapezoidal locking free solid shell formulation by the Assumed Natural Strain (ANS) method in order to account for geometrically nonlinear effects in thin walled applications. A quasi isotropic laminate, constant stiffness and variable stiffness laminates were designed, manufactured and tested to assess the effect of the layup construction on improving the failure load of a flat composite panel with a large cutout by Khani et al. (2017). Priyadharshani et al. (2017) analysed the glass fiber reinforced polymer (GFRP) stiffened composite plates with and without rectangular cutout under axial, lateral and combined axial and lateral loadings using finite element method.

2.5 CRITICAL DISCUSSIONS

The literature that has accumulated on the study of shells indicates that research on shells began with closed form solutions of shell forms under static load. As it was realized that closed form solutions are possible to be formulated for very limited combinations of loading patterns and boundary conditions, engineers started employing the numerical techniques like the finite differences, finite strip and finite element approaches. The numerical analyses have been continued by the researchers later on to study the complicated aspects like stability and failure of laminated composite shells.

As pointed out in Section 2.3, most of the researchers focused their research works on laminated composite cylindrical and spherical shell surfaces for static, dynamic and stability characteristics excluding the failure study. The researchers also realized that closed form solution of shell problems could not be reached except for very simple shell surface geometries (cylindrical and spherical ones), loading and boundary conditions. They noted that the configurations like conical, conoidal, saddle, elliptic and hyperbolic paraboloid and skewed hypar shells do not admit closed form solutions though these shell forms can offer a number of parallel advantages that suit to the requirements of the industry. In fact, in industrial applications, a shell may have complicated boundary condition and may be subjected to complex loading. So the researchers resorted to numerical solutions of shell problems among which the finite element approach is most versatile and widely applied. Researchers like Chakravorty et al. (1995a, 1995b, 1996, 1998), Nayak and Bandyopadhyay (2002, 2005, 2006), Civalek (2007), Kishimoto et al. (2009), Das and Chakravorty (2007, 2008, 2009, 2010), Kumari and Chakravorty (2010, 2011) worked on the complicated conical and conoidal shell forms. The doubly ruled, doubly curved, non-developable skewed hypar shells enjoy a special attention from engineers owing to their aesthetic elegance, high stiffness and ease of fabrication. A number of researchers like, Sahoo and Chakravorty (2004, 2005, 2006a, 2006b, 2007, 2008a, 2008b), Pradyumna and Bandyopadhyay (2008), Nanda and Bandyopadhyay (2009), Neogi et al. (2011), Nanda and Pradyumna (2013), Kumar et al. (2013) reported different static, dynamic and instability behaviours of laminated composite skewed hypar shells.

An examination of research reports (Section 2.4) on laminated composites reveals that failure aspects of composite plates do have enjoyed the focus of the researchers but similar studies on composite shells are few in number. Very few researchers such as Prusty et al. (2001a, b), Adali and Cagdas (2011) studied the first ply failure of composite cylindrical shell panels applying linear finite element approach. First ply failure analysis of industrially important conoidal shell was done by Bakshi and Chakravorty (2012, 2013, 2014a, 2015) using geometrically linear finite element formulation. Gupta et al. (2012, 2015), Bakshi and Chakravorty (2014b) and Chróścielewski et al. (2016) reported the failure analysis of composite cylindrical shell form using geometrically nonlinear approach. On the other hand the failure behaviour of cylindrical shells considering material nonlinearity was reported by Abu-Farsakh and Almasri (2011).

The first ply failure strengths of industrially preferred and aesthetically appealing doubly curved synclastic spherical and doubly ruled anticlastic skewed hypar shell forms were not studied by the researchers. Thus, it is felt that failure characteristics of these forms with different laminations need to be studied in details to apply them efficiently as roofing units and to popularize their use further in the industry. With these findings from the overall review of the literature the actual scope of the present study is presented systematically in the next chapter.

Chapter 3

SCOPE OF PRESENT STUDY

3.1 GENERAL

The detailed review of literature presented in Chapter 2 and the critical discussions of the contents hint towards a number of areas through which future course of research may flow. The present thesis aims to address a part of the detailed scope of work as the present scope of research. Section 3.2 furnishes systematically the present scope of this thesis.

3.2 PRESENT SCOPE

A generalized finite element formulation is presented in Chapter 4 using eight noded isoparametric curved quadratic Serendipity finite element for the first ply failure analysis of industrially important synclastic spherical and anticlastic skewed hypar shell. Well-established failure criteria like maximum stress, maximum strain, Tsai-Hill, Tsai-Wu, Hashin and Puck criteria are used to obtain the first ply failure loads and the failure modes or tendencies. Both linear strains as proposed by Sanders' and nonlinear components of strains as proposed by von – Kármán are used to study the failure load values obtained through linear and nonlinear approaches respectively. A good number of benchmark problems are solved to establish the applicability of the present approach. The primary outcomes of the numerical experimentations are the first ply failure load values, the failure modes or tendencies and the failure zones. The results are post-processed to obtain a number of other relevant data that characterize the failure behaviour of spherical and skewed hypar laminated composite shells. Chapter 5 deals with the linear first ply failure of laminated composite spherical shell roofs with clamped boundary condition. The linear and nonlinear first ply failure loads of laminated composite clamped skewed hypar shells with varying planform and thickness are described in Chapter 6. The first ply failure characteristics considering geometrically linear and nonlinear approaches of the same type of shell roofs with varying curvature including guidelines for non-destructive test monitoring are also furnished in Chapter 7. Chapter 8 reported the nonlinear first ply failure loads of simply supported skewed hypar shells. All the results are critically examined to arrive at conclusions of engineering importance.

The scope of future study is indicated in Chapter 9.

Chapter 4

MATHEMATICAL FORMULATION

4.1 GENERAL

Based on the thesis objectives and present scope, a mathematical formulation is presented in this chapter. A finite element code is developed which can effectively be utilized to calculate the first ply failure load using geometric linear and nonlinear strains of laminated composite spherical and skewed hypar shells under uniformly distributed load. An eight-noded isoparametric curved shell element is used for this present finite element approach. The governing nonlinear equilibrium equations and solution procedure, the shell element, nonlinear strain – displacement relations and the failure theories are furnished in details in the different sections of this chapter. The symbols used to describe the formulation are either explained where they appear first or are explained in the 'List of Notations' given at the beginning of the thesis.

4.2 COMPOSITE SHELL ELEMENT

A doubly curved laminated composite thin shallow shell of uniform thickness *h* made of homogeneous linearly elastic, laminated composite material is considered. Two shell surfaces are considered for the present study. One is the doubly curved synclastic spherical shell (Fig. 4.1) having curvature $1/R$. The other shell surface is the skewed hypar shell (Fig. 4.2) or commonly called hypar shell which is doubly curved and anticlastic having only the cross curvature $1/R_{xy}$ expressed as:

$$1/R_{xy} = \delta^2 z/\delta x\delta y = 4c/ab. \tag{4.1}$$

The projection of the shell on the *XY* plane is a rectangle of dimensions *a* and *b*. The equation of the mid-surface of the hypar shell, referred to the Cartesian coordinate system (*X, Y, Z*) is expressed as:

$$z = (4c/ab)(x - a/2)(y - b/2) \tag{4.2}$$

4.3 DISPLACEMENT FIELD

The first order shear deformation theory is applied here for the analysis of thin shallow shells. According to Vlasov (1958), the shell is considered as shallow when the height to shorter span ratio is kept within 0.2. The following simplifying assumptions are used, which provide a reasonable description of the behaviour of thin elastic shells: (1) the thickness of the shell is small compared to the radii of curvature, (2) the transverse normal stress is negligible and (3) normal to the middle plane of the shell before deformation remains straight but not necessarily normal after deformation (a relaxed form of Kirchhoff-Love's hypothesis). The displacements *U*, *V* and *W* at any point (*x, y, z*) and at a distance *z* from the mid-surface of shell are expressed as:

$$U(x,y,z) = u(x,y) + z\alpha_x(x,y)$$

$$V(x,y,z) = v(x,y) + z\alpha_y(x,y) \tag{4.3}$$

$$W(x,y,z) = w(x,y)$$

where, *u*, *v* and *w* are the mid- surface displacements of the shell and α_x and α_y are the rotations about *Y* and *X* axes, respectively.

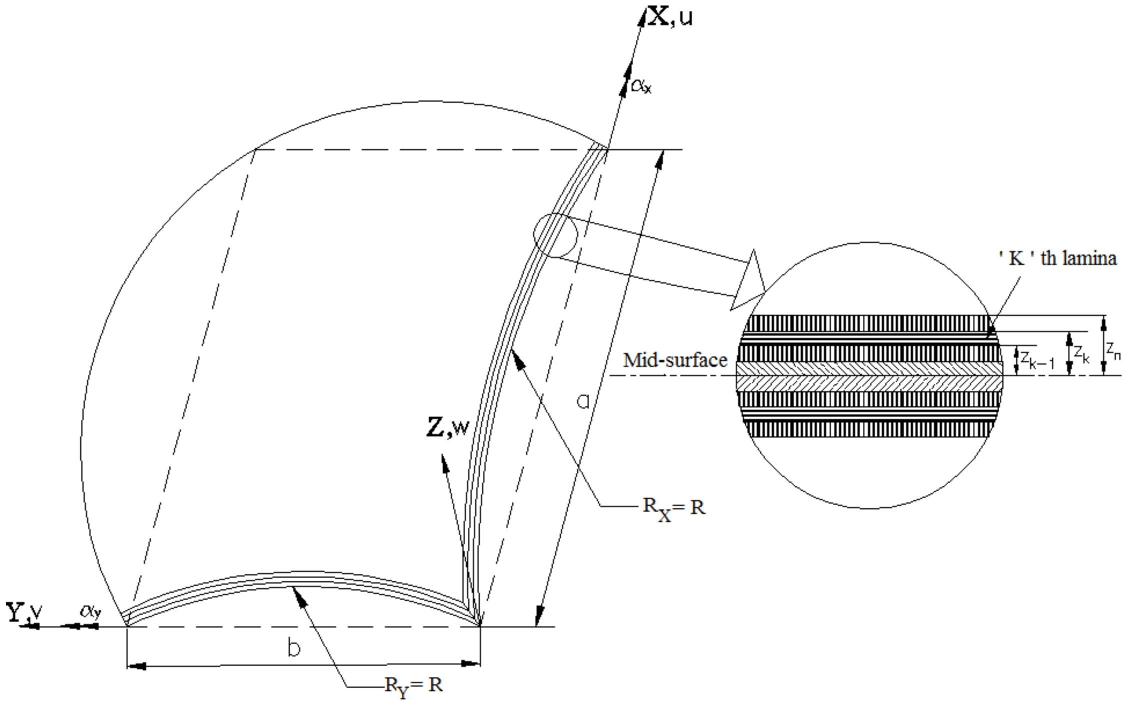


Fig. 4.1 A typical spherical shell surface

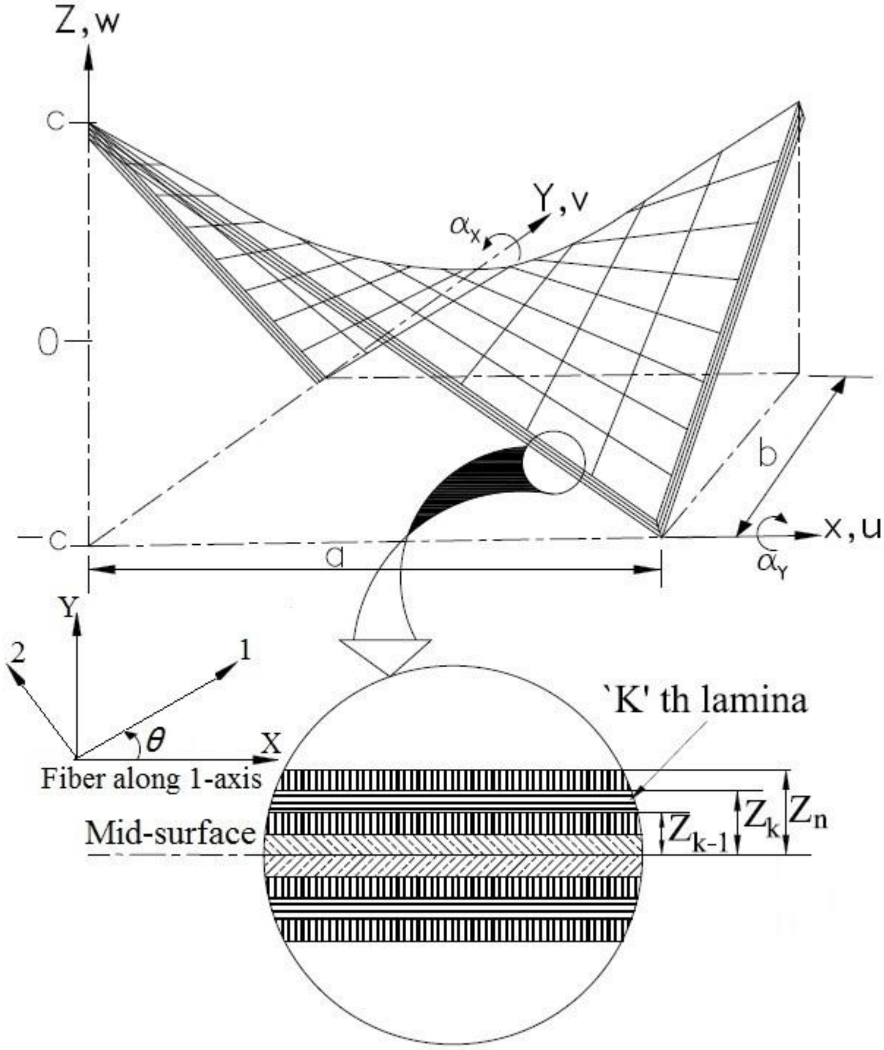


Fig. 4.2 A typical skewed hypar shell surface

4.4 STRAIN – DISPLACEMENT RELATIONSHIPS

The strain displacement relations of the shear deformable theory applied to shells that includes von-Kármán type geometric nonlinearity (Reddy 2004) are derived. Sanders' nonlinear strain displacement relations (Sanders' 1963) associated with the displacement field are used. According to the modified Sanders' first approximation theory for thin shells, the strain displacement relationships at a distance *z* from lamina mid-surface are established as:

$$\{\varepsilon'_x \quad \varepsilon'_y \quad \gamma'_{xy} \quad \gamma'_{xz} \quad \gamma'_{yz}\}^T = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^T + z\{\kappa_x \quad \kappa_y \quad \kappa_{xy} \quad \kappa_{xz} \quad \kappa_{yz}\}^T \tag{4.4}$$

where, the first vector on the right hand side represents the mid-surface strains and the second vector represents the curvatures and are respectively related to the degrees of freedom as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} - \dfrac{w}{R_x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} - \dfrac{u}{R_x}\right)^2 \\ \dfrac{\partial v}{\partial y} - \dfrac{w}{R_y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{v}{R_y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} - \dfrac{2w}{R_{xy}} + \left(\dfrac{\partial w}{\partial x} - \dfrac{u}{R_x}\right)\left(\dfrac{\partial w}{\partial y} - \dfrac{v}{R_y}\right) \\ \dfrac{\partial w}{\partial x} + \alpha_x - \dfrac{u}{R_x} - \dfrac{v}{R_{xy}} \\ \dfrac{\partial w}{\partial y} + \alpha_y - \dfrac{v}{R_y} - \dfrac{u}{R_{xy}} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \kappa_{xz} \\ \kappa_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \alpha_x}{\partial x} \\ \dfrac{\partial \alpha_y}{\partial y} \\ \dfrac{\partial \alpha_x}{\partial y} + \dfrac{\partial \alpha_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix} \tag{4.5}$$

The strain components of Eq. (4.5) are to be considered together for generalised representation of the three dimensional strain field and can be expressed in the form of

$$\{\varepsilon\} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \kappa_x \quad \kappa_y \quad \kappa_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^T = \begin{Bmatrix} \varepsilon_{inplane} \\ \varepsilon_{bending} \\ \varepsilon_{shear} \end{Bmatrix} + \begin{Bmatrix} \varepsilon'_{inplane} \\ 0 \\ 0 \end{Bmatrix} = \{\varepsilon_l\} + \{\varepsilon_{nl}\} \tag{4.6}$$

where, the in-plane, bending and shear strain components are expressed as:

$$\{\varepsilon_{inplane}\} = \left\{ \left(\frac{\partial u}{\partial x} - \frac{w}{R_x}\right) \quad \left(\frac{\partial v}{\partial y} - \frac{w}{R_y}\right) \quad \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{2w}{R_{xy}}\right) \right\}^T,$$

$$\{\varepsilon_{bending}\} = \left\{ \frac{\partial \alpha_x}{\partial x} \quad \frac{\partial \alpha_y}{\partial y} \quad \left(\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x}\right) \right\}^T,$$

$$\{\varepsilon_{shear}\} = \left\{ \left(\frac{\partial w}{\partial x} + \alpha_x - \frac{u}{R_x} - \frac{v}{R_{xy}}\right) \quad \left(\frac{\partial w}{\partial y} + \alpha_y - \frac{v}{R_y} - \frac{u}{R_{xy}}\right) \right\}^T \tag{4.7}$$

Finally, the nonlinear components of in-plane strains $\{\varepsilon'_{inplane}\}$ are defined as:

$$\{\varepsilon'_{inplane}\} = \left\{ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{u}{R_x}\right)^2 \quad \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{v}{R_y}\right)^2 \quad \left(\frac{\partial w}{\partial x} - \frac{u}{R_x}\right)\left(\frac{\partial w}{\partial y} - \frac{v}{R_y}\right) \right\}^T \tag{4.8}$$

4.5 CONSTITUTIVE RELATIONS OF COMPOSITE MATERIALS

The force and moment resultants (Fig. 4.3) for the laminate are obtained as:

$$\{F\} = \{N_x \quad N_y \quad N_{xy} \quad M_x \quad M_y \quad M_{xy} \quad Q_x \quad Q_y\}^T = [E]\{\varepsilon\} \tag{4.9}$$

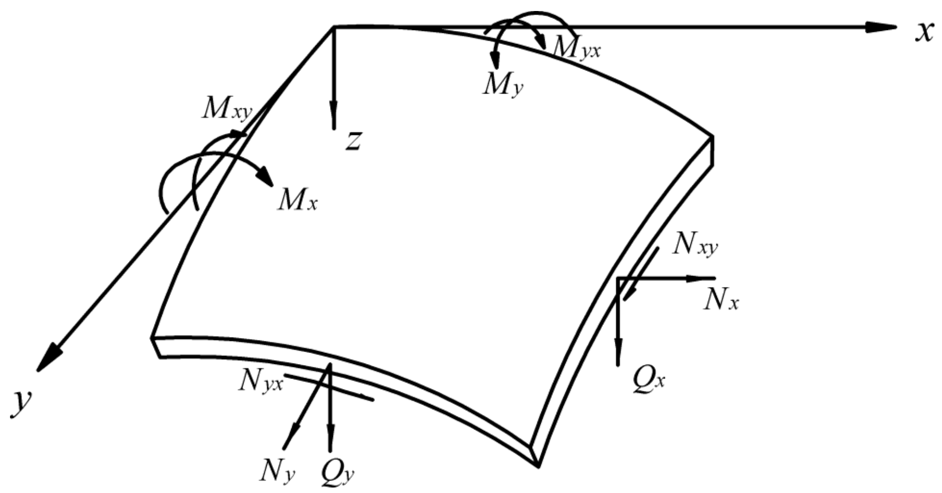


Fig. 4.3 Generalized force and moment resultant vectors

The normal stress resultants of a laminate are,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{np} \left[Q_{ij} \right]_k \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} z dz \right\}; \qquad i, j = 1, 2, 6 \tag{4.10}$$

Bending stress resultants are,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{np} \left[Q_{ij} \right]_k \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} z dz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} z^2 dz \right\}; \qquad i, j = 1, 2, 6 \tag{4.11}$$

Transverse shear resultants are,

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \sum_{k=1}^{np} f \left\{ \left[Q_{ij} \right]_k \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \int_{z_{k-1}}^{z_k} dz \right\}; \qquad i, j = 4, 5 \tag{4.12}$$

By combining the above equations (Eqs. 4.10 – 4.12) and with the help of Eq. (4.9) the laminate stiffness matrix [E] can be expressed as:

$$\left[E \right] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{44} & S_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{45} & S_{55} \end{bmatrix} \tag{4.13}$$

where, $A_{ij} = \sum_{k=1}^{np} (Q_{ij})_k (z_k - z_{k-1})$; $\quad B_{ij} = \frac{1}{2} \sum_{k=1}^{np} (Q_{ij})_k (z_k^2 - z_{k-1}^2)$;

$D_{ij} = \frac{1}{3} \sum_{k=1}^{np} (Q_{ij})_k (z_k^3 - z_{k-1}^3) \ldots$ i, j = 1, 2, 6;

$S_{ij} = \sum_{k=1}^{np} f (Q_{ij})_k (z_k - z_{k-1}) \ldots$ i, j = 4, 5

Here f is the shear correction factor, taken as unity for the class of thin shells and Q_{ij} are elements of the off-axis elastic constant matrix which are given by

$$[Q_{ij}] = [T]^T \begin{bmatrix} (1-\nu_{12}\nu_{21})^{-1}E_{11} & (1-\nu_{12}\nu_{21})^{-1}E_{11}\nu_{21} & 0 \\ (1-\nu_{12}\nu_{21})^{-1}E_{11}\nu_{21} & (1-\nu_{12}\nu_{21})^{-1}E_{22} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} [T]; \qquad i, j = 1, 2, 6 \tag{4.14}$$

$$[Q_{ij}] = [TT]^T \begin{bmatrix} G_{23} & 0 \\ 0 & G_{13} \end{bmatrix} [TT]; \qquad i, j = 4, 5 \tag{4.15}$$

in which, $[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -\sin 2\theta & \sin 2\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$ and $[TT] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

4.6 FINITE ELEMENT FORMULATIONS

4.6.1 Selection of Finite Element and Discretization of the Structure

Geometrically linear and nonlinear finite element formulation is developed here for the first ply failure analysis of laminated composite shells using the first order shear deformation shell theory. An eight noded isoparametric curved quadratic Serendipity element having five degrees of freedom per node (*u*, *v* and *w* are the displacements along *X*, *Y* and *Z* axes respectively and α_x and α_y are the rotations about *Y* and *X* axes respectively) is used for the present analysis. The element configuration is shown in Fig. 4.4. The full surface of the shell is discretized to account for its curvature and skew symmetry. The order of numbering of elements and nodes over the plan area of a typical shell surface is shown in Fig. 4.5.

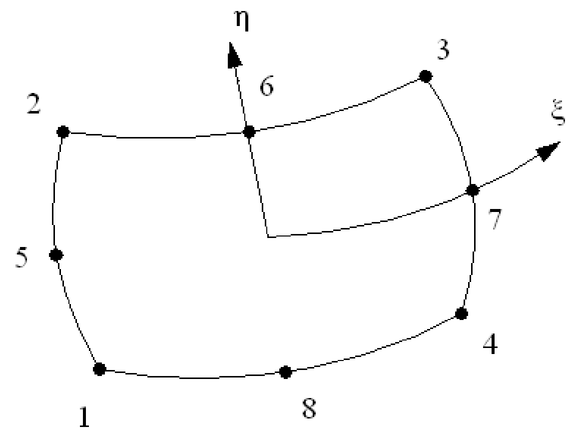


Fig. 4.4 The isoparametric shell element with natural coordinates

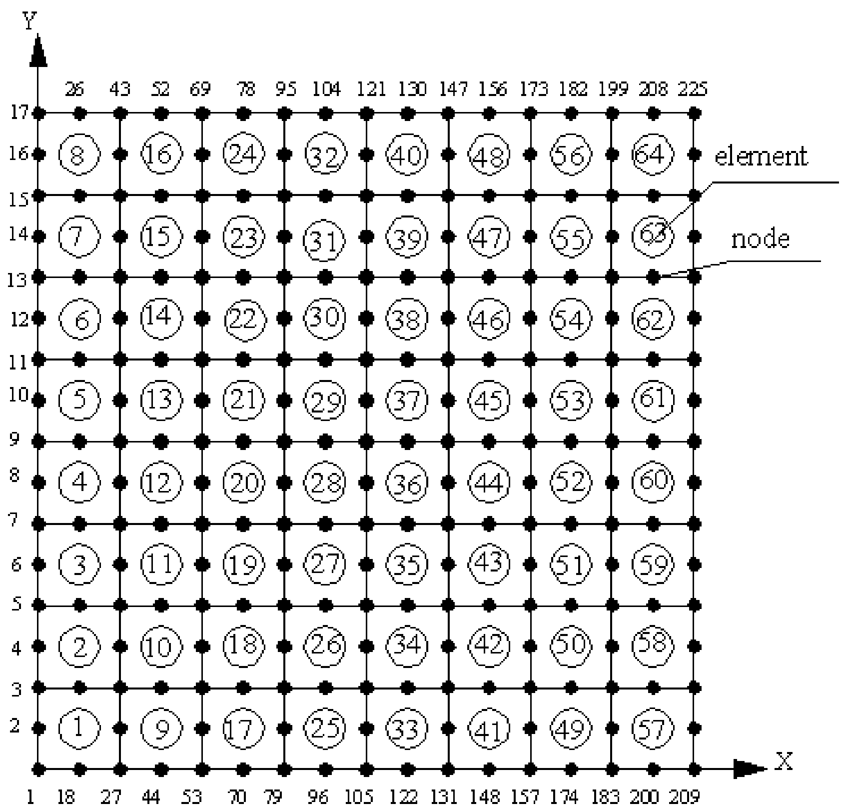


Fig. 4.5 A typical discretization of 8×8 mesh on plan area with element and node numbers

4.6.2 Selection of Shape Functions or Interpolation Functions

The shape functions or the interpolation functions are polynomials of the natural coordinates (ξ, η, ζ) which relate the generalized displacements at any point within an element to the nodal values of the displacements. These are derived from an interpolation polynomial in terms of the natural coordinates so that the displacement fields are satisfactorily represented. For the analysis of thin shells where the final element is assumed to have mid-surface nodes only, the interpolation polynomial is a function of ξ and η and has the following form:

$$u(\xi,\eta) = A_0 + A_1\xi + A_2\eta + A_3\xi^2 + A_4\xi\eta + A_5\eta^2 + A_6\xi^2\eta + A_7\xi\eta^2 \tag{4.16}$$

The shape functions derived from interpolation polynomial are as:

$$N_i = (1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i + \eta\eta_i - 1)/4 \qquad \text{for} \qquad i = 1, 2, 3, 4$$

$$N_i = (1+\xi\xi_i)(1-\eta^2)/2 \qquad \text{for} \qquad i = 5, 7$$

$$N_i = (1+\eta\eta_i)(1-\xi^2)/2 \qquad \text{for} \qquad i = 6, 8 \tag{4.17}$$

where, N_i denotes the shape function at i^{th} node having natural coordinates ξ_i and η_i.

The correctness of the shape functions is checked from the relations

$$\sum N_i = 1, \qquad \sum \frac{\partial N_i}{\partial \xi} = 0 \quad \text{and} \quad \sum \frac{\partial N_i}{\partial \eta} = 0 \tag{4.18}$$

In an isoparametric formulation the generalised displacements and coordinates are interpolated from their nodal values by the same set of shape functions. Hence, the coordinates (x, y) of any point within an element are obtained as:

$$x = \sum_{i=1}^{8} N_i x_i \qquad y = \sum_{i=1}^{8} N_i y_i \,. \tag{4.19}$$

where x_i and y_i are the coordinates of the i^{th} node. The displacement fields are expressed by the shape functions N_i.

$$u = \sum_{i=1}^{8} N_i u_i \; , \quad v = \sum_{i=1}^{8} N_i v_i \; , \quad w = \sum_{i=1}^{8} N_i w_i \; , \; \alpha_x = \sum_{i=1}^{8} N_i \alpha_{xi} \; , \; \alpha_y = \sum_{i=1}^{8} N_i \alpha_{yi} \tag{4.20}$$

The generalized independent element degrees of freedom $\{d\}$ are expressed in terms of their nodal values $\{d_e\}$by the following relationship,

$$\{d\} = \sum_{i=1}^{8} [N_i]\{d_e\}_i \tag{4.21}$$

i.e., $$\begin{Bmatrix} u \\ v \\ w \\ \alpha_x \\ \alpha_y \end{Bmatrix} = \begin{bmatrix} [N_i] & & & & \\ & [N_i] & & & \\ & & [N_i] & & \\ & & & [N_i] & \\ & & & & [N_i] \end{bmatrix}_{i=1,2,3\bullet\bullet\bullet 8} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \alpha_{xi} \\ \alpha_{yi} \end{Bmatrix}_{i=1,2,3\bullet\bullet\bullet 8}$$

The isoparametric element shall be oriented in the natural coordinate system (ξ, η) and shall be transformed to the Cartesian coordinate system using the Jacobian matrix. The derivatives of the shape functions N_i with respect to x and y are expressed in terms of their derivatives with respect to ξ and η by the following relationship

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} \tag{4.22}$$

where [J] is the Jacobian matrix given by $[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$

4.6.3 Establishing Strain – Displacement Matrices

Total strain $\{\varepsilon\}$ of Eq. (4.6) is decomposed into linear and nonlinear parts as

$$\{\varepsilon\} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \quad \kappa_x \quad \kappa_y \quad \kappa_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^T = \{\varepsilon_l\} + \{\varepsilon_{nl}\} \tag{4.23}$$

The linear strain field can be expressed in terms of nodal displacement vector and the resulting relations are,

$$\{\varepsilon_l\} = \left\{\begin{array}{c} \dfrac{\partial u}{\partial x} - \dfrac{w}{R_x} \\ \dfrac{\partial v}{\partial y} - \dfrac{w}{R_y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} - \dfrac{2w}{R_{xy}} \\ \dfrac{\partial \alpha_x}{\partial x} \\ \dfrac{\partial \alpha_y}{\partial y} \\ \dfrac{\partial \alpha_x}{\partial y} + \dfrac{\partial \alpha_y}{\partial x} \\ \dfrac{\partial w}{\partial x} + \alpha_x - \dfrac{u}{R_x} - \dfrac{v}{R_{xy}} \\ \dfrac{\partial w}{\partial y} + \alpha_y - \dfrac{v}{R_y} - \dfrac{u}{R_{xy}} \end{array}\right\} = \left\{\begin{array}{c} \dfrac{\partial \sum N_i u_i}{\partial x} - \dfrac{\sum N_i w_i}{R_x} \\ \dfrac{\partial \sum N_i v_i}{\partial y} - \dfrac{\sum N_i w_i}{R_y} \\ \dfrac{\partial \sum N_i u_i}{\partial y} + \dfrac{\partial \sum N_i v_i}{\partial x} - \dfrac{2\sum N_i w_i}{R_{xy}} \\ \dfrac{\partial \sum N_i \alpha_{xi}}{\partial x} \\ \dfrac{\partial \sum N_i \alpha_{yi}}{\partial y} \\ \dfrac{\partial \sum N_i \alpha_{xi}}{\partial y} + \dfrac{\partial \sum N_i \alpha_{yi}}{\partial x} \\ \dfrac{\partial \sum N_i w_i}{\partial x} + \sum N_i \alpha_{xi} - \dfrac{\sum N_i u_i}{R_x} - \dfrac{\sum N_i v_i}{R_{xy}} \\ \dfrac{\partial \sum N_i w_i}{\partial y} + \sum N_i \alpha_{yi} - \dfrac{\sum N_i v_i}{R_y} - \dfrac{\sum N_i u_i}{R_{xy}} \end{array}\right\}_{i=1\cdots 8}$$

$$= \left\{\begin{array}{c} \sum \dfrac{\partial N_i}{\partial x} u_i - \sum \dfrac{N_i}{R_x} w_i \\ \sum \dfrac{\partial N_i}{\partial y} v_i - \sum \dfrac{N_i}{R_y} w_i \\ \sum \dfrac{\partial N_i}{\partial y} u_i + \sum \dfrac{\partial N_i}{\partial x} v_i - 2\sum \dfrac{N_i}{R_{xy}} w_i \\ \sum \dfrac{\partial N_i}{\partial x} \alpha_{xi} \\ \sum \dfrac{\partial N_i}{\partial y} \alpha_{yi} \\ \sum \dfrac{\partial N_i}{\partial y} \alpha_{xi} + \sum \dfrac{\partial N_i}{\partial x} \alpha_{yi} \\ \sum \dfrac{\partial N_i}{\partial x} w_i + \sum N_i \alpha_{xi} - \sum \dfrac{N_i}{R_x} u_i - \sum \dfrac{N_i}{R_{xy}} v_i \\ \sum \dfrac{\partial N_i}{\partial y} w_i + \sum N_i \alpha_{yi} - \sum \dfrac{N_i}{R_y} v_i - \sum \dfrac{N_i}{R_{xy}} u_i \end{array}\right\}_{i=1\cdots 8} = \sum \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 & -\dfrac{N_i}{R_x} & 0 & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} & -\dfrac{N_i}{R_y} & 0 & 0 \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} & -\dfrac{2N_i}{R_{xy}} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\partial N_i}{\partial y} \\ 0 & 0 & 0 & \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \\ -\dfrac{N_i}{R_x} & -\dfrac{N_i}{R_{xy}} & \dfrac{\partial N_i}{\partial x} & N_i & 0 \\ -\dfrac{N_i}{R_{xy}} & -\dfrac{N_i}{R_y} & \dfrac{\partial N_i}{\partial y} & 0 & N_i \end{bmatrix} \left\{\begin{array}{c} u_i \\ v_i \\ w_i \\ \alpha_{xi} \\ \alpha_{yi} \end{array}\right\}_{i=1\cdots 8}$$

$$\Rightarrow \{\varepsilon_l\} = [B]\{d_e\} \tag{4.24}$$

Now, the generalised strain displacement relations are expressed as:

$$\{\varepsilon\} = ([B] + 0.5[B'])\{d_e\} \tag{4.25}$$

where, $[B]$ is the linear part and $[B']$, dependent on displacement, is the nonlinear part of the strain displacement matrix $[\overline{B}]$ and

$$[\overline{B}] = [B] + 0.5[B'] = [B] + 0.5[A'][G'] \tag{4.26}$$

$$[A'] = \begin{bmatrix} \frac{\partial w}{\partial x} - \frac{u}{R_x} & 0 \\ 0 & \frac{\partial w}{\partial y} - \frac{v}{R_y} \\ \frac{\partial w}{\partial y} - \frac{v}{R_y} & \frac{\partial w}{\partial x} - \frac{u}{R_x} \end{bmatrix}, \quad [G'] = \begin{bmatrix} -\frac{N_i}{R_x} & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & -\frac{N_i}{R_y} & \frac{\partial N_i}{\partial y} & 0 & 0 \end{bmatrix}_{i=1\cdots 8} \tag{4.27}$$

4.7 GOVERNING NONLINEAR EQUILIBRIUM EQUATIONS AND SOLUTION PROCEDURE

The equilibrium equations may be obtained by application of virtual work principle as:

$$\{\varphi\} = \oint_A [\overline{B}]^T \{F\} dA - \{P\} = \{0\} \tag{4.28}$$

where $\{\varphi\}$ denotes the resultants of internal forces and external generalised forces $\{P\}$. The external force is expressed as:

$$\{P\} = \sum_{i=1}^{8} \oint_A [N_i]^T \{q\} dA \tag{4.29}$$

The area integral is evaluated by 2×2 Gauss quadrature rule.

Here, $\{q\} = \{q_x \quad q_y \quad q_z \quad \mu_x \quad \mu_y\}^T$ in which q_x, q_y and q_z are the uniformly distributed forces per unit area along *X*, *Y* and *Z* axes, respectively, and μ_x and μ_y are the moments per unit area along X and Y axes, respectively. In case of shells subjected to transverse loads only, q_x, q_y, μ_x and μ_y vanish and q_z remains.

Now, $\{F\}$ represents the generalised stress resultant vector and with the help of Eqs. (4.9), (4.25) and (4.26), Eq. (4.28) becomes

$$\oint_A ([B]+[B'])^T [E]([B]+0.5[B'])dA\{d_e\}-\{P\}=\{0\} \Rightarrow [K]_s\{d_e\}=\{P\} \tag{4.30}$$

where, the secant stiffness matrix $[K]_s$ is given by

$$[K]_s = \oint_A [B]^T[E][B]dA + 0.5\oint_A [B]^T[E][B']dA + \oint_A [B']^T[E][B]dA + 0.5\oint_A [B']^T[E][B']dA$$

Taking appropriate variation of Eq. (4.28) with respect to $\{d_e\}$ the following may be obtained

$$d\{\varphi\} = \oint_A d[\overline{B}]^T\{F\}dA + \oint_A [\overline{B}]^T d\{F\}dA - d\{P\} = [K]_T\, d\{d_e\} - d\{P\} \tag{4.31}$$

Substituting Eqs. (4.9), (4.25) and (4.26) into Eq. (4.31), one gets

$$\oint_A d([B]+[B'])^T\{F\}dA + \oint_A ([B]+[B'])^T\{E\}([B]+[B'])dA\, d\{d_e\} - d\{P\} = \{0\}$$

$$\Rightarrow [K]_F\, d\{d_e\} + ([K]_L + [K]_{LT})d\{d_e\} = d\{P\} \tag{4.32}$$

where, $[K]_L = \oint_A [B]^T[E][B]dA$, $\qquad [K]_F = \oint_A [G']^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} [G']dA$

$$[K]_{LT} = \oint_A [B]^T[E][B']dA + \oint_A [B']^T[E][B]dA + \oint_A [B']^T[E][B']dA$$

$[K]_F$ is a symmetric matrix dependent on the stress level. This matrix is referred to as the initial stress matrix or the geometric matrix. N_x, N_y and N_{xy} are in-plane force and shear resultants as described by Eq. (4.10). Thus, the tangent stiffness matrix consists of three parts and expressed as:

$$[K]_T = [K]_L + [K]_{LT} + [K]_F \tag{4.33}$$

In order to solve nonlinear equilibrium equation Eq. (4.28) by the Newton – Raphson iteration procedure as depicted in Fig. 4.6, the function $\{\varphi\}(\{d_e\})$ is expressed in terms of Taylor's series and the higher order terms are ignored to get the improved solution of displacement field at $(n+1)^{th}$ iteration as,

$$\{\varphi\}\left(\{d_e\}^{n+1}\right)=\{\varphi\}\left(\{d_e\}^{n}\right)+\left(\frac{d\{\varphi\}}{d\{d_e\}}\right)_n\{\Delta d_e\}^n=0 \tag{4.34}$$

where $\{d_e\}^{n+1}=\{d_e\}^n+\{\Delta d_e\}^n$ (4.35)

In the above equation $\left(\frac{d\{\varphi\}}{d\{d_e\}}\right)_n=[K]_T^n$ (4.36)

Improved values of $\{d_e\}^{n+1}$ are obtained from Eq. (4.35) by calculating

$$\{\Delta d_e\}^{n+1}=\left([K]_T^n\right)^{-1}\left(\{P\}-[K]_S^n\{d_e\}^n\right) \tag{4.37}$$

Now, the convergence of this procedure is checked by using a pre-set tolerance limit (1% is adopted here) which is expressed in Eq. (4.38).

$$\sqrt{\left(\{\varphi\}^T\{\varphi\}\right)/\left(\{P\}^T\{P\}\right)}\times 100\leq Tolerance \tag{4.38}$$

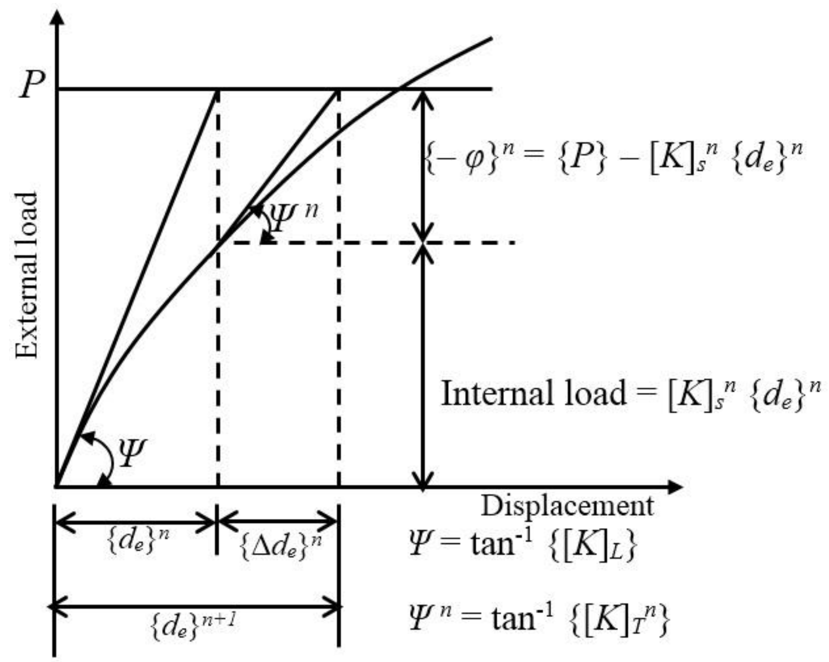


Fig. 4.6 The Newton – Raphson method

The element stiffness matrices and element load vectors are evaluated by numerical integration techniques using 2×2 Gauss quadrature rule. These element matrices are assembled with proper transformations from the curved geometry of the shell to get the global stiffness matrix $[K]$ and global load vectors $\{P\}$ respectively. The basic problem of static equilibrium takes the form, $[K]\{d\} = \{P\}$, where $\{d\}$ is the global displacement vector.

4.8 LAMINA STRESS CALCULATIONS

Lamina strains are transformed from the global axes of the shell to the local axes (1, 2 and 3) of the lamina using transformation matrix (Eq. 4.39). Lamina stresses are obtained using the constitutive relations of the lamina (Eq. 4.40).

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \sin^2\theta & \cos^2\theta & \sin 2\theta \\ \cos^2\theta & \sin^2\theta & -\sin 2\theta \\ -\sin 2\theta & \sin 2\theta & -2\cos 2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.39}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} E_{11}(1-\nu_{12}\nu_{21})^{-1} & E_{11}\nu_{21}(1-\nu_{12}\nu_{21})^{-1} & 0 \\ E_{11}\nu_{21}(1-\nu_{12}\nu_{21})^{-1} & E_{22}(1-\nu_{12}\nu_{21})^{-1} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{4.40}$$

The stress resultants are evaluated at the Gauss points (2×2) considering the shear correction factor as unity for the class of thin shells taken up here. Lamina stresses and strains are used in well accepted failure theories like maximum stress, maximum strain, Tsai-Hill, Tsai-Wu, and Hoffman's failure criterion as well as two other failure mode based criteria proposed by Hashin and Puck, given below, are used to evaluate the first ply failure loads of the composite shells under present study. A lamina is considered as failed if the calculated failure index [refer Reddy and Reddy (1992)] reaches a value very close to unity. The schematic algorithm for computing the first ply failure loads for composite shells is shown in Fig. 4.7 and it is discussed briefly in Section 4.10.

4.9 FAILURE THEORIES

4.9.1 Maximum Stress Failure Criterion

According to maximum stress theory, the failure initiates if at least one of the criteria is satisfied,

Fiber breakage mode: $\dfrac{\sigma_1}{\sigma_{1T}^u} \geq 1$, Fiber buckling mode: $-\sigma_1 \geq \sigma_{1C}^u$

Matrix cracking mode: $\dfrac{\sigma_2}{\sigma_{2T}^u} \geq 1$, Matrix crushing mode: $-\sigma_2 \geq \sigma_{2C}^u$

Matrix shear failure mode: $\dfrac{\tau_{12}}{\tau_{12}^u} \geq 1$ (4.41)

4.9.2 Maximum Strain Failure Criterion

According to maximum strain theory, the failure initiates if at least one of the criteria is satisfied,

Fiber breakage mode: $\dfrac{\varepsilon_1}{\varepsilon_{1T}^u} \geq 1$, Fiber buckling mode: $-\varepsilon_1 \geq \varepsilon_{1C}^u$

Matrix cracking mode: $\dfrac{\varepsilon_2}{\varepsilon_{2T}^u} \geq 1$, Matrix crushing mode: $-\varepsilon_2 \geq \varepsilon_{2C}^u$

Matrix shear failure mode: $\dfrac{\gamma_{12}}{\gamma_{12}^u} \geq 1$ (4.42)

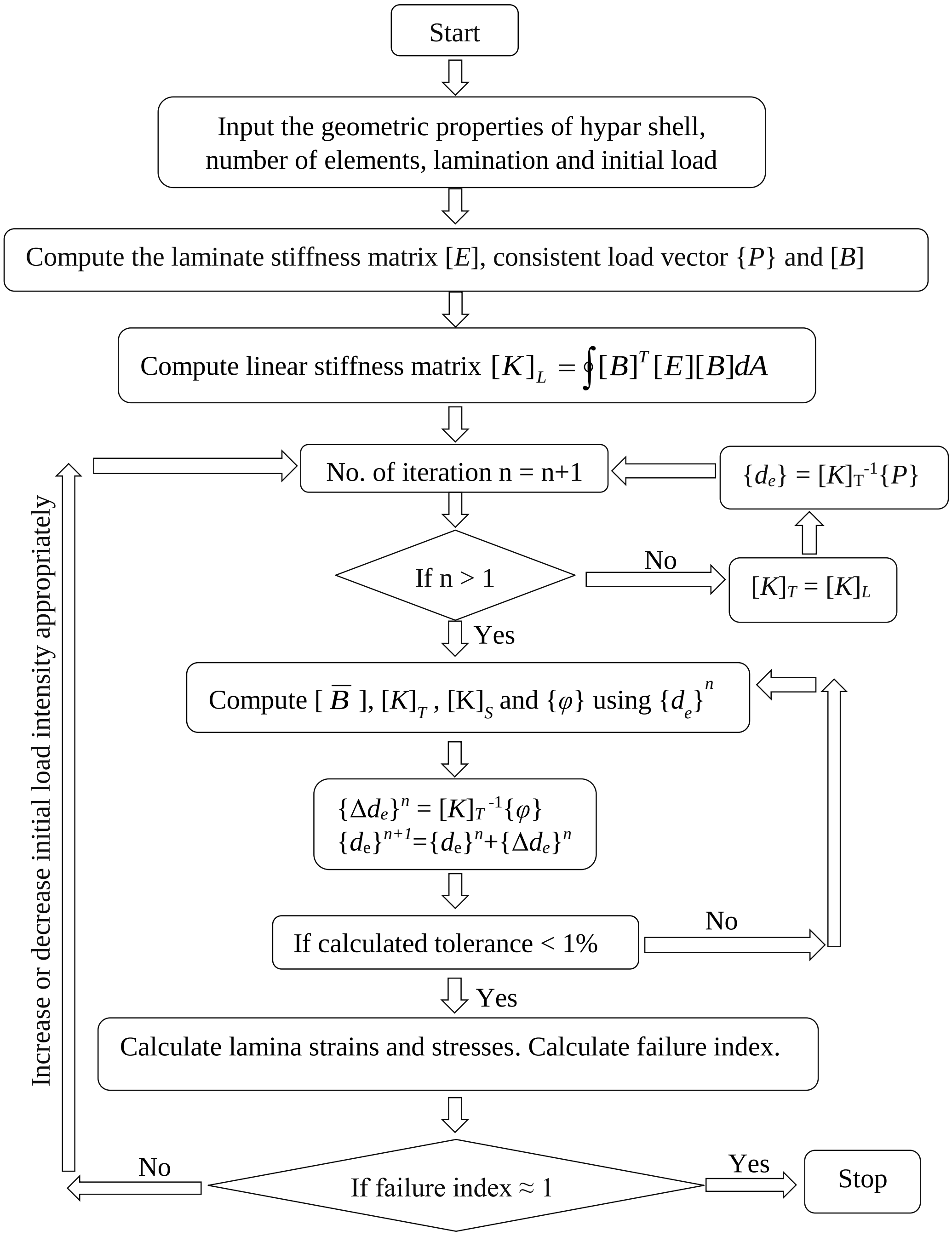


Fig.4.7 Algorithm for evaluating first ply failure load

4.9.3 Tsai-Hill Failure Criterion

According to Tsai-Hill failure theory, a lamina fails if at least one of the following conditions is satisfied,

$$\left(\frac{\sigma_1}{\sigma_{1T}^u}\right)^2+\left(\frac{\sigma_2}{\sigma_{2T}^u}\right)^2-\left(\frac{1}{\sigma_{1T}^u}+\frac{1}{\sigma_{2T}^u}\right)\sigma_1\sigma_2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1 \qquad \text{for } \sigma_1,\sigma_2>0$$

$$\left(\frac{\sigma_1}{\sigma_{1C}^u}\right)^2+\left(\frac{\sigma_2}{\sigma_{2C}^u}\right)^2-\left(\frac{1}{\sigma_{1C}^u}+\frac{1}{\sigma_{2C}^u}\right)\sigma_1\sigma_2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1 \qquad \text{for } \sigma_1,\sigma_2<0 \tag{4.43}$$

4.9.4 Tsai-Wu Failure Criterion

Tsai-Wu criterion can be expressed as,

$$\left(\frac{1}{\sigma_{1T}^u}-\frac{1}{\sigma_{1C}^u}\right)\sigma_1+\left(\frac{1}{\sigma_{2T}^u}-\frac{1}{\sigma_{2C}^u}\right)\sigma_2+\left(\frac{1}{\sigma_{1T}^u\sigma_{1C}^u}\right)\sigma_1^2+\left(\frac{1}{\sigma_{2T}^u\sigma_{2C}^u}\right)\sigma_2^2-\left(\sqrt{\sigma_{1T}^u\sigma_{1C}^u\sigma_{2T}^u\sigma_{2C}^u}\right)\sigma_1\sigma_2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1 \tag{4.44}$$

4.9.5 Hoffman Failure Criterion

Hoffman criterion is expressed as,

$$\frac{1}{2}\left(\frac{1}{\sigma_{1T}^u\sigma_{1C}^u}-\frac{1}{\sigma_{2T}^u\sigma_{2C}^u}\right)\sigma_1^2+\frac{1}{2}\left(\frac{1}{\sigma_{1T}^u\sigma_{1C}^u}+\frac{1}{\sigma_{2T}^u\sigma_{2C}^u}\right)(\sigma_1-\sigma_2)^2+\left(\frac{1}{\sigma_{1T}^u}-\frac{1}{\sigma_{1C}^u}\right)\sigma_1+\left(\frac{1}{\sigma_{2T}^u}-\frac{1}{\sigma_{2C}^u}\right)\sigma_2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1 \tag{4.45}$$

4.9.6 Hashin Failure Criterion

Fiber breakage mode: $\left(\frac{\sigma_1}{\sigma_{1T}^u}\right)^2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1$, Fiber buckling mode: $-\sigma_1\geq\sigma_{1C}^u$

Matrix cracking mode: $\left(\frac{\sigma_2}{\sigma_{2T}^u}\right)^2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1$

Matrix crushing mode: $\left[\left(\frac{\sigma_{2C}^u}{2\tau_{12}^u}\right)^2-1\right]\left(\frac{\sigma_2}{\sigma_{2C}^u}\right)+\left(\frac{\sigma_2}{2\tau_{23}^u}\right)^2+\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2\geq 1$ (4.46)

4.9.7 Puck Failure Criterion

Fiber breakage mode: $\frac{1}{2}\left(\left|\frac{\sigma_1}{\sigma_{1T}^u}\right|+\left|\frac{\varepsilon_1}{\varepsilon_{1T}^u}\right|\right)\geq 1$, Fiber buckling mode: $\frac{1}{2}\left(\left|\frac{\sigma_1}{\sigma_{1C}^u}\right|+\left|\frac{\varepsilon_1}{\varepsilon_{1C}^u}\right|\right)\geq 1$

Matrix cracking mode A $(\sigma_2 > 0)$: $\sqrt{\left(\frac{\tau_{12}}{\tau_{12}^u}\right)^2+\left(1-p_{12}^{(+)}\frac{\sigma_{2T}^u}{\tau_{12}^u}\right)^2\left(\frac{\sigma_2}{\sigma_{2T}^u}\right)^2}+p_{12}^{(+)}\frac{\sigma_2}{\tau_{12}^u}\geq 1$

Matrix crushing mode B $\left[(\sigma_2 < 0);0\leq\left|\frac{\sigma_2}{\tau_{12}}\right|\leq\frac{\tau_{23}^A}{\left|\tau_{12}^C\right|}\right]$: $\frac{1}{\tau_{12}^u}\left[\sqrt{(\tau_{12})^2+\left(p_{12}^{(-)}\sigma_2\right)^2}+p_{12}^{(+)}\sigma_2\right]\geq 1$

Matrix crushing mode C $\left[(\sigma_2 < 0);0\leq\left|\frac{\tau_{12}}{\sigma_2}\right|\leq\frac{\left|\tau_{12}^C\right|}{\tau_{23}^A}\right]$: $\left[\left\{\frac{\tau_{12}}{2\left(1+p_{12}^{(-)}\tau_{23}^A\right)}\right\}^2+\left(\frac{\sigma_2}{\sigma_{2C}^u}\right)^2\right]\frac{\sigma_{2C}^u}{(-\sigma_2)}\geq 1$

(4.47)

where, $p_{12}^{(+)}=0.3$, $p_{12}^{(-)}=0.3$, $\tau_{23}^A=\frac{\tau_{12}^u}{2p_{12}^{(-)}}\left[\sqrt{1+2p_{12}^{(-)}\frac{\sigma_{2C}^u}{\tau_{12}^u}}-1\right]$ and $\tau_{12}^C=\tau_{12}^u\sqrt{1+2p_{12}^{(-)}\frac{\tau_{23}^A}{\tau_{12}^u}}$

In case of interactive failure theories such as Tsai-Hill, Tsai-Wu and Hoffman failure criteria, none of the individual lamina stress components reach the permissible values but their interaction leads to failure. In case of such failures, the individual stress values developed may be compared to their corresponding permissible values to investigate that which stress component contributing to a particular interactive criterion plays the most significant role in the failure. The stress component for which the ratio of the developed to permissible stress is nearest to unity may be identified as the most significant component contributing to the failure following the corresponding failure mode also.

4.10 SOLUTION OF FIRST PLY FAILURE ANALYSIS

Step 1: The static displacements of shells are found for the external load $\{P\}$. The linear displacement fields are used for obtaining the linear first ply failure loads and similarly the converged nonlinear displacement fields are used for getting the nonlinear failure loads.

Step 2: The linear and nonlinear strain vectors are calculated at four Gauss points from the linear and nonlinear displacement fields. The linear and nonlinear strains are combined to obtain the mid-surface strain vector following Eqs. 4.24 and 4.25.

Step 3: The in-plane and transverse strains are transformed from global to local axes system using Eq. 4.39.

Step 4: The lamina stresses are calculated at Gauss points using Eq. 4.40. For linear failure loads, the linear strains are used. The nonlinear failure loads are calculated using the nonlinear strains.

Step 5: The stresses are extrapolated from Gauss point locations to the element node points.

Step 6: The lamina stresses are incorporated in the different failure theories and the failure index is calculated.

Step 7: The failure index is compared with unity. The initial external load is appropriately increased or decreased depending on whether the failure index value is lesser than or greater than unity respectively.

Step 8: Steps 1 to 7 are repeated till the percentage difference between the first ply failure loads of two successive iterations is less than unity.

Chapter 5

LINEAR FIRST PLY FAILURE OF CLAMPED SPHERICAL SHELL ROOFS

5.1 GENERAL

Stiff doubly curved synclastic spherical shells are extremely popular in civil engineering construction industry and hence performances of these shell forms with laminated composite as the material need to be known in depth to the engineering fraternity. This chapter is devoted to study the failure characteristics of laminated composite spherical shells with different practical parametric variations. The developed finite element code is used to solve some benchmark problems for validation and is also used to solve a number of other problems, with different practical parametric variations, to bring out the failure characteristics of graphite – epoxy spherical shells. The results of all the problems and the relevant discussions are presented in Section 5.2. All the results that are presented in this chapter are arrived at after an appropriate mesh convergence study and are represented in forms of tables and figures for clarity. A particular value is assumed to have converged for a particular finite element grid when further refinement of the mesh does not improve the results to an extent exceeding 1%. The salient conclusions which emerged from the present study are presented in Section 5.3 at the end of the chapter.

5.2 NUMERICAL STUDY AND DISCUSSIONS

The non-dimensional transverse displacements of composite spherical shells for different finite element grids are $\overline{w} = [wE_{22}h^3/(qa^4)]$ where w is the transverse displacement in cm and q is the uniformly distributed transverse loading intensity as mentioned by Reddy (1984). The results displayed in Table 5.1 show a good match of the present results with the results obtained by Reddy (1984) using exact method. The table further shows an excellent and monotonic convergence of the displacement values. In order to verify the correctness of the present approach the first ply failure loads evaluated using the present finite element formulation are compared with the linear failure loads reported by Kam et al. (1996) in Table 5.2.

Table 5.1 Nondimensional central deflections ($\overline{w} \times 10^3$) of composite spherical shells

Method	0°/90°	0°/90°/0°	0°/90°/90°/0°
Reddy (1984)	1.1412	1.0443	1.0559
Present finite element model (4×4)	1.1754	1.0151	1.0812
Present finite element model (6×6)	1.1583	1.0568	1.0682
Present finite element model (8×8)	1.1416	1.0445	1.0561
Present finite element model (10×10)	1.1418	1.0450	1.0565

Note: E_{11}/E_{22} = 25, G_{12} = G_{13} = 0.5E_{22}, E_{22} = 10GPa, ν_{12} = 0.25, $a = b$, $a/h = 100$, R/a = 4

Table 5.2 Comparison of linear first ply failure loads in Newton for a ($0^{\circ}_2/90^{\circ}$)S plate

Failure criteria	Length/ plate thickness	Experimental failure load (Kam et al. 1996)	First ply failure loads (Kam et al. 1996)	First ply failure loads (present formulation)
Maximum stress			108.26	112.15
Maximum strain			122.86	127.56
Hoffman	105.26	157.34	106.45	103.36
Tsai-Wu			112.77	110.46
Tsai-Hill			107.06	104.40

Note: Length = 100mm, loading details = a central point of load

All the problems presented in this chapter are solved for clamped spherical shells under uniformly distributed static transverse loading and the results are interpreted from practical engineering standpoint. The material properties of the *Q*–1115 graphite – epoxy composite to fabricate the spherical shells are as follows:

Material properties: E_{11} = 142.5 GPa, $E_{22} = E_{33}$ = 9.79 GPa, $G_{12} = G_{13}$ = 4.72 GPa, G_{23} = 1.192 GPa, $\nu_{12} = \nu_{13}$ = 0.27, ν_{23} = 0.25.

Strengths: σ_{1T}^{u} = 2193.5 MPa, σ_{1C}^{u} = 2457 MPa, σ_{2T}^{u} = 41.3 MPa, σ_{2C}^{u} = 206.8 MPa, τ_{13}^{u} = 61.28 MPa, τ_{12}^{u} = τ_{23}^{u} = 78.78 MPa, ε_{1T}^{u} = 0.01539, ε_{1C}^{u} = 0.01724, ε_{2T}^{u} = 0.00412, ε_{2C}^{u} = 0.02112, γ_{13}^{u} = 0.05141, γ_{12}^{u} = γ_{23}^{u} = 0.01669.

The aspect ratio (*a/b*) of the shell is considered as unity and the value of span to thickness ratio (*a/h*) is taken as 100 for the present study. The *R/a* value is taken as 0.75 here. Plies are numbered from top to bottom of the laminate i.e. the topmost ply is numbered one.

The uniformly distributed first ply failure load (*FL*) values of laminated composite clamped spherical shells are furnished in Tables 5.3 and 5.4 respectively. These first ply failure loads are nondimensionalized as $\overline{FL} = (FL/E_{22})(a/h)^4$. The independent failure criteria such as Maximum Stress and Maximum Strain criteria, the interactive failure criteria like Hoffman, Tsai – Hill and Tsai – Wu criteria and the partially interactive or failure mode based criteria like Hashin and Puck failure criteria are used to evaluate the first ply failure loads for different stacking arrangements of laminated composite spherical shells. The minimum value of the failure load obtained from different failure criteria is considered as the acceptable failure load on which the engineering factor of safety must be imposed.

It is observed from the results presented in Tables 5.3 and 5.4 that the upper bound of the first ply failure load is determined by the interactive or partially interactive failure criteria for angle ply and by the partially interactive failure criteria for cross ply shells and the lower bound is determined from the independent and partially interactive failure criteria for angle and cross ply shells respectively. For the cross ply shells taken up here the upper bound values of failure load are given by Puck's failure criterion while the Hashin's criterion corresponds to the lower bound value. It is interesting to note that this trend is observed for angle ply shells also where the Puck or the Tsai-Hill criteria correspond to the upper bound failures while the Maximum Strain criterion yields the lower bound value of the failure load. With the highest value of failure load as reference the percentage difference between the upper and lower bounds varies from 67% to 74% in the case of cross ply spherical shells and from 42.5% to 51.5% in the case of angle ply spherical shells. The Puck's theory yields quite higher values of failure loads compared to the other criteria for cross ply shells. Even if one excludes the results obtained from this theory for cross ply shells the percentage difference between the highest and lowest failure load values with the highest value of the load as reference varies between 41.5% and 42.15%. These observations simply establish the fact that all the different failure criteria should simultaneously be tried out to get the failure load acceptable from engineering point of view. An exclusion of any of the theories may lead to grossly erroneous estimate of the failure load. It is interesting to note that, for cross ply, this difference is minimum for double layered shells and is maximum for four layered symmetric ply. Again, for angle ply shells, the reverse trend is observed. This phenomenological wide variation of the failure load values obtained from different criteria may be attributed to the difference in basic formulation of the failure theories and such deviations are difficult to justify through physical reason. The results obtained here are in tune with the observations made by Soni (1983) where he concluded that percentage differences between upper and lower bound values of failure load may be as high as 75% when the loading conditions are varied. Figs. 5.1 and 5.2 show the variation of first ply failure loads obtained from different failure criteria.

When the results given in Tables 5.3 and 5.4 are observed it is found that the cross ply shells are better options compared to the angle ply ones because the cross ply shells consistently yield higher values of the failure loads compared to those obtained from angle ply shells. In fact the 0°/90°/90°/0° shell is the weakest among the cross ply arrangements taken up here and still the failure load for this lamination is about 1.7 times the failure load of 45°/-45°/45° shell – the combination which gives the highest failure load among the angle ply combinations taken up here. Naturally, from engineering stand point, for a particular quantity of material consumption the cross ply shells should be preferred compared to the angle ply ones. It is interesting to note that all the cross ply combinations yield their minimum values of failure loads corresponding to Hashin's criteria which involves all the stresses and strains contributing in the failure criteria and hence indicates a more efficient utilization of material strength. On the other hand all the angle ply combinations considered here failed through Maximum Strain criteria by matrix cracking.

Table 5.3. Nondimensionalized first ply uniformly distributed failure load of cross ply spherical shells

Lamination in degree	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
0°/90°	Maximum stress	76221.62	(0.875a,0)	2	Fiber Buckling
	Maximum strain	76602.62	(0.875a,0)	2	Fiber Buckling
	Hoffman	95915.18	(0.875a,0)	2	Fiber Buckling
	Tsai-Hill	79733.37	(0.875a,0)	2	Fiber Buckling
	Tsai-Wu	130297.2	(0.875a,0)	2	Fiber Buckling
	Hashin	*76220.6*	*(0.875a,0)*	*2*	*Fiber Buckling*
	Puck	232327.8	(0,0.875b)	1	Matrix Cracking Mode A
0°/90°/0°	Maximum stress	82197.1	(a,0.75b)	3	Fiber Buckling
	Maximum strain	82623.05	(a,0.75b)	3	Fiber Buckling
	Hoffman	103868.2	(a,0.75b)	3	Fiber Buckling
	Tsai-Hill	86209.36	(a,0.75b)	3	Fiber Buckling
	Tsai-Wu	128834.5	(0.125a,0)	3	Fiber Buckling
	Hashin	*74839.6*	*(0.125a,0)*	*3*	*Matrix Crushing*
	Puck	251378.8	(0.125a,0)	3	Matrix Cracking Mode A
0°/90°/0°/90°	Maximum stress	87211.4	(0.125a,b)	4	Fiber Buckling
	Maximum strain	87634.28	(0.125a,b)	4	Fiber Buckling

Lamination in degree	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
	Hoffman	109876.4	(0.125a,b)	4	Fiber Buckling
	Tsai-Hill	91529.07	(0.125a,b)	4	Fiber Buckling
	Tsai-Wu	140279.8	(a,0.25b)	4	Fiber Buckling
	Hashin	*81150.12*	*(a,0.25b)*	*4*	*Matrix Crushing*
	Puck	268718	(0,0.875b)	1	Matrix Cracking Mode A
0°/90°/90°/0°	Maximum stress	87940.72	(a,0.75b)	4	Fiber Buckling
	Maximum strain	88374.83	(a,0.75b)	4	Fiber Buckling
	Hoffman	112566.9	(a,0.75b)	4	Fiber Buckling
	Tsai-Hill	93861.04	(a,0.75b)	4	Fiber Buckling
	Tsai-Wu	121260.4	(0.875a,0)	4	Fiber Buckling
	Hashin	*70275.76*	*(0.875a,0)*	*4*	*Matrix Crushing*
	Puck	274601.5	(a,0.875b)	1	Matrix Cracking Mode A

Note: *a/b* = 1, *a/h* = 100, *R/a* = 0.75, the least failure loads are shown by italics.

Table 5.4. Nondimensionalized first ply uniformly distributed failure load of angle ply spherical shells

Lamination in degree	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
45°/-45°	Maximum stress	56169.54	(0.875a,b)	1	Matrix Cracking
	Maximum strain	*29611.84*	*(0.875a,b)*	*1*	*Matrix Cracking*
	Hoffman	40940.74	(0.875a,b)	1	Matrix Cracking
	Tsai-Hill	60932.56	(0.875a,b)	1	Matrix Cracking
	Tsai-Wu	36655.75	(0.875a,b)	1	Matrix Cracking
	Hashin	40571.99	(0.875a,b)	2	Matrix Crushing
	Puck	55397.32	(0.875a,b)	1	Matrix Cracking Mode A
45°/-45°/45°	Maximum stress	59774.23	(0.125a,0)	3	Fiber Buckling
	Maximum strain	*41262.49*	*(0.125a,0)*	*1*	*Matrix Cracking*
	Hoffman	56396.3	(0.125a,0)	1	Fiber Buckling
	Tsai-Hill	56748.7	(0.125a,0)	3	Fiber Buckling
	Tsai-Wu	50861.06	(0.125a,0)	1	Fiber Buckling
	Hashin	46448.4	(0.125a,b)	3	Matrix Crushing
	Puck	77757.88	(0.125a,b)	2	Fiber Buckling
45°/-45°/ 45°/-45°	Maximum stress	64278.83	(0.125a,0)	3	Fiber Buckling
	Maximum strain	*35671.08*	*(0.125a,0)*	*1*	*Matrix Cracking*
	Hoffman	49265.55	(0.125a,0)	1	Matrix Cracking
	Tsai-Hill	61675.15	(0.875a,0)	4	Fiber Buckling
	Tsai-Wu	44136.85	(0.125a,0)	1	Matrix Cracking
	Hashin	41148.09	(0.125a,0)	4	Matrix Crushing
	Puck	64351.35	(0.125a,0)	3	Fiber Buckling
45°/-45°/ -45°/45°	Maximum stress	59148.08	(0.125a,0)	4	Fiber Buckling
	Maximum strain	*40330.93*	*(0.125a,0)*	*1*	*Matrix Cracking*

Lamination in degree	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
	Hoffman	55627.15	(0.125*a*,0)	1	Matrix Cracking
	Tsai-Hill	56333.99	(0.125*a*,0)	4	Fiber Buckling
	Tsai-Wu	49885.58	(0.125*a*,0)	1	Matrix Cracking
	Hashin	44628.17	(0.125*a*,*b*)	4	Matrix Crushing
	Puck	70119.48	(0.875*a*,0)	3	Fiber Buckling

Note: *a/b* = 1, *a/h* = 100, *R/a* = 0.75, the least failure loads are shown by italics.

It is well established that the variation of angle of lamination has a pronounced effect on the composite spherical shell characteristics and it is possible to optimally tailor the directions of the individual lamina to extract maximum advantage from a given quantity of material. Hence in the next part of the study typical 0°/θ°/θ°/0° and 0°/θ°/0°/θ° laminations are studied for discrete values of θ to see how the failure loads respond to such variations. The results are furnished in Tables 5.5 and 5.6 and Fig. 5.3. The results show that for any particular value of θ the four layered symmetric and anti-symmetric laminates show comparable results though when θ reaches 90° the 0°/90°/0°/90° shell shows a failure load which is 1.15 times of that corresponding to 0°/90°/90°/0° shell. For both symmetric and anti-symmetric laminates higher values of failure loads are obtained when θ approaches 0° or 90° while for intermediate values of θ the failure loads are of lesser magnitude. This again shows that cross ply combination should be preferred in fabricating four layered composite shells. Since in a laminate all the fibers in different layers are not normally put in same directions, the shell options with only 0° laminations are not practically applicable. Hence 0°/90°/0°/90° and 0°/90°/90°/0° shells are recommended among which again the 0°/90°/0°/90° option is relatively better.

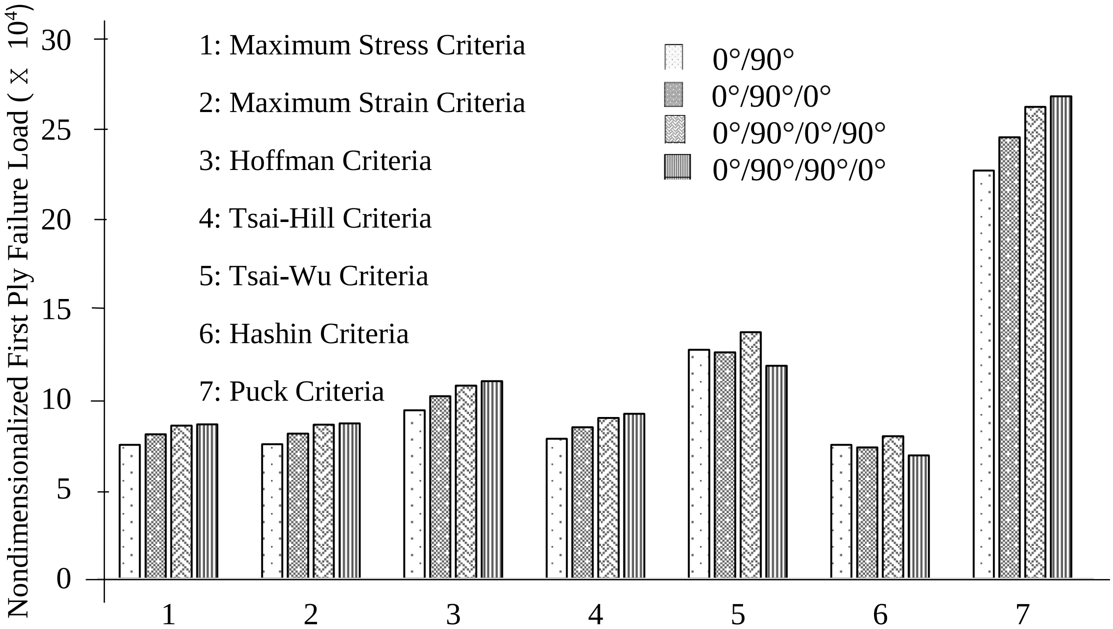


Fig.5.1. The first ply failure loads from different failure criteria for cross ply spherical shells

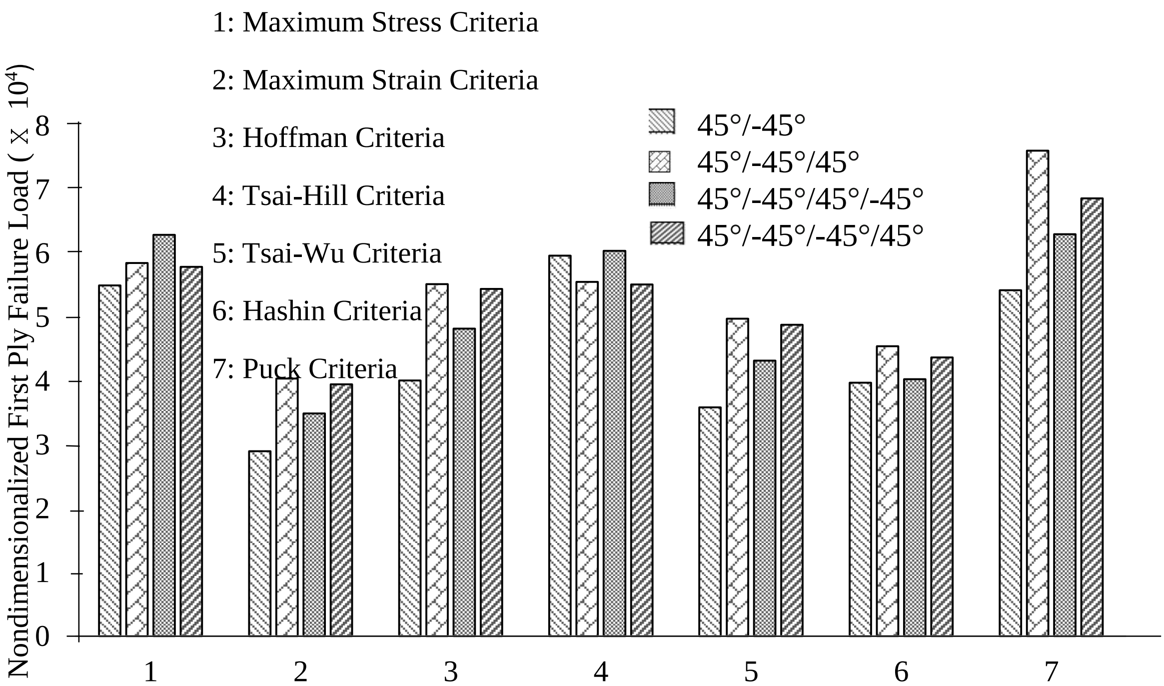


Fig.5.2.The first ply failure loads from different failure criteria for angle ply spherical shells

Failure occurring at the support, as evident from Tables 5.5 and 5.6, is quiet expected because for the clamped boundary condition, the points on the support or in near vicinity are subjected to worst values of bending moment and shear force. When two layered cross and angle ply laminates are compared it is found that they tend to fail through compressive stress along the fibers in a lamina and transverse tension respectively. The three and four layered cross ply laminates are weak in transverse compression. On the other hand, in angle ply laminates the individual lamina always has a set of fibers running along the diagonal in plan, no fiber being present along the perpendicular diagonal. Naturally the shells become vulnerable in diagonal tension in that direction and if the lamina is considered alone, the failure is through transverse cracking of the matrix.

While comparing four layered symmetric and anti-symmetric stacking orders (Tables 5.5 and 5.6) it is very interesting to note that all shells except 0°/0°/0°/0° fail through matrix failure. The lamina where fibers are oriented at angle θ ranging from 15° to 60° are vulnerable in diagonal tension perpendicular to the fiber direction and the failed ply undergoes to the failure mode of transverse cracking of the matrix.

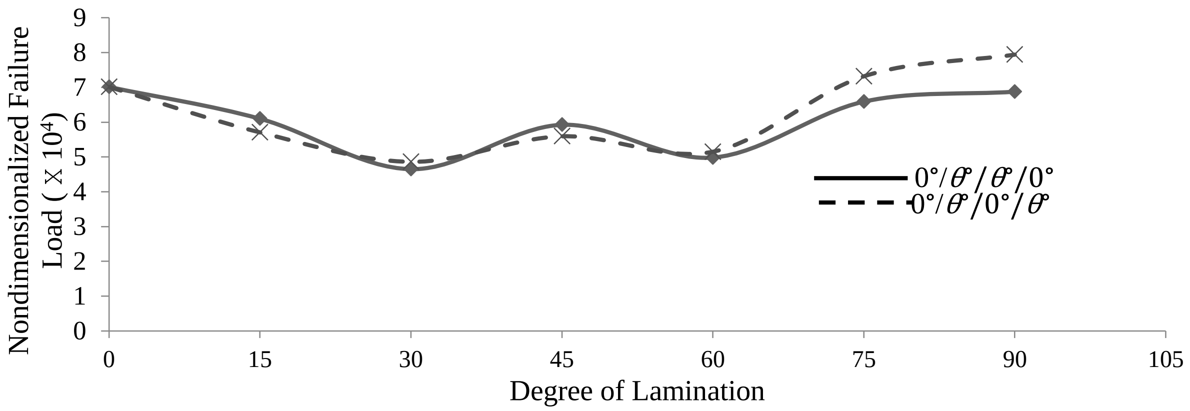


Fig. 5.3. The first ply failure loads vs. degree of laminations for spherical shells

Table 5.5. Nondimensionalized first ply uniformly distributed failure load of 0°/θ°/θ°/0° spherical shells

Variation of θ°	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
0°	Maximum stress	71853.9	(0,0.25b)	4	Fiber Buckling
	Maximum strain	72248.18	(0,0.25b)	4	Fiber Buckling
	Hoffman	87083.72	(0,0.25b)	4	Fiber Buckling
	Tsai-Hill	*71666.97*	*(0,0.25b)*	*4*	*Fiber Buckling*
	Tsai-Wu	118176.7	(a,0.25b)	4	Fiber Buckling

Variation of θ°	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
	Hashin	71853.39	(0,0.25b)	4	Fiber Buckling
	Puck	129213.4	(0,0.25b)	4	Fiber Buckling
15°	Maximum stress	65567.9	(0,0.125b)	3	Fiber Buckling
	Maximum strain	*62337.05*	*(0,0.125b)*	*3*	*Matrix Cracking*
	Hoffman	67447.36	(0,0.125b)	3	Fiber Buckling
	Tsai-Hill	65497.42	(a,0.75b)	4	Fiber Buckling
	Tsai-Wu	65567.9	(0,0.125b)	3	Fiber Buckling
	Hashin	65916.21	(0,0.75b)	4	Fiber Buckling
	Puck	78742.56	(a,0.75b)	3	Fiber Buckling
30°	Maximum stress	65971.37	(0,0.75b)	4	Fiber Buckling
	Maximum strain	*47519.9*	*(0,0.125b)*	*3*	*Matrix Cracking*
	Hoffman	60270.66	(0,0.125b)	3	Fiber Buckling
	Tsai-Hill	67390.16	(a,0.875b)	4	Fiber Buckling
	Tsai-Wu	57124.59	(0,0.125b)	3	Fiber Buckling
	Hashin	65971.37	(0,0.75b)	4	Fiber Buckling
	Puck	68284.95	(a,0.875b)	3	Fiber Buckling
45°	Maximum stress	70638.37	(a,0.25b)	4	Fiber Buckling
	Maximum strain	*60537.26*	*(0.875a,b)*	*2*	*Matrix Cracking*
	Hoffman	79252.26	(0.875a,b)	2	Fiber Buckling
	Tsai-Hill	73210.39	(0,0.125b)	4	Fiber Buckling
	Tsai-Wu	72973.41	(0.875a,b)	2	Fiber Buckling
	Hashin	64165.45	(0.125a,0)	4	Matrix Crushing
	Puck	95321.71	(0.875a,b)	3	Fiber Buckling
60°	Maximum stress	77703.74	(0,0.75b)	4	Fiber Buckling
	Maximum strain	*50923.37*	*(0.125a,0)*	*2*	*Matrix Cracking*
	Hoffman	70005.08	(0.125a,0)	2	Fiber Buckling
	Tsai-Hill	83148.07	(a,0.875b)	4	Fiber Buckling
	Tsai-Wu	63220.6	(0.125a,0)	2	Fiber Buckling
	Hashin	64842.67	(0.125a,0)	4	Matrix Crushing
	Puck	87453.48	(0.125a,0)	3	Fiber Buckling
75°	Maximum stress	84503.54	(a,0.25b)	4	Fiber Buckling
	Maximum strain	84923.35	(a,0.25b)	4	Fiber Buckling
	Hoffman	109737.4	(0,0.75b)	4	Fiber Buckling
	Tsai-Hill	91658.79	(0,0.75b)	4	Fiber Buckling
	Tsai-Wu	112990.8	(0.625a,b)	3	Fiber Buckling
	Hashin	*67349.31*	*(0.875a,0)*	*4*	*Matrix Crushing*
	Puck	136895.8	(0.375a,0)	2	Fiber Buckling
90°	Maximum stress	87940.72	(a,0.75b)	4	Fiber Buckling
	Maximum strain	88374.83	(a,0.75b)	4	Fiber Buckling
	Hoffman	112566.9	(a,0.75b)	4	Fiber Buckling
	Tsai-Hill	93861.04	(a,0.75b)	4	Fiber Buckling
	Tsai-Wu	121260.4	(0.875a,0)	4	Fiber Buckling

Variation of θ°	Failure theory	$\overline{FL}$	Location (x,y)	First failed ply	Failure mode/ failure tendency
	Hashin	*70275.76*	*(0.875a,0)*	*4*	*Matrix Crushing*
	Puck	274601.5	(a,0.875b)	1	Matrix Cracking Mode A

Note: *a/b* = 1, *a/h* = 100, *R/a* = 0.75, the least failure loads are shown by italics.

Table 5.6. Nondimensionalized first ply uniformly distributed failure load of 0°/θ°/0°/θ° spherical shells

Variation of θ°	Failure theory	$\overline{FL}$	Location (x,y) (m, m)	First failed ply	Failure mode/ failure tendency
0°	Maximum stress	71853.9	(0,0.25b)	4	Fiber Buckling
	Maximum strain	72248.18	(0,0.25b)	4	Fiber Buckling
	Hoffman	87083.72	(0,0.25b)	4	Fiber Buckling
	Tsai-Hill	*71666.97*	*(0,0.25b)*	*4*	*Fiber Buckling*
	Tsai-Wu	118176.7	(a,0.25b)	4	Fiber Buckling
	Hashin	71853.39	(0,0.25b)	4	Fiber Buckling
	Puck	129213.4	(0,0.25b)	4	Fiber Buckling
15°	Maximum stress	60404.47	(0,0.25b)	4	Fiber Buckling
	Maximum strain	*58336.03*	*(0,0.125b)*	*4*	*Matrix Cracking*
	Hoffman	60764.02	(0,0.125b)	4	Fiber Buckling
	Tsai-Hill	60649.62	(a,0.75b)	4	Fiber Buckling
	Tsai-Wu	60459.63	(0,0.125b)	4	Fiber Buckling
	Hashin	60404.47	(0,0.25b)	4	Fiber Buckling
	Puck	67496.39	(0,0.125b)	4	Fiber Buckling
30°	Maximum stress	59991.8	(0,0.125b)	4	Fiber Buckling
	Maximum strain	*49682.31*	*(0,0.125b)*	*4*	*Matrix Cracking*
	Hoffman	58170.56	(0,0.125b)	4	Fiber Buckling
	Tsai-Hill	62266.57	(0,0.125b)	4	Fiber Buckling
	Tsai-Wu	57250.23	(0,0.125b)	4	Fiber Buckling
	Hashin	57751.76	(0.125a,b)	4	Matrix Crushing
	Puck	59981.59	(0,0.125b)	4	Fiber Buckling
45°	Maximum stress	78946.85	(0,0.125b)	4	Fiber Buckling
	Maximum strain	78103.13	(0.875a,b)	2	Matrix Cracking
	Hoffman	89245.11	(0,0.125b)	4	Fiber Buckling
	Tsai-Hill	75917.23	(0,0.125b)	4	Fiber Buckling
	Tsai-Wu	89107.21	(0.875a,b)	2	Fiber Buckling
	Hashin	*57176.69*	*(a,0.125b)*	*4*	*Matrix Crushing*
	Puck	108449.4	(0.875a,b)	2	Fiber Buckling
60°	Maximum stress	68839.6	(0.125a,0)	4	Fiber Buckling
	Maximum strain	*52643.49*	*(0.125a,0)*	*2*	*Matrix Cracking*
	Hoffman	69713.96	(0.125a,0)	4	Fiber Buckling
	Tsai-Hill	68986.69	(0.125a,0)	4	Fiber Buckling
	Tsai-Wu	65164.42	(0.125a,0)	2	Fiber Buckling

Variation of θ°	Failure theory	$\overline{FL}$	Location (x,y) (m, m)	First failed ply	Failure mode/ failure tendency
75°	Hashin	63775.25	(0,0.875*b*)	4	Matrix Crushing
	Puck	96543.37	(0.125*a*,0)	2	Fiber Buckling
	Maximum stress	79136.84	(0.125*a*,0)	4	Fiber Buckling
	Maximum strain	79306.4	(0.125*a*,0)	4	Fiber Buckling
	Hoffman	91012.22	(0.125*a*,0)	4	Fiber Buckling
	Tsai-Hill	81264.52	(0.125*a*,0)	4	Fiber Buckling
	Tsai-Wu	102917.2	(0.125*a*,0)	4	Fiber Buckling
	Hashin	*74786.48*	*(0,0.875b)*	*4*	*Matrix Crushing*
90°	Puck	124518.8	(0.25*a*,0)	2	Fiber Buckling
	Maximum stress	87211.4	(0.125*a*,*b*)	4	Fiber Buckling
	Maximum strain	87634.28	(0.125*a*,*b*)	4	Fiber Buckling
	Hoffman	109876.4	(0.125*a*,*b*)	4	Fiber Buckling
	Tsai-Hill	91529.07	(0.125*a*,*b*)	4	Fiber Buckling
	Tsai-Wu	140279.8	(*a*,0.25*b*)	4	Fiber Buckling
	Hashin	*81150.12*	*(a,0.25b)*	*4*	*Matrix Crushing*
	Puck	268718	(0,0.875*b*)	1	Matrix Cracking Mode A

Note: *a/b* = 1, *a/h* = 100, *R/a* = 0.75, the least failure loads are shown by italics.

5.3 CONCLUDING REMARKS

The following conclusions are evident from the present study.

- The finite element code used here can be accepted as a successful tool to explore the first ply failure aspects of composite spherical shells. The solutions obtained for benchmark problems by the present method confirm this fact.
- The failure load values of spherical shells for different laminations are expected to serve as design aids to practicing engineers.
- Among cross and angle ply shells, the cross ply ones are better in performance compared to their angle ply counterparts, for a given number of lamina and shell curvature. In case of cross ply, relatively more efficient material utilization may be achieved.

- The critical values of bending moment and shear force occur at the support for clamped boundary condition. So the failure must initiate at support and the results of the present investigation agree to this fact.
- Four layered symmetric and anti-symmetric spherical shells are weak in diagonal tension or compression and this fact is reflected in their failure modes or tendencies.

Chapter 6

LINEAR AND NONLINEAR FIRST PLY FAILURE OF CLAMPED SKEWED HYPAR SHELL ROOFS OF VARYING PLANFORM AND THICKNESS

6.1 GENERAL

Apart from being esthetically appealing doubly curved anticlastic surfaces, the skewed hypar shells are doubly ruled and may be fabricated and cast easily. No wonder that this form capable of allowing entry of north light is preferred for cladding industrial areas requiring diffused sunlight. In order to cover a particular area in plan an engineer often needs to decide the column grid suited to the functional requirements of the area and hence the cladding units may be of different aspect ratios. The thicknesses of shell units of varying planform are to be appropriately designed depending on the expected value of the superimposed loads and the serviceability criterion mostly in terms of deflection. Keeping the industrial requirements in mind, this chapter presents a study on clamped skewed hypar shells of different planforms and thicknesses using linear and nonlinear strains. The results which are furnished in forms of tables and figures in Sections 6.2 and 6.3 are expected to reflect the relative efficacy of the linear and nonlinear approaches apart from providing useful design results for the clamped boundary condition which is very common in industrial applications. The results that are obtained for validation of the present approach and also to explore the failure characteristics of laminated composite skewed hypar shells are examined meticulously to extract inferences of engineering value. These conclusions are presented in Section 6.4.

6.2 NUMERICAL EXAMPLES

In order to validate the geometrically nonlinear finite element formulation the author studies the nondimensional central displacements of an isotropic simply supported plate under static uniformly distributed surface pressure as presented in Fig. 6.1. The displacement values are compared with those reported by Palazotto and Dennis (1992). The material properties and the geometry of the shell are carefully adjusted by assigning equal values of E_{11} and E_{22} and a very high value of R_{xy} (10^{30}) respectively in the present computer code for analysing the isotropic plate as a special case. The transverse load and central plate displacements are nondimensionalized as $\bar{q} = (q_0 a^4)/(E_{11} h^4)$ and $\overline{w} = w/h$ respectively where w is the transverse displacement and q_0 is intensity of the uniformly distributed surface pressure. The related geometrical and material properties of the plate are taken as: length (*a*) and width (*b*) of the plate = 8 inch [203.2 mm] and the thickness (*h*) = 0.08 inch [2.032 mm], E_{11} = 10 x 10^6 psi [68.95 GPa], ν = 0.316. Apart from this benchmark problem the author also solves one additional nonlinear bending problem of a cylindrical shell surface. For this, the author compares static nonlinear displacement values of clamped isotropic cylindrical shell evaluated by current formulation with the results published by Palazotto and Dennis (1992). The comparison is furnished in Fig. 6.2. The material and geometric properties of the cylindrical shell used are reported as: length of the shell (*a*) = 20 inch [508 mm], shell thickness (*h*) = 0.125 inch [3.175 mm], rise to width (*b*) ratio = 0.0496, radius of curvature =100 inch [2540 mm], semi-sector angle = 0.1 radian, E_{11} = 4.5 × 10^5 psi [3.1 GPa], ν = 0.3.

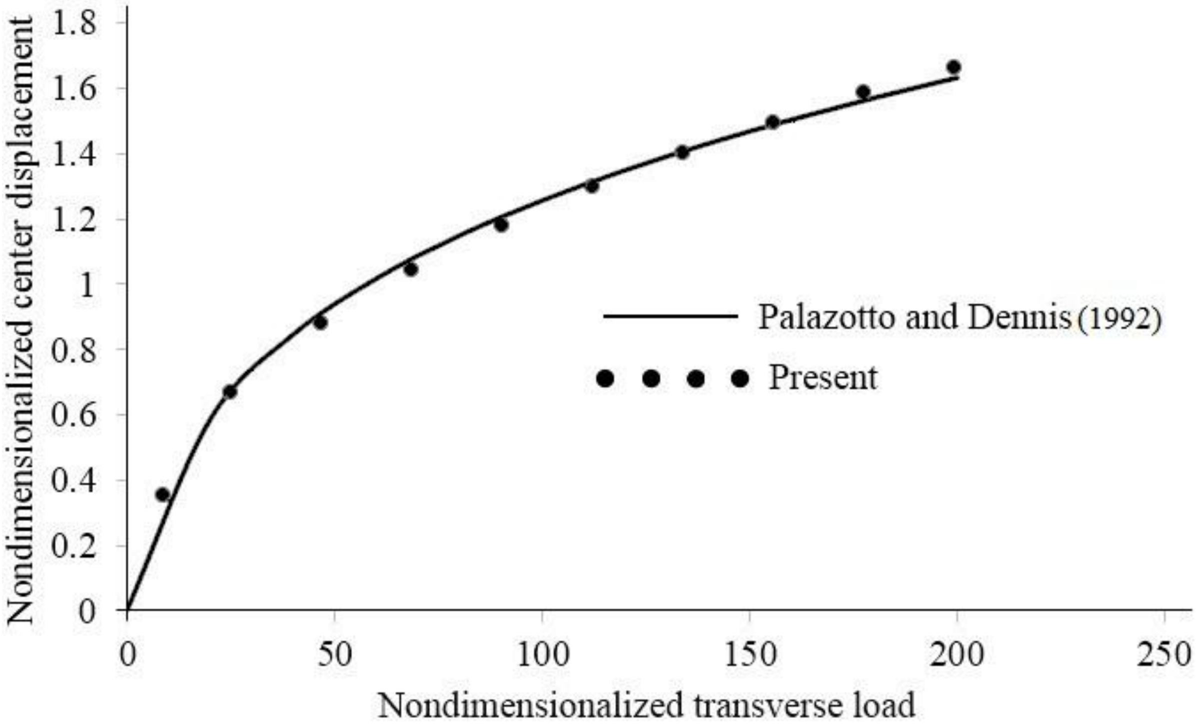


Fig. 6.1 Nonlinear deflection of isotropic plate

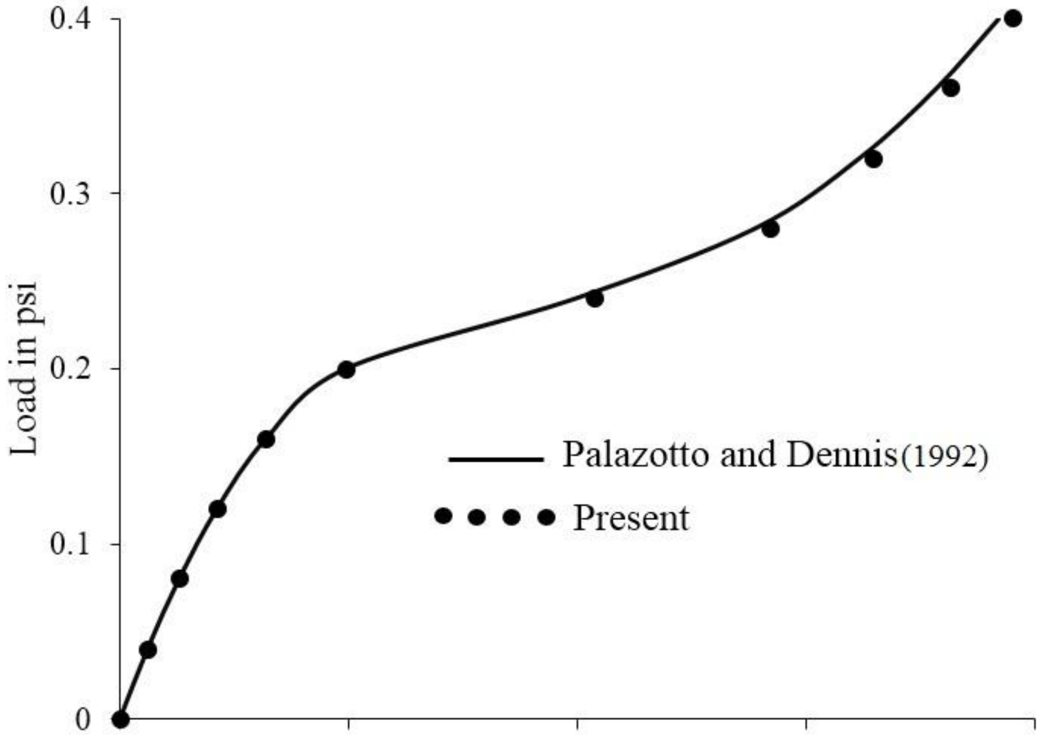


Fig. 6.2 Nonlinear deflection of isotropic cylindrical shell

To validate the first ply failure formulation of laminated composites, the failure loads calculated from present approach are compared with those reported earlier by Kam et al. (1996) for a partially clamped plate. Table 6.1 shows the comparative results. Further, a comparison of the nondimensional fundamental frequencies of twisted plates reported by Qatu and Leissa (1991b) and those obtained by the present formulation are furnished in Table 6.2. This comparison problem is solved to ensure the correct incorporation of the skewed hypar shell geometry in the present computer code.

Table 6.1 Comparison of nonlinear first ply failure loads in Newton for a $(0^{\circ}_{2}/90^{\circ})_S$ plate

Failure criteria	Length/ plate thickness	Experimental failure load (Kam et al. 1996)	First ply failure loads (Kam et al. 1996)	First ply failure loads (present formulation)
Maximum stress			147.61	139.94
Maximum strain			185.31	194.58
Hoffman	105.26	157.34	143.15	137.12
Tsai-Wu			157.78	150.71
Tsai-Hill			144.42	151.22

Note: Length = 100mm, load details = central point load.

Table 6.2 Nondimensional natural frequencies for $(\theta/-\theta/\theta)$ graphite-epoxy twisted plates

Angle of twist	θ	0°	15°	30°	45°	60°	75°	90°
$\phi = 15°$	Qatu and Leissa (1991b)	1.0035	0.9296	0.7465	0.5286	0.3545	0.2723	0.2555
	Present formulation	0.9985	0.9250	0.7444	0.5280	0.3542	0.2720	0.2552
$\phi = 30°$	Qatu and Leissa (1991b)	0.9566	0.8914	0.7205	0.5149	0.3443	0.2606	0.2436
	Present formulation	0.9501	0.8841	0.7180	0.5143	0.3446	0.2611	0.2446

Note: E_{11} = 138 GPa, E_{22} = 8.96 GPa, G_{12} = 7.1GPa, ν_{12} = 0.3, *a/b* = 1, *a/h* = 100.

The present author carries out a number of numerical investigations with different parametric variations of clamped composite shallow (Vlasov 1958) skewed hypar shells under uniformly distributed transverse static pressure and the results are critically discussed from practical engineering point of view. The material properties of Q–115 graphite – epoxy composite and the geometric dimensions of the shells that are used here are reported in Table 6.3. The uniformly distributed first ply failure load values (*FL*) are nondimensionalized as $\overline{FL} = (FL/E_{22})(a/h)^4$. Different stacking sequences like 0°/90°, 0°/90°/0°, 0°/90°/0°/90° or $(0°/90°)_2$ and 0°/90°/90°/0° or $(0°/90°)_S$ laminations are taken up here as cross ply laminates. On the other hand 45°/-45°, 45°/-45°/45°, 45°/-45°/45°/-45° or $(45°/-45°)_2$ and 45°/-45°/-45°/45° or $(45°/-45°)_S$ angle ply laminates are considered. The plies are numbered from top to bottom of the laminate. The developed stresses and strains of the laminae are checked with their permissible values by different well-established failure criteria like maximum stress, maximum strain, Hoffman's, Tsai-Hill, Tsai-Wu, Hashin's and Puck's criteria and minimum of these are taken as the failure loads of collapse on which engineering factor of safety may be imposed. In Tables 6.4 and 6.5 these first ply collapse failure loads are marked by italics and the corresponding failed ply numbers are reported in the parentheses. Moreover, the permissible deflection of skewed hypar shell is taken as shorter plan dimension / 250 and the failure loads corresponding to this deflection are taken as the first ply failure loads from serviceability consideration.

Table 6.3 Material properties of Q–1115 graphite-epoxy

Material constants values	E_{11} 142.5 GPa	$E_{22} = E_{33}$ 9.79 GPa	$G_{12} = G_{13}$ 4.72 GPa	G_{23} 1.192 GPa	$\nu_{12} = \nu_{23}$ 0.27	ν_{13} 0.25
Strengths values	σ_{1T}^u 2193.5 MPa	σ_{1C}^u 2457 MPa	σ_{2T}^u 41.3 MPa	σ_{2C}^u 206.8 MPa	τ_{13}^u 61.28 MPa	$\tau_{12}^u = \tau_{23}^u$ 78.78 MPa
	ε_{1T}^u 0.01539	ε_{1C}^u 0.01724	ε_{2T}^u 0.00412	ε_{2C}^u 0.02112	γ_{13}^u 0.05141	$\gamma_{12}^u = \gamma_{23}^u$ 0.01669

In many practical situations, plan areas of different facilities are covered by repeating the modular forms of shell units with varying aspect ratios. In this chapter the aspect ratio i.e. *a/b* ratio is varied by keeping the value of '*b*' as constant to study the effect of aspect ratio on the failure load values. A shell with *a/b* = 0.5 may be designated as short shell, that with *a/b* =

1.0 is a square shell and a long shell has *a/b* = 2.0. For these three types of shells, the nonlinear first ply failure loads for different cross and angle ply laminations with shorter span to thickness ratio equal to 100, are compared. The results are presented in Fig. 6.3 and Fig. 6.4 respectively. These figures compare the nonlinear collapse failure loads.

Apart from the criterion of collapse the shell thickness in design is also governed by other considerations like controlling the resonating frequency and deflection. In some practical situations even a thick shell may be required to enhance stability of a structure against wind suction forces constructed in cyclonic coastal areas. The numerical experiments are carried out for moderately thick (*a/h*=80), thin (*a/h*=100) and very thin (*a/h*=120) hypar shells. The geometrically nonlinear first ply failure loads of all the angle and cross ply shells corresponding to their governing failure criteria with varying the *a/h* ratio from 80 to 120 are furnished in Fig. 6.5.

6.3 RESULTS AND DISCUSSIONS

6.3.1 Benchmark Problems

The values of static nonlinear displacements of a simply supported isotropic plate and a clamped isotropic cylindrical shell are reported in Fig. 6.1 and Fig. 6.2 respectively which show a close agreement between the present results and the published ones. These prove that the bending formulation involving geometrically nonlinear strains are correctly incorporated in the author's formulation. The results of Table 6.1 show that the agreement of present results with those of Kam et al. (1996) is excellent and the correctness of the present approach for determination of first ply failure loads of laminated composite shell considering geometrically linear and nonlinear strains is established.

The author uses a simple lumped mass matrix scheme together with the undamaged stiffness matrix to solve the eigenvalue problem of twisted plates to compare the nondimensional fundamental frequencies obtained by the present approach with those published by Qatu and Leissa (1991b). The close agreement of results [Table 6.2] establishes the proper incorporation of hypar shell geometry in the formulation which is structurally similar to a twisted plate.

6.3.2 Failure Loads of Thin (*a/h* = 100) Hypar Shells on Varying Planform

It is found from the present numerical investigations that the Puck's and the maximum strain criteria govern the first ply failure loads of the cross and angle ply shells respectively. From Fig. 6.3 and Fig. 6.4 it is interestingly noted that the load carrying capacities of the square shells are much higher than those of short and long shells for both cross and angle ply laminations. In most of the cases, except for four layered cross ply laminates, short shells yield higher values of failure loads in comparison to the long shells. Hence, to cover a large column free open space, a practicing engineer should preferably use a number of square shells or combinations of square and short shells.

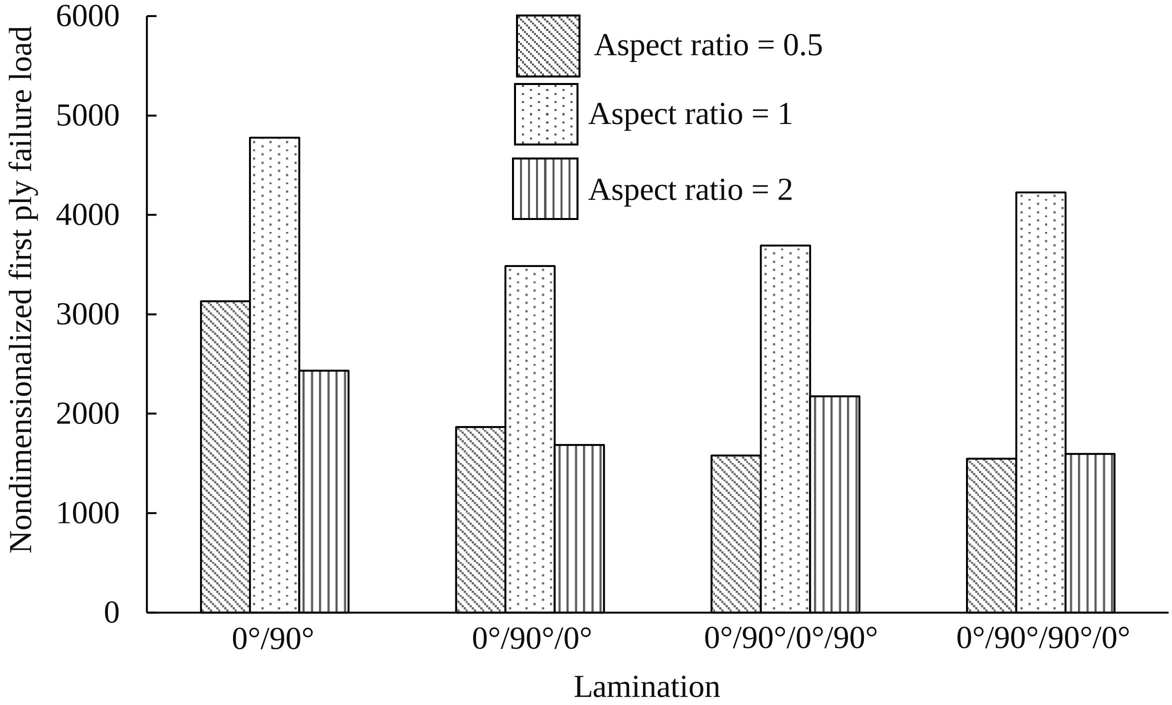


Fig. 6.3 First ply failure loads of cross ply thin shells with different aspect ratios

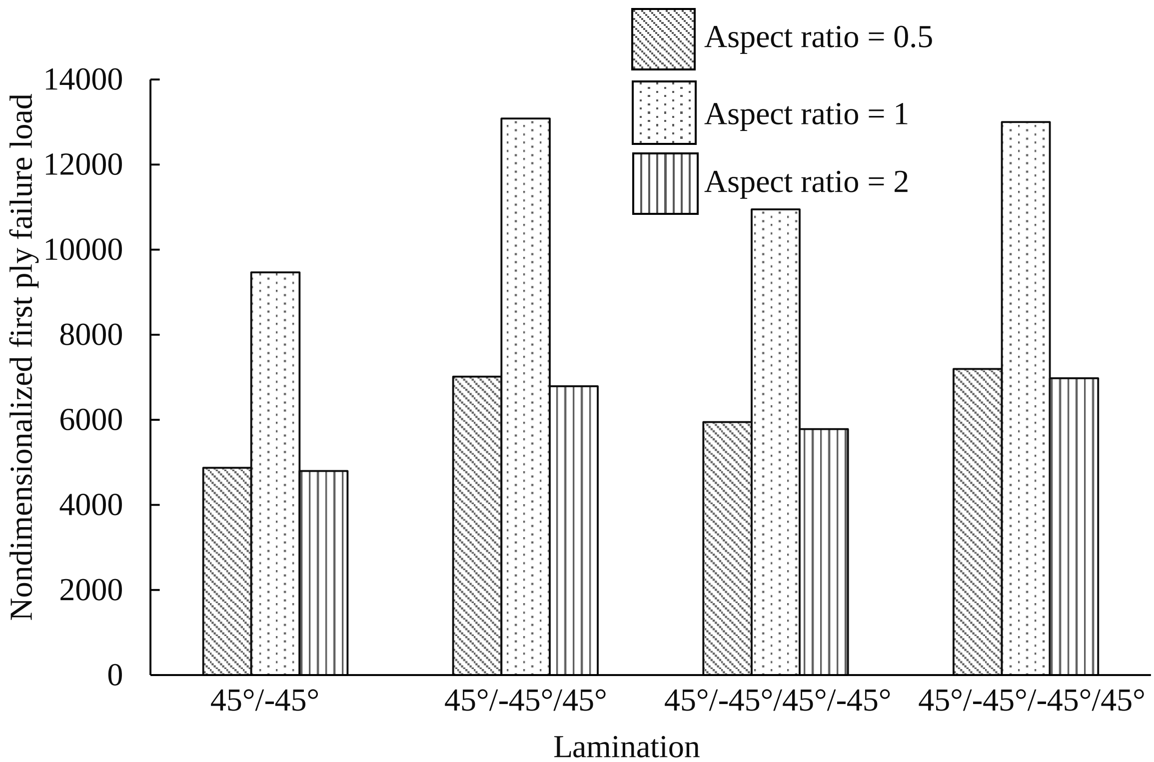


Fig. 6.4 First ply failure loads of angle ply thin shells with different aspect ratio

6.3.3 Failure Loads of Thin (*a/h* = 100) Hypar Shells on Square Planform (*a/b* = 1)

The first ply failure loads of cross and angle ply shallow thin hypar shells are reported in Tables 6.4 and 6.5 respectively. In practical civil engineering, failure of a structure is not only evaluated by comparing the developed stresses and strains with the permissible ones but the maximum deflection value is also to be maintained within a limit (taken as span/250 here) to satisfy the serviceability criterion. All the four cross ply laminates show comparable failure loads when serviceability criterion is considered. This means that periodic non-destructive monitoring of the deflections undergone by the cross ply shells is enough to conclude on their safety.

It is interestingly noted that the minimum collapse failure loads are obtained from Puck's and the maximum strain criteria respectively for cross and angle ply shells. It is noted further that for angle ply laminates Puck's criterion yields failure loads which are very close (8% to 10% deviation taking failure loads from maximum strain criterion as the base) to those obtained from the maximum strain criterion. Since we always impose a factor of safety to the tune of 1.5 to 2 on the ultimate loads to get the working load values it would not be wrong to conclude that Puck's failure criterion may be universally accepted as the only criterion that may be used to obtain the failure loads of all the shell combinations that are considered here.

It is noted further that the failure loads considering the geometric linearity are more compared to that of nonlinear ones but by margins not exceeding 5% with respect to nonlinear failure loads. This trend is observed for two and three layered cross ply laminates. However, for four layered cross ply laminates, the linear failure load is about the 10% less than the nonlinear failure load. This establishes the fact that no unified conclusion may be reached about which approach (linear or nonlinear) should be adopted for obtaining the conservative values of failure loads. This calls for a clear recommendation about the approach applicable for evaluating the failure loads in different specific cases and such guidelines are proposed in the later part of this chapter. The two layered 0°/90° laminate yields the maximum value of the failure load among all the cross ply laminates considered here.

In contrast to cross ply shells for the angle ply shells, the collapse loads are consistently less compared to the loads corresponding to permissible deflection. In all the four cases, the first ply failure loads obtained from the failure criteria are about 47% to 67% of those calculated from serviceability considerations. This observation leads to conclude that there is an apprehension of a brittle failure for this class of shells. The results furnished in Table 6.5 show that the failure loads computed considering geometric linear strains are higher than those obtained from nonlinear computations. So the nonlinear analysis should preferably be carried out for the angle ply shells. By virtue of the geometry of a skewed hypar shell, majority of the loads and moments are transferred along the diagonal directions. This is why the angle ply laminates taken up here, which have their fibers oriented along the diagonals, prove to be convincingly better than the cross ply ones in terms of first ply failure which is evident from the results given in Table 6.5. The three layered 45°/-45°/45° shell yields the maximum failure load among the angle ply options and it is almost equal to 2.7 times than what one gets for the best option of 0°/90° cross ply shell. So, the 45°/-45°/45° stacking sequence may be concluded as the best option among all the combinations of stacking sequences considered in this chapter.

It is observed that the failure modes of cross ply shells are matrix crushing mode C and all the angle ply laminates fail through matrix cracking mode. According to the Puck's criterion, matrix crushing mode C refers the shear failure of the matrix through a 45° failure plane. For cross ply laminates all the fibers run along and perpendicular to the plan direction. So, the cross ply laminates are very weak at its 45° plane because there are no such fibers run along this direction. Hence, the first ply failure loads calculated from Puck's criterion are comparatively lesser than the failure loads obtained from other criteria. On the other hand, for all angle ply laminates the fibers run along diagonal directions and the matrix is reinforced enough by the fibers to prevent its failure mode through matrix crushing mode C. It is interesting to note that the first ply failure loads obtained from Puck's criterion do not differ much from the other criteria for angle ply shells.

Table 6.4 Nondimensionalized first ply failure loads ($\overline{FL}$) and failed ply numbers of thin cross ply skewed hypar shells

Failure criteria	Laminations			
	0°/90°	0°/90°/0°	0°/90°/0°/90°	0°/90°/90°/0°
Maximum stress	11653.73^{L} (1)	10363.64^{L} (1)	10747.70^{L} (1)	9594.48^{L} (1)
	11908.07^{N} (2)	10972.42^{N} (1)	10246.17^{N} (1)	10008.17^{N} (1)
Maximum strain	11653.73^{L} (1)	10272.73^{L} (1)	10724.21^{L} (1)	9493.36^{L} (1)
	11908.07^{N} (2)	10836.57^{N} (1)	10195.10^{N} (1)	9896.83^{N} (1)
Hoffman	10164.45^{L} (2)	10205.31^{L} (1)	10228.80^{L} (4)	9486.21^{L} (1)
	10879.47^{N} (1)	10781.41^{N} (1)	9914.20^{N} (1)	9907.05^{N} (1)
Tsai-Hill	11578.14^{L} (2)	10339.12^{L} (1)	10604.70^{L} (1)	9586.31^{L} (1)
	11646.58^{N} (2)	10890.70^{N} (1)	10130.75^{N} (1)	9988.76^{N} (1)
Tsai-Wu	10263.53^{L} (2)	10248.21^{L} (1)	10396.32^{L} (4)	9520.94^{L} (1)
	11066.39^{N} (2)	10812.05^{N} (1)	9966.29^{N} (1)	9940.76^{N} (1)
Hashin	11389.17^{L} (2)	10273.75^{L} (1)	10506.64^{L} (1)	9534.22^{L} (1)
	11638.41^{N} (2)	10839.63^{N} (1)	10050.05^{N} (1)	9961.19^{N} (1)
Puck	*4922.37^{L} (2)*	*3633.30^{L} (3)*	*3422.88^{L} (2)*	*3815.12^{L} (3)*
	4775.28^{N} (2)	*3485.19^{N} (1)*	*3690.50^{N} (3)*	*4224.72^{N} (3)*
Serviceability	3825.332^{L}	3594.48^{L}	3942.80^{L}	3710.93^{L}
	3883.55^{N}	3656.79^{N}	3964.25^{N}	3779.37^{N}

Note: "L" and "N" indicate the linear and nonlinear failure loads respectively.
Note: Values in the parentheses indicate the failed ply numbers.
Note: a/b = 1, a/h = 100, c/a = 0.2

Table 6.5 Nondimensionalized first ply failure loads ($\overline{FL}$) and failed ply numbers of thin angle ply skewed hypar shells

Failure criteria	Laminations			
	45°/-45°	45°/-45°/45°	45°/-45°/ 45°/-45°	45°/-45°/ -45°/45°
Maximum stress	11189.99^{L} (1)	14979.57^{L} (1)	12441.27^{L} (1)	14687.44^{L} (1)
	10755.87^{N} (1)	14263.53^{N} (1)	12017.36^{N} (1)	14171.60^{N} (1)
Maximum strain	*9799.79^{L} (1)*	*13661.90^{L} (1)*	*11287.03^{L} (1)*	*13417.77^{L} (1)*
	9463.74^{N} (1)	*13082.74^{N} (1)*	*10943.82^{N} (1)*	*12998.98^{N} (1)*
Hoffman	11122.57^{L} (1)	14894.79^{L} (1)	12373.85^{L} (1)	14602.66^{L} (1)
	10692.54^{N} (1)	14184.88^{N} (1)	11953.01^{N} (1)	14090.91^{N} (1)
Tsai-Hill	11727.27^{L} (1)	15393.26^{L} (1)	12818.18^{L} (1)	15079.67^{L} (1)
	11251.28^{N} (1)	14626.15^{N} (1)	12363.64^{N} (1)	14527.07^{N} (1)
Tsai-Wu	10716.04^{L} (1)	14546.48^{L} (1)	12062.31^{L} (1)	14268.64^{L} (1)
	10315.63^{N} (1)	13875.38^{N} (1)	11664.96^{N} (1)	13785.50^{N} (1)

Failure criteria	Laminations			
	45°/-45°	45°/-45°/45°	45°/-45°/ 45°/-45°	45°/-45°/ -45°/45°
Hashin	11159.35^L (1)	14915.22^L (1)	12391.22^L (1)	14622.06^L (1)
	10726.25^N (1)	14203.27^N (1)	11969.36^N (1)	14110.32^N (1)
Puck	11154.24^L (1)	14902.96^L (1)	12382.02^L (1)	14610.83^L (1)
	10721.14^N (1)	14192.03^N (1)	11960.16^N (1)	14098.06^N (1)
Serviceability	20953.01^L	20813.07^L	21186.93^L	20954.03^L
	20007.15^N	19281.92^N	20955.06^N	21259.45^N

Note: “L” and “N” indicate the linear and nonlinear failure loads respectively.
Note: Values in the parentheses indicate the failed ply numbers.
Note: $a/b = 1$, $a/h = 100$, $c/a = 0.2$

6.3.4 Failure Loads of Shells with Varying Thicknesses

It has been discussed earlier in this chapter that the diagonals being the primary load transfer directions in a hypar shell, the cross ply configurations having fiber orientations parallel to the plan directions do not yield high failure load values and for this complex interaction again the variation of the failure loads with *a*/*h* ratio does not show any specific trend. On the other hand the failure load values of angle ply shells monotonically increase with increase of thickness (as *a*/*h* ratio decreases). For all the angle ply laminates considered here working equations may be formulated connecting the failure loads with *a/h* ratios and these working equations are presented in Fig. 6.5.

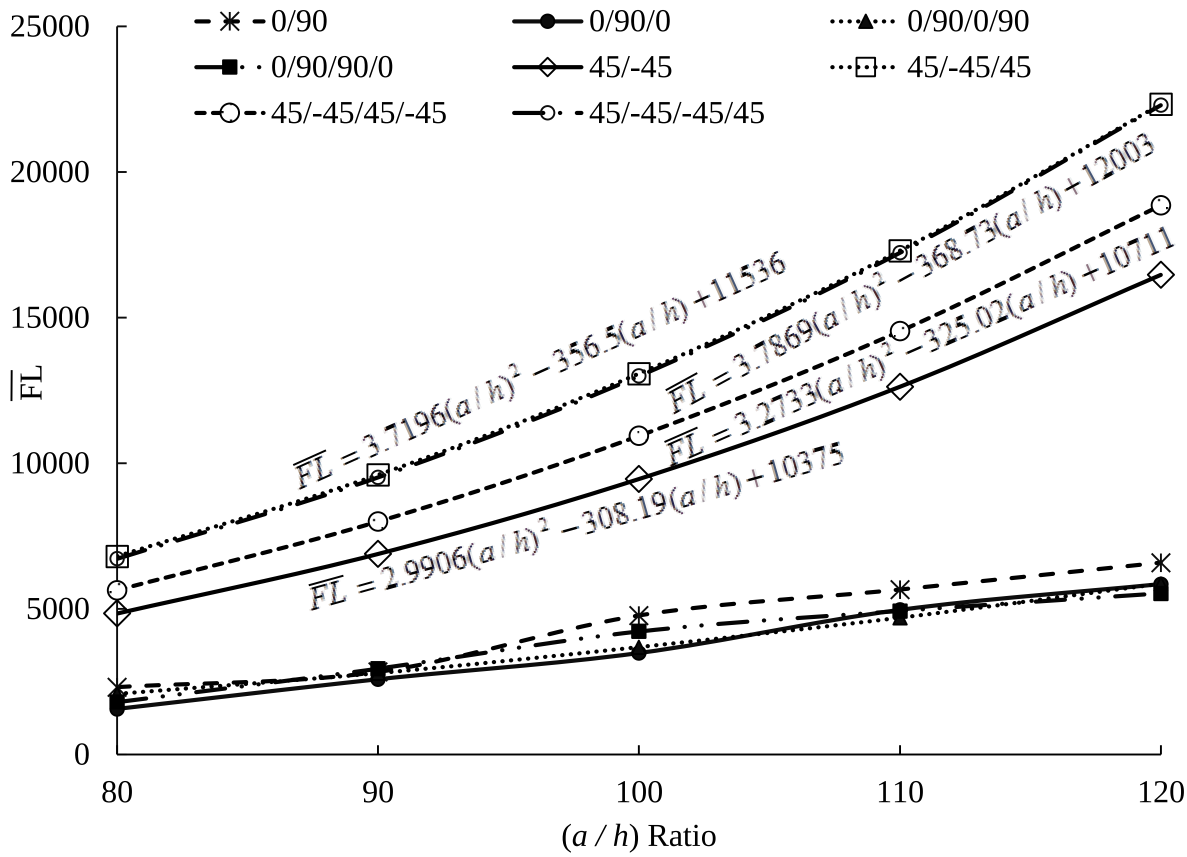


Fig. 6.5 Variation of first ply failure loads with (*a/h*) ratio for different lamination (Degree)

6.3.5 Comparison between Linear and Nonlinear Approaches to Evaluate Failure Loads

The approach considering geometric nonlinearity of strains yields accurate results of failure loads undoubtedly but the linear approach is easy from implementation point of view. Naturally, a structural analyst may be keen to know that for which of the shell options the linear approach is acceptable to be used. To find the answer to this question the percentage differences between linear and nonlinear failure loads with the later as the base are furnished in Table 6.6. The cases where this percentage is negative the linear theory yields a lower value of failure load and may be used for design purpose. In other cases where this percentage is positive an exact analysis through nonlinear approach will give a lower conservative value of the failure load and the nonlinear theory should be adopted. It is noteworthy that whatever may be the approach by which the failure load is obtained a factor of safety to the tune of 1.5 to 2 is to be applied on these loads to get the working load values. Keeping this in mind the present author suggests here a practical relaxation that when the percentage difference as mentioned above is positive but within 5%, use of the linear theory to get the failure load and ultimately applying a factor of safety of 1.5 to 2 on it will safely assess the working load and may be recommended. So recommendation matrix giving a broad guideline regarding the approach of analysis to be adopted is furnished in Table 6.7 for different *a/h* ratio values as a ready reference to structural analysts. The table shows that for both cross and angle ply combinations of thin and moderately thick shells, the linear theory may be adopted but this theory is not applicable for very thin shells with *a/h* = 120. It is concluded that for very thin shell options (*a/h* = 120) a structural engineer should apply the geometrically nonlinear theory for safe assessment of failure load. When the shell becomes thinner, it gains its load carrying capacity primarily from its curved geometry. Theoretically this load carrying capacity is due to from the nonlinear components of strains. Hence the nonlinear components of strains should be considered for numerically computing the failure loads of very thin shells and the results of Table 6.7 reflect this fact.

Table 6.6 Percentage differences between linear and nonlinear first ply failure loads

Lamination	Percentage differences between linear and nonlinear first ply failure loads		
	a/h = 80	*a/h*=100	*a/h*=120
0°/90°	1.664	3.080	5.100
0°/90°/0°	4.205	4.249	1.338
0°/90°/0°/90°	-4.217	-7.251	-2.419
0°/90°/90°/0°	4.572	-9.695	7.139
45°/-45°	3.151	3.551	3.858
45°/-45°/45°	3.822	4.226	5.217
45°/-45°/45°/-45°	2.705	3.136	3.493
45°/-45°/-45°/45°	2.942	3.221	3.448

Note: *a/b* = 1, *c/b* = 0.2

Table 6.7 Recommendation on design

Lamination	Design approach		
	a/h = 80	*a/h*=100	*a/h*=120
0°/90°	Geometric linear	Geometric linear	Geometric nonlinear
0°/90°/0°	Geometric linear	Geometric linear	Geometric nonlinear
0°/90°/0°/90°	Geometric linear	Geometric linear	Geometric nonlinear
0°/90°/90°/0°	Geometric linear	Geometric linear	Geometric nonlinear
45°/-45°	Geometric linear	Geometric linear	Geometric nonlinear
45°/-45°/45°	Geometric linear	Geometric linear	Geometric nonlinear
45°/-45°/45°/-45°	Geometric linear	Geometric linear	Geometric nonlinear
45°/-45°/-45°/45°	Geometric linear	Geometric linear	Geometric nonlinear

6.4 CONCLUDING REMARKS

The following conclusions are evident from the study of present section.

- The finite element code developed here is capable of computing both geometrically linear and nonlinear uniformly distributed first ply failure pressures of laminated composite clamped skewed hypar shells. It is evident from the results of the benchmark problems that are solved here.
- In general the Puck's and maximum strain criteria give the minimum first ply failure loads of cross and angle ply shells respectively on which factor of safety should be imposed to get the working load values. For the angle ply shells, Puck's criterion gives close failure load values in comparison with the loads obtained through maximum strain criterion. So, Puck's criterion may be universally used to get the first ply failure loads of laminated hypar shells.
- The shells square in plan (aspect ratio = 1) carry higher failure loads compared to those on rectangular plan forms. Hence, the square forms may be adopted by the practicing design engineers as far as practicable.
- The periodic non-destructive monitoring of the deflections are enough to ensure the safety of the cross ply shells, but for the angle ply shells brittle failure may occur as the collapse loads are consistently less compared to the serviceability failure loads.

- In general, angle ply shells are better in comparison to cross ply ones in terms of first ply failure. The double layered 0°/90° and three layered 45°/-45°/45° stacking sequences are the best stacking sequences respectively for the cross and angle ply shells among the laminations considered in this chapter.
- A recommendation table is suggested in this chapter for guiding the designer regarding the approach of failure analysis applicable to composite hypar shells. For all moderately thick and thin shells, the geometrically linear theory yields reasonably accurate results and for very thin shells only the geometrically nonlinear theory is required for evaluation of failure loads.
- For all the angle ply shells, working equations, useful to practicing engineers, may be derived correlating first ply failure load with shorter span to total thickness ratio.

Chapter 7

LINEAR AND NONLINEAR FIRST PLY FAILURE OF CLAMPED SKEWED HYPAR SHELL ROOFS OF VARYING CURVATURE INCLUDING GUIDELINES FOR NON-DESTRUCTIVE TEST MONITORING

7.1 GENERAL

The curvature factor in shells is basically responsible for coupling of bending and membrane stiffnesses and thereby increasing the load bearing capacity compare to plates. Introducing curvature in a surface during casting and fabrication requires stringent monitoring. An engineer has to strike an optimum balance between the desired strength and ease of construction. This necessitates determination of optimum curvature and this chapter is devoted to study first ply failure loads against varying curvatures. Design guidelines are also formulated regarding identifying appropriates zones on the shell surface which are prone to failure initiation and are to be kept under the scanner of health monitoring through periodic non-destructive tests. The results are presented in Section 7.3. The obtained results are post-processed to extract important design guidelines which are also presented in Section 7.3. Section 7.4 furnishes the salient engineering conclusions.

7.2 NUMERICAL EXAMPLES

A number of problems on clamped skewed hypar shells under uniformly distributed static transverse loading are solved and the results are interpreted from practical engineering standpoint. The material properties of Q–1115 graphite – epoxy composite that are used here are reported in Table 6.3 of previous chapter. The uniformly distributed first ply failure load values (*FL*) are nondimensionalized as $\overline{FL} = (FL/E_{22})(a/h)^4$ and other related information of laminated composite clamped skewed hypar shells with cross and angle ply laminations are furnished in Tables 7.1 and 7.2 respectively. 0°/90°, 0°/90°/0°, 0°/90°/0°/90° and 0°/90°/90°/0° cross ply laminates and 45°/-45°, 45°/-45°/45°, 45°/-45°/45°/-45° and 45°/-45°/-45°/45° angle ply laminates are considered in the present study. Fig. 7.1 shows the failure zones of the shells in plan. The independent failure criteria such as maximum stress and maximum strain criteria, the interactive failure criteria like Hoffman's, Tsai – Hill and Tsai – Wu criteria and the partially interactive or failure mode based criteria like Hashin's and Puck's failure criteria are used to evaluate the first ply failure loads for different stacking arrangements of laminated composite hypar shells. The minimum value of the failure load obtained from different failure criteria is considered as the acceptable failure load on which the engineering factor of safety should be imposed to get the working load values. These collapse failure loads are shown in italics in the corresponding Tables 7.1 to 7.4. Besides all these collapse criteria, the present author considers the failure load from serviceability point of view with the permissible deflection of the skewed hypar shell taken as shorter span/250. Apart from determining failure loads, the author reports the failure zones, failure modes or tendencies and failed plies in the results furnished in Tables 7.1 and 7.2. The plies are numbered from top to bottom of the laminate.

It is well established that the curvature value of a shell surface has a pronounced effect on its behaviour and introduction of curvature in a plate surface to make it a shell is actually responsible for the enhancement in its load bearing capacity. On the other hand, from practical engineering point of view, shells with deep curvatures are difficult to cast and fabricate and hence the present study considers height to shorter span ratio within 0.2 keeping the shell within the definition of a shallow one (Vlasov 1958). The nondimensionalized failure loads and other related information from different failure criteria and serviceability criterion, in terms of deflection, are furnished in Table 7.1 and Table 7.2 for cross and angle ply, anti-symmetric and symmetric laminates.

7.3 RESULTS AND DISCUSSIONS

7.3.1 Behaviour of Clamped Cross Ply Skewed Hypar Shells with Height to Span Ratio = 0.2

The results furnished in Table 7.1 show that the failure load corresponding to Puck's criterion gives the minimum value of the collapse failure load. If one considers only the material failure, it is found that Puck's partially interactive or failure mode based criterion yields the lowest value of the failure load which should be taken for design. In fact the other independent or fully interactive failure criteria yield much higher values of failure loads and it may be concluded that a study using Puck's criterion taken together with the serviceability limits is enough to assess the safe load which a cross ply shell can support.

For two and three layered cross ply laminates, the linear theory yields a higher value of failure load compared to the nonlinear one but by a margin not more than 5% with respect to nonlinear failure loads. For four layered cross ply laminates, however, the linear failure load is about the 10% less than the nonlinear failure load. These observations lead to conclude that the linear failure theory, which is much easier to be implemented programmatically, may be used for evaluation of failure loads of cross ply shells without sacrificing the engineering safety because in any standard code of practice a minimum factor of safety to the tune of 1.5 to 2 is applied on the ultimate load to get the working load values. It is further observed from Table

7.1 that among the cross ply shells taken up here the two layered 0°/90° laminate turns out to be the best choice yielding the highest value of the failure load if failure through collapse is considered. If, however, serviceability criterion is taken into consideration too, all the four cross ply laminates show comparable results.

It is also interesting to note that for all the cross ply shells taken up here failure is through matrix crushing mode C. The cross ply skewed hypar shell having its curvature along the diagonals but having the fibers oriented parallel to the plan dimensions show such a shear crushing of the matrix.

Table 7.1 Nondimensionalized first ply failure loads $(\overline{FL})$ of cross ply shallow hypar shells

Laminations	Failure criteria	$\overline{FL}$	Failure zone	Failed Ply	Failure mode / failure tendency
0°/90°	Maximum	11653.73^{L}	C	1	Matrix shear Failure
	stress	11908.07^{N}	C	2	Matrix shear failure
	Maximum	11653.73^{L}	C	1	Matrix shear failure
	strain	11908.07^{N}	C	2	Matrix shear failure
	Hoffman	10164.45^{L}	B	2	Matrix shear failure
		10879.47^{N}	A	1	Matrix shear failure
	Tsai-Hill	11578.14^{L}	B	2	Matrix shear failure
		11646.58^{N}	D	2	Matrix shear failure
	Tsai-Wu	10263.53^{L}	B	2	Matrix shear failure
		11066.39^{N}	C	2	Matrix shear failure
	Hashin	11389.17^{L}	B	2	Matrix cracking
		11638.41^{N}	C	2	Matrix cracking
	Puck	*4922.37*^{L}	*A*	*2*	*Matrix crushing mode C*
		4775.28^{N}	*A*	*2*	*Matrix crushing mode C*
	Serviceability	3825.332^{L}	B		
		3883.55^{N}	B		
0°/90°/0°	Maximum	10363.64^{L}	A	1	Matrix cracking
	stress	10972.42^{N}	A	1	Matrix cracking
	Maximum	10272.73^{L}	A	1	Matrix cracking
	strain	10836.57^{N}	A	1	Matrix cracking
	Hoffman	10205.31^{L}	A	1	Matrix cracking
		10781.41^{N}	A	1	Matrix cracking
	Tsai-Hill	10339.12^{L}	A	1	Matrix cracking
		10890.70^{N}	A	1	Matrix cracking
	Tsai-Wu	10248.21^{L}	A	1	Matrix cracking
		10812.05^{N}	A	1	Matrix cracking
	Hashin	10273.75^{L}	A	1	Matrix cracking
		10839.63^{N}	A	1	Matrix cracking

Laminations	Failure criteria	$\overline{FL}$	Failure zone	Failed Ply	Failure mode / failure tendency
	Puck	*3633.30*^{L}	*A*	*3*	*Matrix crushing mode C*
		3485.19^{N}	*A*	*1*	*Matrix crushing mode C*
	Serviceability	3594.48^{L}	B		
		3656.79^{N}	B		
0°/90°/0°/90°	Maximum stress	10747.70^{L}	A	1	Matrix cracking
		10246.17^{N}	A	1	Matrix cracking
	Maximum strain	10724.21^{L}	A	1	Matrix cracking
		10195.10^{N}	A	1	Matrix cracking
	Hoffman	10228.80^{L}	B	4	Matrix shear failure
		9914.20^{N}	A	1	Matrix cracking
	Tsai-Hill	10604.70^{L}	A	1	Matrix cracking
		10130.75^{N}	A	1	Matrix cracking
	Tsai-Wu	10396.32^{L}	B	4	Matrix shear failure
		9966.29^{N}	A	1	Matrix cracking
	Hashin	10506.64^{L}	A	1	Matrix cracking
		10050.05^{N}	A	1	Matrix cracking
	Puck	*3422.88*^{L}	*A*	*2*	*Matrix crushing mode C*
		3690.50^{N}	*A*	*3*	*Matrix crushing mode C*
	Serviceability	3942.80^{L}	C		
		3964.25^{N}	C		
0°/90°/90°/0°	Maximum stress	9594.48^{L}	A	1	Matrix cracking
		10008.17^{N}	A	1	Matrix cracking
	Maximum strain	9493.36^{L}	A	1	Matrix cracking
		9896.83^{N}	A	1	Matrix cracking
	Hoffman	9486.21^{L}	A	1	Matrix cracking
		9907.05^{N}	A	1	Matrix cracking
	Tsai-Hill	9586.31^{L}	A	1	Matrix cracking
		9988.76^{N}	A	1	Matrix cracking
	Tsai-Wu	9520.94^{L}	A	1	Matrix cracking
		9940.76^{N}	A	1	Matrix cracking
	Hashin	9534.22^{L}	A	1	Matrix cracking
		9961.19^{N}	A	1	Matrix cracking
	Puck	*3815.12*^{L}	*A*	*3*	*Matrix crushing mode C*
		4224.72^{N}	*A*	*3*	*Matrix crushing mode C*
	Serviceability	3710.93^{L}	B		
		3779.37^{N}	B		

Note: *a/b* = 1, *a/h* = 100, 'L' and 'N' indicate the linear and nonlinear failure loads respectively.

7.3.2 Behaviour of Clamped Angle Ply Skewed Hypar Shells with Height to Span Ratio = 0.2

In contrast to the behaviour of the cross ply shells, in all four cases of angle ply laminates taken up here, the failure loads obtained from the collapse criteria are less than those from serviceability criterion [Table 7.2]. For the cases of angle ply shells the independent maximum strain failure criterion governs the failure loads and the loads obtained through nonlinear analysis are consistently less from what result from linear computations. This observation directly leads to conclude that for angle ply shells a nonlinear analysis should preferably be carried out, though the linear computations yield higher non-conservative failure load values not more than 10% when compared to the nonlinear ones. It is noted further that the collapse loads are around 65% of the loads corresponding to the permissible deflection and there is an apprehension of a brittle failure for these class of shells.

In the angle ply shells considered here, the fibers are oriented in the diagonal directions along the curvatures of the skewed hypar shells resulting in an enhanced stiffness of the structure. For this reason the failure loads for angle ply shells are much higher than the corresponding cross ply ones and the three layered 45°/-45°/45° and the four layered 45°/-45°/-45°/45° shells, which are the best choices among the angle ply options, yield failure loads almost equal to 2.5 times than what one gets for the best option of 0°/90° cross ply shell. For the same reason the loads corresponding to limiting deflection for 45°/-45°/45° angle ply shell is almost 5 times of the corresponding value for the 0°/90°/0° cross ply shell option. Among the angle ply shells, the 45°/-45°/-45°/45° and 45°/-45°/45° laminations show comparable performances and better than the other two angle ply options and from ease of fabrication point of view the 45°/-45°/45° stacking sequence may be concluded as the best option.

Interestingly, the failure loads obtained from different failure criteria for angle ply shells are comparable in magnitude which means that these shells fail satisfying almost all the independent and interactive criteria simultaneously and this behaviour indicates towards an optimum utilisation of the material strengths considering both fiber and matrix. Due to this efficient utilisation of the material properties the resultant load supporting capacities of the angle ply shells are much higher than their cross ply counterparts.

Table 7.2 Nondimensionalized first ply failure loads $(\overline{FL})$ of angle ply shallow hypar shells

Laminations	Failure criteria	$\overline{FL}$	Failure zone	Failed ply	Failure mode / failure tendency
45°/-45°	Maximum	11189.99^L	A	1	Matrix cracking
	stress	10755.87^N	A	1	Matrix cracking
	Maximum	*9799.79^L*	*A*	*1*	*Matrix cracking*
	strain	*9463.74^N*	*A*	*1*	*Matrix cracking*
	Hoffman	11122.57^L	A	1	Matrix cracking
		10692.54^N	A	1	Matrix cracking
	Tsai-Hill	11727.27^L	A	1	Matrix cracking
		11251.28^N	A	1	Matrix cracking
	Tsai-Wu	10716.04^L	A	1	Matrix cracking
		10315.63^N	A	1	Matrix cracking
	Hashin	11159.35^L	A	1	Matrix cracking
		10726.25^N	A	1	Matrix cracking
	Puck	11154.24^L	A	1	Matrix cracking mode A
		10721.14^N	A	1	Matrix cracking mode A
	Serviceability	20953.01^L	B		
		20007.15^N	B		
45°/-45°/45°	Maximum	14979.57^L	A	1	Matrix cracking
	stress	14263.53^N	A	1	Matrix cracking
	Maximum	*13661.90^L*	*A*	*1*	*Matrix cracking*
	strain	*13082.74^N*	*A*	*1*	*Matrix cracking*
	Hoffman	14894.79^L	A	1	Matrix cracking
		14184.88^N	A	1	Matrix cracking
	Tsai-Hill	15393.26^L	A	1	Matrix cracking
		14626.15^N	A	1	Matrix cracking
	Tsai-Wu	14546.48^L	A	1	Matrix cracking
		13875.38^N	A	1	Matrix cracking
	Hashin	14915.22^L	A	1	Matrix cracking
		14203.27^N	A	1	Matrix cracking
	Puck	14902.96^L	A	1	Matrix cracking mode A
		14192.03^N	A	1	Matrix cracking mode A
	Serviceability	20813.07^L	B		
		19281.92^N	B		

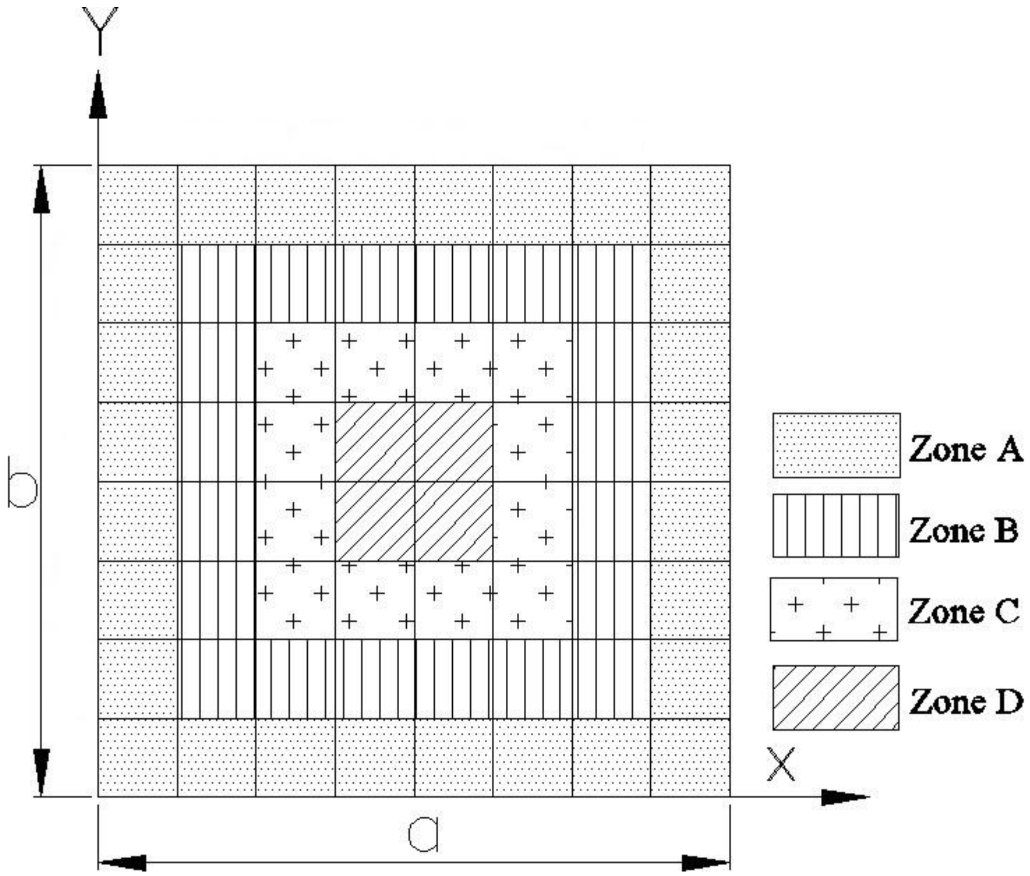
Laminations	Failure criteria	$\overline{FL}$	Failure zone	Failed ply	Failure mode / failure tendency
45°/-45°/45°/-45°	Maximum stress	12441.27^{L}	A	1	Matrix cracking
		12017.36^{N}	A	1	Matrix cracking
	Maximum strain	*11287.03^{L}*	*A*	*1*	*Matrix cracking*
		10943.82^{N}	*A*	*1*	*Matrix cracking*
	Hoffman	12373.85^{L}	A	1	Matrix cracking
		11953.01^{N}	A	1	Matrix cracking
	Tsai-Hill	12818.18^{L}	A	1	Matrix cracking
		12363.64^{N}	A	1	Matrix cracking
	Tsai-Wu	12062.31^{L}	A	1	Matrix cracking
		11664.96^{N}	A	1	Matrix cracking
	Hashin	12391.22^{L}	A	1	Matrix cracking
		11969.36^{N}	A	1	Matrix cracking
	Puck	12382.02^{L}	A	1	Matrix cracking mode A
		11960.16^{N}	A	1	Matrix cracking mode A
	Serviceability	21186.93^{L}	B		
		20955.06^{N}	B		
45°/-45°/-45°/45°	Maximum stress	14687.44^{L}	A	1	Matrix cracking
		14171.60^{N}	A	1	Matrix cracking
	Maximum strain	*13417.77^{L}*	*A*	*1*	*Matrix cracking*
		12998.98^{N}	*A*	*1*	*Matrix cracking*
	Hoffman	14602.66^{L}	A	1	Matrix cracking
		14090.91^{N}	A	1	Matrix cracking
	Tsai-Hill	15079.67^{L}	A	1	Matrix cracking
		14527.07^{N}	A	1	Matrix cracking
	Tsai-Wu	14268.64^{L}	A	1	Matrix cracking
		13785.50^{N}	A	1	Matrix cracking
	Hashin	14622.06^{L}	A	1	Matrix cracking
		14110.32^{N}	A	1	Matrix cracking
	Puck	14610.83^{L}	A	1	Matrix cracking mode A
		14098.06^{N}	A	1	Matrix cracking mode A
	Serviceability	20954.03^{L}	B		
		21259.45^{N}	B		

Note: $a/b = 1$, $a/h = 100$, 'L' and 'N' indicate the linear and nonlinear failure loads respectively.

7.3.3 Failure Zones of Hypar Shell and Guidelines to Non-Destructive Test Monitoring

The different failure zones of a hypar shell in plan from where first ply failure initiates are marked as Zones A, B, C and D and shown in Fig. 7.1. In Tables 7.1 and 7.2 the failure zones corresponding to the failure loads for different cross and angle ply shells are furnished. The location of the first ply failure point is extremely important to be known to a practising engineer because any instrumentation needed for hidden flaw detection should start from that point. For both clamped cross and angle ply shells, failure always initiates from a zone close to the edges (zone A) within a narrow strip of width less than or equal to one-eighth of the plan dimension. This indicates that any non-destructive health monitoring instrumentation should be restricted within this peripheral area.

In practical engineering collapse and serviceability failures should be taken together to conclude on the safe usability of a structure. For cross ply shells it is observed that for all laminations taken up here, the loads corresponding to the limiting deflection (taken as span / 250 here) are less or marginally more than those obtained by using the different failure criteria of collapse. This means that periodic monitoring of the deflections undergone by the cross ply shells is enough to conclude on its safety. But this guideline is not valid for angle ply shell options.



Lengths of each segments along X and Y axes are *a*/8 and *b*/8 respectively.

Fig. 7.1 Failure zones of a hypar shell in plan.

7.3.4 Effect of Change of Curvature on Failure Loads and Suggestions Regarding Design Approaches

The study done so far is about shells with the ratio of height to shorter span is 0.2. The study is extended for shallower shells as well, to explore the variation of the failure loads with curvature. The overall behaviour of a shell depends much on its curvature and hence the curvature is varied in terms of height to shorter span ratio from 0 to 0.2 to see the effect of curvature on the failure load values. The results are furnished in Tables 7.3 and 7.4. Interestingly for all curvature values, except for the plate case with no curvature, the maximum strain criterion governs the failure loads for angle ply shells. On the other hand, Puck's failure criterion gives failure load values for cross ply shells for height to span ratio varying in between 0.1 to 0.2 and in between 0 to 0.1 no such unified conclusion can be drawn. While for angle ply shells the failure loads increase monotonically with increase of curvature, on the other hand, for cross ply shells no such unified correlation is observed as evident from the results in Tables 7.3 and 7.4.

It is interestingly to note that in all four cases of cross and angle ply laminates taken up here, the failure loads obtained from serviceability criterion are less or marginally more than those from collapse criterion except for the cases of height to shorter span ratios equal to 0.2 and 0.15 for only angle ply shells. For these two cases the reverse trend is observed.

It is worth mentioning that the approach considering geometric nonlinearity of strains yields accurate results of failure loads undoubtedly but the linear approach is easy from implementation point of view. Naturally, a structural analyst may be keen to know that for which shell options the linear approach is acceptable to be used. To find the answer to this question the percentage differences between linear and nonlinear failure loads with the later as the base are furnished in Table 7.5. The cases where this percentage is negative the linear theory yields a lower conservative value of failure load and may be used for design purpose. In other cases where this percentage is positive an exact analysis through nonlinear approach will give a lower conservative value of the failure load and the nonlinear theory should be adopted. It is noteworthy that whatever may be the approach by which the failure load is obtained, a factor of safety to the tune of 1.5 to 2 is to be applied on these loads to get the working load values. Keeping this in mind, the present author suggests here a practical relaxation that where the percentage difference as mentioned above is positive but within 5%, use of the linear theory to get the failure load and ultimately applying a factor of safety of 1.5 to 2 on it will safely assess the working load and may be recommended. Keeping this in view a recommendation matrix regarding the use of the approach of analysis is furnished in Table 7.6 for ready reference for a structural analyst. The table shows that for any cross ply combination the linear theory may be adopted while the applicability of this theory for analysis of angle ply shells is case specific. Keeping in mind the fact that the angle ply shells are prominently better than their cross ply counterparts, nonlinear approach has to be applied in many cases to correctly analyse the use of angle ply options.

Table 7.3 Nondimensionalized first ply failure loads $(\overline{FL})$ for different height to span ratios of cross ply shells

Laminations	Failure criteria	$\overline{FL}$				
		c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
0°/90°	Maximum stress	11653.73^{L}	8569.97^{L}	5301.33^{L}	2993.87^{L}	1686.42^{L}
		11908.07^{N}	8060.27^{N}	6342.19^{N}	4932.58^{N}	4469.87^{N}
	Maximum strain	11653.73^{L}	8569.97^{L}	5301.33^{L}	2993.87^{L}	1695.61^{L}
		11908.07^{N}	8060.27^{N}	6342.19^{N}	5082.74^{N}	4641.47^{N}
	Hoffman	10164.45^{L}	8264.56^{L}	5268.64^{L}	2713.99^{L}	*1072.52^{L}*
		10879.47^{N}	7639.43^{N}	5302.35^{N}	*4213.48^{N}*	*2928.49^{N}*
	Tsai-Hill	11578.14^{L}	8563.84^{L}	5300.31^{L}	2947.91^{L}	1701.74^{L}
		11646.58^{N}	8006.13^{N}	5743.62^{N}	4813.08^{N}	4545.46^{N}
	Tsai-Wu	10263.53^{L}	8415.73^{L}	5269.66^{L}	2728.29^{L}	1222.68^{L}
		11066.39^{N}	7649.64^{N}	5347.29^{N}	4635.34^{N}	4127.68^{N}
	Hashin	11389.17^{L}	8545.45^{L}	5296.22^{L}	2940.76^{L}	1686.42^{L}
		11638.41^{N}	7989.79^{N}	5671.09^{N}	4763.02^{N}	4466.80^{N}
	Puck	*4922.37^{L}*	*3039.84^{L}*	*4112.36^{L}*	*1587.33^{L}*	1344.23^{L}
		4775.28^{N}	*3403.47^{N}*	*3940.76^{N}*	4737.48^{N}	4466.80^{N}

Laminations	Failure criteria	$\overline{FL}$				
		c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
	Serviceability	3825.332^{L}	2319.71^{L}	963.23^{L}	260.47^{L}	73.54^{L}
		3883.55^{N}	2290.09^{N}	984.68^{N}	278.86^{N}	88.87^{N}
0°/90°/0°	Maximum	10363.64^{L}	8419.82^{L}	5959.14^{L}	3357.51^{L}	1715.02^{L}
	stress	10972.42^{N}	8942.80^{N}	6350.36^{N}	4040.86^{N}	3054.14^{N}
	Maximum	10272.73^{L}	8419.82^{L}	5959.14^{L}	3311.54^{L}	*1689.48^{L}*
	strain	10836.57^{N}	8658.84^{N}	6243.11^{N}	*3958.12^{N}*	*2948.93^{N}*
	Hoffman	10205.31^{L}	7712.97^{L}	5393.26^{L}	3316.65^{L}	1713.99^{L}
		10781.41^{N}	8205.31^{N}	5517.88^{N}	4012.26^{N}	3052.09^{N}
	Tsai-Hill	10339.12^{L}	8316.65^{L}	5808.99^{L}	3347.29^{L}	1721.14^{L}
		10890.70^{N}	8648.62^{N}	6257.41^{N}	4034.73^{N}	3064.35^{N}
	Tsai-Wu	10248.21^{L}	7740.55^{L}	5396.32^{L}	3325.84^{L}	1717.06^{L}
		10812.05^{N}	8274.77^{N}	5577.12^{N}	4017.37^{N}	3059.24^{N}
	Hashin	10273.75^{L}	8279.88^{L}	5800.82^{L}	3334.01^{L}	1715.02^{L}
		10839.63^{N}	8606.74^{N}	6230.85^{N}	4023.49^{N}	3054.14^{N}
	Puck	*3633.30^{L}*	*2788.56^{L}*	*2520.94^{L}*	*2678.24^{L}*	1715.02^{L}
		3485.19^{N}	*3884.58^{N}*	*3604.70^{N}*	4021.45^{N}	3054.14^{N}
	Serviceability	3594.48^{L}	2100.10^{L}	1021.45^{L}	369.77^{L}	171.60^{L}
		3656.79^{N}	2146.07^{N}	1046.99^{N}	388.15^{N}	187.95^{N}
0°/90°/0°/90°	Maximum	10747.7^{L}	7982.64^{L}	5310.52^{L}	3062.31^{L}	1588.36^{L}
	stress	10246.17^{N}	8017.37^{N}	5413.69^{N}	3625.13^{N}	2751.79^{N}
	Maximum	10724.21^{L}	7982.64^{L}	5310.52^{L}	3027.58^{L}	*1570.99^{L}*
	strain	10195.10^{N}	7959.14^{N}	5342.19^{N}	*3541.37^{N}*	*2686.42^{N}*
	Hoffman	10228.80^{L}	7963.23^{L}	4943.82^{L}	2810.01^{L}	1587.33^{L}
		9914.20^{N}	7835.55^{N}	5329.93^{N}	3604.70^{N}	2750.77^{N}
	Tsai-Hill	10604.70^{L}	7982.64^{L}	5257.41^{L}	3065.37^{L}	1597.55^{L}
		10130.75^{N}	7964.25^{N}	5385.09^{N}	3624.11^{N}	2762.00^{N}
	Tsai-Wu	10396.32^{L}	7963.23^{L}	4951.99^{L}	2858.02^{L}	1593.46^{L}
		9966.29^{N}	7886.62^{N}	5343.21^{N}	3610.83^{N}	2757.92^{N}
	Hashin	10506.64^{L}	7977.53^{L}	5242.08^{L}	3050.05^{L}	1588.36^{L}
		10050.05^{N}	7926.46^{N}	5365.68^{N}	3612.87^{N}	2752.81^{N}
	Puck	*3422.88^{L}*	*3496.43^{L}*	*2698.67^{L}*	*1912.16^{L}*	1588.36^{L}
		3690.50^{N}	*3218.59^{N}*	*3289.07^{N}*	3611.85^{N}	2752.81^{N}
	Serviceability	3942.80^{L}	2165.48^{L}	936.67^{L}	319.71^{L}	140.96^{L}
		3964.25^{N}	2185.90^{N}	956.08^{N}	337.08^{N}	155.26^{N}
0°/90°/90°/0°	Maximum	9594.48^{L}	8125.64^{L}	5489.28^{L}	2827.38^{L}	1462.72^{L}
	stress	10008.17^{N}	7898.88^{N}	5642.49^{N}	3276.81^{N}	2792.65^{N}
	Maximum	9493.36^{L}	8058.22^{L}	5430.03^{L}	2788.56^{L}	*1441.27^{L}*
	strain	9896.83^{N}	7669.05^{N}	5547.50^{N}	*3203.27^{N}*	*2719.10^{N}*
	Hoffman	9486.21^{L}	7966.29^{L}	5179.78^{L}	2813.08^{L}	1461.70^{L}
		9907.05^{N}	7626.15^{N}	5520.94^{N}	3263.53^{N}	2791.62^{N}
	Tsai-Hill	9586.31^{L}	8098.06^{L}	5490.30^{L}	2830.44^{L}	1467.82^{L}
		9988.76^{N}	7865.17^{N}	5615.94^{N}	3278.86^{N}	2799.80^{N}
	Tsai-Wu	9520.94^{L}	7967.31^{L}	5178.75^{L}	2820.23^{L}	1464.76^{L}
		9940.76^{N}	7658.84^{N}	5574.06^{N}	3269.66^{N}	2795.71^{N}
	Hashin	9534.22^{L}	8045.97^{L}	5460.67^{L}	2819.20^{L}	1462.72^{L}
		9961.19^{N}	7669.05^{N}	5599.59^{N}	3269.66^{N}	2792.65^{N}
	Puck	*3815.12^{L}*	*3241.06^{L}*	*2600.61^{L}*	*2165.48^{L}*	1462.72^{L}
		4224.72^{N}	*3007.15^{N}*	*3568.95^{N}*	3269.66^{N}	2792.65^{N}

Laminations	Failure criteria	$\overline{FL}$				
		c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
	Serviceability	3710.93^{L}	2144.03^{L}	1009.19^{L}	354.65^{L}	166.50^{L}
		3779.37^{N}	2180.80^{N}	1027.58^{N}	373.85^{N}	182.84^{N}

Note: *a/b* = 1, *a/h* = 100, 'L' and 'N' indicate the linear and nonlinear failure loads respectively.

Table 7.4 Nondimensionalized first ply failure loads $(\overline{FL})$ for different height to span ratios of angle ply shells

Laminations	Failure criteria	$\overline{FL}$				
		c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
45°/-45°	Maximum stress	11189.99^{L}	9201.23^{L}	6757.92^{L}	3625.13^{L}	542.39^{L}
		10755.87^{N}	8701.74^{N}	6199.18^{N}	3367.72^{N}	1213.48^{N}
	Maximum strain	*9799.79*^{L}	*8127.68*^{L}	*6017.37*^{L}	*3213.48*^{L}	*493.36*^{L}
		9463.74^{N}	*7725.23*^{N}	*5532.18*^{N}	*3010.22*^{N}	*1070.48*^{N}
	Hoffman	11122.57^{L}	9150.15^{L}	6722.17^{L}	3600.61^{L}	538.30^{L}
		10692.54^{N}	8653.73^{N}	6166.50^{N}	3348.32^{N}	1197.14^{N}
	Tsai-Hill	11727.27^{L}	9603.68^{L}	7018.39^{L}	3773.24^{L}	557.71^{L}
		11251.28^{N}	9066.39^{N}	6430.03^{N}	3496.43^{N}	1239.02^{N}
	Tsai-Wu	10716.04^{L}	8839.63^{L}	6511.75^{L}	3484.17^{L}	526.05^{L}
		10315.63^{N}	8372.83^{N}	5982.64^{N}	3246.17^{N}	1163.43^{N}
	Hashin	11159.35^{L}	9173.65^{L}	6735.44^{L}	3610.83^{L}	539.33^{L}
		10726.25^{N}	8676.2^{N}	6178.75^{N}	3355.47^{N}	1199.18^{N}
	Puck	11154.24^{L}	9167.52^{L}	6731.36^{L}	3607.76^{L}	539.33^{L}
		10721.14^{N}	8672.11^{N}	6175.69^{N}	3352.40^{N}	1198.16^{N}
	Serviceability	20953.01^{L}	12413.69^{L}	6075.59^{L}	1405.52^{L}	72.52^{L}
		20007.15^{N}	12185.90^{N}	5984.68^{N}	1403.47^{N}	83.76^{N}
45°/ -45°/45°	Maximum stress	14979.57^{L}	12905.01^{L}	9972.42^{L}	5536.26^{L}	1257.41^{L}
		14263.53^{N}	11937.69^{N}	8735.44^{N}	4254.34^{N}	2739.53^{N}
	Maximum strain	*13661.9*^{L}	*11850.87*^{L}	*9322.78*^{L}	*5429.01*^{L}	*1037.79*^{L}
		13082.74^{N}	*11045.97*^{N}	*8127.68*^{N}	*4011.24*^{N}	2983.66^{N}
	Hoffman	14894.79^{L}	12828.40^{L}	9874.36^{L}	5466.80^{L}	1243.11^{L}
		14184.88^{N}	11870.28^{N}	8686.42^{N}	4203.27^{N}	*2557.71*^{N}
	Tsai-Hill	15393.26^{L}	13216.55^{L}	10000.0^{L}	5500.51^{L}	1355.47^{L}
		14626.15^{N}	12194.08^{N}	8844.74^{N}	4223.70^{N}	2937.69^{N}
	Tsai-Wu	14546.48^{L}	12556.69^{L}	9803.88^{L}	5470.89^{L}	1174.67^{L}
		13875.38^{N}	11643.51^{N}	8533.20^{N}	4202.25^{N}	2723.19^{N}
	Hashin	14915.22^{L}	12844.74^{L}	9905.01^{L}	5494.38^{L}	1257.41^{L}
		14203.27^{N}	11885.60^{N}	8697.65^{N}	4221.66^{N}	2684.37^{N}
	Puck	14902.96^{L}	12834.53^{L}	9892.75^{L}	4949.95^{L}	1257.41^{L}
		14192.03^{N}	11875.38^{N}	8690.50^{N}	4216.55^{N}	2675.18^{N}
	Serviceability	20813.07^{L}	12553.63^{L}	5888.66^{L}	1335.04^{L}	124.62^{L}
		19281.92^{N}	11858.02^{N}	5642.49^{N}	1255.36^{N}	136.87^{N}
45°/-45°/ 45°/-45°	Maximum stress	12441.27^{L}	10522.98^{L}	7953.01^{L}	4287.03^{L}	1004.09^{L}
		12017.36^{N}	10022.47^{N}	7315.63^{N}	3916.24^{N}	1753.83^{N}
	Maximum strain	*11287.03*^{L}	*9622.06*^{L}	*7310.52*^{L}	*4082.74*^{L}	1055.16^{L}
		10943.82^{N}	*9207.35*^{N}	*6790.60*^{N}	*3691.52*^{N}	1860.06^{N}
	Hoffman	12373.85^{L}	10463.74^{L}	7906.03^{L}	4247.19^{L}	*978.55*^{L}
		11953.01^{N}	9966.29^{N}	7279.88^{N}	3869.25^{N}	*1693.57*^{N}

Laminations	Failure criteria	$\overline{FL}$				
		c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
	Tsai-Hill	12818.18^{L}	10799.8^{L}	8140.96^{L}	4310.52^{L}	1027.58^{L}
		12363.64^{N}	10268.64^{N}	7469.87^{N}	3922.37^{N}	1801.84^{N}
	Tsai-Wu	12062.31^{L}	10226.76^{L}	7740.55^{L}	4208.38^{L}	1001.02^{L}
		11664.96^{N}	9753.83^{N}	7159.35^{N}	3841.68^{N}	1742.59^{N}
	Hashin	12391.22^{L}	10477.02^{L}	7916.24^{L}	4259.45^{L}	993.87^{L}
		11969.36^{N}	9978.55^{N}	7283.96^{N}	3884.58^{N}	1727.27^{N}
	Puck	12382.02^{L}	10467.82^{L}	7909.09^{L}	4254.34^{L}	991.83^{L}
		11960.16^{N}	9970.38^{N}	7279.88^{N}	3878.45^{N}	1722.17^{N}
	Serviceability	21186.93^{L}	12946.88^{L}	6182.84^{L}	1320.74^{L}	135.85^{L}
		20955.06^{N}	12869.25^{N}	6159.35^{N}	1326.86^{N}	147.09^{N}
45°/-45°/	Maximum	14687.44^{L}	12747.70^{L}	9993.87^{L}	5367.72^{L}	1313.59^{L}
-45°/45°	stress	14171.6^{N}	12056.18^{N}	8924.41^{N}	4426.97^{N}	2512.77^{N}
	Maximum	*13417.77*^{L}	*11716.04*^{L}	*9196.12*^{L}	*5070.48*^{L}	*1170.58*^{L}
	strain	*12998.98*^{N}	*11144.02*^{N}	*8307.46*^{N}	*4097.04*^{N}	3101.12^{N}
	Hoffman	14602.66^{L}	12671.09^{L}	9911.13^{L}	5296.22^{L}	1232.89^{L}
		14090.91^{N}	11984.68^{N}	8870.28^{N}	4417.77^{N}	*2363.64*^{N}
	Tsai-Hill	15079.67^{L}	13049.03^{L}	10030.64^{L}	5346.27^{L}	1397.34^{L}
		14527.07^{N}	12314.61^{N}	9088.87^{N}	4541.37^{N}	2787.54^{N}
	Tsai-Wu	14268.64^{L}	12406.54^{L}	9727.27^{L}	5309.50^{L}	1306.44^{L}
		13785.5^{N}	11753.83^{N}	8718.08^{N}	4324.82^{N}	2488.25^{N}
	Hashin	14622.06^{L}	12687.44^{L}	9943.82^{L}	5326.86^{L}	1292.14^{L}
		14110.32^{N}	12000.00^{N}	8882.53^{N}	4426.97^{N}	2479.06^{N}
	Puck	14610.83^{L}	12444.33^{L}	9100.10^{L}	4768.13^{L}	1288.05^{L}
		14098.06^{N}	11989.79^{N}	8875.38^{N}	4426.97^{N}	2479.06^{N}
	Serviceability	20954.03^{L}	12821.25^{L}	6031.67^{L}	1308.48^{L}	137.90^{L}
		21259.45^{N}	13214.50^{N}	6064.35^{N}	1347.29^{N}	150.15^{N}

Note: *a/b* = 1, *a/h* = 100, 'L' and 'N' indicate the linear and nonlinear failure loads respectively.

Table 7.5 Percentage differences between linear and nonlinear first ply failure loads

Laminations	Percentage differences between linear and nonlinear first ply failure loads				
	c/b=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
0°/90°	3.0802	-10.6843	4.3545	-62.3273	-63.3764
0°/90°/0°	4.2497	-28.2146	-30.0652	-32.3355	-42.7087
0°/90°/0°/90°	-7.2515	8.6321	-17.9503	-46.0052	-41.5209
0°/90°/90°/0°	-9.6953	7.7785	-27.1322	-32.398	-46.9947
45°/-45°	3.5509	5.2095	8.7703	6.7526	-53.9122
45°/-45°/45°	4.4269	7.2868	14.7040	35.3450	-59.4249
45°/-45°/45°/-45°	3.1360	4.5041	7.6564	10.5976	-42.2195
45°/-45°/-45°/45°	3.2217	5.1329	10.6971	23.7596	-50.4754

Note: *a/b* = 1, *a/h* = 100

Table 7.6 Recommendation on design

Laminations	Approach				
	c/b=0.2	c/b=0.15	c/b=0.1	c/b=0.05	c/b=0
0°/90°	Geometric linear	Geometric nonlinear	Geometric linear	Geometric linear	Geometric linear
0°/90°/0°	Geometric linear	Geometric nonlinear	Geometric linear	Geometric linear	Geometric linear
0°/90°/0°/90°	Geometric linear	Geometric nonlinear	Geometric linear	Geometric linear	Geometric linear
0°/90°/90°/0°	Geometric linear	Geometric nonlinear	Geometric linear	Geometric linear	Geometric linear
45°/-45°	Geometric linear	Geometric nonlinear	Geometric nonlinear	Geometric nonlinear	Geometric linear
45°/-45°/45°	Geometric linear	Geometric nonlinear	Geometric nonlinear	Geometric nonlinear	Geometric linear
45°/-45°/45°/-45°	Geometric linear	Geometric nonlinear	Geometric nonlinear	Geometric nonlinear	Geometric linear
45°/-45°/-45°/45°	Geometric linear	Geometric nonlinear	Geometric nonlinear	Geometric nonlinear	Geometric linear

7.3.5 Suggesting Partial Factor of Safety on Failure Loads for Design Purpose

The shell combinations can be categorized into three distinct classes on carefully observing the results presented in Tables 7.1 to 7.4 as explained below. In some cases (say Class – I), the first ply failure loads are less than those corresponding to serviceability failures and from serviceability point of view these shells may be described as brittle ones as they fail even before the limit of serviceability failure is reached. In the second class of shells (say Class – II), the first ply failure loads are greater than those of corresponding to serviceability failure, but the ratio of the collapse load to that corresponding to serviceable limit of deflection does not exceed 3. The author chooses a practical ratio of 3 up to which the shells exhibit a ductile mode of failure and a practical factor of safety in between 1 to 3 may be imposed on the first ply failure load to get the safe values of working load. For the third category of shells (say Class – III), the above mentioned ratio exceeds 3, at places marginally and at places conspicuously to exhibit a highly ductile mode of failure. For these shells, the first ply failure loads need not be evaluated at all. The ratios of collapse failure loads to the loads corresponding to permissible deflection values for all type of shells are furnished in Table 7.7. Since the nonlinear approach of analysis is the more reliable approach, Table 7.8 post-processes the data furnished in Table 7.7 and furnishes the factor of safety values equal to 1 for Class – I shells. For Class – II shells, conservative rounded off figures are furnished and no factor of safety values are suggested for Class – III shells.

Table 7.7 Ratio of collapse load to load corresponding to permissible deflection

Laminations	*c/b*=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
0°/90°	1.2296	1.4862	4.0021	15.1097	32.9525
0°/90°/0°	0.9531	1.8100	3.4429	10.1974	15.6900
0°/90°/0°/90°	0.9309	1.4724	3.4402	10.5060	17.3027
0°/90°/90°/0°	1.1178	1.3789	3.4732	8.5683	14.8715
45°/-45°	0.4730	0.6339	0.9243	2.1448	12.7803
45°/-45°/45°	0.6785	0.9315	1.4404	3.1953	18.6871
45°/-45°/45°/-45°	0.5223	0.7155	0.8174	2.7821	11.5138
45°/-45°/-45°/45°	0.6114	0.8433	1.3699	3.0409	15.7419

Note: a/b = 1, a/h = 100

Table 7.8 Suggested partial factor of safety for different height to span ratio values to be imposed on first ply failure loads

Laminations	*c/b*=0.2	*c/b*=0.15	*c/b*=0.1	*c/b*=0.05	*c/b*=0
0°/90°	1.25	1.5			
0°/90°/0°	1.0	2.0			
0°/90°/0°/90°	1.0	1.5			
0°/90°/90°/0°	1.25	1.5			
45°/-45°	1.0	1.0	1.0	2.25	
45°/-45°/45°	1.0	1.0	1.5		
45°/-45°/45°/-45°	1.0	1.0	1.0	3.0	
45°/-45°/-45°/45°	1.0	1.0	1.5		

Note: No factor of safety values have been suggested for Class – III shells

7.4 CONCLUDING REMARKS

The present work leads to the following conclusions.

- The failure load values of skewed hypar shells for different parametric variations reported here are expected to serve as valuable design aids to practicing structural engineers.
- Among the cross ply shells taken up here, the two layered $0^0/90^0$ laminate turns out to be the best choice, while $45^0/-45^0/45^0$ stacking sequence may be considered as the best option in terms of first ply failure loads among the angle ply shells taken up here.
- By virtue of geometry of a skewed hypar shell, majority of the loads and moments are transferred along the diagonal directions. This is why the angle ply laminates taken up here, which have their fibers oriented along the diagonals, prove to be convincingly better than the cross ply ones in terms of first ply failure. The strength of a laminate depends on individual contribution of the fibers and the matrix. The angle ply configurations optimally utilize the matrix and fiber strengths to yield higher first ply failure load values compared to the cross ply counterparts. Thus these stacking orders may be preferred by the practicing engineers for design purpose.
- For angle ply shells, as the failure load corresponding to the limiting deflection (shorter span/250) is higher than the first ply collapse load, there is a possibility of a sudden brittle failure of these shells due to overloading without undergoing appreciable deflection. A designer must be cautious about this while using these laminates.
- The critical values of bending moment and shear force occur at the support for clamped boundary condition. So the failure initiates at the region adjacent to the support and any non-destructive health monitoring measurements may be restricted within this peripheral area only.

- The recommendation table which points out the cases where a relatively simpler linear theory may be used in place of the more involved nonlinear theory to get the first ply failure loads shall be helpful to the researchers.
- A design engineer may readily use the factor of safety values to be imposed on the first ply failure loads to get the working load values from the table given in this chapter. The nondimensional first ply failure loads corresponding to practical values of the parameters representing the shell curvature as used by design engineers practically are furnished in Tables 7.1 to 7.4. Further, indication about the failure zones has also been given which will be useful in non-destructive monitoring of shells at service.

Chapter 8

NONLINEAR FIRST PLY FAILURE CHARACTERISTICS OF SIMPLY SUPPORTED HYPAR SHELL ROOFS

8.1 GENERAL

The different edge conditions with which the boundaries of a shell surfaces are restrained include the commonly used simply supported boundary. This chapter presents a study carried out on first ply failure characteristics of simply supported laminated composite skewed hypar shells for different stacking orders of the composite. The results are post processed to extract guidelines suitable for practicing engineers engaged in design of composite shells. For non-destructive health monitoring of a shell surface, an engineer has to first pinpoint the vulnerable area from where failure may initiate. This chapter presents clear guidelines indicating the vulnerable areas. The results are presented systematically in Section 8.3 and the pinpointed conclusions are mentioned precisely in Section 8.4.

8.2 NUMERICAL EXAMPLES

The uniformly distributed first ply failure load values (*FL*) are nondimensionalized as $\overline{FL} = (FL/E_{22})(a/h)^4$. Various cross ply combinations including symmetric and anti-symmetric laminates such as 0°/90°, 90°/0°, 0°/90°/0°, 90°/0°/90°, 0°/90°/0°/90° or $(0°/90°)_2$, 90°/0°/90°/0° or $(90°/0°)_2$, 0°/90°/90°/0° or $(0°/90°)_S$ and 90°/0°/0°/90° or $(90°/0°)_S$ are taken up here. On the other hand 45°/-45°, 45°/-45°/45°, 45°/-45°/45°/-45° or $(45°/-45°)_2$ and 45°/-

45°/-45°/45° or $(45°/-45°)_S$ angle ply laminates are also considered here. The failure load values ($\overline{FL}$) and other failure related information like failure zones, modes or tendencies and first failed ply numbers are reported in Tables 8.1 and 8.2. Plies are numbered from top to bottom of the laminates. For any particular stacking sequence the minimum failure load value obtained from the different failure criteria is accepted as the first ply failure load from collapse point of view and it is marked in italics in Tables 8.1 and 8.2. Besides these failure loads corresponding to collapse point of view, the author also reports the failure loads corresponding to the permissible deflection of hypar shell taken as shorter span/250 in Tables 8.1 and 8.2. This load value is termed as the first ply failure load from serviceability standpoint. The failure study is carried out for various cross ply combinations which are obtained by repeating the 0°/90° or the 90°/0° units. These results are furnished in Fig. 8.1.

8.3 RESULTS AND DISCUSSIONS

8.3.1 First Ply Failure Behaviour of Cross Ply Hypar Shells for Different Stacking Sequences

Due to the effect of geometry of a hypar shell, majority of the loads are transferred along its diagonal directions while all the fibers of the cross ply laminates run along and perpendicular to the plan direction of the shell. Naturally, the matrix of a cross ply composite is weak in diagonal direction. Hence, all the cross ply laminates fail through in-plane shear failure of the matrix and results of Table 8.1 prove this fact. The Hoffman failure criterion yields the least value of the collapse failure loads in all the cases considered here. Hence, it is safely concluded that the Hoffman failure criterion may be accepted for the failure analysis of cross ply shells.

When two layered anti-symmetric cross ply laminates are compared, it is found that the 0°/90° shell is convincingly better than the 90°/0° one as the former yields failure load value about 10% more than that of the later. Among the four layered anti-symmetric laminates, the failure load of $(0°/90°)_2$ shell is about 6% more than that of $(90°/0°)_2$ shell. On the other hand, symmetric three and four layered cross ply shells show comparable performances in terms of first ply failure. From fabrication point of view, an engineer must attempt to maximize the failure load value against a fixed cost of production. So, a study of the comparative values of the failure loads as discussed above will give a clue for selecting the most efficient stacking sequence for a given consumption of material. The 90° lamina lying above the mid surface is the most vulnerable layer in the cross ply shells as this ply fails first in all the cases considered here except for 0°/90° shell. For this exceptional case, the top ply fails first. This information may be utilised for non-destructive health monitoring.

Safe and economical design means satisfying both collapse and serviceability criteria. So, the present author assesses the maximum allowable load from serviceability point of view with the permissible deflection of the shell taken as shorter span/250. These failure loads are given in Table 8.1. The serviceability failure load values are much lower than the failure loads obtained through collapse criteria. This indicates that the cross ply laminates behave like ductile materials. This is definitely an advantage of using cross ply shells because one gets sufficient warning before these shells fail due to overloading. Further it is noted that the maximum deflection occurs at the central node of the shell surface as expected for simply supported boundary condition.

Table 8.1. Nondimensionalized first ply failure loads $(\overline{FL})$ of cross ply skewed hypar shells

Lamination	Failure theory	$\overline{FL}$	Failure zone	First failed ply	Failure mode/ failure tendency
0°/90°	Maximum stress	7990.81	A	1	Matrix shear failure
	Maximum strain	7990.81	A	1	Matrix shear failure
	Hoffman	*7540.35*	*A*	*1*	*Matrix shear failure*
	Tsai-Hill	7941.78	A	1	Matrix shear failure
	Tsai-Wu	7557.71	A	1	Matrix shear failure
	Hashin	7932.58	A	1	Matrix cracking
	Puck	7804.90	A	1	Matrix cracking mode A
	Serviceability	2642.49	D		
90°/0°	Maximum stress	7931.56	A	1	Matrix shear failure
	Maximum strain	7932.58	A	1	Matrix shear failure
	Hoffman	*6633.30*	*A*	*1*	*Matrix shear failure*
	Tsai-Hill	7321.76	A	1	Matrix shear failure
	Tsai-Wu	6638.41	A	1	Matrix shear failure
	Hashin	7310.52	A	1	Matrix cracking
	Puck	7045.97	A	1	Matrix cracking mode A
	Serviceability	2643.51	D		
0°/90°/0°	Maximum stress	7431.05	A	1	Matrix shear failure
	Maximum strain	7431.05	A	1	Matrix shear failure
	Hoffman	*6842.70*	*A*	*2*	*Matrix shear failure*
	Tsai-Hill	7386.11	A	1	Matrix shear failure
	Tsai-Wu	6846.78	A	2	Matrix shear failure
	Hashin	7379.98	A	1	Matrix cracking
	Puck	7263.53	A	1	Matrix cracking mode A
	Serviceability	2636.36	D		
90°/0°/90°	Maximum stress	8326.86	A	1	Matrix shear failure
	Maximum strain	8326.86	A	1	Matrix shear failure
	Hoffman	*6963.23*	*A*	*1*	*Matrix shear failure*
	Tsai-Hill	7688.46	A	1	Matrix shear failure
	Tsai-Wu	6968.34	A	1	Matrix shear failure
	Hashin	7675.18	A	1	Matrix cracking
	Puck	7398.37	A	1	Matrix cracking mode A
	Serviceability	2651.69	C		
0°/90°/0°/90° or $(0°/90°)_2$	Maximum stress	7718.08	A	1	Matrix shear failure
	Maximum strain	7718.08	A	1	Matrix shear failure
	Hoffman	*6836.57*	*A*	*2*	*Matrix shear failure*
	Tsai-Hill	7540.35	A	2	Matrix shear failure
	Tsai-Wu	6839.63	A	2	Matrix shear failure
	Hashin	7530.13	A	2	Matrix cracking
	Puck	7260.47	A	2	Matrix cracking mode A
	Serviceability	2485.19	D		
90°/0°/90°/0° or $(90°/0°)_2$	Maximum stress	7695.61	A	1	Matrix shear failure
	Maximum strain	7695.61	A	1	Matrix shear failure
	Hoffman	*6436.16*	*A*	*1*	*Matrix shear failure*
	Tsai-Hill	7107.25	A	1	Matrix shear failure
	Tsai-Wu	6439.22	A	1	Matrix shear failure

Lamination	Failure theory	$\overline{FL}$	Failure zone	First failed ply	Failure mode/ failure tendency
	Hashin	7096.02	A	1	Matrix cracking
	Puck	6839.63	A	1	Matrix cracking mode A
	Serviceability	2486.21	D		
0°/90°/90°/0° or $(0°/90°)_S$	Maximum stress	7481.10	A	1	Matrix shear failure
	Maximum strain	7481.10	A	1	Matrix shear failure
	Hoffman	*6693.56*	*A*	*2*	*Matrix shear failure*
	Tsai-Hill	7379.98	A	2	Matrix shear failure
	Tsai-Wu	6697.65	A	2	Matrix shear failure
	Hashin	7368.74	A	2	Matrix cracking
	Puck	7106.23	A	2	Matrix cracking mode A
	Serviceability	2562.82	D		
90°/0°/0°/90° or $(90°/0°)_S$	Maximum stress	8045.97	A	1	Matrix shear failure
	Maximum strain	8045.97	A	1	Matrix shear failure
	Hoffman	*6725.23*	*A*	*1*	*Matrix shear failure*
	Tsai-Hill	7426.97	A	1	Matrix shear failure
	Tsai-Wu	6729.32	A	1	Matrix shear failure
	Hashin	7416.75	A	1	Matrix cracking
	Puck	7148.11	A	1	Matrix cracking mode A
	Serviceability	2587.33	D		

Note: a/b = 1, a/h = 100, c/b = 0.2, minimum collapse failure loads are indicated by italics.

8.3.2 First Ply Failure Behaviour of Angle Ply Hypar Shells for Different Stacking Sequences

The nondimensionalized nonlinear first ply failure load values are reported for angle ply hypar shells in Table 8.2. The symmetric angle ply shells perform better than the anti-symmetric ones in terms of first ply failure loads and $(45°/-45°)_S$ laminate gives the maximum failure load among all the angle ply options taken up here. This shell option also gives approximately 68.87% higher failure load than the failure load of 0°/90° shell which is the best option among all the cross ply shells considered presently. The maximum strain failure criterion may be accepted as the governing failure criterion for angle ply shells as it gives the minimum failure loads among all the collapse failure criteria considered here for all the angle ply cases.

It is interestingly noted that the top layers of all the angle ply shells fail first and it is a useful input to the practicing engineers for designing these shells. Since the skewed hypar shell transfers loads along its diagonal direction and for angle ply shells, the fibers also run along the diagonals, it is expected that the load bearing capacities of angle ply shells are higher than those of cross ply shells. The results of Tables 8.1 and 8.2 prove the correctness of this expectation.

In contrast to the behaviour of the cross ply shells, the first ply failure loads obtained from the serviceability criterion are higher than the first ply failure loads from collapse point of view for all the angle ply shells. So there is a chance of brittle failure for this kind of shell and the user may not get adequate warning before failure.

Table 8.2. Nondimensionalized first ply failure loads $(\overline{FL})$ of angle ply skewed hypar shells

Lamination	Failure theory	$\overline{FL}$	Failure zone	First failed ply	Failure mode/ failure tendency
45°/-45°	Maximum stress	8372.83	A	1	Matrix cracking
	Maximum strain	*6309.50*	*A*	*1*	*Matrix cracking*
	Hoffman	8229.83	A	1	Matrix cracking
	Tsai-Hill	8921.35	A	1	Matrix cracking
	Tsai-Wu	7515.83	A	1	Matrix cracking
	Hashin	8357.51	A	1	Matrix cracking
	Puck	8355.46	A	1	Matrix cracking mode A
	Serviceability	9751.79	D		
45°/-45°/45°	Maximum stress	12233.91	A	1	Matrix cracking
	Maximum strain	*10103.17*	*A*	*1*	*Matrix cracking*
	Hoffman	12052.09	A	1	Matrix cracking
	Tsai-Hill	13162.41	A	1	Matrix cracking
	Tsai-Wu	11398.37	A	1	Matrix cracking
	Hashin	12213.48	A	1	Matrix cracking
	Puck	12210.42	A	1	Matrix cracking mode A
	Serviceability	16377.94	C		
45°/-45°/45°/-45°	Maximum stress	12503.58	A	1	Matrix cracking
or $(45°/-45°)_2$	Maximum strain	*10440.25*	*A*	*1*	*Matrix cracking*
	Hoffman	12343.21	A	1	Matrix cracking
	Tsai-Hill	13385.09	A	1	Matrix cracking
	Tsai-Wu	11712.97	A	1	Matrix cracking
	Hashin	12478.04	A	1	Matrix cracking
	Puck	12472.93	A	1	Matrix cracking mode A

Lamination	Failure theory	$\overline{FL}$	Failure zone	First failed ply	Failure mode/ failure tendency
	Serviceability	15145.05	C		
45°/-45°/-45°/45° or (45°/-45°)$_S$	Maximum stress	16116.45	A	1	Matrix cracking
	Maximum strain	*12733.40*	*A*	*1*	*Matrix cracking*
	Hoffman	15703.78	A	1	Matrix cracking
	Tsai-Hill	17593.46	A	1	Matrix cracking
	Tsai-Wu	14664.96	A	1	Matrix cracking
	Hashin	16100.10	A	1	Matrix cracking
	Puck	16097.04	A	1	Matrix cracking mode A
	Serviceability	19389.17	C		

Note: *a/b* = 1, *a/h* = 100, *c/b* = 0.2, minimum collapse failure loads are indicated by italics.

8.3.3 Guidelines for Non-Destructive Test Monitoring from a Prior Knowledge of Probable Failure Zones

For non-destructive health monitoring of a shell roof, a practicing engineer must have a prior knowledge of the vulnerable zones on the shell surface where from failure may initiate. The author has divided the plan area of the shell into distinct zones namely A, B, C and D which is depicted in Fig. 7.1 of Chapter 7. These failure zones are indicated against each shell combinations in Tables 8.1 and 8.2. It is noted that in all the cases taken up here failure initiates from Zone A and hence for detecting any hidden flaw appropriate instrumentations may be restricted within Zone A only.

8.3.4 First Ply Failure Loads of Laminates (0°/90°)$_n$ and (90°/0°)$_n$ Units

This chapter reports how the failure loads change if the 0°/90° or the 90°/0° units are repeated a number of times within a fixed thickness in Fig. 8.1. It is interestingly observed that the 0°/90° units yield higher failure load values in compare to the 90°/0° units for any given number of repetitions. All the shell options fail through the mode of in-plane shear failure of matrix and the reason is discussed earlier of this section. When the number of repetition becomes one, the shells yield maximum failure load for both the units (0°/90° and 90°/0°) and the failure loads decrease gradually when the units are repeated more than once. This observation leads to conclude that there is no point in repeating the 0°/90° units more than once through involved fabrication process because there is no corresponding gain from first ply failure load point of view.

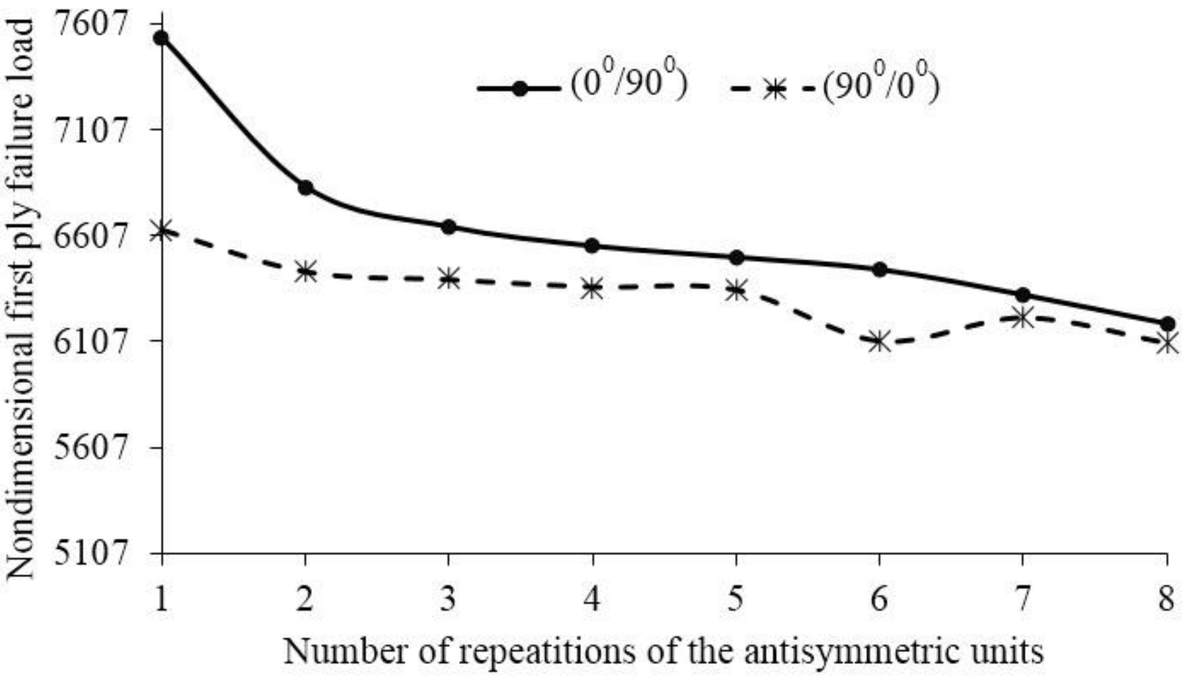


Fig.8.1. Failure loads of skewed hypar shells by repeating anti-symmetric units

8.3.5 Suggesting Partial Factor of Safety Values on First Ply Failure Loads

In every civil engineering design, the concept of working load is an important point of interest. The failure loads corresponding to serviceability failure are considered as working loads here. The author reports the values of the ratios of first ply failure loads to working loads in Table 8.3. These values rounded up to nearest quarter of an integer may be proposed as the partial factor of safety values. In some cases, for angle ply shells, the failure loads due to collapse point of view are lower than the failure loads corresponding to serviceability failure. Naturally, for these cases, the above mentioned ratio works out to be less than unity and the partial factor of safety may be suggested as unity.

Table 8.3. Partial factors of safety for composite simply supported skewed hypar shells

Lamination	Ratios of first ply failure loads to working loads	Proposed partial factor of safety
0°/90°	2.85	3.00
90°/0°	2.51	2.75
0°/90°/0°	2.60	2.75
90°/0°/90°	2.63	2.75
0°/90°/0°/90° or $(0°/90°)_2$	2.75	2.75
90°/0°/90°/0° or $(90°/0°)_2$	2.59	2.75
0°/90°/90°/0° or $(0°/90°)_S$	2.61	2.75
90°/0°/90°/0° or $(90°/0°)_S$	2.60	2.75
45°/-45°	0.65	1.0
45°/-45°/45°	0.62	1.0
45°/-45°/45°/-45° or $(45°/-45°)_2$	0.69	1.0
45°/-45°/-45°/45° or $(45°/-45°)_S$	0.66	1.0

Note: a/b = 1, a/h = 100, c/b = 0.2

8.4 CONCLUDING REMARKS

The following conclusions are evident from the present study.

- The nondimensionalized first ply failure loads of skewed hypar shells of simply supported boundary condition that are presented here are expected to serve as very useful design guidelines to the practicing engineers.
- The matrix of a cross ply composite is very weak in diagonal direction when it is used to fabricate the hypar shell. So all the cross ply laminates fail through in-plane shear failure of the matrix.
- The angle ply shells give higher first ply failure loads than the cross ply shells. The $(45°/-45°)_S$ and 0°/90° shell options turn out to be the best among angle and cross ply shells in terms of first ply failure load values respectively.

- Though angle ply shells are convincingly better than the cross ply ones in terms of first ply failure but these angle ply shells exhibit a brittle failure which may be attributed as one of their disadvantages. On the other hand, the nature of failure of cross ply shells is quite gradual and this ductility is preferred in many practical cases despite the fact that these shells fail under relatively lower values of surface pressure.
- There is no significant gain from first ply failure loads point of view when 0°/90° or 90°/0° units are repeated more than once within a given shell thickness.
- Guidelines for non-destructive monitoring of skewed hypar shells and the suggested partial factor of safety values which are to be applied on the first ply failure load to get the working loads are expected to be extremely valuable inputs to practicing engineers.

Chapter 9

SCOPE FOR FUTURE RESEARCH

The first ply failure of laminated composite shell roofs are studied in the present investigation considering the geometrically linear and nonlinear strain terms. Among the various types of shell geometries, the spherical and skewed hypar shell forms are examined numerically in terms of their first ply failure behaviours in this research work. Though the pin pointed scope of the present study is reported in Chapter 3 but the present work is not claimed to be complete in all respects. Several other areas where future researches may be extended to have become apparent through review of literature and also during the course of this research work. Some of the areas are indicated below.

First ply failure study using linear and nonlinear strains may be carried out for other shell configurations like the elliptic paraboloidal and conoidal ones which are aesthetically attractive. This study may be extended for different values of height to short span ratios. Failure study under dynamic loads may be investigated also. First ply and progressive failure studies on shells having different boundary conditions along the different edges are also interesting areas for future researchers. Shell configurations with cutouts, stiffeners – concentric or eccentric have received relatively less attention and hence failure study on these shell combinations may be carried out. Shells of smart composites and functionally graded materials may be taken up for future study.

The above areas are only indicative but not exhaustive. Deeper search within the literature is expected to unfold many other areas which need attention but have not been studied and reported.

Appendix

REFERENCES

1. Abe, A., Kobayashi, Y. and Yamada, G., 2000, 'Non-linear vibration characteristics of clamped laminated shallow shells', Journal of Sound and Vibration, Vol. 234, No. 3, pp. 405-426.
2. Abel, J. F. and Billington, D. P., 1972, 'Stability analysis of cooling towers – A review of current methods', Shell Structures and Climatic Influences, Proceedings of International Association for Shell Structures Symposium, July, University of Calgary, Alberta, Canada.
3. Abu-Farsakh, G. A. and Almasri, A. H., 2011, 'A composite finite element to predict failure progress in composite laminates accounting for nonlinear material properties', Structural Control And Health Monitoring, Vol. 18, pp. 752-768.
4. Adali, S. and Cagdas, I.U., 2011, 'Failure analysis of curved composite panels based on first-ply and buckling failures', Procedia Engineering, Vol. 10, pp. 1591-1596.
5. Ahmad, S., Irons, B. M. and Zienkiewicz, O. C., 1970, 'Analysis of thick and thin shell structures of curved finite elements', International Journal Numerical Methods in Engineering, Vol.2, pp. 419-450.
6. Ahmed, S. and Sluys, L. J., 2014, 'Implicit/explicit elasto-dynamics of isotropic and anisotropic plates and shells using a solid-like shell element', European Journal of Mechanics A/Solids, Vol. 43, pp. 118-132.
7. Aksu, T., 1997, 'A finite element formulation for free vibration analysis of shells of general shape', Computers and Structures, Vol. 65, No. 5, pp. 687-694.
8. Alijani, F. and Amabili, M., 2013, 'Nonlinear vibrations of thick laminated circular cylindrical panels,' Composite Structures, Vol. 96, pp. 643-660.

9. Alijani, F. and Amabili, M., 2014, ‘Nonlinear vibrations of shells: a literature review from 2003 to 2013’, International Journal of Nonlinear Mechanics, Vol. 58, pp. 233-257.

10. Amabili, M. and Reddy, J. N., 2010, ‘A new nonlinear higher order shear deformation theory for large-amplitude vibrations of laminated doubly curved shells’, International Journal of Nonlinear Mechanics, Vol. 45, pp. 409-418.

11. Amabili, M., 2011, ‘Nonlinear vibrations of laminated circular cylindrical shells: comparison of different shell theories’, Composite Structures, Vol. 94, pp. 207-220.

12. Amabili, M., 2013, ‘A new nonlinear higher order shear deformation theory with thickness variation for large-amplitude vibrations of laminated doubly curved shells,’ Journal of Sound and Vibration, Vol. 332, pp. 4620-4640.

13. Amabili, M., 2014, ‘A nonlinear higher-order thickness variation and shear deformation theory for large-amplitude vibrations of laminated doubly curved shells’, International Journal of Nonlinear Mechanics, Vol. 58, pp. 57-75.

14. Amabili, M., and Païdoussis, M. P., 2003, ‘Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid structure interaction’, Applied Mechanics Reviews, Vol. 56, pp. 349-381.

15. Ambartsumyan, S. A., 1953, ‘Calculation of laminated anisotropic shells’, Izvestiia Akademiia Nauk Armenskoi SSR, Ser. Fiz. Mat. Est. Tekh. Nauk., Vol.6, No. 3, pp. 15.

16. Anlas, G. and Goker, G., 2001, ‘Vibration analysis of skew fibre-reinforced composite laminated plates’, Journal of Sound and Vibration, Vol. 242, No. 2, pp. 265-276.

17. Apeland, K. and Popov, E. P., September 1961, ‘Analysis of bending stresses in translational shells’, Proceedings of the Colloquium on Simplified calculation methods, Brussels, pp. 9-43.

18. Arciniega, R. A., and Reddy, J. N., 2007, 'Tensor-based finite element formulation for geometrically nonlinear analysis of shell structures', Computer Methods in Applied Mechanics and Engineering, Vol. 196, pp. 1048-1073.

19. Aron, H., 1874, 'Das Gleichgewitcht und die Bewegund einer underlich dunner, beliebig, gekrummten, elastischen Schale', J. fur reine und ang. Math., 78.

20. Avramov, K.V., 2012, 'Nonlinear modes of vibrations for simply supported cylindrical shell with geometrical nonlinearity', Acta Mechanica, Vol. 223, pp. 279-292.

21. Azzi, V. D. and Tsai, S. W., 1965, 'Anisotropic strength of components', Experimental Mechanics, Vol. 5, No. 9, pp. 286-288.

22. Bakshi, K. and Chakravorty, D., 2012, 'Delamination and first ply failure study of composite conoidal shells', Bonfring International Journal of Industrial Engineering and Management Science, Vol. 4, pp. 21-26.

23. Bakshi, K. and Chakravorty, D., 2013, 'First ply failure study of composite conoidal shells used as roofing units in civil engineering', Journal of Failure Analysis and Prevention, Vol. 13, No. 5, pp. 624-633.

24. Bakshi, K. and Chakravorty, D., 2014a, 'First ply failure study of thin composite conoidal shells subjected to uniformly distributed loading', Thin Walled Structures, Vol. 76, pp. 1-7.

25. Bakshi, K. and Chakravorty, D., 2014b, 'Geometrically linear and nonlinear first ply failure loads of composite cylindrical shells', ASCE Journal of Engineering Mechanics, Vol. 140, No. 12.

26. Bakshi, K. and Chakravorty, D., 2015, 'First ply failure loads of composite conoidal shell roofs with varying lamination', Mechanics of Advanced Materials and Structures, Vol. 22, pp. 978-987.

27. Bandyopadhyay, J. N. and Ray, D. P., 1971-72, ‘Reinforced concrete hyperbolic paraboloid and cylindrical shells, Part-I, behavior of hyperbolic paraboloid shells in elastic and ultimate ranges’, Proceedings of Indian Society of Theoretical and Applied Mechanics, Allahabad, pp. 117-127.

28. Beltrami, E., 1881, ‘Sullequilibrio delle super ficite flessibilied in esterdibill’, Mem R. Acad. Sci. di Bologna.

29. Bert, C. W. and Kumar, M., 1982, ‘Vibration of cylindrical shells of bimodulus composite materials’, Journal of Sound and Vibration, Vol.81, No.1, pp. 107-121.

30. Bert, C. W. and Reddy, V. S., 1982, ‘Cylindrical shells of bimodulus material’, Journal of Engineering Mechanics Division, ASCE, Vol. 108, No. 5, pp. 675-688.

31. Bhaskar, K. and Varadan, T. K., 1993, ‘Interlaminar stresses in composite cylindrical shells under transient loads’, Journal of Sound and Vibration, Vol.168, No.3, pp. 469-477.

32. Bich, D. H. and Nguyen, N. X., 2012, ‘Nonlinear vibration of functionally graded circular cylindrical shells based on improved Donnell equations’, Journal of Sound and Vibration, Vol. 331, pp. 5488-5501.

33. Bongard, W., 1959, ‘Zurtheorite and berechnung von scalentragwerken in form gleichseltiger hyparbolischer paraboloide’, Bautechnik-Archiv, Vol.15.

34. Brebbia, C. A. and Hadid, H. A., 1971, ‘Analysis of plates and shells using rectangular curved elements’, University of Southampton Civil Engineering, Report No. CE/5/71.

35. Brebbia, C. A., 1966, ‘An experimental and theoretical investigation into hyperbolic paraboloid shells’, University of Southampton, Civil Engineering, Report No. CE/2/66.

36. Budiansky, B. and Sanders, J.L., 1963, ‘On the best first-order linear shell theory’, Progress in Applied Mechanics, The Prager Anniversary Volume, Macmillan, pp. 129-140.

37. Burton, W. S. and Noor, A. K., 1995, ‘Assessment of computational models for sandwich panels and shells’, Computer Methods in Applied Mechanics and Engineering, Vol.124, pp. 125-151.

38. Chakravorty, D., Bandyopadhyay, J. N. and Sinha, P. K., 1996, ‘Finite element free vibration analysis of doubly curved laminated composite shells’, Journal of Sound and Vibration, Vol. 191, No. 4, pp. 491-504.

39. Chakravorty, D., Bandyopadhyay, J. N. and Sinha, P.K., 1995a, ‘Free vibration analysis of point-supported laminated composite doubly curved shells – a finite element approach’, Computers and Structures, Vol. 54, No. 2, pp. 191-198.

40. Chakravorty, D., Bandyopadhyay, J. N. and Sinha, P.K., 1995b, ‘Finite element free vibration analysis of conoidal shells’, Computers and Structures, Vol. 56, No. 6, pp. 975-978.

41. Chakravorty, D., Bandyopadhyay, J. N. and Sinha, P.K., 1998, ‘Applications of FEM on free and forced vibration of laminated shells’ Journal of Engineering Mechanics, ASCE, Vol. 124, No. 1, pp. 1-8.

42. Chandrashekhara, K., 1989, ‘Free vibrations of anisotropic laminated doubly curved shells’, Computers and Structures, Vol. 33, No. 1-3, pp. 435-440.

43. Chang, R. R. and Chiang, T. H., 2010, ‘Theoretical and experimental predictions of first ply failure of a laminated composite elevated floor plate’, Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering, Vol. 224, No. 4, pp. 233-245.

44. Chang, T.P. and Chang, H.C., 2000, ‘Nonlinear vibration analysis of geometrically nonlinear shell structures’, Mechanics Research Communications, Vol. 27, No. 2, pp. 173-180.

45. Chao, C. C. and Tung, T. P., 1989, ‘Step presence and blast response of clamped orthotropic hemispherical shells’, International Journal of Impact Engineering, Vol. 8, No. 3, pp. 191-207.

46. Chaudhuri, R.A. 2008, ‘A nonlinear zigzag theory for finite element analysis of highly shear-deformable laminated anisotropic shells’, Composite Structures, Vol. 85, No. 4, pp. 350-359.

47. Chen, J. F., Morozov, E.V. and Shankar, K., 2012, ‘A combined elasto-plastic damage model for progressive failure analysis of composite materials and structures’, Composite Structures, Vol. 94, No. 12, pp. 3478-3489.

48. Chen, J. F., Morozov, E.V. and Shankar, K., 2014, ‘Simulating progressive failure of composite laminates including in-ply and delamination damage effects’, Composites: Part A, Vol. 61, pp. 185-200.

49. Chetty, S. K. M. and Tottenham, H., 1964, ‘An investigation into the bending analysis of hyperbolic paraboloid shells’, Indian Concrete Journal, Vol. 38, pp.248-158.

50. Chia, C. Y. and Chia, D. S., 1992, ‘Nonlinear vibration of moderately thick antisymmetric angle-ply shallow spherical shell’, Computers and Structures, Vol. 44, No. 4, pp. 797-805.

51. Choi, C. K. and Schnobrich, W. C., 1970, ‘Finite element analysis of translation shells’, University of Illinois, Structural Research No. 368.

52. Chróścielewski J., Sabik, A., Sobczyk, B. and Witkowski, W., 2016, ‘Nonlinear FEM 2D failure onset prediction of composite shells based on 6-parameter shell theory’, Thin – Walled Structures, Vol. 105, pp. 207-219.

53. Chun, L. and Lam, K. Y., 1998, ‘Dynamic response of fully-clamped laminated composite plates subjected to low-velocity impact of a mass’, International Journal of Solids and Structures, Vol. 35, No. 11, pp. 963-979.

54. Civalek O., 2007, 'Numerical analysis of free vibrations of laminated composite conical and cylindrical shells: Discrete singular convolution (DSC) approach', Journal of Computational and Applied Mathematics, Vol. 205, pp. 251-271.

55. Coelho, A. M. G., Mottram, J. T. and Harries, K. A., 2015, 'Finite element guidelines for simulation of fibre – tension dominated failures in composite materials validated by case studies', Composite Structures, Vol. 126, pp. 299-313.

56. Coleby, J. R. and Majumdar, 1982, 'Vibrations of simply supported shallow shells on elliptical bases', Journal of Applied Mechanics, Transactions of ASME, Vol. 49, pp. 227-229.

57. Connor, J. J. and Brebbia, C. A., 1967, 'Stiffness matrix for shallow rectangular shell element', Journal of Engineering Mechanics Division, Proceedings of ASCE, Vol. 93, No. 5, pp. 43-66.

58. Crossland, J. A. and Dickinson, S.M., 1997, 'The free vibration of thin rectangular planform shallow shells with slits', Journal of Sound and Vibration, Vol. 199, No. 3, pp. 513-521.

59. Das, H. S. and Chakravorty, D., 2007, 'Design aids and selection guidelines for composite conoidal shell roofs – a finite element application', Journal of Reinforced Plastics and Composites, Vol. 26, No. 17, pp. 1793-1819.

60. Das, H. S. and Chakravorty, D., 2008, 'Natural frequencies and mode shapes of composite conoids with complicated boundary conditions', Journal of Reinforced Plastics and Composites, Vol. 27, No. 13, pp. 1397-1415.

61. Das, H. S. and Chakravorty, D., 2009, 'Composite full conoidal shell roofs under free vibration', Advances in Vibration Engineering, Vol. 8, No. 4, pp. 303-310.

62. Das, H. S. and Chakravorty, D., 2010, ‘Finite element application in analysis and design of point supported composite conoidal shell roofs suggesting selection guidelines’, Journal of Strain Analysis for Engineering Design, Vol. 45, No. 3, pp. 165-177.
63. Dey, A., Bandyopadhyay, J. N. and Sinha, P. K., 1994, ‘Behaviour of paraboloid of revolution shell using cross-ply and anti-symmetric angle-ply laminates’, Computers and Structures, Vol. 52, No.6, pp. 1301-1308.
64. Dhatt, G. S., 1970, ‘An efficient triangular finite element’, AIAA Journal Vol. 8, No. 11, pp. 2100-2102.
65. Djoudi, M.S. and Bahai, H., 2003, ‘A shallow shell finite element for the linear and nonlinear analysis of cylindrical shells’, Engineering Structures, Vol. 25, pp. 769-778.
66. Dogan, A. and Arslan, H.M., 2009, ‘Effects of curvature on free vibration characteristics of laminated composite cylindrical shallow shells’, Scientific Research and Essay, Vol.4, No. 4, pp. 226-238.
67. Dong, H., Wanga, J. and Karihaloo, B. L., 2014, ‘An improved Puck’s failure theory for fibre reinforced composite laminates including the in situ strength effect’, Composite Science and Technology, Vol. 98, pp. 86-92.
68. Dong, S. B., 1968, ‘Free vibrations of laminated orthotropic cylindrical shells’, Journal of Acoustic Society, America, Vol. 44, No. 6, pp. 1628-1635.
69. Dong, S. B., Pister, K. S. and Taylor, R. L., 1962, ‘On the theory of laminated anisotropic shells and plates’, Journal of Aerospace Sciences, Vol. 29, pp. 969-975.
70. Donnell, L. H., 1933, ‘Stability of thin walled tubes in torsion’, NACA Report 479.
71. Dulaska, E., 1969, ‘Vibration and stability of anisotropic shallow shells’, Acta Tech., Vol. 65, No. 3/4 pp. 225-260.

72. Ellul, B., Camilleri, D. and Betts, J. C., 2014, ‘A progressive failure analysis applied to fiber-reinforced composite plates subject to out-of-plane bending’, Mechanics of Composite Materials, Vol. 49, No. 6, pp. 605-620.

73. Ergatoudis, I., Irons, B. M. and Zienkiewicz, O. C., 1968, ‘Curved isoparametric quadrilateral elements for finite element analysis’, International Journal of Solids and Structures, Vol. 4, pp. 31-42.

74. Falzon, B. G. and Apruzzese, P., 2010, ‘Numerical analysis of failure mechanisms in composite structures Part I: FE implementation’, Composite Structures, Vol. 93, No. 2, pp. 1039-1046.

75. Falzon, B. G. and Apruzzese, P., 2011, ‘Numerical analysis of intra-laminar failure mechanisms in composite structures Part II: applications’, Composite Structures, Vol. 93, No. 2, pp. 1047-1053.

76. Ferreira, A. J. M. and Barbosa, J. T., 2000, ‘Buckling behaviour of composite shells’, Composite Structures, Vol. 50, pp. 93-98.

77. Ferreira, A. J. M., Roque, C. M. C. and Jorge, R. M. N., 2006, ‘Modeling cross-ply laminated elastic shells by a higher-order theory and multiquadrics’, Computers and Structures, Vol. 84, pp. 1288-1299.

78. Fischer, L., 1960, ‘Determination of membrane stresses on elliptic paraboloid using polynomials’, Journal of American Concrete Institute, Vol. 32, No. 4, pp. 433-441.

79. Flügge, W. and Conrad, D. A., 1956, ‘Singular solution in the theory of shallow shells’, Tech. Report No. 101, Division of Engineering Mechanics, Stanford University.

80. Flügge, W. and Geyling, F. T., 1957, ‘A general theory of deformation of membrane shells’, International Association for Bridge and Structural Engineering, Vol. 17, pp. 23.

81. Gadade, A. M., Lal A. and Singh B. N., 2016a, 'Accurate stochastic initial and final failure of laminated plates subjected to hygro-thermo-mechanical loadings using Puck's failure criteria', International Journal of Mechanical Sciences, Vol. 114, pp. 177-206.

82. Gadade, A. M., Lal A. and Singh B. N., 2016b, 'Finite element implementation of Puck's failure criterion for failure analysis of laminated plate subjected to biaxial loadings', Aerospace Science and Technology, Vol. 55, pp. 227-241.

83. Gallagher, R.H., 1969, 'Analysis of plate and shell structures – applications of finite element method in engineering', Vanderbilt University, ASCE Publication.

84. Ganapathi, M., Patel, B. P. and Pawargi, D. S., 2002, 'Dynamic analysis of laminated cross-ply composite non-circular thick cylindrical shells using higher-order theory', International Journal of Solids and Structures', Vol. 39, pp. 5945-5962.

85. Ganapathi, M., Patel, B. P., Gupta, S. S., Khatri, K. N., Sambandam, C. T. and Giri, S. N., 2004, 'Free-vibration characteristics of laminated angle-ply non-circular cylindrical shells', Defense Science Journal, Vol. 54, No. 4, pp. 429-442.

86. Ganapathi, M., Patel, B. P., Patel, H. G. and Pawargi, D. S., 2003, 'Vibration analysis of laminated cross-ply oval cylindrical shells', Journal of Sound and Vibration, Vol. 262, pp. 65-86.

87. Ganapathi, M., Vardan, T. K. and Balamurugan, V., 1994, 'Dynamic instability of laminated composite curved panels using finite element method', Computers and Structures, Vol. 53, No. 2, pp. 335-342.

88. Ganesan, R. and Liu, D.Y., 2008, 'Progressive failure and post-buckling response of tapered composite plates under uni-axial compression', Composite Structures, Vol. 82, pp. 159-176.

89. Garai, J. and Ray, C., 2005, ‘Initial failure analysis of laminated composite plates under humid conditions’, Journal of Reinforced Plastics and Composites, Vol. 24, No. 11, pp. 1203-1212.

90. Gautham, B. P. and Ganesan, N., 1997, ‘Free vibration characteristics of isotropic and laminated orthotropic spherical caps’, Journal of Sound and Vibration, Vol. 204, No. 1, pp. 17-40.

91. Gendy, A. S., Saleeb, A. F. and Mikhail, S. N., 1997, ‘Free vibrations and stability analysis of laminated composite plates and shells with hybrid/mixed formulation’, Computers and Structures, Vol. 63, No. 6, pp. 1149-1163.

92. Gergely, P., 1972, ‘Buckling of orthotropic hyperbolic paraboloid shells’, Journal of Structural Division, Proceedings of ASCE, Vol. 98, No. ST1, pp. 613-699.

93. Ghosh, B. and Bandyopadhyay, J. N., 1989, ‘Bending analysis of conoidal shells using curved quadratic isoparametric element’, Computers and Structures, Vol. 33, No. 3, pp. 717-728.

94. Gohari, S., Golshan, A., Mostakhdemin, M., Mozafari, F. and Momenzadeh, A., 2012, ‘Failure strength of thin – walled cylindrical GFRP composite shell against static internal and external pressure for various volumetric fiber fraction’, International Journal of Applied Physics and Mathematics, Vol. 2, No. 2, pp. 111-116.

95. Gohari, S., Sharifi, S., Vrcelj, Z. and Yahya, M. Y., 2015, ‘First ply failure prediction of an unsymmetrical laminated ellipsoidal woven GFRP composite shell with incorporated surface-bounded sensors and internally pressurized’, Composite – Part B, Vol. 77, pp. 502-518.

96. Goldenveizer, A. L., 1968, ‘Method for justifying and refining the theory of shells’, Journal of Applied Mathematics and Mechanics, Vol. 32, No. 4, pp. 704-718.

97. Grafton, P. E. and Strome, D. R., 1963, ‘Analysis of axi-symmetric shells by the direct stiffness method’, AIAA Journal, Vol. 1, pp. 2342-2347.

98. Greene, B. E., Jones, R. E. and Strome, D. R., 1968, ‘Dynamic analysis of shells using doubly curved finite elements’, Proceedings of 2nd Conference on Matrix Methods Structural Mechanics, AFFDL-TR-68-150, pp. 185-212.

99. Greene, B.E., Strome, D. R. and Weikel, R.C., 1961, ‘Application of the stiffness method of analysis of shell structures’, Proceedings Aviation Conference of ASME, Los Angeles, C.A.

100. Gulati, S. T. and Essenberg, F., 1967, ‘Effects of anisotropy in axi-symmetric cylindrical shells’, Journal of Applied Mechanics, Vol. 34, pp. 650-666.

101. Günay, E., 1999, ‘Finite element analysis of laminated stiffened cylindrical shallow shell’, Applied Composite Materials, Vol. 6, pp. 381-395.

102. Gupta, A. K., Patel, B. P. and Nath, Y., 2012, ‘Continuum damage mechanics approach to composite laminated shallow cylindrical/conical panels under static loading’, Composite Structures, Vol. 94, No. 5, pp. 1703-1713.

103. Gupta, A. K., Patel, B. P. and Nath, Y., 2013, ‘Nonlinear static analysis of composite laminated plates with evolving damage’, Acta Mechanica, Vol. 224, No. 6, pp. 1285-1298.

104. Gupta, A. K., Patel, B. P. and Nath, Y., 2015, ‘Progressive damage of laminated cylindrical/conical panels under meridional compression’, European Journal of Mechanics A/Solids, Vol. 53, pp. 329-341.

105. Han, S. C., Tabiei, A. and Park, W. T. 2008, ‘geometrically nonlinear analysis of laminated composite thin shells using a modified first order shear deformable element based Lagrangian shell element’, Composite Structures, Vol. 82, pp. 465-474.

106. Hashin, Z., 1980, ‘Failure criteria for unidirectional fiber composites’, Journal of Applied Mechanics, Vol. 47, pp. 329-334.

107. He, K., Hoa, S. V. and Ganesan, R., 2000, ‘The study of tapered laminated composite structures: a review’, Composites Science and Technology, Vol. 60, No. 14, pp. 2643-2657.

108. Hildebrand, F. B., 1949, ‘Notes on the foundations of the theory of small displacements of orthotropic shells,’ NACA – TN – 1833.

109. Hill, R., 1948, ‘A theory of the yielding and plastic flow of anisotropic metals’, Proceedings of the Royal Society London, Vol. 193, No. 1033, pp. 281-297.

110. Hoffman, O., 1967, ‘The brittle strength of orthotropic materials’, Journal of Composite Materials, Vol. 1, No. 2, pp. 200-206.

111. Hohe, J. and Librescu, J., 2003, ‘A nonlinear theory for doubly curved anisotropic sandwich shells with transversely compressible core’, International Journal of Solids and Structure, Vol. 40, pp. 1059-1088.

112. Hoppmann II, W. H., 1961, ‘Frequencies of vibration of shallow spherical shells’, Journal of Applied Mechanics, Transactions of ASME, Vol. 28, pp. 306-307.

113. Hu, H. T. and Tsai, J. Y., 1999a, ‘Maximization of the fundamental frequencies of laminated cylindrical shells with respect to fibre orientations’, Journal of Sound and Vibration, Vol. 225, No. 4, pp. 723-740.

114. Hwang, D. Y. and Foster Jr., W. A., 1992, ‘Analysis of axi-symmetric free vibration of isotropic shallow spherical shells with a circular hole’, Journal of Sound and Vibration, Vol. 157, No. 2, pp. 331-343.

115. Iyengar, K. T. S. and Srinivasan, R. S., 1968, ‘Bending analysis of hyperbolic paraboloid shells’, The Structural Engineer, Vol. 46, No. 12, pp. 397-401.

116. Jing, Hung-Sying and Teng, Kuan-Goang, 1995, ‘Analysis of thick laminated anisotropic cylindrical shells using a refined shell theory’, International Journal of Solids and Structures, Vol. 32, No. 10, pp. 1459-1476.

117. Johnson, M. W., and Reissner, E., 1958, ‘On transverse vibrations of shallow spherical shells’, Quarterly of Applied Mathematics, Vol. 15, No. 4, pp. 365-380.

118. Kabir, H. R. H., 2002, ‘Application of linear shallow shell theory of Reissner to frequency response of thin cylindrical panels with arbitrary lamination’, Composite Structures, Vol. 56, No. 1, pp. 35-52.

119. Kalnins, A. and Naghdi, P. M., 1960, ‘Axi-symmetric vibrations of shallow elastic spherical shells’, Journal of Acoustic Society, America, Vol. 32, pp. 342-347.

120. Kam, T. Y. and Chang, E. S., 1997, ‘Reliability formulation for composite laminates subjected to first-ply failure’, Composite Structures, Vol. 38, No. l-4, pp. 441-452.

121. Kam, T. Y. and Jan, T. B., 1995, ‘First-ply failure analysis of laminated composite plates based on the layer-wise linear displacement theory’, Composite Structures, Vol. 32, pp. 583-591.

122. Kam, T. Y. and Lai, F. M., 1999, ‘Experimental and theoretical predictions of first-ply failure strength of laminated composite plates’, International Journal of Solids and Structures, Vol. 25, pp. 2379-2395.

123. Kam, T. Y. and Sher, H. F., 1995, ‘Nonlinear and first ply failure analyses of laminated composite cross – ply plates’, Journal of Composite Materials, Vol. 29, No. 4, pp. 463-482.

124. Kam, T. Y., Sher, H. F., Chao, T. N. and Chang, R. R., 1996, ‘predictions of deflection and first-ply failure load of thin laminated composite plates via the finite element approach’, International Journal of Solids and Structure, Vol. 33, No. 3, pp. 375-398.

125. Kandasamy S. and Singh A. V., 2006, ‘Free vibration analysis of skewed open circular cylindrical shells’, Journal of sound and Vibration, Vol. 290, No.3-5, pp.1100-1118.

126. Kant, T., Kumar, S. and Singh, U. P., 1994, ‘Shell dynamics with three-dimensional degenerate finite elements’, Computers and Structures, Vol. 50, No. 1, pp. 135-146.

127. Kapania, R. K. and Raciti, S., 1989a, ‘Recent advances in analysis of laminated beams and plates part I: shear effects and buckling’, AIAA Journal, Vol. 27, No. 7, pp. 923-934.

128. Kapania, R. K. and Raciti, S., 1989b, ‘Recent advances in analysis of laminated beams and plates part II: vibration and wave propagation’, AIAA Journal, Vol. 27, No. 7, pp. 935-946.

129. Kapania, R. K., 1989, ‘A review on the analysis of laminated shells’, Journal of Pressure Vessel Technology, Vol. 111, pp. 88-96.

130. Kelly, G. and Hallström, S., 2005, ‘Strength and failure mechanisms of composite laminates subject to localised transverse loading’, Composite Structures, Vol. 69, No. 3, pp. 301-314.

131. Khani, A., Abdalla, M. M., Gürdal, Z., Sinke, J., Buitenhuis, A. and Van Tooren, M. J. L., 2017, ‘Design, manufacturing and testing of a fibre steered panel with a large cut-out’, Composite Structures, Vol. 180, pp. 821-830.

132. Khatri, K. N. and Asnani, N. T., 1996, ‘Vibration and damping analysis of fibre reinforced composite material conical shells’, Journal of Sound and Vibration, Vol. 193, No. 3, pp. 581-595.

133. Kirchhoff, G., 1876, ‘Vor elesungen Uber Mathematische Physik’, Vol. 1, Mechanik.

134. Kistler, L. S. and Waas, A. M., 1999, ‘On the response of curved laminated panels subjected to transverse impact loads’, International Journal of Solids and Structures, Vol. 36, pp. 1311-1327.

135. Koiter, W. T., 1960, 'A consistent first-approximation in the general theory of thin elastic shells', Proc. Symposium on Theory of Thin Elastic Shells, IUTAM Delft, 24-28, Aug. 1953, North Holland Publishing Co., Amsterdam, pp. 12-33.

136. Korjakin, A., Rikards, R., Altenbach, H. and Chate, A., 1998, 'Analysis of free damped vibrations of laminated composite conical shells', Composite Structures, Vol. 41, pp. 39-47.

137. Korjakin, A., Rikards, R., Altenbach, H. and Chate, A., 2001, 'Free damped vibrations of sandwich shells of revolution', Journal of Sandwich Structures And Materials, Vol. 3, pp. 171-196.

138. Kremer, T. and Schürmann, H., 2008, 'Buckling of tension-loaded thin-walled composite plates with cut-outs', Composites Science and Technology, Vol. 68, pp. 90-97.

139. Kumar, A., Chakrabarti, A. and Ketkar, M., 2013, 'Analysis of laminated composite skew shells using higher order shear deformation theory', Latin American Journal of Solids and Structures, Vol. 10, pp. 891-919.

140. Kumar, Y. V. S. and Srivastava, A., 2003, 'First ply failure analysis of laminated stiffened plates', Composite Structures, Vol. 60, pp. 307-315.

141. Kumari, S. and Chakravorty, D., 2010, 'On the bending characteristics of damaged composite conoidal shells – a finite element approach', Journal of Reinforced Plastics and Composites, Vol. 29, No. 21, pp. 3287-3296.

142. Kumari, S. and Chakravorty, D., 2011, 'Bending of delaminated composite conoidal shells under uniformly distributed load', Journal of Engineering Mechanics – ASCE, Vol. 137, pp. 660-668.

143. Kundu, C. K. and Sinha, P. K., 2007, 'Post buckling analysis of laminated composite shells', Composite Structures, Vol. 78, pp. 316-324.

144. Lakshminarayana, H. V. and Viswanath, S., 1976, ‘Ahigh precision triangular laminated anisotropic cylindrical shell finite element’, Computers and Structures, Vol. 8, pp. 633-640.

145. Lal, A., Singh, B. N. and Patel, D., 2012, ‘Stochastic nonlinear failure analysis of laminated composite plates under compressive transverse loading’, Composite Structures, Vol. 94, No. 3, pp. 1211-1223.

146. Lame, G. and Clapeyron, B. P. E., 1833, ‘Memoire Sur I’equilibre interieur des corps solids’, Mem. Pres. Par. Div. Savants 4.

147. Laura, P. A. A., Avalos, D. R. and Larrondo, H. A., 1999, ‘Forced vibrations of simply supported anisotropic rectangular plates’, Journal of Sound and Vibration, Vol. 220, No. 1, pp. 178-185.

148. Lee, C. S., Kim, J. H., Kim, S. K., Ryu, D. M. and Lee, J. M., 2015, ‘Initial and progressive failure analyses for composite laminates using Puck failure criterion and damage-coupled finite element method’, Composite Structures, Vol. 121, pp. 406-419.

149. Lee, S. J. and Han, S. E., 2001, ‘Free-vibration analysis of plates and shells with a nine-node assumed natural degenerated shell element’, Journal of Sound and Vibration, Vol. 241, No. 4, pp. 605-633.

150. Lee, S.J., Reddy, J.N. and Abadi, F.S., 2006, ‘Nonlinear finite element analysis of laminated composite shells with actuating layers’, Finite Elements in Analysis and Design, Vol. 43, pp. 1-21.

151. Leissa, A. W., Lee, J. K. and Wang, A. J., 1981, ‘Vibrations of cantilevered shallow cylindrical shells of rectangular planform’, Journal of Sound and Vibration, Vol.78, No. 3, pp. 311-328.

152. Leissa, A. W., Lee, J. K. and Wang, A. J., 1983, 'Vibrations of cantilevered doubly-curved shallow shells', International Journal of Solids and Structures, Vol. 19, No. 5, pp. 411-424.

153. Li F. M., Kishimoto, K. and Huang, W. H., 2009, 'The calculations of natural frequencies and forced vibration responses of conical shell using the Rayleigh – Ritz method', Mechanics Research Communications, Vol. 36, pp. 595-602.

154. Liew, K. M. and Lim, C. W., 1994, 'Vibration of perforated doubly curved shallow shells with rounded corners', International Journal of Solids and Structures, Vol. 31, No. 11, pp. 1519-1536.

155. Liew, K. M., Hu, Y. G., Zhao, X. and Ng, T. Y., 2006, 'Dynamic stability analysis of composite laminated cylindrical shells via the mesh-free kp-Ritz method', Computer Methods in Applied Mechanics and Engineering, Vol. 196, No. 1-3, pp. 147-160.

156. Lim, C. W. and Liew, K. M., 1995, 'A higher order theory for vibration of shear deformable cylindrical shallow shells', International Journal of Mechanical Sciences, Vol. 37, No. 3, pp. 277-295.

157. Lim, C. W., Liew, K. M. and Kitipornchai, S., 1998, 'Vibration of cantilevered laminated composite shallow conical shells', International Journal of Solids and Structures, Vol.35, No.15, pp. 1695-1707.

158. Lin, S. C., Kam, T. Y. and Chu, K. H., 1998, 'Evaluation of buckling and first-ply failure probabilities of composite laminates', International Journal of Solids and Structures, Vol. 35, No. 13. pp. 1395-1410.

159. Liu, P. F. and Zheng, J. Y., 2010, 'Recent developments on damage modelling and finite element analysis for composite laminates: a review', Materials and Design, Vol. 31, pp. 3825-3834.

160. Love, A. E. H., 1888, ‘On the small free vibrations and deformations of thin elastic shells’, Phil. Trans. Royal Society, 179(A).

161. Maimí, P., Camanho, P. P., Mayugo, J. A. and Dávila, C. G., 2007, ‘A continuum damage model for composite laminates: Part I – constitutive model’, Mechanics of Materials, Vol. 39, No. 10, pp. 897-908.

162. Maimí, P., Camanho, P. P., Mayugo, J. A. and Turon, A., 2011, ‘Matrix cracking and delamination in laminated composites. Part I: Ply constitutive law, first ply failure and onset of delamination’, Mechanics of Materials, Vol. 43, pp. 169-185.

163. Marguerre, K., 1938, ‘Zurtheorie der gekrummten platte grossen formanderung’, Proceedings of Fifth International Congress of Applied Mechanics, pp. 93-101.

164. Matthias, D. H. and Kröplin, B., 2012, ‘Finite element implementation of Puck’s failure theory for fibre-reinforced composites under three-dimensional stress’, Journal of Composite Materials, Vol. 46, No. 19-20, pp. 2485-2513.

165. Mizusawa, T. and Kito, H., 1995, ‘Vibration of antisymmetric angle-ply laminated cylindrical panels by the spline finite strip method’, Computers and Structures, Vol. 56, No.4, pp. 589-604.

166. Mohd, S. and Dawe, D. J., 1993, ‘Finite strip vibration analysis of composite prismatic shell structures with diaphragm ends’, Computers and Structures, Vol. 49, No. 5, pp. 753-765.

167. Munro, J., 1961, ‘The linear analysis of thin shallow shells’, The Institute of Civil Engineering, Proceedings, Vol. 19, pp. 291-306.

168. Naghdi, P. M., 1963, ‘Foundations of elastic shell theory’, Progress in Solid Mechanics, Vol. IV, North Holland Publishing Co., Amsterdam.

169. Nali, P. and Carrera, E., 2012, ‘A numerical assessment on two-dimensional failure criteria for composite layered structures’, Composite – Part B, Vol. 43, pp. 280-289.

170. Nanda, N. and Bandyopadhyay, J. N., 2007, 'Nonlinear free vibration analysis of laminated composite cylindrical shells with cutouts', Journal of Reinforced Plastics and Composites, Vol. 26, No. 14, pp. 1413-1427.

171. Nanda, N. and Bandyopadhyay, J. N., 2008, 'Nonlinear transient response of laminated composite shells', Journal of Engineering Mechanics, Vol. 134, No. 11, pp. 983-990.

172. Nanda, N. and Bandyopadhyay, J. N., 2009, 'Geometrically nonlinear transient analysis of laminated composite shells using the finite element method', Journal of Sound and Vibration, Vol. 325, pp.174-185.

173. Nanda, N. and Pradyumna, S., 2011, 'Nonlinear dynamic response of laminated shells with imperfections in hygrothermal environments', Journal of Composite Materials, Vol. 45, No. 20, pp. 2103-2112.

174. Narita, Y. and Leissa, A. W., 1984, 'Vibrations of corner supported shallow shells of rectangular planform', Earthquake Engineering and Structural Dynamics, Vol. 12, pp. 651-661.

175. Nayak, A. N. and Bandyopadhyay, J. N., 2002, 'Free vibration analysis and design aids of stiffened conoidal shells', Journal of Engineering Mechanics, Vol. 128, No. 4, pp. 419-427.

176. Nayak, A. N. and Bandyopadhyay, J. N., 2005, 'Free vibration analysis of laminated stiffened shells, Journal of Engineering Mechanics, Vol. 131, No.1, pp. 100-105.

177. Nayak, A. N. and Bandyopadhyay, J. N., 2006, 'Dynamic response analysis of stiffened conoidal shells', Journal of Sound and Vibration, Vol. 291, No. 3-5, pp. 1288-1297.

178. Nazarov, A. A., 1949 (English Translation in NASA TN 1426, 1956), 'On the theory of thin shells', Prikl. Mat. Mek., Vol. 13, pp. 547-550.

179. Neogi, S. D., Karmakar, A. and Chakravorty, D., 2011, ‘Impact response of simply supported skewed hypar shell roofs by finite element’, Journal of Reinforced Plastics and Composites, Vol. 30, No. 21, pp. 1795-1805.

180. Noor, A.K. and Burton, W.S., 1989, ‘Assessment of shear deformation theories for multilayered composite plates’, Applied Mechanics Reviews, Vol. 42, No. 1, pp. 1-12.

181. Noor, A.K. and Burton, W.S., 1990, ‘Assessment of computational models for multi-layered composite shells’, Applied Mechanics Reviews, Vol. 43, No. 4, pp. 67-97.

182. Olson, M. D. and Lindberg, G. M., 1968, ‘Vibration analysis of cantilevered curved-plates using a new cylindrical shell finite element’, Proceedings 2nd Conference on Matrix Methods Structural Mechanics, AFFDL-TR-68-150, pp. 247-269.

183. Orifici, C. A., Shah, A. S., Herszberg, I., Kotler, A. and Weller, T., 2008, ‘Failure analysis in post buckled composite T-sections’, Composite Structures, Vol. 86, pp. 146-153.

184. Oterkus, E., Madenci, E., Weckner, O., Silling, S., Bogert, P. and Tessler, A., 2012, ‘Combined finite element and peridynamic analyses for predicting failure in a stiffened composite curved panel with a central slot’, Composite Structures, Vol. 94, pp. 839-850.

185. Palazotto, A. N. and Dennis, S. T., 1992, ‘Nonlinear analysis of shell structures’, AIAA Education Series, American Institute of Aeronautics and Astronautics (AIAA) Washington, DC, pp. 131-154.

186. Palazotto, A. N. and Dennis, S. T., 1992, ‘Nonlinear analysis of shell structures’, AIAA Education Series, American Institute of Aeronautics and Astronautics (AIAA) Washington, DC, pp. 195-232.

187. Parisch, H., 1979, ‘A critical survey of the 9 node degenerated shell element with special emphasis on thin shell application and reduced integration’, Computer Methods in Applied Mechanics and Engineering, Vol. 20, pp. 323-350.

188. Parme, A. L., 1956, ‘Hyperbolic paraboloid and other shells of double curvatures’, Journal of Structural Division, Proceedings of ASCE, Vol. 82, No. ST5, pp. 1057-1/1057- 32.

189. Piskunov, V. G., Verijenko, V. E., Adali, S. and Tabakov, P. Y., 1994, ‘Transverse shear and normal deformation higher-order theory for the solution of dynamic problems of laminated plates and shells’, Computers and Structures, Vol. 31, No. 24, pp. 3345-3374.

190. Poore, A. L., Barut, A. and Madenci, E., 2008, ‘Free vibration of laminated cylindrical shells with a circular cutout’, Journal of Sound and Vibration, Vol. 312, pp. 55-73.

191. Pradyumna, S. and Bandyopadhyay, J.N., 2008, ‘Static and free vibration analyses of laminated shells using a higher order theory’, Journal of Reinforced Plastics and Composites, Vol. 27, No. 2, pp. 167-186.

192. Pradyumna, S. and Nanda, N., 2013, ‘Geometrically nonlinear transient response of functionally graded shell panels with initial geometric imperfection’, Mechanics of Advanced Materials and Structures, Vol. 20, pp. 217-226.

193. Priyadharshani, S. A., Prasad A. M. and Sundaravadivelu R., 2017, ‘Analysis of GFRP stiffened composite plates with rectangular cutout’, Composite Structures, Vol. 169, pp. 42-51.

194. Prusty, B. G., Satsangi, S. K. and Ray, C., 2001a, ‘First ply failure analysis of laminated panels under transverse loading’, Journal of Reinforced Plastics and Composites, Vol. 20, No. 8, pp. 671-684.

195. Prusty, B. G., Satsangi, S. K. and Ray, C., 2001b, 'First ply failure analysis of stiffened panels - a finite element approach', Composite Structures, Vol. 51, pp. 73-81.

196. Puck, A. and Schürmann, H., 1998, 'Failure analysis of FRP laminates by means of physically based phenomenological models', Composite Science and Technology, Vol. 58, pp. 1045-1067.

197. Qatu, M. S. and Leissa, A. W., 1991a, 'Natural frequencies for cantilevered doubly-curved laminated composite shallow shells', Composite Structures, Vol. 17, pp. 227-255.

198. Qatu, M. S. and Leissa, A. W., 1991b, 'Vibration studies for laminated composite twisted cantilever plates', International Journal of Mechanical Sciences, Vol. 33, No. 11, pp. 927-940.

199. Qatu, M. S. and Leissa, A. W., 1993, 'Vibrations of shallow shells with two adjacent edges clamped and others free', Journal of Mechanics of Structural Machines, Vol. 21, No. 3, pp. 285-301.

200. Qatu, M. S., 1989, 'Free vibration and static analysis of laminated composite shallow shells', Ph.D. Dissertation, Ohio State University.

201. Qatu, M. S., 1992, 'Review of shallow shell vibration research', Shock and Vibration Digest, Vol. 24, No.9, pp. 3-15.

202. Qatu, M. S., 1996, 'Vibration analysis of cantilevered shallow shells with triangular and trapezoidal planforms', Journal of Sound and Vibration, Vol. 191, No. 2, pp. 219-231.

203. Qatu, M. S., 1999, 'Accurate equations for laminated composite deep thick shells', International Journal of Solids and Structures, Vol.36, pp. 2917-2941.

204. Qatu, M. S., 2002a, 'Recent research advances in the dynamic behaviour of shells: 1989-2000, Part1: laminated composite shells', ASME Applied Mechanics Reviews, Vol. 55, No. 4, pp. 325-349.

205. Qatu, M. S., 2002b, 'Recent research advances in the dynamic behaviour of shells: 1989-2000, Part2: homogeneous shells', ASME Applied Mechanics Reviews, Vol. 50, No. 5, pp. 415-434.

206. Ram, K. S. S and Babu, T. S., 2002, 'Free vibration of composite spherical shell cap with and without a cutout', Computers and Structures, Vol. 80, No. 23, pp. 1749-1756.

207. Ramaswamy, G. S. and Rao, K., May 1961, 'The membrane theory applied to hyperbolic paraboloid shells', Indian Concrete Journal, Vol. 35, pp. 156-171.

208. Ramaswamy, G. S., 1971, 'Design and construction of concrete shell roofs', Tata McGraw-Hill, New Delhi.

209. Ramtekkar, G. S., Desai, Y. M. and Shah, A. H., 2004, 'First ply failure of laminated composite plates – a mixed finite element approach', Journal of Reinforced Plastics and Composites, Vol. 23, No. 3, pp. 291-315.

210. Rao, K.P., 1978, 'A rectangular laminated anisotropic shallow thin shell finite element', Computer Methods in Applied Mechanics and Engineering, Vol. 15, pp. 13-33.

211. Ray, C. and Dey, M., 2008, 'Failure analysis of laminated composite plates under linearly varying temperature', Journal of Reinforced Plastics and Composites, Vol. 28, No. 1, pp. 99-107.

212. Reddy, A. R. K. and Palaninathan, R., 1999, 'Free vibration of skew laminates', Computers and Structures, Vol. 70, pp. 415-423.

213. Reddy, J. N. and Chao, W. C., 1981, 'Large deflection and large amplitude free vibrations of laminated composite material plates', Computers and Structures, Vol. 13, No. 1 – 3, pp. 341-347.

214. Reddy, J. N., 1984, 'Exact solutions of moderately thick laminated shells', Journal of Engineering Mechanics, Vol. 110, No. 5, pp. 794-809.

215. Reddy, J. N., 2004, 'Mechanics of laminated composite plates and shells: Theory and Analysis', CRC Press, Second Edition, Boca Raton, Florida.

216. Reddy, J. N., and Pandey, A. K., 1987, 'A first-ply failure analysis of composite laminates', Computers and Structures, Vol. 25, No. 3, pp. 371-393.

217. Reddy, J. N., December 1981, 'Finite element modeling of layered anisotropic composite plates and shells', Shock and Vibration Digest, Vol. 13, No. 12, pp. 3-12.

218. Reddy, Y. N. S., Moorthy, C. M. D. and Reddy, J. N., 1995, 'Nonlinear progressive failure analysis of laminated composite plates', International Journal of Nonlinear Mechanics, Vol. 30, No. 5, pp. 629-649.

219. Reddy, Y. S. N. and Reddy, J. N., 1992, 'Linear and nonlinear failure analysis of composite laminates with transverse shear', Composites Science and Technology, Vol. 44, pp. 227-255.

220. Reinoso, J. and Blázquez, A., 2016, 'Application and finite element implementation of 7-parameter shell element for geometrically nonlinear analysis of layered CFRP composites', Composite Structures, Vol. 139, pp. 263-276.

221. Reinoso, J., Catalanotti, G., Blázquez, A., Areias, P., Camanho, P. P. and París, F., 2017, 'A consistent anisotropic damage model for laminated fiber-reinforced composites using the 3D-version of the Puck failure criterion', International Journal of Solids and Structures, Vol. 126, pp. 37-53.

222. Reisman, H. and Culowski, P. M., 1968, 'Forced axi-symmetric motion of shallow spherical shells', Journal of Engineering Mechanics Division, Vol. 94, pp. 653-670.

223. Reissner, E., 1946, 'On vibrations of shallow spherical shells', Journal of Applied Physics, Vol. 17, pp. 1038-1042.

224. Reissner, E., 1955, 'On transverse vibrations of thin shallow elastic shells', Quarterly of Applied Mathematics, Vol. 13, pp. 169-176.

225. Ribeiro, P., 2008, ‘Forced large amplitude periodic vibrations of cylindrical shallow shells’, Finite Elements in Analysis and Design, Vol. 44, pp. 657-674.

226. Romanowicz, M., 2014, ‘Determination of the first ply failure load for a cross ply laminate subjected to uniaxial tension through computational micromechanics’, International Journal of Solids and Structures, Vol. 51, pp. 2549-2556.

227. Rougui, M., Moussaoui, F. and Benamar, R., 2007, ‘Geometrically nonlinear free and forced vibrations of simply supported circular cylindrical shells: a semi-analytical approach’, International Journal of Nonlinear Mechanics, Vol. 42, pp. 1102-1115.

228. Russel, R. R. and Gerstle, K. H., 1967, ‘Bending of hyperbolic paraboloid structures’, Journal of Structural Division, Proceedings of ASCE, Vol. 93, No. 3, pp. 181-199.

229. Russel, R. R. and Gerstle, K. H., 1968, ‘Hyperbolic paraboloid structures on four supports’, Journal of Structural Division, Proceedings of ASCE., Vol. 94, No. 4.

230. Sahoo, S. and Chakravorty, D., 2004, ‘Finite element bending behaviour of composite hyperbolic paraboloidal shells with various edge conditions’, Journal of Strain Analysis for Engineering Design, Vol. 39, No. 5, pp. 1-15.

231. Sahoo, S. and Chakravorty, D., 2005, ‘Finite element vibration characteristics of composite hypar shallow shells with various edge supports’, Journal of Vibration and Control, Vol. 11, No. 10, pp. 1291-1309.

232. Sahoo, S. and Chakravorty, D., 2006a, ‘Deflections, forces and moments of composite stiffened hypar shell roofs under concentrated load’, Journal of Strain Analysis for Engineering Design, Vol. 41, No. 1, pp. 81-97.

233. Sahoo, S. and Chakravorty, D., 2006b, ‘Stiffened composite hypar shell roofs under free vibration: Behaviour and optimization aids’, Journal of Sound and Vibration, Vol. 295, pp. 362-377.

234. Sahoo, S. and Chakravorty, D., 2007, 'Static bending of point supported composite hypar shell roofs', Journal of Structural Engineering, Vol. 34, No. 2, pp. 169-176.

235. Sahoo, S. and Chakravorty, D., 2008a, 'Bending of composite stiffened hypar shell roofs under point load', ASCE Journal of Engineering Mechanics, Vol. 134, No. 6, pp. 441-454.

236. Sahoo, S. and Chakravorty, D., 2008b, 'Free vibration characteristics of point supported hypar shells – a finite element approach', Advances in Vibration Engineering, Vol. 7, No. 2, pp. 197-205.

237. Sanders, J.L. (Jr.), 1959, 'An improved first approximation theory for thin shells, NASA TR-R24.

238. Sathyamoorthy, M., 1994, 'Vibrations of moderately thick shallow spherical shells at large amplitudes', Journal of Sound and Vibration, Vol. 172, No. 1, pp. 63-70.

239. Sathyamoorthy, M., 1995, 'Nonlinear vibrations of moderately thick orthotropic shallow spherical shells', Computers and Structures, Vol. 57, No. 1, pp. 59-65.

240. Schokker, A., Sridharan, S. and Kasagi, A., 1996, 'Dynamic buckling of composite shells', Computers and Structures, Vol. 59, No. 1, pp. 43-53.

241. Sciuva, M. D., Icardi, U. and Villani, M., 1998, 'Failure analysis of composite laminates under large deflection', Composite Structures, Vol. 40, No 3-4, pp. 239-255.

242. Sechler, E. E., 1974, 'The historical developments of shell research and design, thin shell structures, theory, experiments and design', Edited by Y.C. Fung and E. E. Sechler, Prentice-Hall Inc., Englewood Cliffs, New Jersey, pp. 3-25.

243. Shao, Z. S. and Ma, G. W., 2007, 'Free vibration analysis of laminated cylindrical shells by using Fourier series expansion method', Journal of Thermoplastic Composite Materials, Vol. 20, pp. 551-572.

244. Sheinman, I. and Reichman, Y., 1992, 'A study of buckling and vibration of laminated shallow curved panels', International Journal Solids and Structures, Vol. 29, No. 11, pp. 1329-1338.

245. Shin, D. K., 1997, 'Large amplitude free vibration behaviour of doubly curved shallow open shells with simply-supported edges', Computers and Structures, Vol. 62, No. 1, pp. 35-49.

246. Singh, S. B. and Kumar, A., 1998, 'Post-buckling response and failure of symmetric laminates under in-plane shear', Composites Science and Technology, Vol. 58, pp. 1949-1960.

247. Sivakumar, K., Iyengar, N. G. R. and Deb, K., 1999, 'Free vibration of laminated composite plates with cutout', Journal of Sound and Vibration, Vol. 221, No. 3, pp. 443-470.

248. Sivasubramonian, B., Kulkarni, A. M., Rao, G. V. and Krishnan, A., 1997, 'Free vibration of curved panels with cutouts', Journal of Sound and Vibration, Vol. 200, No. 2, pp. 227-234.

249. Sivasubramonian, B., Rao, G. V. and Krishnan, A., 1999, 'Free vibration of longitudinally stiffened curved panels with cutout', Journal of Sound and Vibration, Vol. 226, No. 1, pp. 41-55.

250. Soare, M., February 1966, 'A numerical approach to the bending theory of hyperbolic shells – 1 and 2', Indian Concrete Journal, pp. 63-69 and pp. 113-119.

251. Stavsky, Y., 1963, 'Thermoelasticity of heterogeneous aelotropic plates', Journal of Engineering Mechanics Division, ASCE, Vol. 89, No. 2, pp. 89-105.

252. Suzuki, K., Shikanai, G. and Leissa, A.W., 1996, 'Free vibrations of laminated composite non-circular thick cylindrical shells', International Journal of Solids and Structures, Vol. 33, No. 27, pp. 4079-4100.

253. Sze, K. Y., Liu, X. H. and Lo, S. H., 2004, 'Popular benchmark problems for geometric nonlinear analysis of shells', Finite Elements in Analysis and Design, Vol. 40, pp. 1551-1569.

254. Tan, D. Y., 1998, 'Free vibration analysis of shells of revolution', Journal of Sound and Vibration', Vol. 213, No. 1, pp. 15-33.

255. Tessler, A., Tsui, T. and Saether, E., 1995, 'A(1,2) – order theory for elasto dynamic analysis of thick orthotropic shells', Computers and Structures, Vol. 32, No. 22. pp. 3237-3260.

256. Toorani, M. H. and Lakis, A. A., 2000, 'General equations of anisotropic plates and shells including transverse shear deformations, rotary inertia and initial curvature effects', Journal of Sound and Vibration, Vol. 237, No. 4, pp. 561-615.

257. Touratier, M., 1992, 'A refined theory of laminated shallow shells', International Journal of Solids and Structures, Vol. 29, No. 11, pp. 1401-1415.

258. Tsai, C. T. and Palazotto, A. N., 1991, 'On the finite element analysis of non-linear vibration for cylindrical shells with high order shear deformation theory', International Journal of Non-linear Mechanics, Vol. 26, No. 3/4, pp. 379-388.

259. Tsai, S. W. and Wu, E. M., 1971, 'A general theory of strength for anisotropic materials', Journal of Composite Materials, Vol. 5, pp. 58-80.

260. Turkmen, H. S., 2002, 'Structural response of laminated composite shells subjected to blast loading: comparison of experimental and theoretical methods', Journal of Sound and Vibration, Vol. 249, No. 4, pp. 663-678.

261. Turvey, G. J., 1980, 'An initial flexural failure analysis of symmetrically laminated cross-ply rectangular plates', International Journal of Solids and Structure, Vol. 16, pp. 451-463.

262. Van, H. N., Hoai, N. N., Dinh, T. C. and Thoi, T. N., 2014, ‘Geometrically nonlinear analysis of composite plates and shells via a quadrilateral element with good coarse-mesh accuracy’, Composite Structures, Vol. 112, pp. 327-338.

263. Vinson, J. R. and Sierakowski, R. L., 1986, ‘The behaviour of structures composed of composite materials’, Nijhoff, The Hague, Springer Netherlands.

264. Vlasov, V. Z., 1947, ‘Membrane theory of thin shells formed by second order surfaces’, Prikl. Mat. Mekh., Akademiya Nauk., S.S.S.R., XI, 4.

265. Vlasov, V. Z., 1958, ‘Allgemeine Schalentheorie und Ihre Anwendung in der Technik’, Akademie – Verlag GmbH Berlin.

266. Vlasov, V. Z., 1964, ‘General theory of shells and its application in engineering’, English translation of 1949 Russian Edition, NASA, USA, No. N64-19883.

267. Wagner, W. and Balzani, C., 2010, ‘Prediction of the post buckling response of composite airframe panels including ply failure’, Engineering Fracture Mechanics, Vol. 77, No. 18, pp. 3648-3657.

268. Wang, C. T., 1953, ‘Applied elasticity’, McGraw-Hill Book Company, New York.

269. Wang, J. and Schweizerhof, K., 1997, ‘Free vibration of laminated anisotropic shallow shells including transverse shear deformation by the boundary-domain element method’, Computers and Structures, Vol. 62, No. 1, pp. 151-156.

270. Whitney, J. M. and Sun, C. T., 1973, ‘A higher order theory for extensional motion of laminated isotropic shells and plates’, Journal of Sound and Vibration, Vol. 30, pp. 85.

271. Whitney, J. M. and Sun, C. T., 1974, ‘A refined theory for laminated anisotropic cylindrical shells’, Journal of Applied Mechanics, Vol. 41, pp. 471-476.

272. Widera, G. E. O. and Chung, S. W., 1970, ‘A theory for non-homogeneous isotropic cylindrical shells’, Journal of Applied Mathematics and Physics (JAMP), Vol. 21, pp. 378-399,.

273. Widera, G. E. O. and Logan, D. L., 1980, ‘Refined theories for non-homogeneous anisotropic cylindrical shells: Part I – Derivation’, Journal of Engineering Mechanics Division, ASCE, Vol. 106, No. 6, pp. 1053-1074.

274. Woo, J. and Meguid, S. A., 2001, ‘Nonlinear analysis of functionally graded plates and shallow shells’, International Journal of Solids and Structure, Vol. 38, pp. 7409-7421.

275. Xiang-Sheng, C., 1985, ‘Forced vibrations of elastic shallow shells due to moving loads’, Applied Mathematics and Mechanics, Vol. 6, No. 3, pp. 233-240.

276. Xiao-ping, S., 1996, ‘An improved simple higher-order theory for laminated composite shells’, Computers and Structures, Vol. 60, No. 3, pp. 343-350.

277. Xue, J., Ding, Y., Han, F. and Liu, R., 2013, ‘An extension of Kármán- Donnell’s theory for non-shallow, long cylindrical shells undergoing large deflection’, Composite Structures, Vol. 112, pp. 327-338.

278. Yadav, D. and Verma, N., 2001, ‘Free vibration of composite circular cylindrical shells with random material properties. Part II: Applications’, Composite Structures, Vol. 51, pp. 371-380.

279. Yang, H. T. Y., Saigal, S., Masud, A. and Kapania, R. K., 2000, ‘A survey of recent shell finite elements’, International Journal for Numerical Methods in Engineering, Vol. 47, pp. 101-127.

280. Ye, Jianqiao and Soldatos, K. P., 1995, ‘Three dimensional buckling analysis of laminated composite hollow cylinders and cylindrical panels’, International Journal of Solids and Structures, Vol. 32, No. 13, pp. 1949-1962.

281. Ye, Z. and Han, R. P. S., 1995, ‘On the nonlinear analysis of orthotropic shallow shells of revolution’, Computers and Structures, Vol. 55, No. 2, pp. 325-331.

282. Zahari, R. and El-Zafrany, A., 2006, ‘Progressive damage analysis of composite layered plates using a mesh reduction method’, International Journal of Engineering and Technology, Vol. 3, No. 1, pp. 21-36.

283. Zhang, X. M., 2001, ‘Vibration analysis of cross-ply laminated composite cylindrical shells using the wave propagation approach’, Applied Acoustics, Vol. 62, pp. 1221-1228.

284. Zhao, X. and Liew, K. M., 2009, ‘geometrically nonlinear analysis of functionally graded shells’, International Journal of Mechanical Sciences, Vol. 51, pp. 131-144.

285. Zienkiewicz, O. C., 1971, ‘The Finite Element Method’, McGraw-Hill, Third Edition, London, 1971.

286. Zukas, J. A. and Vinson, J. R., 1971, ‘Laminated transversely isotropic cylindrical shells’, Journal of Applied Mechanics, Vol. 38, pp. 400-407.