BACHELOR OF SCIENCE EXAMINATION, 2018

(Final Year, 2nd Semester)

PHYSICS (HONOURS)

Quantum Mechanics - II

Paper : HO - 12

Time : Two hours **Full Marks : 50**

Use separate Answer Script for each group

GROUP - A (25 marks)

Answer any *four* questions.

(1). Consider the following 2×2 matrix given by:

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Find its eigenvectors and eigenvalues, and hence diagonalize it.

(2). Consider the stationary states of a particle in an infinitely deep square (potential) well (of width $2L$, and stretching from $x = -L$ to $x = L$) and let the initial ($t = 0$) wave-function of the particle be given by:

$$\Psi(x, t = 0) = A\,[u_0(x) + u_1(x)]$$

where u_0 and u_1 are the normalized ground state and the first excited states of that particle, Obain the wave-function $\Psi(x, t)$ for any later time $t > 0$ and show that the probability $|\Psi(x,t)|^2$ is an oscillating function of time.

(3). In the operator approach to the $1d$- quantum harmonic oscillator, the n^{th} excited state is *created* from its ground state $|0>$ by the prescription :

$$|n> \;=\; \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0>

Show that the states thus generated are orthonormal, ie $< m|n > = \delta_{mn}$ (Here the symbols have their usual meaning and the second quantization operators obeys the commutation relation: $[a, a^{\dagger}] = 1$

(4). What do you understand by the *Parity* of a State? What are the properties of the Parity operator P in quantum mechanics? If P commutes with the Hamiltonian show that the eigen-states of the system must also have definite parity. Are the angular monentum operators parity invariant?

(5). At a given time, a particle is in a state $|\psi_o>$, given by

$$|\psi_0> = \left(\frac{1}{\pi a^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2a^2}}$$

A measurement of its momentum is made. What is the probability of getting an answer within a small range $\Delta p = \frac{\hbar}{100a}$ and centred around the value $\hbar/a$?

(6). Define an Hermitian operator, and show that its eigenvalues are always real. Starting from the fundamental commutation relation: $[\hat{x}, \hat{p_x}] = i\hbar$, calculate the value of $[\hat{x}, \hat{p_x}^N]$, where N is an integer.

(7). Starting from the time-dependent Schrödinger equation, obtain the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

where $\rho = \int d^3x\, |\Psi(x,t)|^2$ and the current density $\vec{j}$ have their usual meanings. In the light of this equation of continuity, discuss the interpretation of $|\Psi|^2$ in quantum mechanics.

Ref. No.: EX/B.SC/PHY/32/H12/GR-B/79/2018

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(Final year, 2nd Semester)

PHYSICS

H 12

Use separate answer-script for this group.

Group B

(Statistical Mechanics - II)

Full Marks: 25

Answer any five questions.

1. In the grand canonical ensemble theory, (a) derive the expression of probability ($P_{ss'}$) of the system having energy E_s and particle number $N_{s'}$ when it is in equilibrium with the reservoir. (b) Show that entropy can be expressed as $S = -k_B \sum_{ss'} P_{ss'} \ln P_{ss'}$. [3+2=5]

2. Let the mixed state density matrix, ρ_{mix}, of a mixed state be defined as a weighted sum of ρ_i, like $\rho_{\text{mix}} = \sum_i p_i \rho_i$, where ρ_i be a pure state density matrix of a pure state vector, $|\psi_i\rangle$ and $\sum_i p_i = 1$. (a) Hence, show that $\text{Tr}\,(\rho^2_{\text{mix}}) \leq 1$. (b) Prove that the mixed state density matrix, ρ_{mix}, satisfies the von Neumann equation. [3+2=5]

3. (a) By using the expression of grand partition function,

$$Q = \sum_{N=0}^{\infty} z^N \sum_{\{n_\epsilon\}} e^{-\beta \sum_\epsilon \epsilon n_\epsilon}, \quad \left(\text{where} \quad \sum_\epsilon n_\epsilon = N\right),$$

for ideal gases or by any other means show that the expression of mean occupation number $\bar{n}_\epsilon$ of energy level ϵ is

$$\bar{n}_\epsilon = \frac{1}{z^{-1} e^{\beta \epsilon} + a}.$$

(b) Derive the expression of relative mean square fluctuation:

$$\frac{\overline{n^2_\epsilon} - \bar{n}^2_\epsilon}{\overline{n^2_\epsilon}} = \frac{1}{\bar{n}_\epsilon} - a.$$

Explain the significance of two terms in the right hand side. [2+(2+1)=5]

4. (a) By assuming that the chemical reaction occurs at constant temperature, T and pressure, P and by minimizing the total Gibbs's potential due to the constituent reactants derive the condition of chemical equilibrium. (b) Derive the law of mass action to achieve the equilibrium of chemical reaction in gaseous mixture where the grand partition function for the m-component mixture is:

$$Q = \prod_i^m Q_i, \; Q_i = e^{(Z_i(1) e^{\beta \mu_i})}, \; Z_i(1) = \frac{V}{\lambda_i^3} Z_i^{\text{int}}(1).$$

Symbols have the usual meanings. [2+3=5]

5. (a) Show that the total partition function of a single hydrogen atom is $Z^{\mathrm{H}}(1) = \frac{V}{\lambda_{\mathrm{H}}^3} e^{\beta R}$, where R is the Rydberg constant. (b) Consider the ionized hydrogen gas at high enough temperature where ionization and recombination lead to the equilibrium governed by $\mathrm{H} \rightleftharpoons \mathrm{p}^+ + \mathrm{e}^-$. Derive the Saha equation in the equilibrium,

$$\frac{n_{\mathrm{p}}\, n_{\mathrm{e}}}{n_{\mathrm{H}}} = \frac{e^{-\beta R}}{\lambda_{\mathrm{e}}^3}.$$

Symbols have their usual meanings. [2+3=5]

6. By using the following relationships valid for the ideal Bose gas

$$\frac{PV}{k_{\mathrm{B}}T} = -\sum_{\epsilon} \ln\left(1 - z\, e^{-\beta\epsilon}\right) \quad \text{and} \quad N = \sum_{\epsilon} \frac{1}{z^{-1}\, e^{\beta\epsilon} - 1},$$

where ϵ is the single particle energy, $z = e^{\mu/k_{\mathrm{B}}T}$ and $\beta^{-1} = k_{\mathrm{B}}T$, show that when the volume V is very large, the above relationships can be expressed as,

$$\frac{P}{k_{\mathrm{B}}T} = \frac{g_{\frac{5}{2}}(z)}{\lambda^3} \quad \text{and} \quad \frac{N - N_0}{V} = \frac{g_{\frac{3}{2}}(z)}{\lambda^3},$$

where $\lambda = h/\sqrt{2\pi m k_{\mathrm{B}}T}$ and the Bose-Einstein function, $g_{\nu}(z)$ is defined as

$$g_{\nu}(z) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} \frac{x^{\nu-1}\, dx}{z^{-1}\, e^{x} - 1}.$$

Other symbols have the usual meanings. [3+2=5]

7. For the ideal Fermi gas we know that $N/V = g f_{3/2}(z)/\lambda^3$ and $E = \frac{3}{2} N k_{\mathrm{B}} T f_{5/2}(z)/f_{3/2}(z)$. By using the following asymptotic expansions of $f_{\nu}(z)$ valid at low temperature:

$$f_{5/2}(z) = \frac{8(\ln z)^{5/2}}{15\sqrt{\pi}} \left[1 + \frac{5}{8}\pi^2 (\ln z)^{-2} + \cdots\right], \quad f_{3/2}(z) = \frac{4(\ln z)^{3/2}}{3\sqrt{\pi}} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots\right],$$

show that (a) $k_{\mathrm{B}}T \ln z = \epsilon_{\mathrm{F}}\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_{\mathrm{F}}}\right)^2\right)$, (b) $\frac{C_V}{N k_{\mathrm{B}}} = \frac{\pi^2}{2}\frac{T}{T_{\mathrm{F}}}$, where $\epsilon_{\mathrm{F}} = \left(\frac{3N}{4\pi g V}\right)^{2/3}\left(\frac{h^2}{2m}\right) = k_{\mathrm{B}} T_{\mathrm{F}}$. Symbols have their usual meanings. [3+2=5]

8. Derive the expressions of both $\langle \sigma_z \rangle$ and $\langle \sigma_x \rangle$ for a spin-$\frac{1}{2}$ particle in a magnetic field along the z direction, *i. e.*, $\vec{B} = B\,\hat{z}$. [4+1=5]