Master of Science Examination, 2018

(2nd Year, 1st semester)

Physics, Dynamical Systems

Paper - PHY/TE/202

Time - Two hours

Full Marks : 40

(20 marks for each group)

Answer any **four** questions from group **A**

(Use separate answer script for each group)

Group A

1. The two dimensional linear system is described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Analyze the equation and identify the type of fixed points depending on the determinant ($\Delta = \det(\mathrm{A})$) and trace ($\tau = \mathrm{trace}(\mathrm{A})$). (5)

2. (a) Define a limit cycle.

(b) The motion of a van der Pol oscillator is described by,

$$\frac{d^2x}{dt^2} + \mu\,(x^2 - 1)\frac{dx}{dt} + x = 0\,, \quad \mu > 0.$$

Argue the possibility of existence of a limit cycle and its nature.

(c) If in the above equation, we allow $\mu < 0$, what changes in the phase diagram are expected ? Can it represent a viable mechanical or electrical system ? (1+2+2)

3. Analyse the nature of bifurcation, and identify the bifurcation point $\mu = \mu_c$, for the following two dimensional dynamical system,

(a) $$\frac{dr}{dt} = \mu\, r - r^3\,, \quad \frac{d\theta}{dt} = \omega + b\, r^2.$$

(b) $\dfrac{dr}{dt} = r\,(4 - r^2)\,, \qquad \dfrac{d\theta}{dt} = \mu - \sin(\theta).$ (2 + 3)

4. Show that the following two dimensional dynamical system undergoes two successive bifurcations as μ changes from positive values to sufficiently negative values.

$$\frac{dr}{dt} = \mu\, r + r^3 - r^5 \qquad \frac{d\theta}{dt} = \omega + b\, r^2.$$

Find at what values of μ a bifurcation occur, and identify the nature of bifurcation. (2+3)

5. Lorenz equations are given by,

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= xy - bz, \end{aligned}$$

where the parameters σ, r, and b are all positive. Find the fixed points. Show that a volume in phase space contracts under the flow (time evolution) given above. (2+3)

6. (a) Give a pictorial representation of a cantor set, and find its similarity dimension.

   (b) Describe a von-Koch curve with diagrams. Find its similarity dimension. (1+1+1+2)

7. Find the fixed point(s) and their stability for the one dimensional map

$$x_{n+1} = x_n^2.$$

Consider a iterative map,

$$x_{n+1} = \cos(x_n).$$

Discuss the behavious of the solution as $n \to \infty$. (3+2)

GROUP - B

Answer any TWO questions

1. (a) Write the statement of the Poincare-Bendixson theorem.

   (b) Now consider the system

$$\begin{aligned}\dot{x} &= 1-(b+1)x+ax^2y\\ \dot{y} &= bx-ax^2y\end{aligned}$$

where $a, b > 0$ and $x, y \geq 0$. Solve the system for the fixed points and identify them by drawing the nullclines. Justify by drawing flow-lines how Poincare-Bendixson theorem ascertains that there will be a limit cycle for this system. Draw a relevant graph in the (a, b) parameter space that serves as a borderline between a limit cycle zone and a fixed-point zone. 3+7

2. (a) Consider the dynamical system

$$\begin{aligned}\dot{x} &= \mu\left[y-\left(\frac{x^3}{3}-x\right)\right]\\ \dot{y} &= -\frac{{\omega_0}^2}{\mu}x.\end{aligned}$$

Show that by appropriate change of variables this system reduces to the Van der Pol equation.

   (b) Treating $\mu >> 1$, draw the phase plot and explain that the system falls on a limit cycle with very slow and very fast motions periodically repeating in every cycle.

   (c) Find the period of the limit cycle approximately. 2+5+3

3. (a) Explain the kind of bifurcations you obtain in the system

$$\ddot{x} - \alpha x + \gamma \dot{x} + x^3 = 0$$

when $\alpha = 0$, for the cases

i. $\gamma > 0$, and α going from negative to positive values,

ii. $\gamma < 0$, and α going from negative to positive values.

(b) Apply Poincare-Bendixson theorem to the following dynamical system to identify some annular zone in the phase space where a stable (or unstable) limit cycle is present. You have to pin down that zone and justify whether there is a stable limit cycle or an unstable one within that annulus:–

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + \frac{3}{2}y^2) \\ \dot{y} &= x + y - y(x^2 + \frac{1}{2}y^2). \end{aligned}$$

Marks: (3 + 3) + 4 = 10